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A Multifractal Formalism

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Let X be a metric space and μ a Borel probability measure on X. For $q, t \in \mathbb{R}$ and $E \subseteq X$ write

$$\overline{\mathscr{H}}_{\mu}^{q,t}(E) = \sup_{\delta > 0} \inf \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \;\middle|\; (B(x_i, r_i))_i \text{ is a centered } \delta\text{-covering of } E \right\}$$

$$\overline{\mathscr{P}}_{\mu}^{q,t}(E) = \inf_{\delta > 0} \sup \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \;\middle|\; (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E \right\}$$

and put

$$\mathscr{H}_{\mu}^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathscr{H}}_{\mu}^{q,t}(F), \qquad \mathscr{P}_{\mu}^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}_{\mu}^{q,t}(E_i).$$

Then $\mathscr{H}_{\mu}^{q,t}$ and $\mathscr{P}_{\mu}^{q,t}$ are Borel measures. $\mathscr{H}_{\mu}^{q,t}$ is a multifractal generalization of the centered Hausdorff measure and $\mathscr{P}_{\mu}^{q,t}$ is a multifractal generalization of the packing measure. The measures $\mathscr{H}_{\mu}^{q,t}$ and $\mathscr{P}_{\mu}^{q,t}$ define, for a fixed q, in the usual way a generalized Hausdorff dimension $\dim_{\mu}^{q}(E)$ and a generalized packing dimension $\operatorname{Dim}_{\mu}^{q}(E)$ of subsets E of X. We study the functions

$$b_{\mu}: q \to \dim_{\mu}^{q}(\operatorname{supp} \mu), \qquad B_{\mu}: q \to \operatorname{Dim}_{\mu}^{q}(\operatorname{supp} \mu)$$

and their relation to the so-called multifractal spectra functions of μ:

$$f_{\mu}(\alpha) = \dim \left\{ x \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu B(x, r)}{\log r} = \alpha \right\}, \qquad F_{\mu}(\alpha) = \operatorname{Dim} \left\{ x \;\middle|\; \lim_{r \searrow 0} \frac{\log \mu B(x, r)}{\log r} = \alpha \right\}.$$

We prove, among other things, that $f_{\mu}(F_{\mu})$ is bounded from above by the Legendre transform of $b_{\mu}(B_{\mu})$ and that equality holds for graph directed self-similar measures and "cookie-cutter" measures. Finally we discuss the connection with generalized Rényi dimensions. © 1995 Academic Press, Inc.

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1. Introduction

In recent papers theoretical physicists [Bo, Co1, Fr, Gr1, Gr2, Ha, He, Pa, Te1, Te2] and mathematicians [Av, Bo, Ca, Co1, Ed, Ki, Lo1, Lo2, Str, Ra] have studied the so-called multifractal theory. A number of claims have been made on the basis of heuristics and physical intuition. The purpose of this paper is to determine to what extent rigorous arguments can be provided for this theory.

If X is a metric space then $\mathscr{P}(X)$ denotes the set of Borel probability measures on X. If $x \in X$ and $r > 0$ then $B(x, r)$ will denote the closed ball with center x and radius $r > 0$. Now fix $\mu \in \mathscr{P}(X)$. The upper resp. lower local dimension of μ at a point $x \in X$ is defined by

$$\bar{\alpha}_\mu(x) = \limsup_{r \searrow 0} \frac{\log \mu B(x, r)}{\log r}$$

resp.

$$\underline{\alpha}_\mu(x) = \liminf_{r \searrow 0} \frac{\log \mu B(x, r)}{\log r}.$$

If $\bar{\alpha}_\mu(x)$ and $\underline{\alpha}_\mu(x)$ agree we refer to the common value as the local dimension of μ at x and denote it by $\alpha_\mu(x)$. Upper and lower local dimensions have been investigated by a large number of authors, cf. e.g. [Bi1, Bi2, Cu1, Cu2, Fro, Haa, Yo].

For $\alpha \geq 0$ write

$$\bar{X}^\alpha = \{x \in \operatorname{supp} \mu \mid \bar{\alpha}_\mu(x) \leq \alpha\}$$

$$\bar{X}_\alpha = \{x \in \operatorname{supp} \mu \mid \alpha \leq \bar{\alpha}_\mu(x)\}$$

$$\underline{X}^\alpha = \{x \in \operatorname{supp} \mu \mid \underline{\alpha}_\mu(x) \leq \alpha\}$$

$$\underline{X}_\alpha = \{x \in \operatorname{supp} \mu \mid \alpha \leq \underline{\alpha}_\mu(x)\}$$

Also write

$$X(\alpha) = \underline{X}_\alpha \cap \bar{X}^\alpha$$

where $\operatorname{supp} \mu$ denotes the topological support of μ. One should think of the family $\{X(\alpha) \mid \alpha \geq 0\}$ as a multifractal decomposition of the support of μ—i.e. we have decomposed the (perhaps fractal) set $\operatorname{supp} \mu$ into a family $\{X(\alpha) \mid \alpha \geq 0\}$ of subfractals according to the measure μ and indexed by $\alpha \in \mathbb{R}_+$.

Now, the main problem in multifractal theory is to estimate the size of $X(\alpha)$. This is done by introducing the functions f_μ and F_μ defined by

$$f_\mu(\alpha) = \dim\{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\}$$
$$= \dim(\underline{X}_\alpha \cap \overline{X}^\alpha)$$
$$F_\mu(\alpha) = \operatorname{Dim}\{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\}$$
$$= \operatorname{Dim}(\underline{X}_\alpha \cap \overline{X}^\alpha)$$

for $\alpha \geqslant 0$, and where dim and Dim denote the Hausdorff dimension and packing dimension respectively. These and similar functions are generically known as the "the multifractal spectrum of μ", "the singularity spectrum of μ", "the spectrum of scaling indices" or simply "the $f(\alpha)$-spectrum". The function $f(\alpha) = f_\mu(\alpha)$ was first explicitly defined by the physicists Halsey et al. in 1986 in their seminal paper [Ha].

There are (apart from trivial cases) so far only four types of measures μ for which the f_μ function has been rigorously determined, namely

(1) graph directed self-similar measures in $\mathbb{R}^d$ with totally disconnected support, cf. Cawley & Mauldin [Ca] and Edgar & Mauldin [Ed];

(2) self affine measures in $\mathbb{R}^2$ whose support satisfies a certain disjointness condition, cf. King [Ki];

(3) "Cookie-Cutters" (i.e. Gibbs states on 0-dimensional hyperbolic attractors in $\mathbb{R}$), cf. Bohr & Rand [Bo], Rand [Ra] and Collet et al. [Col];

(4) invariant measures of maximal entropy for rational maps of the complex plan, cf. Lopes [Lo1, Lo2].

In all four cases it turns out that there exist numbers $\underline{a} \leqslant \bar{a}$ such that $f_\mu(\alpha) = 0$ for $\alpha \in [0, \infty[\setminus[\underline{a}, \bar{a}]$ and f_μ is concave and smooth on $]\underline{a}, \bar{a}[$. The proofs in [Ca, Ed, Ki] are based on the ergodic theorem and some combinatoric geometric arguments whereas the proofs in [Bo, Ra, Col, Lo1, Lo2] are based on the thermodynamic formalism developed by Bowen [Bow] and Ruelle [Ru]. (See also Note Added in Proof (1) at the end of this paper.)

The concepts underlying the above mentioned multifractal decompositions go back to two early papers by Mandelbrot [Ma1, Ma2] from 1972 and 1974 respectively. Mandelbrot [Ma1, Ma2] suggests that the bulk of intermittent dissipation of energy in a highly turbulent fluid flow occurs over a set of fractal dimension. The ideas introduced in [Ma1, Ma2] were taken up by Frisch and Parisi [Fr] and Benzi et al. [Ben] in 1985 and 1984 respectively. Frisch and Parisi [Fr] replaced the very complicated multifractal formalism introduced in [Ma2] with a simpler (and hence also less general) formalism, whereas Benzi et al. extended this formalism to include dynamical systems (and not just intermittent dissipation of energy in turbulent fluids). Finally, the major breakthrough from a physically point of view occured in 1985 when Hasley et al. [Ha] introduced the above mentioned $f(\alpha)$ function. A parallel (but otherwise independent) set of ideas based on Rényi-entropies (introduced by Rényi [Re1, Re2, Re3] in 1960) were introduced by Hentschel & Procaccia [He], Grassberger & Procaccia [Gr1] and Grassberger [Gr2] during the period 1982-1984. In [He, Gr1, Gr2] Hentschel et al., Grassberger et al. and Grassberger defined a one-parameter family of numbers $(D_q)_{q \in \mathbb{R}}$ known as the generalized Rényi-dimension. A related one-parameter family of numbers was introduced by Badii and Politi [Ba] in 1985. However, it was "proved" by Halsey et al. [Ha] that the $f(\alpha)$ function and the generalized Rényi-dimensions $(D_q)_{q \in \mathbb{R}}$ can be derived from each other (i.e. if $f(\alpha)$ is known then it is possible to determine $(D_q)_{q \in \mathbb{R}}$ and vice versa), and the two approaches are thus equivalent from a physical and heuristical point of view (but, as we shall see later, not from a rigorous mathematically point of view).

The popularity of multifractal theory and the $f(\alpha)$ function is basically due to two facts: (1) the $f(\alpha)$ function is usually a smooth function of α and (2) there seems to be a remarkable agreement between experimental observations in a large number of different physical systems and $f(\alpha)$ functions computed by simple theoretical models.

