ADVANCES IN MATHEMATICS **117**, 294–318 (1996)
ARTICLE NO. 0014

A Basis for the Homology of the d-Divisible Partition Lattice

MICHELLE L. WACHS*

Department of Mathematics, University of Miami, Coral Gables, Florida 33124

Received March 9, 1992

INTRODUCTION

A partition of a set in which each block size is divisible by some fixed positive integer d shall be called a *d-divisible* partition. For n a multiple of d, let Π_n^d be the lattice of d-divisible partitions of the set $\{1, 2, ..., n\}$ ordered by refinement, with a bottom element $\hat{0}$ adjoined when $d > 1$. Clearly Π_n^d is a join sublattice of $\Pi_n = \Pi_n^1$, the lattice of partitions of $\{1, 2, ..., n\}$.

It is well-known that Π_n is a geometric lattice. Björner [Bj1] showed that geometric lattices are shellable, and hence their order complexes have the homotopy type of a wedge of spheres. It had been proved earlier by Folkman [F] that the order complex of a geometric lattice has vanishing simplicial homology in all dimensions except the top dimension. In the top dimension, the homology group is free of rank $|\mu(L)|$ where μ denotes the Möbius function. Björner [Bj2, Bj3] developed a general theory for constructing a basis for the top homology group of a geometric lattice in terms of its NBC bases. This theory yields a natural explicit basis for the top homology group of Π_n, which was used by Barcelo [Ba] to study the action of the symmetric group $\mathscr{S}_n$ on the top homology group.

For $d > 1$, Π_n^d is not a geometric lattice. Hence, one cannot use the machinery of geometric lattice theory as developed in [Bj1], [Bj2] and [Bj3] to study the structure of $\Delta(\Pi_n^d)$, the order complex of Π_n^d. In 1983, in response to a question posed by Hanlon, the author showed that Π_n^d is shellable for all $d > 1$ (cf. [Sa]). Stanley [St1] had previously shown that $|\mu(\Pi_n^d)| = a_n^d$, where a_n^d denotes the number of permutations in $\mathscr{S}_{n-1}$ with descent set $\{d, 2d, ..., n-d\}$. Hence all homology groups of $\Delta(\Pi_n^d)$ vanish except the top homology group $H(\Pi_n^d)$ which is free of rank a_n^d. This result was needed by Calderbank, Hanlon, and Robinson [CHR] in their study

* Research supported in part by National Science Foundation Grant DMS 9102760. E-mail address: wachs@math.miami.edu.

0001-8708/96 $12.00

of the action of the symmetric group on $H(\Pi_n^d)$. One of the results appearing in [CHR] is a proof of a conjecture of Stanley, relating the character of $\mathscr{S}_n$ acting on $H(\Pi_n^d)$ to a certain skew character.

In this paper, we construct an explicit natural basis for $H(\Pi_n^d)$. For $d>1$, each basis element turns out to be the fundamental cycle of the barycentric subdivision of the boundary of an $(n/d-1)$-dimensional cube. Moreover, these cycles correspond in a natural way to permutations in $\mathscr{S}_{n-1}$ with descent set $\{d, 2d, ..., n-d\}$. The basis constructed here yields a direct combinatorial derivation of the above mentioned result of Stanley, Calderbank, Hanlon, and Robinson. It also enables us to give a purely combinatorial construction of the matrices representing the action of $\mathscr{S}_n$ on $H(\Pi_n^d)$.

For $d=1$, we have a new basis for the partition lattice, one that is different from the Björner basis. Just as for the Björner basis, each basis element is the fundamental cycle of the barycentric subdivision of the boundary of an $(n-2)$-simplex. Moreover, our basis elements are directly indexed by permutations in $\mathscr{S}_{n-1}$. The new basis turns out to be the natural basis for combinatorially explaining a result of Stanley [St2] which states that the restriction of the action of $\mathscr{S}_n$ on $H(\Pi_n)$ to $\mathscr{S}_{n-1}$ is the regular representation.

In Section 1, we review some standard notation and terminology. Sections 2 through 5 deal with the $d>1$ case. In Section 2, we construct the basis. The action of $\mathscr{S}_{n-1}$ on $H(\Pi_n^d)$ is studied in Section 3. Representation matrices for the action of $\mathscr{S}_n$ are constructed in Section 4. In Section 5, we use the theory of lexicographical shellability to give a bijective proof of the fact that $|\mu(\Pi_n^d)|=a_n^d$. More precisely, we construct an EL-labeling of Π_n^d, identify its decreasing chains and show that these chains correspond in a natural way with permutations in $\mathscr{S}_{n-1}$ with descent set $\{d, 2d, ..., n-d\}$. These decreasing chains provide a natural explicit basis for the top portion of the Stanley-Reisner ring, or equivalently the top cohomology module, of Π_n^d. In Section 6, the $d=1$ case is discussed.

1. Preliminaries

We begin by reviewing some standard poset and simplicial complex terminology. Let P be a graded poset of rank r with top element $\hat{1}$ and bottom element $\hat{0}$. For $x \leqslant y$ in P, the interval $\{z \in P \mid x \leqslant z \leqslant y\}$ is denoted by $[x, y]$. We shall use the symbol $\rightarrow$ to denote the covering relation in P. That is, $x \rightarrow y$ means that y covers x in P. We shall also let $\mathscr{M}(P)$ denote the set of all maximal chains of P.

The *order complex* of P, denoted by $\Delta(P)$, is defined to be the simplicial complex whose vertices are the elements of $P-\{\hat{0}, \hat{1}\}$ and whose faces are the chains of $P-\{\hat{0},\hat{1}\}$. Since P is graded, $\Delta(P)$ is a pure $(r-2)$-dimensional complex.

Let $\mathbf{k}$ be a field or the ring of integers $\mathbf{Z}$. Let $C_i(P)$ denote the ith *chain group* of $\Delta(P)$, that is $C_i(P)$ is the free $\mathbf{k}$-module on the basis of i-chains $x_0<x_1<\cdots<x_i$ of $P-\{\hat{0},\hat{1}\}$. Let $H(P)$ denote the top reduced simplicial homology module $\tilde{H}_{r-2}(\Delta(P),\mathbf{k})$.

We shall view a permutation σ in the symmetric group $\mathcal{S}_n$ as a word $\sigma(1)\,\sigma(2)\cdots\sigma(n)$. The *identity* permutation $123\cdots n$ will be denoted by ε. Our convention in multiplying permutations is right to left composition. That is, for $\sigma,\tau\in\mathcal{S}_n$, and $i=1,2,...,n$, $\sigma\tau(i)=\sigma(\tau(i))$. We denote an adjacent transposition $(i,i+1)$ in $\mathcal{S}_n$ by s_i.

A *descent* of a permutation σ is a position i such that $1\leqslant i\leqslant n-1$ and $\sigma(i)>\sigma(i+1)$. We denote the set of descents of σ by $\operatorname{des}(\sigma)$. For example, $\operatorname{des}(324615)=\{1,4\}$.

If d divides n then we define a d-segment of $\sigma\in\mathcal{S}_n$ to be a subword of the form $\sigma(id+1)\sigma(id+2)\cdots\sigma(id+d)$, where $0\leqslant i\leqslant n/d-1$.

2. The $d>1$ Case

To *split* a permutation $\sigma\in\mathcal{S}_n$ at position j, means to divide σ into two subwords $\sigma(1)\,\sigma(2)\cdots\sigma(j)$ and $\sigma(j+1)\,\sigma(j+2)\cdots\sigma(n)$. To *switch and split* at position j, means to split the word σs_j at position j, i.e. to divide σ into the subwords $\sigma(1)\,\sigma(2)\cdots\sigma(j-1)\,\sigma(j+1)$ and $\sigma(j)\,\sigma(j+2)\cdots\sigma(n)$. More generally, applying a sequence of split operations and switch and split operations results in a collection of complementary subwords of σ. If we view these subwords as subsets then we obtain a partition of $\{1,2,...,n\}$. For example, if

$$\sigma=12345678$$

and we split at position 2 and switch and split at position 6 then we obtain the partition

$$\pi=12\mid 3457\mid 68.$$

If

$$\sigma=35271864$$

and we split at position 2 and switch and split at position 4 and 6, then we obtain the partition

$$\pi=35\mid 21\mid 76\mid 84.$$

We shall say that a partition $\pi \in \Pi_n^d$ is *d-compatible* with a permutation $\sigma \in \mathscr{S}_n$ if π is obtained from σ by applying split operations and switch and split operations *only* at d-divisible positions. In the examples given in the previous paragraph, π is 2-compatible with σ. For $\sigma \in \mathscr{S}_n$, let Π_σ denote the subposet of Π_n^d consisting of $\hat{0}$ and all partitions that are d-compatible with σ. For $d=2$ and $\sigma = 123456$, Π_σ is given in Fig. 2.1. It is not difficult to see that Π_σ is a familiar lattice.