Multifractal theory and diffusion-limited aggregation (DLA) have been discussed by numerous authors. In DLA one first places a particle at the origin as a "seed". Then let another particle start from far away and diffuse by a random walk process. The wandering particle sticks to the "seed" when it reaches it. Repeat this process many times. This type of aggregation process produces clusters which have a typically dendritical appearance. Next define a probability measure P on the DLA structure in such a way that $P(v)$ is the probability that a wandering particle will reach v (i.e. P is the "harmonic measure" of the DLA structure). The $f(\alpha)$ function of P can be computed numerically, cf. [Mea1, Mea2, Mea3] and Amitrano et al. [Am]. Matsushita et al. [Mats] have observed DLA like structures when zinc diffuses through an aqueous zinc sulfate and n-butul acetate electrolyte and eventually deposites on an electrode. DLA like structures, known as viscous fingers, are observed when a low viscosity fluid is injected into a high viscosity fluid, cf. Meakin [Mea2, Mea3] and Måløy et al. [Må1, Må2]. Define a probability distribution μ on viscous finger such that $\mu(v)$ is the probability that the viscous finger will expand at v if more low viscous fluid is injected. The distribution μ can be determined experimentally and the $f(\alpha)$ function of μ can then be computed numerically, cf. [Mea1]. It turns out that there is a remarkably good agreement between the $f(\alpha)$ curve of μ and P, cf. Amitrano et al. [Am] and Nittmann et al. [Ni].

The connection between Rayleigh-Bernard convection and multifractals is studied in e.g. Jensen et al. [Je]. Jensen et al. [Je] compute the $f(\alpha)$ function of the distribution of the time fluctuations of the temperature at the bottom of a small cell of mercury exposed to a vertical temperature gradient and an alternating horizontal magnetic field. Jensen et al. [Je] show that the experimentally determined $f(\alpha)$ curve fits remarkably well to the $f(\alpha)$ function corresponding to the invariant measure of the "circle map".

In Meneveau and Sreenivasan [Men] it is shown that the observed multifractal $f(\alpha)$ curve of the dissipation field of fully developed turbulence is very well described by the $f(\alpha)$ curve of a certain self-similar measure.

The reader is referred to Feder [Fe] for a more thorough discussion concerning the applications of multifractals to physics, chemistry, meteorology and other natural sciences.

The purpose of this paper is to introduce and develop a mathematical rigorous multifractal formalism based on a natural multifractal generalization of the centered Hausdorff measure and of the packing measure. These generalizations are motivated by the heuristics of Halsey et al. [Ha]. If μ is a (Borel) probability measure on $\mathbb{R}^d$, then Halsey et al. [Ha, formula (2.8)] "prove", in a very heuristical way, that for each $q \in \mathbb{R}$ there exists a unique number $\tau(q)$ such that

$$\lim_{l \to 0} \sum_i \frac{p_i^q}{l_i^\tau} = \begin{cases} \infty & \text{for} \quad \tau > \tau(q) \\ 0 & \text{for} \quad \tau < \tau(q) \end{cases} \tag{1.1}$$

where $(E_i)_i$ is a partition of $\operatorname{supp} \mu$ with $\operatorname{diam} E_i < l$, $p_i = \mu(E_i)$ and $l_i = \operatorname{diam} E_i$. The main purpose of this paper is to formalize this notion in a rigorous mathematical way and to investigate the relation between the introduced "dimension" functions and the multifractal spectrum of μ. This formalisation yields a very general multifractal formalism which we will study.

We first recall the definition of the Hausdorff measure, the centered Hausdorff measure and the packing measure. Let X be a metric space, $E \subseteq X$ and $\delta > 0$. A countable family $\mathscr{B} = (B(x_i, r_i))_i$ of closed balls in X is called a centered δ-covering of E if $E \subseteq \bigcup_i B(x_i, r_i)$, $x_i \in E$ and $0 < r_i < \delta$ for all i. The family $\mathscr{B}$ is called a centered δ-packing of E if $x_i \in E$, $0 < r_i < \delta$

and $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$ for all $i \neq j$. Let $E \subseteq X$, $s \geqslant 0$ and $\delta > 0$. Now put

$$\mathscr{H}^s_\delta(E) = \inf\left\{\sum_i \operatorname{diam}(E_i)^s \,\middle|\, E \subseteq \bigcup_{i=1}^{\infty} E_i, \operatorname{diam} E_i < \delta\right\}.$$

The s-dimensional Hausdorff measure $\mathscr{H}^s(E)$ of E is defined by

$$\mathscr{H}^s(E) = \sup_{\delta > 0} \mathscr{H}^s_\delta(E).$$

The reader is referred to [Fa1] for more information on $\mathscr{H}^s$. Next we define the centered Hausdorff measure introduced by Raymond & Tricot in [Ray]. Put

$$\bar{\mathscr{C}}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} (2r_i)^s \,\middle|\, (B(x_i, r_i))_i \text{ is a centered } \delta\text{-covering of } E\right\}.$$

The s-dimensional centered pre-Hausdorff measure $\bar{\mathscr{C}}^s(E)$ of E is defined by

$$\bar{\mathscr{C}}^s(E) = \sup_{\delta > 0} \bar{\mathscr{C}}^s_\delta(E).$$

The set function $\bar{\mathscr{C}}^s$ is not necessarily monotone, and hence not necessarily an outer measure, cf. [Ray, pp. 137–138]. But $\bar{\mathscr{C}}^s$ give rise to a Borel measure, called the s-dimensional centered Hausdorff measure $\mathscr{C}^s(E)$ of E, as follows

$$\mathscr{C}^s(E) = \sup_{F \subseteq E} \bar{\mathscr{C}}^s(F).$$

It is easily seen (c.f. [Ray, Lemma 3.3]) that

$$2^{-s}\mathscr{C}^s \leqslant \mathscr{H}^s \leqslant \mathscr{C}^s$$

We will now define the packing measure. Write

$$\bar{\mathscr{P}}^s_\delta(E) = \sup\left\{\sum_{i=1}^{\infty} (2r_i)^s \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E\right\}.$$

The s-dimensional prepacking measure $\bar{\mathscr{P}}^s(E)$ of E is defined by

$$\bar{\mathscr{P}}^s(E) = \inf_{\delta > 0} \bar{\mathscr{P}}^s_\delta(E).$$

The set function $\bar{\mathscr{P}}^s$ is not necessarily countably subadditive, and hence not necessarily an outer measure, c.f. [Ta] or [Fa2]. But $\bar{\mathscr{P}}^s$ give rise to a

Borel measure, namely the s-dimensional packing measure $\mathscr{P}^s(E)$ of E, as follows

$$\mathscr{P}^s(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_{i=1}^{\infty} \overline{\mathscr{P}}^s(E_i).$$

The packing measure was introduced by Taylor and Tricot in [Ta] using centered δ-packings of open balls, and by Raymond and Tricot in [Ray] using centered δ-packings of closed balls.

Also recall that the Hausdorff dimension $\dim(E)$, the packing dimension $\mathrm{Dim}(E)$ and the logarithmic index $\Delta(E)$ of E are defined by

$$\dim(E) = \sup\{s \geqslant 0 \mid \mathscr{H}^s(E) = \infty\}$$

$$\mathrm{Dim}(E) = \sup\{s \geqslant 0 \mid \mathscr{P}^s(E) = \infty\}$$

$$\Delta(E) = \sup\{s \geqslant 0 \mid \overline{\mathscr{P}}^s(E) = \infty\}.$$

We refer the reader to [Tr] and [Ray] for more information on the centered Hausdorff measure, the packing measure and the packing dimension.

We will now define multifractal generalizations of the centered Hausdorff measure and of the packing measure. For $q \in \mathbb{R}$ define $\varphi_q\colon [0, \infty[\to \overline{\mathbb{R}}_+ = [0, \infty]$ by

$$\varphi_q(x) = \begin{cases} \infty & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} \quad \text{for } q < 0$$

$$\varphi_q(x) = 1 \qquad \text{for } q = 0$$

$$\varphi_q(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^q & \text{for } 0 < x \end{cases} \quad \text{for } 0 < q$$

For $\mu \in \mathscr{P}(X)$, $E \subseteq X$, $q, t \in \mathbb{R}$ and $\delta > 0$ write

$$\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E) = \inf\Big\{\sum_i \varphi_q(\mu(B(x_i, r_i)))(2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centred } \delta\text{-covering of } E\Big\}, \qquad E \neq \varnothing$$

$$\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(\varnothing) = 0$$

$$\overline{\mathscr{H}}^{q,t}_{\mu}(E) = \sup_{\delta > 0} \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E)$$

$$\mathscr{H}^{q,t}_{\mu}(E) = \sup_{F \subseteq E} \overline{\mathscr{H}}^{q,t}_{\mu}(F).$$

We also make the dual definitions

$$\overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E) = \sup\left\{\sum_i \varphi_q(\mu(B(x_i, r_i)))(2r_i)^t \mid (B(x_i, r_i))_i \text{ is a centered } \delta\text{-packing of } E\right\}, \qquad E \neq \varnothing$$

$$\overline{\mathscr{P}}^{q,t}_{\mu,\delta}(\varnothing) = 0$$

$$\overline{\mathscr{P}}^{q,t}_{\mu}(E) = \inf_{\delta>0} \overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E)$$

$$\mathscr{P}^{q,t}_{\mu}(E) = \inf_{E\subseteq \bigcup_i E_i} \sum_i \overline{\mathscr{P}}^{q,t}_{\mu}(E_i).$$

Below we prove that $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$ are measures on the family of Borel subsets of X. The measure $\mathscr{H}^{q,t}_{\mu}$ is of course a multifractal generalisation of the centered Hausdorff measure, whereas $\mathscr{P}^{q,t}_{\mu}$ is a multifractal generalisation of the packing measure. In fact, it is easily seen that the following holds for $t \geqslant 0$,

$$\begin{gathered} \mathscr{C}^t = \mathscr{H}^{0,t}_{\mu} \\ 2^{-t}\mathscr{H}^{0,t}_{\mu} \leqslant \mathscr{H}^t \leqslant \mathscr{H}^{0,t}_{\mu} \\ \mathscr{P}^t = \mathscr{P}^{0,t}_{\mu} \\ \overline{\mathscr{P}}^t = \overline{\mathscr{P}}^{0,t}_{\mu} \end{gathered} \tag{1.2}$$

The next result shows that the measures $\mathscr{H}^{q,t}_{\mu}$, $\mathscr{P}^{q,t}_{\mu}$ and the pre-measure $\overline{\mathscr{P}}^{q,t}_{\mu}$ in the usual way assign a dimension to each subset E of X.