THEOREM 2.1. *For each $\sigma \in \mathscr{S}_n$, Π_σ is isomorphic to the face lattice of an $(n/d-1)$-cube.*

Proof. Let L_n be the lattice of faces of the n-cube $\{(x_1, x_2, ..., x_n) \in \mathbf{R}^n \mid 0 \leqslant x_i \leqslant 1, i = 1, 2, ..., n\}$. The k dimensional faces, $k \geqslant 0$, can be viewed as n-tuples over $\{0, 1, *\}$ with k entries equal to $*$. We can also encode partitions in Π_σ as $(n/d-1)$-tuples over $\{0, 1, *\}$. Indeed, let $\phi\colon \Pi_\sigma \to L_{n/d-1}$ be the map defined by letting $\phi(\hat{0})$ be the empty face and $\phi(\pi)$ be the $(n/d-1)$-tuple whose jth entry is 0, 1, or $*$, according to whether π is obtained from σ by performing a split, a switch and split, or nothing at position jd. For example, if $d=2, \sigma = 12345678$ and $\pi = 1234 \mid 57 \mid 68$ then $\phi(\pi) = (*, 0, 1)$. It is easy to verify that ϕ is a lattice isomorphism.

A consequence of Theorem 2.1 is that the order complex $\Delta(\Pi_\sigma)$ is the barycentric subdivision of the boundary of an $(n/d-1)$-cube. Hence it is a triangulation of $(n/d-2)$-sphere. Let ρ_σ be a fundamental cycle of the spherical complex $\Delta(\Pi_\sigma)$. We now ask whether the collection of cycles

$$\{\rho_\sigma \mid \sigma \in \mathscr{S}_n\}$$

spans $H(\Pi_n^d)$. According to the following theorem, not only do these cycles span the space, but the cycles indexed by the permutations enumerated by a_n^d form a basis.

THEOREM 2.2. *Let*

$$A_n^d = \{\sigma \in \mathscr{S}_n \mid \sigma(n) = n, \quad \operatorname{des}(\sigma) = \{d, 2d, ..., n-d\}\}.$$

Then

$$\{\rho_\sigma \mid \sigma \in A_n^d\}$$

is a basis for $H(\Pi_n^d)$.

Proof. For each $\sigma \in \mathscr{S}_n$, let m_σ be the maximal chain of $\Pi_\sigma - \{\hat{0}, \hat{1}\}$,

$$\pi_{n/d}(\sigma) \to \pi_{n/d-1}(\sigma) \to \cdots \to \pi_2(\sigma), \tag{2.1}$$

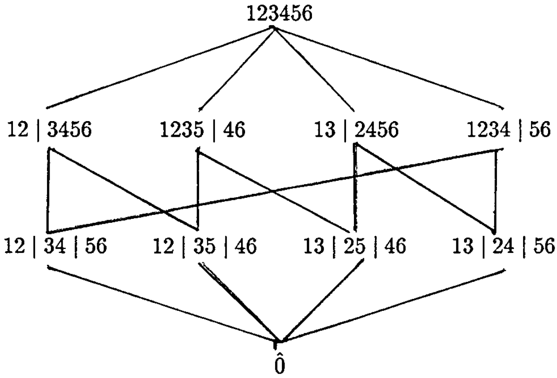

FIGURE 2.1

where $\pi_k(\sigma)$ is the k-block partition obtained by splitting σ at $d, 2d, ..., (k-1)\,d$. For example, if $n=8$, $d=2$ and $\sigma=12345678$ then m_σ is the maximal chain

$$12 \mid 34 \mid 56 \mid 78 \rightarrow 12 \mid 34 \mid 5678 \rightarrow 12 \mid 345678.$$

Linear independence of $\{\rho_\sigma \mid \sigma \in A_n^d\}$ is a consequence of Lemma 2.3 below. Indeed, let $\sum_{\sigma \in A^d{}_n} c_\sigma \rho_\sigma = 0$, where $c_\sigma \in \mathbf{k}$. Suppose, by way of contradiction, that α is the lexicographically last element of A_n^d for which $c_\alpha \neq 0$. Then by Lemma 2.3,

$$0 = \sum_{\sigma \in A_n^d} c_\sigma \rho_\sigma \mid_{m_\alpha} = \sum_{\substack{\sigma \in A_n^d \\ \sigma \leqslant \alpha}} c_\sigma \rho_\sigma \mid_{m_\alpha} = c_\alpha \rho_\alpha \mid_{m_\alpha} = \pm c_\alpha,$$

since in a fundamental cycle of a spherical complex the coefficient of every facet is ± 1.

Since, as was stated in the introduction, the rank of $H(\Pi_n^d)$ is $a_n^d = |A_n^d|$ (see also Corollary 5.3), we may conclude that $\{\rho_\sigma \mid \sigma \in A_n^d\}$ is a basis, when $\mathbf{k}$ is a field.

For $\mathbf{k} = \mathbf{Z}$, the linear independence of $\{\rho_\sigma \mid \sigma \in A_n^d\}$ implies that $\{\rho_\sigma \mid \sigma \in A_n^d\}$ is linearly independent over the rationals $\mathbf{Q}$ which implies that it spans $H(\Pi_n^d)$ over $\mathbf{Q}$. If $\rho \in H(\Pi_n^d)$, then

$$\rho \in \sum_{\sigma \in A_n^d} c_\sigma \rho_\sigma,$$

where $c_\sigma \in \mathbf{Q}$. We shall show that $c_\sigma \in \mathbf{Z}$ for all $\sigma \in A_n^d$. If α is the lexicographically last element of A_n^d for which $c_\alpha \neq 0$, then by Lemma 2.3 below,

$$\rho\,|_{m_\alpha} = \sum_{\substack{\sigma \in A_n^d \\ \sigma \leqslant \alpha}} c_\sigma \rho_\sigma\,|_{m_\alpha} = c_\alpha \rho_\alpha\,|_{m_\alpha} = \pm c_\alpha.$$

Since $\rho \in H(\Pi_n^d)$, we have $\rho|_{m_\alpha} \in \mathbf{Z}$. Hence $c_\alpha \in \mathbf{Z}$ and $\rho - c_\alpha \rho_\alpha \in H(\Pi_n^d)$. We can then apply the above argument to $\rho - c_\alpha \rho_\alpha$ to obtain $c_\beta \in \mathbf{Z}$ and $\rho - c_\alpha \rho_\alpha - c_\beta \rho_\beta \in H(\Pi_n^d)$, where β is the next to the last element, lexicographically, of A_n^d for which $c_\beta \neq 0$. Continuing this way allows us to conclude that $c_\sigma \in \mathbf{Z}$ for all $\sigma \in A_n^d$. Hence $\{\rho_\sigma \in \sigma \in A_n^d\}$ spans $H(\Pi_n^d)$ over $\mathbf{Z}$ and is therefore a basis for $H(\Pi_n^d)$ over $\mathbf{Z}$. ▮

LEMMA 2.3. *If $\alpha, \beta \in A_n^d$ and m_α is a chain in Π_β then $\alpha \leqslant \beta$ in lexicographical order.*

Lemma 2.3 is a consequence of two lemmas which are stated and proved below. For $\sigma \in \mathscr{S}_n$, let Λ_σ denote the subposet of Π_n^d consisting of partitions obtained from σ by splitting the word σ only at positions divisible by d. For example, if $\sigma = 123456$ and $d = 2$ then Λ_σ is the poset,

$$\begin{tikzcd} & 123456 & \\ 12 \mid 3456 \arrow[ur, dash] & & 1234 \mid 56 \arrow[ul, dash] \\ & 12 \mid 34 \mid 56. \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Note that Λ_σ is an atomic interval of Π_σ, i.e., an interval of the form $[a, \hat{1}]$, where a is an atom of Π_σ.

For each $\sigma \in \mathscr{S}_n$, let D_n^d be the subgroup of $\mathscr{S}_n$ generated by the adjacent transpositions $s_{di} = (di, di+1)$, $i = 1, 2, ..., n/d - 1$. Clearly, the generators s_{di} commute and each $\sigma \in D_n^d$ has a unique representation as

$$\sigma = s_{di_1} s_{di_2} \cdots s_{di_k}, \quad k \geqslant 0, \quad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant \frac{n}{d} - 1.$$

For example, if $d = 3$ and $n = 9$ then

$$\begin{aligned} D_n^d &= \{\varepsilon, s_3, s_6, s_3 s_6\} \\ &= \{123456789, 124356789, 123457689, 124357689\}. \end{aligned}$$

LEMMA 2.4. *Let* $\beta \in \mathcal{S}_n$. *Then* $\{\mathcal{M}(\Lambda_\gamma) \mid \gamma \in \beta D_n^d\}$ *is a partition of* $\mathcal{M}(\Pi_\beta - \{\hat{0}\})$.

Proof. Each permutation $\gamma \in \beta D_n^d$ corresponds bijectively to an atom $\bar{\gamma}$ of Π_β, namely $\bar{\gamma}$ is the partition obtained by splitting γ at all positions divisible by d. Moreover Λ_γ is the atomic interval $[\bar{\gamma}, \hat{1}]$ of Π_β. Since every maximal chain of $\Pi_\beta - \{\hat{0}\}$ is a maximal chain of a unique atomic interval, the result follows.

For $\sigma \in \mathcal{S}_n$, let $\sigma\uparrow$ denote the permutation obtained from σ by sorting each d-segment of σ, $\sigma(dj+1), \sigma(dj+2), ..., \sigma(dj+d), j=0, 1, 2, ...n/d-1$ in increasing order. For example, if $n=9, d=3$ and $\sigma=253941867$ and $\sigma\uparrow=235149678$.

LEMMA 2.5. *Let* $\alpha, \gamma \in \mathcal{S}_n$ *be such that* $\alpha(n)=\gamma(n)=n$. *If* m_α *is a chain of* Λ_γ *then* $\alpha\uparrow=\gamma\uparrow$.

Proof. We leave the straight forward proof to the reader. ∎

Proof of Lemma 2.3. By Lemma 2.4, $m_\alpha \cup \{\hat{1}\} \in \mathcal{M}(\Lambda_\gamma)$ for some $\gamma \in \beta D_n^d$. It follows from Lemma 2.5 that $\alpha\uparrow=\gamma\uparrow$. Since $\alpha \in A_n^d$, $\alpha=\alpha\uparrow$. Hence $\alpha=\gamma\uparrow$. Clearly $\gamma\uparrow \leqslant \gamma$. It is also easy to see that $\gamma \leqslant \beta$ since $\gamma \in \beta D_n^d$ and β has descents at all positions divisible by d. Putting this together we have

$$\alpha=\gamma\uparrow \leqslant \gamma \leqslant \beta. \quad ∎$$

3. THE ACTION OF THE SYMMETRIC GROUP

A permutation $\sigma \in \mathcal{S}_n$ acts on a partition in Π_n by replacing each element of each block by its image under σ. Clearly σ acts as an automorphism on the lattice Π_n^d. This induces an action on $H(\Pi_n^d)$ which turns $H(\Pi_n^d)$ into an $\mathcal{S}_n$-module over $\mathbf{k}$ (from now on assume $\mathbf{k}$ is a field). Moreover, by Theorem 2.2, $H(\Pi_n^d)$ is a cyclic $\mathcal{S}_n$-module generated by ρ_ε. Given ρ_ε, we can choose an orientation for each ρ_σ, $\sigma \in \mathcal{S}_n$, so that

$$\rho_\sigma = \sigma\rho_\varepsilon.$$

Then for $\sigma, \tau \in \mathcal{S}_n$, we have

$$\sigma\rho_\tau = \rho_{\sigma\tau}.$$

In [CHR], Calderbank, Hanlon, and Robinson obtain results on the character of $\mathcal{S}_n$ acting on $H(\Pi_n^d)$. One result, in particular, is the following theorem which was originally conjectured by Stanley.

THEOREM 3.1. *For $d > 1$, the restriction of the character of $\mathscr{S}_n$ acting on $H(\Pi_n^d)$ to $\mathscr{S}_{n-1}$ is the skew character indexed by the skew hook shape λ given in Fig.* 3.1.

The Calderbank, Hanlon, and Robinson proof involves symmetric function theory and cycle index manipulations. Here, we give a direct combinatorial proof by means of Specht modules (see [Ja] or [JK] for the definitions, terminology, and basic results relating to Specht modules). The Specht module indexed by λ is an $\mathscr{S}_{n-1}$-module and is denoted by S^λ. Its character is the skew character indexed by λ. Hence, Theorem 3.1 means that $H(\Pi_n^d)$ and S^λ are isomorphic $\mathscr{S}_{n-1}$-modules. We shall give an explicit isomorphism between these two modules.

We begin by reviewing the construction of the Specht module S^λ, where λ is the skew hook given in Figure 3.1. An equivalence relation is defined on tableaux of shape λ by $t_1 \sim t_2$ if t_2 can be obtained from t_1 by permuting the entries of each row of t_1. An equivalence class $[t]$ is called a *tabloid.* Let M^λ be the vector space over $\mathbf{k}$ whose basis elements are the λ-tabloids. Let $\sigma \in \mathscr{S}_{n-1}$ act on a λ-tableau by replacing each entry by its image under σ and let σ act an a λ-tabloid $[t]$ by $\sigma[t] = [\sigma t]$. This action of $\mathscr{S}_{n-1}$ on λ-tabloids turns M^λ into an $\mathscr{S}_{n-1}$-module.

Let t be a tableau of shape λ. The *polytabloid* e_t associated with t is defined by

$$e_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)[\sigma t],$$

where C_t is the column stabilizer of t. The action of $\sigma \in \mathscr{S}_{n-1}$ on polytabloids is given by $\sigma e_t = e_{\sigma t}$. This turns the space spanned by

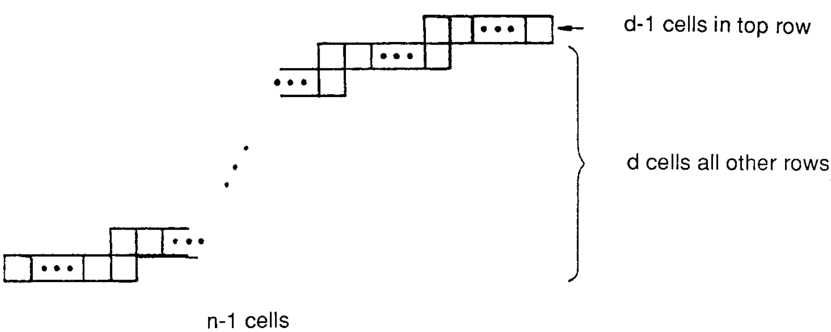

FIGURE 3.1

polytabloids into a submodule of M^λ called the *Specht module* and denoted by S^λ. The collection of standard polytabloids,

$$\{e_t \mid t \text{ is a standard tableau of shape } \lambda\},$$

forms a basis for S^λ.

Let t_0 be the tableau of shape λ obtained by filling in the cells from left to right (and bottom to top) with entries $1, 2, ..., n-1$ in the given order. For example, when $n=6$ and $d=2$,

$$t_0 = \begin{array}{ccc} & & 5 \\ & 3 & 4 \\ 1 & 2 & \end{array}$$

Clearly, each tableau t can be expressed as σt_0, where $\sigma \in \mathscr{S}_{n-1}$. Note that t is a standard tableau of shape λ if and only if $t = \sigma t_0$, where $\sigma \in A_n^d$. (We are viewing the permutation $\sigma \in \mathscr{S}_{n-1}$ as a permutation in $\mathscr{S}_n$ with $\sigma(n) = n$.) It follows that the standard polytabloid basis for S^λ can be expressed as

$$\{e_{\sigma t_0} \mid \sigma \in A_n^d\}.$$

This leads to an obvious bijection between the basis for S^λ and the basis for $H(\Pi_n^d)$, which extends by linearity to a vector space isomorphism between the two spaces. The following theorem asserts that this vector space isomorphism is in fact a module isomorphism.

THEOREM 3.2. *Let ψ: $S^\lambda \to H(\Pi_n^d)$ be the linear map defined on the basis of standard polytabloids by*

$$e_{\sigma t_0} \mapsto \rho_\sigma, \quad \sigma \in A_n^d.$$

Then ψ is an $\mathscr{S}_{n-1}$-module isomorphism.

This theorem is proved below by showing that ψ is the restriction of an isomorphism between M^λ and a module which contains $H(\Pi_n^d)$ as a submodule.

Recall the poset Λ_σ, defined in Section 2. It is not difficult to see that for each $\sigma \in \mathscr{S}_n$, Λ_σ is isomorphic to the face lattice of an $(n/d-2)$-simplex. Hence the order complex $\Delta(\Lambda_\sigma)$ is the barycentric subdivision of the boundary of the simplex. Let v_σ be a fundamental cycle of the spherical complex $\Delta(\Lambda_\sigma)$. Given v_ε, an orientation for each v_σ is determined by

$$v_\sigma = \sigma v_\varepsilon.$$

For each $\sigma \in \mathscr{S}_n$, let $\bar{\sigma}$ be the bottom element of Λ_σ, i.e., $\bar{\sigma}$ is the partition obtained by splitting σ at all positions divisible by d. Let $\bar{\sigma} * \Delta(\Lambda_\sigma)$ be the cone on $\Delta(\Lambda_\sigma)$ with vertex $\bar{\sigma}$. Note that $\bar{\sigma} * \Delta(\Lambda_\sigma)$ is simply the simplicial complex of chains of $\Lambda_\sigma - \{\hat{1}\}$. Each element v of the chain group $C_{n/d-3}(\Delta(\Lambda_\sigma))$ corresponds bijectively to an element $\bar{\sigma} * v$ of the chain group $C_{n/d-2}(\bar{\sigma} * \Delta(\Lambda_\sigma))$. That is, $\bar{\sigma} * v$ is obtained from v by adjoining $\bar{\sigma}$ to each chain involved in v. Let

$$\mu_\sigma = \bar{\sigma} * v_\sigma.$$

We clearly have $\mu_\sigma = \sigma\mu_\varepsilon$. Hence the vector space L_n^d spanned by $\{\mu_\sigma \mid \sigma \in \mathscr{S}_n\}$ is a cyclic $\mathscr{S}_n$-module.