PROPOSITION 1.1. (i) *There exists a unique number* $\Delta^q_{\mu}(E) \in [-\infty, \infty]$ *such that*

$$\overline{\mathscr{P}}^{q,t}_{\mu}(E) = \begin{cases} \infty & \textit{for} \quad t < \Delta^q_{\mu}(E) \\ 0 & \textit{for} \quad \Delta^q_{\mu}(E) < t \end{cases}$$

(ii) *There exists a unique number* $\operatorname{Dim}^q_{\mu}(E) \in [-\infty, \infty]$ *such that*

$$\mathscr{P}^{q,t}_{\mu}(E) = \begin{cases} \infty & \textit{for} \quad t < \operatorname{Dim}^q_{\mu}(E) \\ 0 & \textit{for} \quad \operatorname{Dim}^q_{\mu}(E) < t \end{cases}$$

(iii) *There exists a unique number* $\dim^q_{\mu}(E) \in [-\infty, \infty]$ *such that*

$$\mathscr{H}^{q,t}_{\mu}(E) = \begin{cases} \infty & \textit{for} \quad t < \dim^q_{\mu}(E) \\ 0 & \textit{for} \quad \dim^q_{\mu}(E) < t. \end{cases}$$

Proof. Follows easily from the definitions. ▌

The results in Proposition 1.1 are obvious mathematically rigorous analogues of (1.1). The number $\dim_\mu^q(E)$ is an obvious multifractal analogue of the Hausdorff dimension $\dim(E)$ of E whereas $\mathrm{Dim}_\mu^q(E)$ and $\Delta_\mu^q(E)$ are obvious multifractal analogues of the packing dimension $\mathrm{Dim}(E)$ and the logarithmic index $\Delta(E)$ of E respectively. In fact, it follows immediately from (1.2) that

$$\begin{aligned} \dim(E) &= \dim_\mu^0(E) \\ \mathrm{Dim}(E) &= \mathrm{Dim}_\mu^0(E) \\ \Delta(E) &= \Delta_\mu^0(E). \end{aligned} \tag{1.3}$$

It is also readily seen that

$$\begin{aligned} 0 \leqslant \dim_\mu^q(E) \quad &\text{for} \quad q \leqslant 1 \quad \text{and} \quad \mu(E) > 0 \\ \Delta_\mu^q(E) \leqslant 0 \quad &\text{for} \quad 1 \leqslant q \end{aligned} \tag{1.4}$$

Since $\dim_\mu^q$ and Dim_μ^q are defined in terms of outer measures we conclude that

(1) $\dim_\mu^q$, Dim_μ^q are monotone, i.e.

$$\begin{aligned} \dim_\mu^q(E) \leqslant \dim_\mu^q(F) \quad &\text{for} \quad E \subseteq F \\ \mathrm{Dim}_\mu^q(E) \leqslant \mathrm{Dim}_\mu^q(F) \quad &\text{for} \quad E \subseteq F. \end{aligned}$$

(2) $\dim_\mu^q$, Dim_μ^q are σ-stable, i.e.

$$\begin{aligned} \dim_\mu^q\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \sup_{n \in \mathbb{N}} \dim_\mu^q(E_n) \\ \mathrm{Dim}_\mu^q\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \sup_{n \in \mathbb{N}} \mathrm{Dim}_\mu^q(E_n). \end{aligned}$$

These properties will be used tactically throughout the paper.

If X is a metric space, $E \subseteq X$ and $\mu \in \mathscr{P}(X)$ then we define functions $b_{\mu,E}$, $B_{\mu,E}$ and $\Lambda_{\mu,E}$ by

$$\begin{aligned} b_{\mu,E}(q) &= \dim_\mu^q(E), & b(q) &= b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu) \\ B_{\mu,E}(q) &= \mathrm{Dim}_\mu^q(E), & B(q) &= B_\mu(q) = \mathrm{Dim}_\mu^q(\operatorname{supp}\mu) \\ \Lambda_{\mu,E}(q) &= \Delta_\mu^q(E), & \Lambda(q) &= \Lambda_\mu(q) = \Delta_\mu^q(\operatorname{supp}\mu). \end{aligned}$$

Our main point is that the functions

$$b_\mu : q \to \dim_\mu^q(\operatorname{supp} \mu)$$

$$B_\mu : q \to \operatorname{Dim}_\mu^q(\operatorname{supp} \mu)$$

are related to the multifractal spectrum of μ, whereas the function

$$\Lambda_\mu : q \to \Delta_\mu^q(\operatorname{supp} \mu)$$

is related to the generalized Rényi dimensions of μ.

Equation (1.4) and Proposition 2.4 imply that

$$\begin{aligned} 0 \leqslant b_\mu(q) \leqslant B_\mu(q) \leqslant \Lambda_\mu(q) & \quad \text{for} \quad q < 1 \\ b_\mu(1) = B_\mu(1) = \Lambda_\mu(1) = 0 & \\ b_\mu(q) \leqslant B_\mu(q) \leqslant \Lambda_\mu(q) \leqslant 0 & \quad \text{for} \quad 1 < q \end{aligned} \tag{1.5}$$

for $\mu \in \mathscr{P}(\mathbb{R}^d)$. Also (by (1.3))

$$b_\mu(0) = \dim(\operatorname{supp} \mu)$$

$$B_\mu(0) = \operatorname{Dim}(\operatorname{supp} \mu)$$

$$\Lambda_\mu(0) = \Delta(\operatorname{supp} \mu).$$

S.J. Taylor [Tay1, Tay2] defined a fractal to be any subset E of a metric space X which satisfies

$$\dim E = \operatorname{Dim} E.$$

Our multifractal formalism contains a natural extension of Taylor's definition to the case of measures. A Borel probability measure $\mu \in \mathscr{P}(X)$ on a metric space X is called a Taylor multifractal measure if

$$b_\mu = B_\mu. \tag{1.6}$$

We show (cf. Chapter 3, Example 4) that there exist measures $\mu \in \mathscr{P}(\mathbb{R}^d)$ such that $b_\mu(q) < B_\mu(q)$ for all $q \in \mathbb{R} \setminus \{1\}$ (of course we always have $b_\mu(1) = B_\mu(1)$ by (1.5)).

We will now give a brief description of the organization of the paper. In Section 2 we define the setting and formulate our main results. Section 3 contains some examples which illustrate the multifractal formalism developed in this paper. Section 4 contains the proofs of the results stated in Section 2. Section 5 gives a multifractal analysis of graph directed self-similar measures in $\mathbb{R}^d$ using our setting. Section 6 gives a multifractal analysis of "cookie-cutter" measures in $\mathbb{R}$ using our setting. Finally, Section 7 contains some further remarks and questions.

Note. After this paper was competed we were informed that Pesin [Pe1–Pe3] has considered a measure and dimension somewhat similar to $\mathscr{H}_{\mu}^{q,t}$ and $\dim_{\mu}^{q}$. However, Pesin's approach is dynamical whereas we have adopted an almost entirely measure theoretic approach.

2. Definitions and Statement of Results

This section contains the basic definitions and states the main results. The proofs will be given in Section 4.

If $f: \mathbb{R} \to \mathbb{R}$ is a real-valued function, let $f^*: \mathbb{R} \to [-\infty, \infty[$ denote the following Legendre transform of f,

$$f^*(x) = \inf_y (xy + f(y)), \qquad x \in \mathbb{R}.$$

Observe that f^* is concave.

A widespread folklore theorem (among physicists) states that the function τ introduced in (1.1) is decreasing, smooth and convex, and that the multifractal spectrum f_μ is equal to the Legendre transform τ^* of τ, cf. [Ba, Fa2, Fe, Gr1, Gr2, Ha, He, Pa]. That is, we have the following two (heuristic and partially incorrect) folklore theorems.

Folklore Theorem 1. *Let τ be the function in (1.1). Then the following hold:*

(i) *τ is decreasing, convex, and smooth.*

(ii) *τ has affine asymptotes as $q \to \pm\infty$.*

(iii) $\tau(1) = 0$.

(iv) *The line with slope 1 passing through the origin is a tangent to the graph of τ^* (see Fig. 1).*

Folklore Theorem 2. *Let τ be the function in (1.1). Then there exist numbers $0 \leqslant \underline{a} \leqslant \bar{a}$ such that*

$$f_\mu(\alpha) = \begin{cases} \tau^*(\alpha) & \alpha \in [\underline{a}, \bar{a}] \\ 0 & \alpha \notin [\underline{a}, \bar{a}] \end{cases}$$

(*see Fig.* 2).

Finally, a third folklore theorem (among physicists) states that τ can be computed (numerically) by a certain box counting argument, cf. [Ba, Fa2, Fe, Gr1, Gr2, Ha, He, Pa].

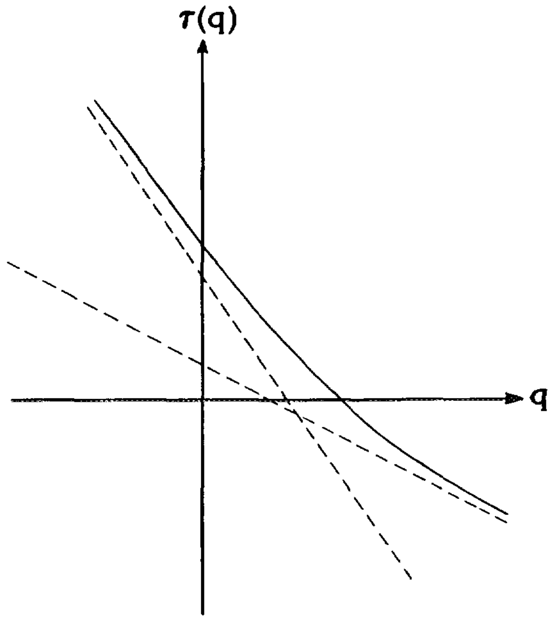

FIG. 1. The typical shape of the function τ in (1.1) according to Folklore Theorem 1.

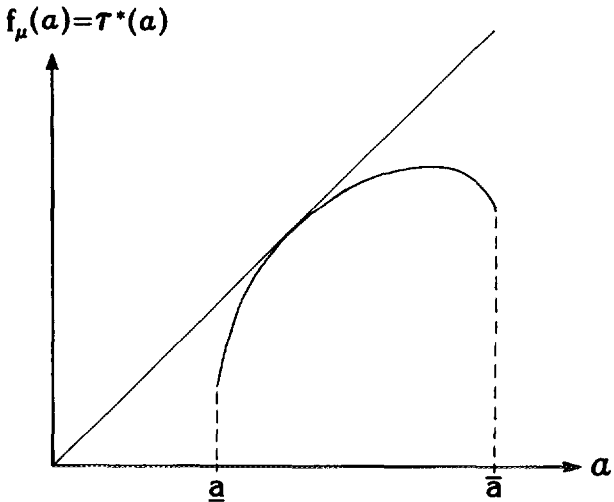

FIG. 2. The typical shape of the multifractal spectrum $f_\mu = \tau^*$ according to Folklore Theorem 1 and Folklore Theorem 2. However, we emphasize that the situation is not as simple as indicated by this figure; especially, we note that the multifractal spectrum need not be concave.

FOLKLORE THEOREM 3. *Let τ be the function in (1.1). Then*

$$\tau(q)=\lim_n \frac{\log(\sum_{C\in \mathscr{C}_n}\mu(C)^q)}{-\log 2^{\ n}}$$

for $\mu\in\mathscr{P}(\mathbb{R}^d)$, where

$$\mathscr{C}_n=\left\{\prod_{i=1}^{d}\left[\frac{k_i}{2^n},\frac{k_i+1}{2^n}\right[\ \middle|\ k_i\in\mathbb{Z}\right\},\qquad n\in\mathbb{N}.$$

Since the functions b_μ, B_μ and Λ_μ are intended as rigorous mathematical analogues of τ, we must now prove that results similar to Folklore Theorem 1, Folklore Theorem 2 and Folklore Theorem 3 actually hold for b_μ, B_μ and Λ_μ.

Section 2.4 contains our results concerning Folklore Theorem 1. These results show that B_μ and Λ_μ behave as stated in Folklore Theorem 1, with the notable exception of smoothness. Several examples in Chapter 3 show that the functions b_μ, B_μ and Λ_μ need not be smooth. However, *all* the functions b_μ, B_μ and Λ_μ are smooth (and coincide) in the case where μ is a graph directed self-similar measure in $\mathbb{R}^d$ with totally disconnected support or a "cookie-cutter" measure in $\mathbb{R}$, cf. Chapter 5 and Chapter 6. Also, the function b_μ need not be convex.

Section 2.6 contains results related to Folklore Theorem 2. We state two theorems which show that b_μ, B_μ and Λ_μ (nearly) behave as stated in Folklore Theorem 2. The functions b_μ^* and B_μ^* are always upper bounds for the multifractal spectra functions f_μ and F_μ respectively, i.e. there exist numbers $0\leq \underline{a}\leq \bar{a}$ such that

$$f_\mu(\alpha)=\begin{cases}\leq b_\mu^*(\alpha) & \alpha\in\,]\underline{a},\bar{a}[\\ 0 & \alpha\notin[\underline{a},\bar{a}]\end{cases}$$

$$F_\mu(\alpha)=\begin{cases}\leq B_\mu^*(\alpha) & \alpha\in\,]\underline{a},\bar{a}[\\ 0 & \alpha\notin[\underline{a},\bar{a}]\end{cases}$$

(cf. Theorem 2.17), but these inequalities can be strict—this is basically due to the fact that the multifractal spectrum f_μ is not necessarily concave, and the situation is therefore not as simple as described by Folklore Theorem 2 and indicated in figure 2.2 (cf. the discussion in Section 2.6). However, if μ satisfies a certain Gibbs state condition then $f_\mu=b_\mu^*=B_\mu^*$ (cf. Theorem 2.18).

Finally, Sections 2.7 and 2.8 contain our results concerning Folklore Theorem 3. These results show that $\Lambda_\mu(q)$ can be obtained by a box counting argument similar to that in Folklore Theorem 3, whereas this is not necessarily the case for $b_\mu(q)$ and $B_\mu(q)$.

2.1. *Definition of $\mathscr{P}_0$ and $\mathscr{P}_1$*

We will frequently in the following have to impose some geometrical constraints on μ. The most important constraint will be defined below. For $\mu \in \mathscr{P}(X)$, $E \subseteq \operatorname{supp} \mu$ and $a > 1$ write

$$T_a(E) = \limsup_{r \searrow 0} \left(\sup_{x \in E} \frac{\mu B(x, ar)}{\mu B(x, r)} \right)$$

We will write $T_a(x) = T_a(\{x\})$ for $x \in \operatorname{supp} \mu$.

The next lemma shows that the precise value of the number a in $T_a(E)$ is unimportant.

Lemma 2.1. *Let $\mu \in \mathscr{P}(X)$ and $E \subseteq \operatorname{supp} \mu$. Then the following statements are equivalent*

(i) $T_a(E) < \infty$ *for some* $a > 1$.

(ii) $T_a(E) < \infty$ *for all* $a > 1$.

Proof. (ii) $\Rightarrow$ (i). This is obvious.

(i) $\Rightarrow$ (ii). Choose $a > 1$ such that $T_a(E) < \infty$ and let $b > 1$. Pick $n \in \mathbb{N}$ with $b \leqslant a^n$. Then clearly

$$\begin{aligned}
T_b(E) \leqslant T_{a^n}(E) &= \limsup_{r \searrow 0} \left(\sup_{x \in E} \frac{\mu B(x, a^n r)}{\mu B(x, r)} \right) \\
&= \limsup_{r \searrow 0} \left(\sup_{x \in E} \prod_{i=1}^{n} \frac{\mu B(x, a(a^{i-1} r))}{\mu B(x, a^{i-1} r)} \right) \\
&\leqslant \prod_{i=1}^{n} \limsup_{r \searrow 0} \left(\sup_{x \in E} \frac{\mu B(x, a(a^{i-1} r))}{\mu B(x, a^{i-1} r)} \right) \\
&= T_a(E)^n < \infty. \quad \blacksquare
\end{aligned}$$

For $E \subseteq \operatorname{supp} \mu$ put

$$\mathscr{P}_0(X, E) = \{\mu \in \mathscr{P}(X) \mid \exists a > 1 \colon \forall x \in E \colon T_a(x) < \infty\}$$

$$\mathscr{P}_1(X, E) = \{\mu \in \mathscr{P}(X) \mid \exists a > 1 \colon T_a(E) < \infty\},$$

and write $\mathscr{P}_0(X, \operatorname{supp} \mu) = \mathscr{P}_0(X)$ and $\mathscr{P}_1(X, \operatorname{supp} \mu) = \mathscr{P}_1(X)$. It follows from Lemma 2.1 that $\mathscr{P}_0(X, E)$ and $\mathscr{P}_1(X, E)$ are well defined (i.e. independent of the number $a > 1$ that appears in the definition). Some differentiation results for measures μ satisfying $T_5(x) < \infty$ for $x \in \operatorname{supp} \mu$ (and consequently $T_a(x) < \infty$ for all $a > 1$) appear in [Fed, pp. 160–163] and [Mat].

2.2. *The Multifractal Measures* $\mathscr{H}_{\mu}^{q,t}$ *and* $\mathscr{P}_{\mu}^{q,t}$

Observe that the pre-measure $\bar{\mathscr{H}}_{\mu}^{q,t}$ is countably subadditive (but not necessarily monotone) and that the pre-packing measure $\bar{\mathscr{P}}_{\mu}^{q,t}$ is monotone (but not necessarily countable subadditive)---these facts will be used frequently in the subsequent parts of the paper. However, $\mathscr{H}_{\mu}^{q,t}$ and $\mathscr{P}_{\mu}^{q,t}$ are measures on the Borel algebra.

PROPOSITION 2.2. *The set function* $\mathscr{H}_{\mu}^{q,t}$ *is a metric outer measure, and thus a measure on the Borel algebra.*

PROPOSITION 2.3. *The set function* $\mathscr{P}_{\mu}^{q,t}$ *is a metric outer measure, and thus a measure on the Borel algebra.*

As in the non multifractal case, the Hausdorff dimension $\dim_{\mu}^{q}$ is majorized by the packing dimension $\operatorname{Dim}_{\mu}^{q}$.