LEMMA 3.3. *Let*

$$B_n^d = \{\sigma \in \mathscr{S}_n \mid \sigma(n) = n, \quad \mathrm{des}(\sigma) \subseteq \{d, 2d, ..., n-d\}\}.$$

Then

$$\{\mu_\sigma \mid \sigma \in B_n^d\}$$

is a basis for L_n^d.

Proof. We need to recall the chain m_σ defined by (2.1). If follows from Lemma 2.5 that for all $\alpha, \beta \in B_n^d$, if m_α is a chain in Λ_β then $\alpha = \beta$. The independence of $\{\mu_\sigma \mid \sigma \in B_n^d\}$ is a consequence of this.

To complete the proof, we compute the dimension of L_n^d. Let A be the set of atoms of Π_n^d. For each $a \in A$, let $L(a)$ be the linear span of

$$\{\mu_\sigma \mid \sigma \in \mathscr{S}_n, \bar{\sigma} = a\}.$$

Clearly, we can decompose L_n^d into a direct sum

$$L_n^d = \bigoplus_{a \in A} L(a). \tag{3.1}$$

We assert that $L(a)$ is isomorphic to a subspace of $H([a, \hat{1}])$. Indeed, each element of $L(a)$ is of the form $a * v$ where v is a linear combination of v_σ such that $\bar{\sigma} = a$. Since v_σ is a fundamental cycle of $\Delta(\Lambda_\sigma)$, v_σ is a cycle of $H([a, \hat{1}])$. Therefore v is a cycle of $H([a, \hat{1}])$. It follows that the map $a * v \mapsto v$ is an injective linear map from $L(a)$ to $H([a, \hat{1}])$, which proves the assertion.

Note that $[a, \hat{1}]$ is isomorphic to the partition lattice $\Pi_{n/d}$. Hence $H([a, \hat{1}])$ is isomorphic to $H(\Pi_{n/d})$. We use the well-known fact that

$|\mu(\Pi_n)| = (n-1)!$, to conclude that the dimension of $H([a, \hat{1}])$ is $(n/d-1)!$. Since $L(a)$ is isomorphic to a subspace of $H([a, \hat{1}])$, we have

$$\dim(L(a)) \leqslant \left(\frac{n}{d} - 1\right)!.$$

It now follows from (3.1) that

$$\dim(L_n^d) \leqslant |A| \left(\frac{n}{d} - 1\right)!. \tag{3.2}$$

For each atom $a \in A$ there are $(n/d-1)!$-permutations $\sigma \in B_n^d$ such that $\bar{\sigma} = a$. Indeed, σ can be obtained from a by writing the elements of each block of a in increasing order and then arranging these n/d increasing words so that the word containing n comes last. Clearly, there are $(n/d-1)!$ ways to do this. We thus have $|A|(n/d-1)! = |B_n^d|$ and (3-2) becomes $\dim(L_n^d) \leqslant |B_n^d|$. ∎

We remark there that all that will be needed for the proof of Theorem 3.2 is the linear independence of $\{\mu_\sigma \mid \sigma \in B_n^d\}$. However for the sake of completeness we showed that the set is actually a basis. Moreover, the dimension argument given in the proof of Lemma 3.3 shows that L_n^d decomposes into the direct sum $\bigoplus_{a \in A} a * H([a, \hat{1}])$, which implies that L_n^d is isomorphic to the direct sum of $|A|$ copies of $H(\Pi_{n/d})$.

The fact that $H(\Pi_n^d)$ is an $\mathscr{S}_n$-submodule of L_n^d is a consequence of the following lemma.

LEMMA 3.4. *$\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \mu_\alpha$ is a fundamental cycle of $\Delta(\Pi_\varepsilon)$. Consequently, we may orient ρ_ε so that*

$$\rho_\varepsilon = \sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \mu_\alpha. \tag{3.3}$$

Proof. By Lemma 2.4, every facet of $\Delta(\Pi_\varepsilon)$ has coefficient ± 1 in the sum. Hence, to show that the sum is a fundamental cycle we need only show that

$$\partial\left(\sum_{a \in D_n^d} \operatorname{sgn}(\alpha)\, \mu_\alpha\right) = 0. \tag{3.4}$$

Since ν_α is a fundamental cycle, we have (cf. [M, Chapter 1.8])

$$\partial(\mu_\alpha) = \nu_\alpha.$$

Hence (3.4) reduces to

$$\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, v_\alpha = 0. \tag{3.5}$$

Since $v_\alpha = \alpha v_\varepsilon$, (3.5) is equivalent to the assertion that the group algebra element

$$\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha$$

annihilates v_ε. To prove this, we claim that each facet f of $\Delta(\Lambda_\varepsilon)$ is stabilized by some adjacent transposition $s_{jd} = (jd, jd+1)$, where $1 \leqslant j \leqslant n/d - 1$. Indeed, the minimum element π of the chain f is the partition obtained by splitting ε at all positions divisible by d except for one position jd. Thus jd and $jd+1$ are in the same block of π. It follows that jd and $jd+1$ are in the same block of each partition in f, which proves our claim. Consequently, the group algebra element $\varepsilon - s_{jd}$ annihilates f. Since

$$\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha = \prod_{j=1}^{n/d-1} (\varepsilon - s_{jd})$$

and the factors $\varepsilon - s_{jd}$ commute, we have

$$\left(\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha \right) f = 0$$

for all facets f of $\Delta(\Lambda_\varepsilon)$. Since v_ε is a linear combination of facets in $\Delta(\Lambda_\varepsilon)$, we have proved that $\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha$ annihilates v_ε, thereby establishing that $\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \mu_\alpha$ is a fundamental cycle of $\Delta(\Pi_\varepsilon)$. ∎

Lemma 3.5. *Let $\Psi\colon M^\lambda \to L_n^d$ be the linear map defined on the basis of tabloids by*

$$[\sigma t_0] \mapsto \mu_\sigma, \quad \sigma \in B_n^d.$$

Then Ψ is an $\mathscr{S}_{n-1}$-module isomorphism.

Proof. Each λ-tabloid has a unique representation as $[\sigma t_0]$, where $\sigma \in B_n^d$. It follows from this and Lemma 3.3, that the map Ψ is a well-defined vector space isomorphism. To verify that Ψ respects the action of $\mathscr{S}_{n-1}$, we note that $\Psi([\sigma t_0]) = \mu_\sigma$ for all $\sigma \in \mathscr{S}_{n-1}$. This implies that

$$\Psi(\alpha[\sigma t_0]) = \Psi([\alpha\sigma t_0]) = \mu_{\sigma\alpha} = \alpha\mu_\sigma = \alpha\Psi([\sigma t_0]). \quad ∎$$

Proof of Theorem 3.2. We show that the restriction of the isomorphism Ψ of Lemma 3.5 to S^λ is ψ. Since S^λ is a cyclic submodule of M^λ generated by e_{t_0} and $H(\Pi_n^d)$ is a cyclic submodule of L_n^d generated by ρ_ε, we need only prove that $\Psi(e_{t_0}) = \rho_\varepsilon$. To this end, we have

$$\begin{aligned}\Psi(e_{t_0}) &= \Psi\left(\sum_{\sigma \in C_{t_0}} \operatorname{sgn}(\sigma)[\sigma t_0]\right)\\ &= \sum_{\sigma \in C_{t_0}} \operatorname{sgn}(\sigma)\, \mu_\sigma\\ &= \rho_\varepsilon,\end{aligned}$$

by Lemma 3.4, since $C_{t_0} = D_n^d$. ▮

4. The Representation Matrices

In this section we give an explicit purely combinatorial construction of the matrices representing the action of $\mathscr{S}_n$ on $H(\Pi_n^d)$ in much the same way as Garsia and Wachs [GW] did for the skew representations of $\mathscr{S}_n$. In fact, our matrices reduce to matrices of [GW] when the action is restricted to $\mathscr{S}_{n-1}$. This gives an alternative combinatorial proof of Theorems 3.1 and 3.2.

We begin by introducing some notation. For $\beta \in \mathscr{S}_{n/d}$, let

$$\tilde{\beta} = w_{\beta(1)} * w_{\beta(2)} * \cdots * w_{\beta(n/d)},$$

where $*$ denotes concatenation and w_i is the word $d(i-1)+1$, $d(i-1)+2, ..., di$. Let R_n be the set of permutations

$$\{\sigma \in \mathscr{S}_n \mid \forall i = 1, 2, ..., n, \sigma(i) \text{ is either smaller or greater than}$$
$$\text{all the letters that follow it}\}.$$

In other words R_n is the set of all permutations in $\mathscr{S}_n$ which avoid the patterns 213 and 231. For example,

$$R_3 = \{123, 132, 312, 321\}.$$

For $\sigma \in \mathscr{S}_n$, define $\operatorname{sgn}^*(\sigma)$ to be $\operatorname{sgn}(\sigma(1)\,\sigma(2)\cdots\sigma(n-1))$. We will also need to recall the operator $\uparrow$ whose definition precedes Lemma 2.5.