PROPOSITION 2.4. *Let* $\mu \in \mathscr{P}(\mathbb{R}^d)$. *Then the following hold for* $q, t \in \mathbb{R}$,

(i) $\mathscr{P}_{\mu}^{q,t} \leqslant \bar{\mathscr{P}}_{\mu}^{q,t}$

(ii) $\mathscr{H}_{\mu}^{q,t} \leqslant \mathscr{P}_{\mu}^{q,t}$ *for* $q \leqslant 0$

(iii) *If* $\mu \in \mathscr{P}_0(\mathbb{R}^d)$ *then* $\mathscr{H}_{\mu}^{q,t} \leqslant \mathscr{P}_{\mu}^{q,t}$ *for* $0 < q$.

(iv) *There exists an integer* $\zeta \in \mathbb{N}$ *such that* $\mathscr{H}_{\mu}^{q,t} \leqslant \zeta \mathscr{P}_{\mu}^{q,t}$.

In particular

$$\dim_{\mu}^{q} \leqslant \operatorname{Dim}_{\mu}^{q} \leqslant \Delta_{\mu}^{q}.$$

Examples in Chapter 3 show that the inequalities in Proposition 2.4 can be strict.

2.3. *Auxiliary Inequalities Involving* $\mathscr{H}^t$, $\mathscr{P}^t$, $\mathscr{H}_{\mu}^{q,t}$ *and* $\mathscr{P}_{\mu}^{q,t}$

Below we collect the main technical lemmas in this section. Let X be a metric space and $\mu \in \mathscr{P}(X)$. Fix $\alpha \geqslant 0$, $q, t \in \mathbb{R}$ and $\delta > 0$ with

$$0 < \delta \leqslant \alpha q + t.$$

Then the following inequalities hold.

PROPOSITION 2.5. (i) $\mathscr{H}^{\alpha q + t + \delta}(\bar{X}^{\alpha}) \leqslant 2^{\alpha q + \delta} \mathscr{H}_{\mu}^{q,t}(\bar{X}^{\alpha})$ *for* $0 \leqslant q$.

(ii) $\mathscr{H}^{\alpha q + t + \delta}(\underline{X}_{\alpha}) \leqslant 2^{\alpha q + \delta} \mathscr{H}_{\mu}^{q,t}(\underline{X}_{\alpha})$ *for* $q \leqslant 0$.

(iii) *If* $0 \leqslant \alpha q + b(q)$ *then*

$$\dim(\bar{X}^{\alpha}) \leqslant \alpha q + b(q) \qquad \textit{for} \quad 0 \leqslant q$$

$$\dim(\underline{X}_{\alpha}) \leqslant \alpha q + b(q) \qquad \textit{for} \quad q \leqslant 0.$$

(iv) *If* $0 \leqslant \alpha q + B(q)$ *and* $X = \mathbb{R}^d$ *then*

$$\dim(\bar{X}_\alpha) \leqslant \alpha q + B(q) \qquad \text{for} \quad q \leqslant 0$$

$$\dim(\underline{X}^\alpha) \leqslant \alpha q + B(q) \qquad \textit{for} \quad 0 \leqslant q.$$

PROPOSITION 2.6. (i) $\mathscr{P}^{\alpha q + t + \delta}(\bar{X}^\alpha) \leqslant 2^{\alpha q + \delta} \mathscr{P}^{q,t}_\mu(\bar{X}^\alpha)$ *for* $0 \leqslant q$.

(ii) $\mathscr{P}^{\alpha q + t + \delta}(\underline{X}_\alpha) \leqslant 2^{\alpha q + \delta} \mathscr{P}^{q,t}_\mu(\underline{X}_\alpha)$ *for* $q \leqslant 0$.

(iii) *If* $0 \leqslant \alpha q + B(q)$ *then*

$$\operatorname{Dim}(\bar{X}^\alpha) \leqslant \alpha q + B(q) \qquad \textit{for} \quad 0 \leqslant q$$

$$\operatorname{Dim}(\underline{X}_\alpha) \leqslant \alpha q + B(q) \qquad \textit{for} \quad q \leqslant 0.$$

PROPOSITION 2.7. (i) *If* $A \subseteq \bar{X}^\alpha$ *is Borel then* $\mathscr{H}^{q,t}_\mu(A) \leqslant 2^t \mathscr{H}^{\alpha q + t - \delta}(A)$ *for* $q \leqslant 0$.

(ii) *If* $A \subseteq \underline{X}_\alpha$ *is Borel then* $\mathscr{H}^{q,t}_\mu(A) \leqslant 2^t \mathscr{H}^{\alpha q + t - \delta}(A)$ *for* $0 \leqslant q$.

PROPOSITION 2.8. (i) *If* $A \subseteq \bar{X}^\alpha$ *is Borel then* $\mathscr{P}^{q,t}_\mu(A) \leqslant 2^{-\alpha q + \delta} \mathscr{P}^{\alpha q + t - \delta}(A)$ *for* $q \leqslant 0$.

(ii) *If* $A \subseteq \underline{X}_\alpha$ *is Borel then* $\mathscr{P}^{q,t}_\mu(A) \leqslant 2^{-\alpha q + \delta} \mathscr{P}^{\alpha q + t - \delta}(A)$ *for* $0 \leqslant q$.

Observe that Propositions 2.5 through 2.8 yield the following well known result.

COROLLARY 2.9. *Let* X *be a metric space and* $\mu \in \mathscr{P}(X)$. *Then the following hold*

(i) $\dim(\bar{X}^\alpha) \leqslant \alpha$, $\operatorname{Dim}(\bar{X}^\alpha) \leqslant \alpha$.

(ii) *If* $A \subseteq \underline{X}_\alpha$ *and* $\mu(A) > 0$ *then* $\alpha \leqslant \dim A$, $\alpha \leqslant \operatorname{Dim} A$.

Proof. (i) Follows immediately from Proposition 2.5 (iii) and Proposition 2.6 (iii) by considering the case $q = 1$.

(ii) It is easily seen that $\mu \leqslant \mathscr{H}^{1,0}_\mu$ and Proposition 2.7 therefore implies that $0 < \mu(A) \leqslant \mathscr{H}^{1,0}_\mu(A) \leqslant \mathscr{H}^{\alpha - \delta}(A)$, i.e. $\alpha - \delta \leqslant \dim(A)$ for all $\delta > 0$. Mutatis mutandis $\alpha \leqslant \operatorname{Dim}(A)$. ∎

The results in Corollary 2.9 appear in [Bi1, Bi2, Cu1, Cu2, Fro, Haa, Yo]. Our results may thus be viewed as multifractal generalizations of Billingsley's Theorem [Bi1, Bi2] and Frostman's Lemma [Fro].

2.4. *The Multifractal Dimension Functions* b_μ, B_μ *and* Λ_μ

The next propositions summarize most of the elementary properties of $b_{\mu,E}$, $B_{\mu,E}$ and $\Lambda_{\mu,E}$.

PROPOSITION 2.10. *The following statements hold*

(i) $\overline{\mathscr{P}}^{q,t}_\mu \geqslant \overline{\mathscr{P}}^{p,t}_\mu$ *for* $q \leqslant p$, $\overline{\mathscr{P}}^{q,s}_\mu \geqslant \overline{\mathscr{P}}^{q,t}_\mu$ *for* $s \leqslant t$.

(ii) $\Lambda_{\mu,E}$ *is decreasing.*

(iii) *The map* $(q,t) \to \overline{\mathscr{P}}^{q,t}_\mu$ *is logarithmic convex, i.e.*

$$\overline{\mathscr{P}}^{\alpha p+(1-\alpha)q,\,\alpha t+(1-\alpha)s}_\mu(E) \leqslant (\overline{\mathscr{P}}^{p,t}_\mu(E))^\alpha\,(\overline{\mathscr{P}}^{q,s}_\mu(E))^{1-\alpha}$$

for all $\alpha \in [0,1]$, $p, q, t, s \in \mathbb{R}$ *and* $E \subseteq X$.

(iv) $\Lambda_{\mu,E}$ *is convex.*

(v) $\mathscr{P}^{q,t}_\mu \geqslant \mathscr{P}^{p,t}_\mu$ *for* $q \leqslant p$, $\mathscr{P}^{q,t}_\mu \geqslant \mathscr{P}^{q,s}_\mu$ *for* $t \leqslant s$.

(vi) $B_{\mu,E}$ *is decreasing and convex.*

(vii) $\mathscr{H}^{q,t}_\mu \geqslant \mathscr{H}^{p,t}_\mu$ *for* $q \leqslant p$, $\mathscr{H}^{q,t}_\mu \geqslant \mathscr{H}^{q,s}_\mu$ *for* $t \leqslant s$.

(viii) $b_{\mu,E}$ *is decreasing.*

The map $b_{\mu,E}$ need not be convex for $\mu \in \mathscr{P}_1(X)$. Section 3 contains an example (viz. Example 4) where we construct a measure $\mu \in \mathscr{P}_1(\mathbb{R})$ such that

$$b_\mu(q) = d(1-q) \wedge D(1-q),$$

for $0 < d < D < 1$. However, the next proposition shows that if $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$, then $b := b_{\mu,E}$ satisfies a "weak" form for convexity: instead of $b(\alpha p + (1-\alpha)q) \leqslant \alpha b(p) + (1-\alpha)\,b(q)$ then the following inequality holds $b(\alpha p + (1-\alpha)q) \leqslant \alpha B(p) + (1-\alpha)\,b(q)$; i.e. we have replaced the smaller number $b(p)$ with the (perhaps) somewhat larger number $B(p) := B_{\mu,E}(p)$ (here $p, q \in \mathbb{R}$ and $\alpha \in [0,1]$).

PROPOSITION 2.11. *Let* $\mu \in \mathscr{P}(\mathbb{R}^d)$, $E \subseteq \mathbb{R}^d$, $p, q \in \mathbb{R}$ *and* $\alpha \in [0,1]$.

(i) *If* $\alpha p + (1-\alpha)q \leqslant 0$ *then*

$$b_{\mu,E}(\alpha p + (1-\alpha)q) \leqslant \alpha B_{\mu,E}(p) + (1-\alpha)\,b_{\mu,E}(q)$$

(ii) *If* $0 < \alpha p + (1-\alpha)q$ *and in addition* $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$ *then*

$$b_{\mu,E}(\alpha p + (1-\alpha)q) \leqslant \alpha B_{\mu,E}(p) + (1-\alpha)\,b_{\mu,E}(q)$$

PROPOSITION 2.12. *Let* $\mu \in \mathscr{P}(\mathbb{R}^d)$.

(i) $b^*_\mu(\alpha) \leqslant B^*_\mu(\alpha) \leqslant \alpha$ *for all* $\alpha \geqslant 0$.

(ii) *If* $\alpha \in \mathbb{R}$ *and* $\mu(\{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\}) > 0$ *then*

$$b_\mu{}^*(\alpha) = B^*_\mu(\alpha) = \alpha.$$

The next proposition investigates the behaviour of $B_\mu(q)$ when $|q|$ is large. We will in fact prove that B_μ has affine asymptotes as $q \to \pm\infty$.

Now write

$$\underline{a}_\mu = \underline{a} := \sup_{0<q} -\frac{b(q)}{q} \qquad \bar{a}_\mu = \bar{a} := \inf_{q<0} -\frac{b(q)}{q}$$

$$\underline{A}_\mu = \underline{A} := \sup_{0<q} -\frac{B(q)}{q} \qquad \bar{A}_\mu = \bar{A} := \inf_{q<0} -\frac{B(q)}{q}$$

and observe that

$$\underline{A} \leqslant \underline{a}, \qquad \bar{a} \leqslant \bar{A}.$$

Note that Example 4 in Section 3 shows that there exist measures μ such that $\bar{a} < \underline{a}$, $\underline{A} < \underline{a}$, and $\bar{a} < \bar{A}$.

Also write

$$I_+ = I_+(\mu) = \left\{ -\frac{B(q)}{q} \,\middle|\, 0 < q \right\} \qquad \text{and} \qquad I_- = I_-(\mu) = \left\{ -\frac{B(q)}{q} \,\middle|\, q < 0 \right\}.$$

If A is a subset of a topological space X then A' denotes the derived set.

PROPOSITION 2.13. (i) *If $\underline{A} \in I'_+$ then the function $q \to B(q) + \underline{A}q$ is decreasing and*

$$\underline{E} := \lim_{q\to\infty} (B(q) + \underline{A}q) \geqslant 0.$$

(ii) *If $\underline{A} \notin I'_+$ then there exists $q_0 \in \mathbb{R}$ such that*

$$B(q) = -\underline{A}q \qquad \textit{for} \quad q_0 < q.$$

(iii) *If $\bar{A} \in I'_-$ then the function $q \to B(q) + \bar{A}q$ is increasing and*

$$\bar{E} := \lim_{q\to-\infty} (B(q) + \bar{A}q) \geqslant 0.$$

(iv) *If $\bar{A} \notin I'_-$ then there exists $q_1 \in \mathbb{R}$ such that*

$$B(q) = -\bar{A}q \qquad \textit{for} \quad q < q_1.$$

2.5. *Densities*

It is well known that density theorems play a major role in geometric measure theory. We will now prove some density theorems for the multifractal Hausdorff measure $\mathscr{H}^{q,t}_\mu$ and the multifractal packing measure $\mathscr{P}^{q,t}_\mu$.

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$. For $x \in \operatorname{supp} \mu$ and $q, t \in \mathbb{R}$ we define the upper and lower (q, t)-density of ν at x w.r.t. μ by

$$\bar{d}_\mu^{q,t}(x, \nu) = \limsup_{r \searrow 0} \frac{\nu(B(x, r))}{\mu(B(x, r))^q (2r)^t},$$

and

$$\underline{d}_\mu^{q,t}(x, \nu) = \liminf_{r \searrow 0} \frac{\nu(B(x, r))}{\mu(B(x, r))^q (2r)^t},$$

respectively. Below, we state our main density theorems. We note that our density theorems are inspired by the density theorems in [Ray].

Let $\nu, \mu \in \mathscr{P}(\mathbb{R}^d)$, $E \subseteq \operatorname{supp} \mu$ be a Borel subset of $\operatorname{supp} \mu$ and $q, t \in \mathbb{R}$.

THEOREM 2.14. (i) *If $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$ and $\mathscr{H}_\mu^{q,t}(E) < \infty$ then*

$$\mathscr{H}_\mu^{q,t}(E) \inf_{x \in E} \bar{d}_\mu^{q,t}(x, \nu) \leqslant \nu(E).$$

(ii) *If $\mathscr{H}_\mu^{q,t}(E) < \infty$ then*

$$\nu(E) \leqslant \mathscr{H}_\mu^{q,t}(E) \sup_{x \in E} \bar{d}_\mu^{q,t}(x, \nu).$$

THEOREM 2.15. *If $\mathscr{P}_\mu^{q,t}(E) < \infty$ then*

$$\mathscr{P}_\mu^{q,t}(E) \inf_{x \in E} \underline{d}_\mu^{q,t}(x, \nu) \leqslant \nu(E) \leqslant \mathscr{P}_\mu^{q,t}(E) \sup_{x \in E} \underline{d}_\mu^{q,t}(x, \nu).$$

COROLLARY 2.16. *If $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$ and $\mathscr{P}_\mu^{q,t}(E) < \infty$, then the following statements are equivalent.*

(i) $\mathscr{H}_\mu^{q,t}(E) = \mathscr{P}_\mu^{q,t}(E)$.

(ii) $\underline{d}_\mu^{q,t}(x, \mathscr{H}_\mu^{q,t} \mid E) = 1 = \bar{d}_\mu^{q,t}(x, \mathscr{H}_\mu^{q,t} \mid E)$ *for $\mathscr{P}_\mu^{q,t}$-a.e. $x \in E$.*

(iii) $\underline{d}_\mu^{q,t}(x, \mathscr{P}_\mu^{q,t} \mid E) = 1 = \bar{d}_\mu^{q,t}(x, \mathscr{P}_\mu^{q,t} \mid E)$ *for $\mathscr{P}_\mu^{q,t}$-a.e. $x \in E$.*

2.6. *Upper and Lower Bounds for the Multifractal Spectrum*

The next two results give upper and lower bounds for f_μ and F_μ in terms of b_μ and B_μ.

We wll first introduce some notation. If $f, g\colon \mathbb{R} \to \mathbb{R}$ are real-valued functions on $\mathbb{R}$ then we write

$$f \,\Box\, g = f \cdot 1_{]-\infty, 0[} + (f(0) \vee g(0)) \cdot 1_{\{0\}} + g \cdot 1_{]0, \infty[}$$

(i.e. $(f \,\square\, g)(x)$ is equal to $f(x)$ for $x<0$, to $f(0) \vee g(0)$ for $x=0$ and to $g(x)$ for $0<x$). Also, if $f\colon \mathbb{R} \to \mathbb{R}$ is a convex function then we will denote the left and right derivative of f by f'_- and f'_+ respectively.

Finally, put

$$\operatorname{dom} B'_\mu := \{q \in \mathbb{R} \mid B_\mu \text{ is differentiable at } q\},$$

$$\operatorname{ran} B'_\mu := \{B'_\mu(q) \mid q \in \operatorname{dom} B'_\mu\}.$$

THEOREM 2.17 (Upper Bound Estimate). *Let X be a metric space, $\mu \in \mathscr{P}(X)$ and $a \geqslant 0$. Then the following assertions hold*

(i)
$$\underline{a} \leqslant \inf \bar{\alpha}_\mu(x) \leqslant \sup \bar{\alpha}_\mu(x) \leqslant \bar{A},$$
$$\underline{A} \leqslant \inf \underline{\alpha}_\mu(x) \leqslant \sup \underline{\alpha}_\mu(x) \leqslant \bar{a}.$$

(ii)
$$\lim_{\varepsilon \searrow 0} \dim(\bar{X}_{\alpha-\varepsilon} \cap \bar{X}^{\alpha+\varepsilon}) = \begin{cases} \leqslant (B \square b)^*(\alpha) & \alpha \in]\underline{a}, \bar{A}[\\ =0 & \alpha \in \mathbb{R}_+ \backslash [\underline{a}, \bar{A}] \end{cases}$$

(iii)
$$\lim_{\varepsilon \searrow 0} \dim(\underline{X}_{\alpha-\varepsilon} \cap \underline{X}^{\alpha+\varepsilon}) = \begin{cases} \leqslant (b \square B)^*(\alpha) & \alpha \in]\underline{A}, \bar{a}[\\ =0 & \alpha \in \mathbb{R}_+ \backslash [\underline{A}, \bar{a}] \end{cases}$$

(iv)
$$\dim(\underline{X}_\alpha \cap \bar{X}^\alpha) = \begin{cases} \leqslant b^*(\alpha) & \alpha \in]\underline{a}, \bar{a}[\\ 0 & \alpha \in \mathbb{R}_+ \backslash [\underline{a}, \bar{a}] \end{cases}$$

(v)
$$\operatorname{Dim}(\underline{X}_\alpha \cap \bar{X}^\alpha) = \begin{cases} \leqslant B^*(\alpha) & \alpha \in]\underline{a}, \bar{a}[\\ 0 & \alpha \in \mathbb{R}_+ \backslash [\underline{a}, \bar{a}]. \end{cases}$$

THEOREM 2.18 (Lower Bound Estimate). *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. For each $q \in \mathbb{R}$ let $t_q \in \mathbb{R}$, r_q, $\underline{K}_q$, $\bar{K}_q \in]0, \infty[$, $\nu_q \in \mathscr{P}(\operatorname{supp} \mu)$, and $\varphi_q\colon \mathbb{R}_+ \to \mathbb{R}$ be a function. For each $q \in \mathbb{R}$ let $(r_{q,n})_n$ be a sequence in $]0, 1[$ satisfying $r_{q,n} \searrow 0$, $\log r_{q,n+1}/\log r_{q,n} \to 1$ and $\sum_n r^\varepsilon_{q,n} < \infty$ for all $\varepsilon > 0$.*

For each $q \in \mathbb{R}$ consider the conditions

(1) $\forall x \in \operatorname{supp} \mu$: $\forall r \in]0, r_q[$: $\underline{K}_q \leqslant \nu_q(B(x,r))/(\mu(B(x,r))^q (2r)^{t_q} e^{\varphi_q(r)}) \leqslant \bar{K}_q$.