For $\sigma, \tau \in \mathscr{S}_n$, define

$$c(\sigma, \tau) = \begin{cases} \operatorname{sgn}(\alpha) \operatorname{sgn}^*(\beta), & \text{if} \quad \sigma\uparrow = (\tau\alpha\tilde{\beta}) \uparrow \text{ for some } \alpha \in D_n^d,\ \beta \in R_{n/d} \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

It is not difficult to see that α and β are uniquely determined by σ and τ. Hence $c(\sigma, \tau)$ is well-defined.

LEMMA 4.1 *Let $\sigma, \tau \in \mathscr{S}_n$ and $\sigma(n) = n = \tau(n)$. Then*

$$c(\sigma, \tau) = \begin{cases} \operatorname{sgn}(\alpha), & \text{if} \quad \sigma\uparrow = (\tau\alpha) \uparrow \textit{ for some } \alpha \in D_n^d \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Note that $\tilde{\beta}$ of (4.1) permutes the d-segments of $\tau\alpha$, i.e. the segments of the form $\tau\alpha(di+1)\, \tau\alpha(di+2) \cdots \tau\alpha(di+d)$. Since the last letter of both $\tau\alpha$ and σ is n, $\tilde{\beta}$ fixes the last d-segment of $\tau\alpha$. Hence $\beta(n/d) = n/d$. It follows from this and the fact that $\beta \in R_{n/d}$, that β is the identity of $\mathscr{S}_{n/d}$. Hence $\tilde{\beta}$ is the identity of $\mathscr{S}_n$.

We now order the permutations in A_n^d lexicographically and let σ_i be the ith permutation for $i = 1, 2, ..., a_n^d$. Then for any permutation $\sigma \in \mathscr{S}_n$, we let $C(\sigma)$ denote the matrix whose i, j-entry is given by

$$C_{ij}(\sigma) = c(\sigma_i, \sigma\sigma_j).$$

PROPOSITION 4.2. *The matrix $C(\varepsilon)$ is upper triangular with diagonal elements equal to* 1.

Proof. It is obvious that the diagonal elements are equal to 1. To show that $C(\varepsilon)$ is upper triangular, we need to show that if $\sigma, \tau \in A_n^d$ and $c(\sigma, \tau) \neq 0$ then $\sigma \leqslant \tau$ lexicographically. By Lemma 4.1, if $c(\sigma, \tau) \neq 0$ then $\sigma = (\tau\alpha)\uparrow$ for some $\alpha \in D_n^d$. Clearly $(\tau\alpha)\uparrow \leqslant \tau\alpha$. Also since τ has descents at all d-divisible positions and $\alpha \in D_n^d$, we have $\tau\alpha \leqslant \tau$. Combining the inequalities gives

$$\sigma = (\tau\alpha)\uparrow \leqslant \tau\alpha \leqslant \tau. \quad \blacksquare$$

THEOREM 4.3. *For $i = 1, 2, ..., a_n^d$, let ρ_i denote the basis element ρ_{σ_i} of $H(\Pi_n^d)$. Then for all $\sigma \in \mathscr{S}_n$,*

$$(\sigma(\rho_1), \sigma(\rho_2), ..., \sigma(\rho_{a_n^d})) = (\rho_1, \rho_2, ..., \rho_{a_n^d})\, C(\varepsilon)^{-1}\, C(\sigma).$$

In other words the matrix representing the action of $\sigma \in \mathscr{S}_n$ on $H(\Pi_n^d)$ is given by $C(\varepsilon)^{-1}\, C(\sigma)$.

Proof. Let $L(\{m_\sigma \mid \sigma \in \mathcal{S}_n\})$ be the submodule of the chain group $C_{n/d-2}(\Delta(\Pi_n^d))$ spanned by the chains m_σ defined in (2.1). Define

$$\gamma\colon C_{n/d-2}(\Delta(\Pi_n^d)) \to L(\{m_\sigma \mid \sigma \in \mathcal{S}_n\}),$$

to be the linear map which is the identity on chains in $\{m_\sigma \mid \sigma \in \mathcal{S}_n\}$ and 0 on all other facets of $\Delta(\Pi_n^d)$. Note that γ is an $\mathcal{S}_n$-module homomorphism.

Consider the restriction of γ to the submodule $H(\Pi_n^d)$, that is,

$$\gamma\colon H(\Pi_n^d) \to L(\{m_\sigma \mid \sigma \in \mathcal{S}_n\}).$$

Our goal is to compute $\gamma(\rho_\sigma)$, for all $\sigma \in \mathcal{S}_n$. We have

$$\begin{aligned} \gamma(\rho_\sigma) &= \sigma\gamma(\rho_\varepsilon) \\ &= \sigma\gamma\left(\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \mu_\alpha\right) \qquad \text{(by(3.3))} \\ &= \sigma\gamma\left(\sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha\mu_\varepsilon\right) \\ &= \sigma \sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha\gamma(\mu_\varepsilon). \end{aligned} \tag{4.2}$$

We now assert that

$$\gamma(\mu_\varepsilon) = \sum_{\substack{\beta \in R_{n/d} \\ \beta(n/d-1) < \beta(n/d)}} \operatorname{sgn}^*(\beta)\, m_{\tilde{\beta}}. \tag{4.3}$$

To prove this assertion we use the fact that Λ_ε is isomorphic to the lattice of subsets of the set $\{1, 2, ..., n/d-1\}$, which we denote by $\mathcal{B}(n/d-1)$. Indeed, if $S \subseteq \{1, 2, ..., n/d-1\}$, then a partition $\pi(S)$ in Λ_ε can be obtained by splitting ε at all positions di where $i \in S$. Clearly, $S \mapsto \pi(S)$ defines an isomorphism from $\mathcal{B}(n/d-1)$ to the dual of Λ_ε.

Next we use the well-known, easily verified fact that $\sum_{\sigma \in \mathcal{S}_m} \operatorname{sgn}(\sigma)\, c_\sigma$, is a fundamental cycle of the spherical complex $\Delta(\mathcal{B}(m))$, where c_σ is the maximal chain of $\mathcal{B}(m) - \{\hat{0}, \hat{1}\}$ given by

$$\{\sigma(1)\} \to \{\sigma(1), \sigma(2)\} \to \{\sigma(1), \sigma(2), \sigma(3)\} \to \cdots \to \{\sigma(1), \sigma(2), ..., \sigma(m-1)\}.$$

Since ν_ε is a fundamental cycle of $\Delta(\Lambda_\varepsilon)$, we can set $\nu_\varepsilon = \sum_{\sigma \in \mathscr{S}_{n/d-1}} \operatorname{sgn}(\sigma)\, \pi(c_\sigma)$. It follows that

$$\mu_\varepsilon = \sum_{\sigma \in \mathscr{S}_{n/d-1}} \operatorname{sgn}(\sigma)(\bar{\varepsilon} * \pi(c_\sigma)). \tag{4.4}$$

Note that $\bar{\varepsilon} * \pi(c_\sigma)$ is the chain in Λ_ε given by

$$\pi_{n/d-1} \to \pi_{n/d-2} \to \cdots \to \pi_1,$$

where π_i is obtained by splitting ε at positions $\sigma(1)\, d$, $\sigma(2)d$, ..., $\sigma(i)\, d$. It follows that $\bar{\varepsilon} * \pi(c_\sigma) = m_\alpha$ for some $\alpha \in \mathscr{S}_n$ if and only if $\sigma \in R_{n/d-1}$. Indeed, $\sigma \in R_{n/d-1}$ if and only if $\bar{\varepsilon} * \pi(c_\sigma)$ is the chain obtained by successively "slicing" off blocks of size d from the initial block $\{1, 2, ..., n\}$. In this case, $\alpha = \tilde{\beta}$ or $\alpha = \widetilde{\beta s_{n/d-1}}$, where for each $i = 1, 2, ..., n/d-2$,

$$\beta(i) = \begin{cases} \sigma(i), & \text{if} \quad \sigma(i) < \sigma(j) \quad \text{for all} \quad j > i, \\ \sigma(i)+1, & \text{if} \quad \sigma(i) > \sigma(j) \quad \text{for all} \quad j > i, \end{cases}$$

and

$$\beta\left(\frac{n}{d}-1\right) = \sigma\left(\frac{n}{d}-1\right)$$

$$\beta\left(\frac{n}{d}\right) = \sigma\left(\frac{n}{d}-1\right)+1.$$

Note that $\beta \in R_{n/d}$, $\beta(n/d-1) < \beta(n/d)$, $\operatorname{sgn}^*(\beta) = \operatorname{sgn}(\sigma)$ and $m_{\tilde{\beta}} = m_{\widetilde{\beta s_{n/d-1}}}$.

Hence, after applying γ to (4.4) we obtain (4.3) as asserted.