(2) $\varphi_q(r) = o(\log r)$ *as* $r \searrow 0$.

(3) $c_{q,n}(p) := (1/-\log r_{q,n}) \log(\int_{\operatorname{supp}\mu} \mu(B(x, r_{q,n}))^p \, d\nu_q(x))$ *is finite for all* $n \in \mathbb{N}$ *and* $q, p \in \mathbb{R}$.

(4) $c_q(p) := \lim_n c_{q,n}(p)$ *exists and is finite for all* $q, p \in \mathbb{R}$.

Then the following hold

(i) *Let* $q \in \mathbb{R}$ *and assume that* (1), (2), (3) *and* (4) *are satisfied and write* $c = c_q$. *Then*

$$\left.\begin{array}{ll} \text{For} & q \leqslant 0,\ b_\mu^*(-c'_{-}(0)) \leqslant -c'_{-}(0)\, q + b_\mu(q) \leqslant -c'_{-}(0)\, q + \Lambda_\mu(q) \\ \text{For} & 0 \leqslant q,\ b_\mu^*(-c'_{+}(0)) \leqslant -c'_{+}(0)\, q + b_\mu(q) \leqslant -c'_{+}(0)\, q + \Lambda_\mu(q) \end{array}\right\}$$
$$\leqslant \dim(\underline{X}_{c'_{+}(0)} \cap \bar{X}^{c'_{-}(0)})$$

where for $\alpha \geqslant 0$,

$$\underline{X}_\alpha = \{x \in \operatorname{supp}\mu \mid \alpha \leqslant \underline{\alpha}_\mu(x)\}, \qquad \bar{X}^\alpha = \{x \in \operatorname{supp}\mu \mid \bar{\alpha}_\mu(x) \leqslant \alpha\}.$$

(ii) *Let* $q \in \mathbb{R}$ *and assume that* (1), (2), (3) *and* (4) *are satisfied and write* $c = c_q$. *If* c *is differentiable at* 0, *then*

$$f_\mu(-c'(0)) = b_\mu^*(-c'(0)) = B_\mu^*(-c'(0)) = \Lambda_\mu^*(-c'(0)).$$

Assume further that $0 < \liminf_n r_{q,n+1}/r_{q,n} \leqslant \limsup_n r_{q,n+1}/r_{q,n} < \infty$. *Then the following hold*

(iii) *Assume* (1), (2), (3) *and* (4) *are satisfied for all* $q \in \mathbb{R}$. *Then*

$$\alpha_\mu = -B'_\mu(q)\ \nu_q\text{-a.e.} \qquad \text{for} \quad q \in \operatorname{dom} B'_\mu$$
$$-\operatorname{ran} B'_\mu \subseteq \alpha_\mu(\operatorname{supp}\mu).$$

(iv) *Assume* (1), (2), (3) *and* (4) *are satisfied for all* $q \in \mathbb{R}$. *Then*

$$b_\mu^* = f_\mu = B_\mu^* \qquad \text{on} \quad -\operatorname{ran} B'_\mu.$$

The proof of Theorem 2.17 is based on some Vitali type arguments, whereas the proof of Theorem 2.18 is inspired by some large deviations theorems in [El1, El2], in particular [El2, Theorem II.6.1, Theorem II.6.3 and Theorem II.6.4]. We note that somewhat similar arguments have been used previously by Collet et al. [Col].

We note that condition (1) in Theorem 2.18 obviously is motivated by the theory of Gibbs states (cf. [Bow, Ru]) and is satisfied in the case of graph directed self-similar measures with totally disconnected support (Lemma 5.4 and Lemma 5.5) and "cookie-cutter" measures (Lemma 6.7). Theorem 2.18 shows that b_μ and B_μ contain more information than f_μ provided that the conditions in Theorem 2.18 are satisfied. The functions b_μ and B_μ contain according to Theorem 2.18.iv the same information as f_μ, whereas Theorem 2.18.iii shows that B_μ in addition contains information about the size of $\alpha_\mu(\operatorname{supp}\mu)$.

We would like to emphasize that the upper bounds obtained in Theorem 2.17 are in general not exact values. This is basically due to the fact that the multifractal spectrum f_μ is not necessarily convex.

EXAMPLE. Let $p_1 = p_2 = \frac{1}{2}$. Put $r_1 = p_1^4$ and $r_2 = p_2^2$ and define maps $f_1, f_2 : [0, 1] \to [0, 1]$ by $f_1(x) = r_1 x$ and $f_2(x) = r_2 x + (1 - r_2)$. For $n \in \mathbb{N}$ and $i_1, ..., i_n \in \{1, 2\}$ write

$$K_{i_1 \cdots i_n} = f_{i_1} \circ \cdots \circ f_{i_n}([0, 1])$$

and define a probability measure ν on $[0, 1]$ by the requirement

$$\nu(K_{i_1 \cdots i_n}) = p_{i_1} \cdots p_{i_n}$$

for all $n \in \mathbb{N}$ and $i_1, ..., i_n \in \{1, 2\}$.

Next, put $s_1 = p_1^2$ and $s_2 = p_2$ and define maps $g_1, g_2 : [2, 3] \to [2, 3]$ by $g_1(x) = s_1 x + 2(1 - s_1)$ and $g_2(x) = s_2 x + 3(1 - s_2)$. For $n \in \mathbb{N}$ and $i_1, ..., i_n \in \{1, 2\}$ write

$$L_{i_1 \cdots i_n} = g_{i_1} \circ \cdots \circ g_{i_n}([2, 3])$$

and define a probability measure λ on $[2, 3]$ by the requirement

$$\lambda(L_{1_1 \cdots i_n}) = p_{i_1} \cdots p_{i_n}$$

for all $n \in \mathbb{N}$ and $i_1, ..., i_n \in \{1, 2\}$. The measures ν and λ are, in fact, graph directed self-similar measures. The multifractal structure of graph directed self-similar measures will be treated in detail in Chapter 5 using our formalism.

It follows immediately from [Ca, pp. 202–206] (or Theorem 5.1 in Section 5) that the following hold

(1) $\operatorname{supp} f_\nu = [\frac{1}{4}, \frac{1}{2}]$, $\operatorname{supp} f_\lambda = [\frac{1}{2}, 1]$.

(2) f_ν is strictly concave on $[\frac{1}{4}, \frac{1}{2}]$ and f_λ are strictly concave on $[\frac{1}{2}, 1]$.

(3) $f_\nu(\frac{1}{4}) = 0 = f_\nu(\frac{1}{2})$, $f_\lambda(\frac{1}{2}) = 0 = f_\lambda(1)$.

Now put $\mu = \frac{1}{2}(\nu + \lambda) \in \mathscr{P}(\mathbb{R})$. Since $\operatorname{dist}(\operatorname{supp}\nu, \operatorname{supp}\lambda) \geqslant 1 > 0$, $\alpha_\mu = \alpha_\nu 1_{\operatorname{supp}\nu} + \alpha_\lambda 1_{\operatorname{supp}\lambda}$, whence

$$\begin{aligned} f_\mu(\alpha) &= \dim\{x \in \operatorname{supp}\mu \mid \alpha_{1/2(\nu+\lambda)}(x) = \alpha\} \\ &= \dim(\{x \in \operatorname{supp}\nu \mid \alpha_\nu(x) = \alpha\} \cup \{x \in \operatorname{supp}\lambda \mid \alpha_\lambda(x) = \alpha\}) \\ &= f_\nu(\alpha) \vee f_\lambda(\alpha). \end{aligned}$$

Properties (1) through (3) therefore imply that f_μ is non-concave and consequently (since $f_\mu \leqslant b_\mu^*$ on supp μ by Theorem 2.17)

$$f_\mu(\alpha) < b_\mu^*(\alpha)$$

for α belonging to a non-degenerate interval.

EXAMPLE. Let $a \in]-1, 0[$ and define $f_a: \mathbb{R} \to \mathbb{R}$ by $f_a(x) = x^a$ for $x \in]0, 1[$ and $f_a(x) = 0$ otherwise. Then $\mu := (a+1) f_a \, dx \in \mathscr{P}(\mathbb{R})$ and supp $\mu = [0, 1]$. It follows from Example 2 in Section 3 that

$$b_\mu(q) = B_\mu(q) = -(a+1)\, q \vee (1-q).$$

Hence

$$b_\mu^*(\alpha) = B_\mu^*(\alpha) = \begin{cases} -\dfrac{1}{a}\alpha + \left(1 + \dfrac{1}{a}\right) & \alpha \in [\underline{a}, \bar{a}] \\ -\infty & \alpha \in \mathbb{R}_+ \setminus [\underline{a}, \bar{a}], \end{cases}$$

where $\underline{a} = a + 1$ and $\bar{a} = 1$. Moreover, it is easily seen that

$$\alpha_\mu(x) = 1 \qquad \text{for} \quad x \in]0, 1[, \qquad \alpha_\mu(0) = a + 1, \tag{2.1}$$

whence

$$f_\mu(\alpha) = F_\mu(\alpha) = \begin{cases} 1 & \alpha = 1 \\ 0 & \alpha \in \mathbb{R}_+ \setminus \{1\}. \end{cases}$$

Hence in this example we clearly have $f_\mu(\alpha) < b_\mu^*(\alpha)$ for $\alpha \in [\underline{a}, \bar{a}[$. However, we believe that the functions b_μ and B_μ contain more information about the measure μ than the spectra functions f_μ and F_μ: b_μ shows that there exist points $x \in \operatorname{supp} \mu$ such that $\mu B(x, r) \sim r^{\alpha+1}$ for $r \approx 0$ (viz. $x = 0$) and that there exist points $x \in \operatorname{supp} \mu$ such that $\mu B(x, r) \sim r$ for $r \approx 0$ (viz. $x \in]0, 1]$), whereas f_μ and F_μ do not contain this information.