Substituting (4.3) into (4.2) yields

$$\begin{aligned} \gamma(\rho_\sigma) &= \sigma \sum_{\alpha \in D_n^d} \operatorname{sgn}(\alpha)\, \alpha \sum_{\substack{\beta \in R_{n/d} \\ \beta(n/d-1) < \beta(n/d)}} \operatorname{sgn}^*(\beta)\, m_{\tilde{\beta}} \\ &= \sum_{\substack{\alpha \in D_n^d \\ \beta \in R_{n/d}}} \operatorname{sgn}(\alpha) \operatorname{sgn}^*(\beta)\, m_{\sigma\alpha\tilde{\beta}}. \end{aligned} \tag{4.5}$$

We now define

$$\psi\colon L(\{m_\sigma \mid \sigma \in \mathscr{S}_n\}) \to L(\{m_\sigma \mid \sigma \in A_n^d\})$$

to be the linear map which is the identity on chains in $\{m_\sigma \mid \sigma \in A_n^d\}$ and 0 on all other chains in $\{m_\sigma \mid \sigma \in \mathcal{S}_n\}$. The composition

$$\gamma \circ \psi \colon H(\Pi_n^d) \to L(\{m_\sigma \mid \sigma \in A_n^d\})$$

is a vector space isomorphism since the dimensions of both spaces are equal and by Lemma 2.3 the composition is injective. Let m_i denote the chain m_{σ_i}, $i = 1, 2, ..., a_n^d$. To compute the coefficient of m_i in $\gamma \circ \psi(\rho_\sigma)$, first observe that $m_i = m_{\sigma\alpha\tilde{\beta}}$ if and only if $\sigma_i = (\sigma\alpha\tilde{\beta})\uparrow$ or $(\sigma\alpha\,\widetilde{\beta s_{n/d-1}})\uparrow$. Hence, the coefficient of m_i in (4.5) is $\operatorname{sgn}(\alpha)\ \operatorname{sgn}^*(\beta)$, where $\alpha \in D_n^d$, $\beta \in R_{n/d}$, $\beta(n/d-1) < \beta(n/d)$ are determined by $\sigma_i = (\sigma\alpha\tilde{\beta})\uparrow$ or $(\sigma\alpha\,\widetilde{\beta s_{n/d-1}})\uparrow$. Since $\operatorname{sgn}^*(\beta) = \operatorname{sgn}^*(\beta s_{n/d-1})$, the coefficient of m_i in (4.5) is $\operatorname{sgn}(\alpha)\operatorname{sgn}^*(\beta) = c(\sigma_i, \sigma)$. Therefore, for any $\sigma \in \mathcal{S}_n$,

$$\gamma \circ \psi(\rho_\sigma) = \sum_{i=1}^{a_n^d} c(\sigma_i, \sigma)\, m_i.$$

It follows that

$$\gamma \circ \psi(\sigma(\rho_1), \sigma(\rho_2), ..., \sigma(\rho_{a_n^d})) = (m_1, m_2, ..., m_{a_n^d})\ C(\sigma). \tag{4.6}$$

Let $B(\sigma)$ be the matrix representing the action of $\sigma \in \mathcal{S}_n$ on $H(\Pi_n^d)$. Then

$$\sigma(\rho_1, \rho_2, ..., \rho_{a_n^d}) = (\rho_1, \rho_2, ..., \rho_{a_n^d})\ B(\sigma).$$

Substituting this into (4.6) yields

$$\begin{aligned}(m_1, m_2, ..., m_{a_n^d})\ C(\sigma) &= \gamma \circ \psi(\sigma(\rho_1, \rho_2, ..., \rho_{a_n^d}))\\ &= \gamma \circ \psi((\rho_1, \rho_2, ..., \rho_{a_n^d})\ B(\sigma))\\ &= (m_1, m_2, ..., m_{a_n^d})\ C(\varepsilon)\ B(\sigma).\end{aligned}$$

Consequently, $B(\sigma) = C(\varepsilon)^{-1} C(\sigma)$. ∎

5. Lexicographical Shellability

We assume a knowledge of the definitions and terminology related to the notion of lexicographical shellability (see [BW1] or [BGS]). The lattice Π_n was one of the first known examples of a lexicographically shellable poset (cf. [Bj1]). An explicit EL-labeling of the edges of the Hasse diagram of Π_n was first constructed by Gessel. The decreasing chains of this labeling are readily seen to be enumerated by $(n-1)!$. A well-known consequence of this is that the order complex $\Delta(\Pi_n)$ is homotopy equivalent to a wedge of $(n-1)!$ $(n-3)$-spheres.

For $d > 1$, it is known that Π_n^d admits a CL-labeling (cf. [Sa]). This was originally proved using the technique of recursive atom ordering, without actually constructing an explicit labeling. In this section, we construct an explicit EL-labeling of Π_n^d for the purpose of identifying the decreasing maximal chains and thereby recovering the result of Stanley that $\mu(\Pi_n^d) = (-1)^{n/d} a_n^d$. The decreasing maximal chains are naturally indexed by permutations in A_n^d and have a nice combinatorial description. Stanley originally computed the Möbius function by means of generating functions [St1]. In addition to obtaining the Möbius function, the EL-labeling allows us to conclude that the order complex $\Delta(\Pi_n^d)$ is homotopy equivalent to a wedge of a_n^d $(n/d - 2)$-spheres. The decreasing chains also provide a basis for the top dimensional cohomology $H^*(\Pi_n^d)$.

Given any permutation $\sigma \in \mathscr{S}_n^d$, let $1 \leqslant t_1, t_2, ..., t_{n/d-1} \leqslant n/d - 1$ be such that

$$\sigma(dt_1) > \sigma(dt_2) > \cdots > \sigma(dt_{n/d-1}). \tag{5.1}$$

Set r_σ equal to the maximal chain of Π_n^d whose k-block partition is obtained by splitting σ at $dt_1, dt_2, ..., dt_{k-1}$. For example, if $\sigma = 23175648$ and $d = 2$, then r_σ is the chain

$$\hat{0} \to 23 \mid 17 \mid 56 \mid 48 \to 2317 \mid 56 \mid 48 \to 2317 \mid 5648 \to 23175648.$$

LEMMA 5.1. *If σ, τ are distinct permutations in A_n^d then r_σ and r_τ are distinct maximal chains.*

Proof. We leave the straight forward verification to the reader. ∎

THEOREM 5.2. *There is an EL-labeling of Π_n^d whose set of decreasing maximal chains is $\{r_\sigma \mid \sigma \in A_n^d\}$.*

Proof. The usual definition of EL-shellability requires that each interval has a unique *weakly* increasing maximal chain. Here, we use a modified formulation in which "weakly increasing" is replaced by "strictly increasing". It is not difficult to see that the two formulations are equivalent. Indeed, just replace each label $\lambda(x, y)$ in the "strict" version by the label $(\lambda(x, y), -r(x))$, where $r(x)$ denotes the rank of x, and order the collection of pairs lexicographically. Then strict ascents remain strict ascents and weak descents become strict descents. In the "strict" formulation of EL-shellability, the Möbius function is computed by counting weakly decreasing chains rather than strictly decreasing chains.

Recall that for $\sigma \in \mathscr{S}_n$, $\bar{\sigma}$ is the atom of Π_n^d obtained by splitting σ at all positions divisible by d. For any atom a of Π_n^d, let $\tilde{a}$ be the permutation that is lexicographically first among all permutations in $\{\sigma \in \mathscr{S}_n \mid \bar{\sigma} = a\}$.

Clearly, $\tilde{a}$ is obtained by writing each block of a in increasing order and then arranging the blocks so that the minimum elements of the blocks are increasing. Order the atoms of $\Pi_n^d, a_1, a_2, ..., a_m$, so that $\tilde{a}_1 < \tilde{a}_2 < \cdots < \tilde{a}_m$, lexicographically.

We shall label the covering relations of Π_n^d with labels in the totally ordered set

$$\{-n < -(n-1) < \cdots < -1 < 0_1 < 0_2 < \cdots < 0_m < 1 < 2 < \cdots < n\}.$$

First we set

$$\lambda(\hat{0}, a_i) = 0_i,$$

for each $i = 1, 2, ..., m$. If y covers x in $\Pi_n^d - \{\hat{0}\}$, then y is obtained by merging two blocks B_1 and B_2 of x. Let $\max(B_1) < \max(B_2)$. Now set

$$\lambda(x, y) = \begin{cases} -\max(B_1) & \text{if} \quad \max(B_1) > \min(B_2), \\ \max(B_2) & \text{if} \quad \max(B_1) < \min(B_2). \end{cases}$$

We claim that λ is an EL-labeling of Π_n^d. Let $x < y$ in Π_n^d. We need to show that the lexicographically first maximal chain of $[x, y]$ is the only strictly increasing maximal chain of $[x, y]$. We prove this assertion for $y = \hat{1}$ and leave it to the reader to modify our argument for general y.

Case 1. Let $x = \hat{0}$. Note that the lexicographically first maximal chain of $[\hat{0}, \hat{1}]$ is the chain whose k-block partition is obtained by splitting the permutation ε at $n-d, n-2d, ..., n-(k-1)\,d$. The label sequence of this chain is $0_1, 2d, ..., n$. We leave the easy verification that this chain is the only strictly increasing maximal chain of $[\hat{0}, \hat{1}]$ to the reader.

Case 2. Let $x \neq \hat{0}$. Let $B_1, B_2, ..., B_k$ be the blocks of x, where $\max(B_i) < \max(B_{i+1})$ for all $i = 1, 2, ..., k-1$.

Suppose that $\max(B_i) < \min(B_{i+1})$ for all $i = 1, 2, ..., k-1$. The lexicographically first maximal chain from x to $\hat{1}$ is obtained by fist merging B_1 and B_2 and then successively merging in $B_3, B_4, ..., B_k$. This chain has increasing label sequence, $\max(B_2), \max(B_3), ..., \max(B_k)$. It is clearly the only strictly increasing maximal chain from x to $\hat{1}$.

Now suppose that $\max(B_i) > \min(B_{i+1})$ for some i. Let j be the largest such i. We set x_1 equal to the partition covering x, obtained by merging B_j with B_{j+1}. We then have $\lambda(x, x_1) = -\max(B_j)$. It is easy to see that the lexicographically first maximal chain c from x to $\hat{1}$ must contain x_1. Let c be

$$x \to x_1 \to x_2 \to \cdots \to \hat{1}.$$

Clearly the lexicographically first maximal chain in $[x_1, \hat{1}]$ is then

$$x_1 \to x_2 \to \cdots \to \hat{1}.$$

By induction, it is also the only strictly increasing maximal chain of $[x_1, \hat{1}]$.

We claim that c is the only strictly increasing maximal chain of $[x, \hat{1}]$. Indeed, it is not difficult to see that a strictly increasing maximal chain must contain x_1. Hence, if c is strictly increasing, then it is the only strictly increasing maximal chain of $[x, \hat{1}]$. Therefore, to verify the claim we need only show that

$$\lambda(x, x_1) < \lambda(x_1, x_2). \tag{5.2}$$

If $\lambda(x_1, x_2) > 0$ then clearly (5.2) holds since $\lambda(x, x_1) < 0$. Suppose then that $\lambda(x_1, x_2) < 0$. Let B and B' be the blocks of x_1 that are merged to obtain x_2, where $\max(B) < \max(B')$. Note that $\max(B) = \max(B_i)$ for some $i < j$. We therefore have

$$\lambda(x_1, x_2) = -\max(B) = -\max(B_i) > -\max(B_j) = \lambda(x, x_1).$$

We may now conclude that λ is an EL-labeling of Π_n^d, as asserted.

One can readily check that for any $\sigma \in A_n^d$, r_σ is a decreasing maximal chain of Π_n^d under the labeling λ. Indeed, the label sequence is 0_k, $-\sigma(dt_{n/d-1})$, $-\sigma(dt_{n/d-2})$, ..., $-\sigma(dt_1)$, where $t_1, t_2, ..., t_{n/d-1}$ is given by (5.1) and k is such that $\bar{\sigma} = a_k$.

Conversely, let

$$\hat{0} \to x_1 \to x_2 \to \cdots \to x_{n/d} = \hat{1},$$

be a decreasing maximal chain. Let $B_1, B_2, ..., B_{n/d}$ be the blocks of x_1, where $\max(B_i) < \max(B_{i+1})$ for all $i = 1, 2, ..., n/d-1$. Since every chain starts with 0_k for some k, the remaining labels on a decreasing chain must be negative. Consequently, the label sequence is

$$0_k,\ -\max(B_1),\ -\max(B_2), ...,\ -\max(B_{n/d-1}).$$

If follows that

$$\lambda(x_i, x_{i+1}) = -\max(B_i),$$

for all $i = 1, 2, ..., n/d-1$.

We prove the following assertion:

Assertion. Given any $i = 1, 2, ..., n/d$, x_i consists of blocks C_1, $C_2, ..., C_{n/d-i+1}$, satisfying

(1) $\{\max(C_j) \mid j=1, 2, ..., n/d-i+1\} = \{\max(B_j) \mid j=i, i+1, ..., n/d\}$,

(2) $\max(C_j) > \min(C_{j+1})$ for all $j=1, 2, ..., n/d-i$,

(3) $n \in C_{n/d-i+1}$.

We use induction on i, starting with $i=n/d$, to prove the assertion. For $i=n/d$, the assertion is trivial. Let $i<n/d$. Assume, by induction, that x_{i+1} consists of blocks $C_1, C_2, ..., C_{n/d-i}$ which satisfy the conditions given by the assertion. Since x_{i+1} covers x_i, one of the blocks of x_{i+1}, say C_k, is the union of two blocks of x_i, C'_k and C''_k. Let $\max(C'_k) < \max(C''_k)$. Then $\max(C_k) = \max(C''_k)$. Consequently,

$$\max(C''_k) > \min(C_{k+1}) \qquad \text{if} \quad k < n/d - i,$$

and $n \in C''_k$ if $k = n/d - i$. Since $\lambda(x_i, x_{i+1}) < 0$, we also have

$$\max(C'_k) > \min(C''_k)$$

and

$$\max(C'_k) = -\lambda(x_i, x_{i+1}) = \max(B_i).$$

By (1) of the induction hypothesis,

$$\max(C_{k-1}) = \max(B_j)$$

for some $j \geqslant i+1$. Therefore

$$\max(C_{k-1}) > \max(B_i) = \max(C'_k) > \min(C'_k).$$

Hence the sequence of blocks $C_1, C_2, C_{k-1}, C'_k, C''_k, C_{k+1}, ..., C_{n/d-i}$ satisfies the conditions given in the assertion for i. By induction, the assertion is valid.

Now let $C_1, C_2, ..., C_{n/d}$ be the blocks of x_1 given by the assertion. Let σ equal the concatenation of the words $w(C_1), w(C_2), ..., w(C_{n/d})$ where $w(C_i)$ is the word obtained by writing the elements of C_i in increasing order. By (2) and (3) of the assertion, $\sigma \in A^d_n$. It is not difficult to see that the decreasing chain $\hat{0} \to x_1 \to \cdots \to x_{n/d}$ is precisely r_σ. We may therefore conclude that the set of decreasing chains is $\{r_\sigma \mid \sigma \in A^d_n\}$. ∎

COROLLARY 5.3. *The Möbius function of the lattice Π^d_n is given by $\mu(\Pi^d_n) = (-1)^{n/d} a^d_n$. Moreover, Π^d_n has the homotopy type of a wedge of a^d_n $(n/d-2)$-spheres. Consequently, the rank of $H(\Pi^d_n)$ is a^d_n.*

Proof. The first statement is an application of Stanley's well-known technique for computing the Möbius function by counting chains according to descent set (see [BGS, Theorem 2.2] or [Bj1, Theorem 2.7]). The second statement is a direct application of Theorem 1.3 of [Bj4]. ∎

COROLLARY 5.4. *Removal of the facets* $r_\sigma - \{\hat{0}, \hat{1}\}$, $\sigma \in A_n^d$ *from* $\Delta(\Pi_n^d)$ *results in an acyclic simplicial complex.*

Proof. This is an application of Lemma 7.7.1 of [Bj3]. ∎

COROLLARY 5.5. *The set* $\{r_\sigma \mid \sigma \in A_n^d\}$ *forms a basis for the top graded portion of the Stanley–Reisner ring of* Π_n^d.

Proof. This is an application of a result of Garsia [G, Theorem 4.2]. ∎

COROLLARY 5.6. *The set* $\{r_\sigma - \{\hat{0}, \hat{1}\} \mid \sigma \in A_n^d\}$ *forms a basis for the top dimensional cohomology* $H^*(\Pi_n^d)$.

Proof. This follows form the preceding corollary and the fact that the top graded portion of a certain quotient of the Stanley–Reisner ring of a ranked poset is isomorphic to the top dimensional cohomology module of the poset (see [BG, Theorem 5.1]). It also follows more directly from [Bj3, Theorem 7.7.2]. ∎

Remark. In [Bj3, Section 7.7], a general method for constructing a basis for the top homology of a shellable poset is given. A basis obtained by this method is said to be *induced* by the shelling. It is not difficult to determine that the basis $\{\rho_\sigma \mid \sigma \in A_n^d\}$ is not induced by the EL-shelling given in the proof of Theorem 5.2.

6. THE $d=1$ CASE

When $d=1$, Λ_σ, defined in Section 2, becomes the sublattice of Π_n consisting of all partitions obtained by splitting σ at any of its $n-1$ positions. In this case, Λ_σ is isomorphic to the face lattice of an $(n-2)$-simplex. Just as in Section 3, let v_σ be a fundamental cycle of the spherical complex $\Delta(\Lambda_\sigma)$.

THEOREM 6.1. *The set* $\{v_\sigma \mid \sigma \in \mathcal{S}_n, \sigma(n)=n\}$ *forms a basis for* $H(\Pi_n)$.

Proof. Since the rank of $H(\Pi_n)$ is $(n-1)!$, we need only prove the independence of $\{v_\sigma \mid \sigma \in \mathcal{S}_n, \sigma(n)=n\}$. Recall the maximum chains m_σ defined in (2.1). For $d=1$, Lemma 2.5 asserts that if $\alpha, \gamma \in \mathcal{S}_n$ are such that $\alpha(n)=n=\gamma(n)$ and m_α is a chain of Λ_γ, then $\alpha=\gamma$. The independence of $\{v_\sigma \mid \sigma \in \mathcal{S}_n, \sigma(n)=n\}$ follows from this. ∎

Just as in the $d>1$ case, $H(\Pi_n)$ is a cyclic $\mathscr{S}_n$-module generated by v_ε. For $\sigma, \tau \in \mathscr{S}_n$, we have

$$\sigma v_\tau = v_{\sigma\tau}.$$

If $\sigma(n)=n$ and $\tau(n)=n$ then σ sends the basis element v_τ to the basis element $v_{\sigma\tau}$. It is immediate from this that the restriction of the action of $\mathscr{S}_n$ on $H(\Pi_n)$ to $\mathscr{S}_{n-1}$ is the regular representation. This result was originally proved by Stanley [S2] by considering the character of the representation.

Theorem 5.2 and its corollaries have $d=1$ versions. For $\sigma \in \mathscr{S}_n$ where $\sigma(n)=n$, let r_σ be the maximal chain of Π_n whose k-block partition is obtained by splitting σ at $\sigma^{-1}(n-1), \sigma^{-1}(n-2), ..., \sigma^{-1}(n-k+1)$. For example, if $\sigma = 34125$ then r_σ is the maximal chain

$$3 \mid 4 \mid 1 \mid 2 \mid 5 \rightarrow 3 \mid 4 \mid 12 \mid 5 \rightarrow 3 \mid 4 \mid 125 \rightarrow 34 \mid 125 \rightarrow 34125.$$

THEOREM 6.2. *The sets* $\{r_\sigma \mid \sigma \in \mathscr{S}_n, \sigma(n)=n\}$ *and* $\{m_\sigma \cup \{\hat{1}\} \mid \sigma \in \mathscr{S}_n, \sigma(n)=n\}$ *are the sets of decreasing maximal chains for EL-labelings of* Π_n.

Proof. We use the modified version of EL-shellability discussed in the proof of Theorem 5.2. Let y cover x in Π_n. Then y is obtained from x by merging two blocks B_1, B_2, where $\max(B_1) < \max(B_2)$. Set

$$\lambda_1(x, y) = -\max(B_1)$$

$$\lambda_2(x, y) = \max(B_2).$$

We leave it to the reader to confirm that λ_1 and λ_2 are EL-labelings of Π_n and that the respective sets of decreasing maximal chains are $\{r_\sigma \mid \sigma \in \mathscr{S}_n, \sigma(n)=n\}$ and $\{m_\sigma \cup \{\hat{1}\} \mid \sigma \in \mathscr{S}_n, \sigma(n)=n\}$. The EL-labeling λ_1 is essentially the Gessel labeling that appears in [Bj1]. ∎

There is a remarkable connection between the representations of $\mathscr{S}_n$ on $H(\Pi_n)$ and on the free Lie algebra (cf. [S2], [K], [Jo]). Barcelo [Ba] combinatorially explains this connection by uncovering a fundamental relationship between the Björner basis for $H(\Pi_n)$ and the Lyndon basis for the free Lie algebra. In [W1] an analogous relationship between the new basis for $H(\Pi_n)$ and the "right comb" basis for the free Lie algebra is established.

Remark. This work was presented in 1991 at a workshop on Combinatorics and Discrete Geometry at MSI at Cornell University. For other work pertaining to the homology of partition posets with restricted block size, done at about the same time or more recently, see, e.g., [BL, BW2, BWe, HW, SW, Su1, Su2, SuW, SuWe, W_2].

Acknowledgments

I express my thanks to Adriano Garsia, Helene Barcelo, and Sheila Sundaram for may inspiring discussions and to Phil Hanlon for originating my interest in the d-divisible partition lattice. I am also grateful to Anders Björner for pointing out an error in an earlier version of this paper.

References

[BG] K. Baclawski and A. M. Garsia, Combinatorial decompositions of a class of rings, *Adv. in Math.* **39** (1981), 155–184.

[Ba] H. Barcelo, On the action of the symmetric group on the free Lie algebra and the partition lattice, *J. Combin. Theory Ser. A* **55** (1990), 93–129.

[Bj1] A. Björner, Shellable and Cohen–Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* **260** (1980), 159–183.

[Bj2] A. Björner, On the homology of geometric lattices, *Algebra Universalis* **14** (1982), 107–128.

[Bj3] A. Björner, Homology and shellability of geometric lattices, *in* "Matroid Applications" (N. White, Ed.), pp. 226–283, Cambridge Univ. Press, Cambridge, 1992.

[Bj4] A. Björner, Some combinatorial and algebraic properties of Coxeter complexes and Tits building, *Adv. in Math.* **52** (1984), 173–212.

[BGS] A. Björner, A. M. Garsia, and R. P. Stanley, An introduction to Cohen–Macaulay partially ordered sets, *in* "Ordered Sets" (I. Rival, Ed.), pp. 583–615, Reidel, Dordrecht, 1982.

[BL] A. Björner and L. Lovász, Linear decision trees, subspace arrangements and Möbius functions, *J. Amer. Math. Soc.* **7** (1994), 677–706.

[BW1] A. Björner and M. L. Wachs, On lexicographically shellable posets, *Trans. Amer. Math. Soc.* **277** (1983), 323–341.

[BW2] A. Björner and M. L. Wachs, Shellable nonpure complexes and posets, I, preprint, 1994; *Trans. Amer. Math. Soc.*, to appear.

[BWe] A. Björner and V. Welker, The homology of "*k*-equal" manifolds and related partition lattices, *Adv. in Math.* **110** (1995), 277–313.

[CHR] A. R. Calderbank, P. Hanlon, and R. W. Robinson, Partitions into even and odd block size and some unusual characters of the symmetric groups, *Proc. London Math. Soc.* (3) **53** (1986), 288–320.

[F] J. Folkman, The homology groups of a lattice, *J. Math. Mech.* **15** (1966), 631–636.

[G] A. M. Garsia, Combinatorial methods in the theory of Cohen–Macaulay rings, *Adv. in Math.* **38** (1980), 229–266.

[GW] A. M. Garsia and M. L. Wachs, Combinatorial aspects of skew representations of the symmetric group, *J. Combin. Theory Ser. A* **50** (1989), 47–81.

[HW] P. Hanlon and M. L. Wachs, On Lie *k*-algebras, *Adv. in Math.* **113** (1995), 206–236.

[Ja] G. D. James, "The Representation Theory of the Symmetric Groups," Springer Lecture Notes, Vol. 682, Springer-Verlag, Berlin/Heidelberg/New York, 1978.

[JK] G. D. James and A. Kerber, The representation theory of the symmetric group, *in* "Encyclopedia of Math.," Vol. 16, Addison–Wesley, Reading, MA, 1981.

[Jo] A. Joyal, "Foncteurs analytiques et espèces de structure," Springer Lecture Notes, Vol. 1234, Springer-Verlag, Berlin/Heidelberg/New York, 1986.

[K] A. A. Klyachco, Lie elements in the tensor algebra, *Siberian Math. J.* **15** (1974), 1296–1304.

[M] J. R. Munkres, "Elements of Algebraic Topology," Benjamin/Cummings, Menlo Park, CA, 1984.

[Sa] B. E. Sagan, Shellability of exponential structures, *Order* **3** (1986), 47–54.

[SW] A. Sanders and M. L. Wachs, The (co)homology of the lattice of partitions with lower bounded block size, in preparation.

[St1] R. P. Stanley, Exponential structures, *Studies Appl. Math.* **59** (1978), 73–82.

[St2] R. P. Stanley, Some aspects of groups acting on finite posets, *J. Combin. Theory Ser. A* **32** (1982), 132–161.

[Su1] S. Sundaram, The homology representations of the symmetric group on Cohen–Macaulay subposets of the partition lattice, *Adv. in Math.* **104** (1994), 255–296.

[Su2] S. Sundaram, Applications of the Hopf trace formula to computing homology representations, *in* "Proceedings of Jerusalem Combinatorics Conference, 1993" (H. Barcelo and G. Kalai, Eds.), Contemporary Math., Vol. 178, pp. 277–309, Amer. Math. Soc., Providence, RI, 1994.

[SuW] S. Sundaram and M. L. Wachs, The homology representations of the k-equal partition lattice, preprint, 1994; *Trans. Amer. Math. Soc.*, to appear.

[SuWe] S. Sundaram and V. Welker, Group actions on arrangements of linear subspaces and applications to configuration spaces, preprint, 1994; *Trans. Amer. Math. Soc.*, to appear.

[W_1] M. L. Wachs, On the (co)homology of the partition lattice and the free Lie algebra, preprint, 1994; *Discrete Math.*, to appear.

[W_2] M. L. Wachs, Whitney homology of semipure shellable posets, in preparation.