Another reason for preferring b_μ and B_μ rather than f_μ and F_μ is that two measures may have different multifractal structure but still possessing the same spectra functions, whereas different multifractal structure often is displayed in b_μ and B_μ. Let μ be as above and let λ denote the restriction of the Lebesque measure to $[0, 1]$. Then clearly

$$f_\mu = f_\lambda = F_\mu = F_\lambda,$$

eventhough μ and λ have different multifractal structure: μ has points with different local dimension (cf. (2.1)), whereas $\alpha_\lambda(x) = 1$ for all $x \in [0, 1]$. The difference in the multifractal structure between μ and λ is on the other hand apparent in b_μ and b_λ since (cf. Section 3, Ex. 1)

$$b_\mu(q) = -(a+1)\, q \vee (1-q),$$

and

$$b_\lambda(q) = 1-q.$$

Hence we believe that the functions b_μ and B_μ, in some cases, are more fundamental in multifractal analysis than the spectra functions f_μ and F_μ, cf. the discussion after Theorem 2.18.

However, we do prove that the functions b_μ^* and B_μ^* are the exact values of f_μ and F_μ in the following cases (and not just upper bounds as asserted by Theorem 2.17):

Case 1. For graph directed measures in $\mathbb{R}^d$ with totally disconnected support. Let $G = (V, E, (r_e)_e, (T_e)_e, (p_e)_e)$ be a strongly connected MW-graph with probabilities, and let $(K_u)_{u \in V}$ and $(\mu_u)_{u \in V}$ be the self-similar invariant sets and measures associated with G respectively (details will be given in Section 5). Let β be the auxiliary function that appears in [Ca, Ed] and $\alpha = -\beta'$. Put $K_u(a) = \{x \in K_u \mid \alpha_{\mu_u}(x) = a\}$ for $a \geqslant 0$. We then prove the following theorem in Section 5. Let Δ be the separation constant defined in equation (5.1) in Section 5.

THEOREM 5.1. *Assume* $\Delta > 0$. *Then*

(i) *For each* $q \in \mathbb{R}$,

$$0 < \mathscr{H}_{\mu_u}^{q,\beta(q)}(K_u(\alpha(q))) \leqslant \mathscr{P}_{\mu_u}^{q,\beta(q)}(K_u(\alpha(q))) \leqslant \overline{\mathscr{P}}_{\mu_u}^{q,\beta(q)}(K_u) < \infty.$$

(ii) *For each* $q \in \mathbb{R}$ *there exists a number* $c_q \in \,]0, \infty[$ *such that*

$$\mathscr{H}_{\mu_u}^{q,\beta(q)} \mid \operatorname{supp} \mu_u = c_q \mathscr{P}_{\mu_u}^{q,\beta(q)} \mid \operatorname{supp} \mu_u.$$

(iii) $\alpha_{\mu_u}(x) = \alpha(q)$ *for* $\mathscr{H}_{\mu_u}^{q,\beta(q)} \mid \operatorname{supp} \mu_u$-a.a. x,

$\alpha_{\mu_u}(x) = \alpha(q)$ *for* $\mathscr{P}_{\mu_u}^{q,\beta(q)} \mid \operatorname{supp} \mu_u$-a.a. x.

(iv) *If* $q, p \in \mathbb{R}$ *and* $\alpha(q) \neq \alpha(p)$ *then*

$$(\mathscr{H}_{\mu_u}^{q,\beta(q)} \mid \operatorname{supp} \mu_u) \perp (\mathscr{H}_{\mu_u}^{p,\beta(p)} \mid \operatorname{supp} \mu_u),$$

$$(\mathscr{P}_{\mu_u}^{q,\beta(q)} \mid \operatorname{supp} \mu_u) \perp (\mathscr{P}_{\mu_u}^{p,\beta(p)} \mid \operatorname{supp} \mu_u).$$

(v) *For each* $q \in \mathbb{R}$

$$b_{\mu_u}(q) = B_{\mu_u}(q) = \Delta_{\mu_u}^q(K_u) = C_{\mu_u}^q(K_u) = (1-q)\, D_{\mu_u}^q = \beta(q).$$

(vi) $\underline{a}_{\mu_u} = \underline{A}_{\mu_u} = \inf_{x \in K_u} \underline{\alpha}_{\mu_u}(x) := \underline{a}$, $\bar{a}_{\mu_u} = \bar{A}_{\mu_u} = \sup_{x \in K_u} \bar{\alpha}_{\mu_u}(x) := \bar{a}$.

(vii) $\dim K_u(\alpha) = \operatorname{Dim} K_u(\alpha) = b_{\mu_u}^*(\alpha) = B_{\mu_u}^*(\alpha) = \beta^*(\alpha)$ *for* $\alpha \in \,]\underline{a}, \bar{a}[$.

Here $C^q_{\mu_u}(K_\mu)$ denotes the multifractal q-box dimension of K_u w.r.t. μ_u (cf. Section 2.7), and $D^q_{\mu_u}$ denotes the generalized Rényi dimension of μ_u (cf. Section 2.8). We note that the result in (ii) was first proved by Spear [Sp], in a slightly more general setting, for the case $q=0$. We also note that the results in (v) and (vii) are minor extensions of the results in [Ca, Ed]. In [Ca] and [Ed] it was proved that $f_{\mu_u}=F_{\mu_u}$ (in a slightly more general setting), whereas we also prove that $f_{\mu_u}=F_{\mu_u}=(C^q_{\mu_u}(K_u))^*=((1-q)\,D^q_{\mu_u})^*$.

Finally we note that a result very similar to the equation $\beta(q)=C^q_{\mu_u}(K_u)=(1-q)\,D^q_{\mu_u}$ has been proved in a recent paper by Strichartz [Str, Theorem 3.2] for the case $1<q<\infty$.

It is an open problem whether the equations

$$f_{\mu_u}=\beta^*, \qquad F_{\mu_u}=\beta^*$$

hold in the case where the support of μ_u is not necessarily totally disconnected, cf. [Ca, p. 215] and [Ed, Section 5.3, Question (d)]. Cf. also Section 7.8 in Chapter 7 and Note Added in Proof (2) at the end of this paper.

Case 2. For "cookie-cutter" measures in $\mathbb{R}$. Let g be a "cookie-cutter" map in $\mathbb{R}$ with invariant set $\Lambda(g)=\Lambda$. Let $\varphi\colon \Lambda\to\mathbb{R}$ be a Hölder continuous function and let ν be the "cookie-cutter" measure associated with φ (details will be given in Chapter 6). Let τ be the auxiliary function that appears in [Ra] and $\alpha=-\tau'$. Put $\Lambda(a)=\{x\in\Lambda \mid \alpha_\nu(x)=a\}$ for $a\geqslant 0$. We then prove the following theorem in Section 6.

THEOREM 6.1. *The following assertions hold*

(i)
$$0<\mathscr{H}^{q,\tau(q)}_\nu(\Lambda(\alpha(q)))\leqslant\mathscr{P}^{q,\tau(q)}_\nu(\Lambda(\alpha(q)))\leqslant\overline{\mathscr{P}}^{q,\tau(q)}_\nu(\Lambda)<\infty.$$

(ii) *For each* $q\in\mathbb{R}$ *there exists a number* $c_q\in\,]0,\infty[$ *such that*

$$\mathscr{H}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu\leqslant\mathscr{P}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu\leqslant c_q\,\mathscr{H}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu.$$

(iii)
$$\alpha_\nu(x)=\alpha(q) \quad \textit{for} \quad \mathscr{H}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu\text{-a.a. } x,$$
$$\alpha_\nu(x)=\alpha(q) \quad \textit{for} \quad \mathscr{P}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu\text{-a.a. } x.$$

(iv) *If* $q,p\in\mathbb{R}$ *and* $\alpha(q)\neq\alpha(p)$ *then*

$$(\mathscr{H}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu)\perp(\mathscr{H}^{p,\beta(p)}_\nu \mid \operatorname{supp}\nu),$$
$$(\mathscr{P}^{q,\beta(q)}_\nu \mid \operatorname{supp}\nu)\perp(\mathscr{P}^{p,\beta(p)}_\nu \mid \operatorname{supp}\nu).$$

(v) *For each* $q \in \mathbb{R}$,

$$b_\nu(q) = B_\nu(q) = \Delta_\nu^q(\Lambda(g)) = C_\nu^q(\Lambda(g)) = (1-q)\, D_\nu^q = \tau(q).$$

(vi) $\underline{a}_\nu = \underline{A}_\nu := \underline{a},\ \bar{a}_\nu = \bar{A}_\nu := \bar{a}.$

(vii) $\dim \Lambda(\alpha) = \operatorname{Dim} \Lambda(\alpha) = b_\nu^*(\alpha) = \tau^*(\alpha)$ *for* $\alpha \in \left]\underline{a}, \bar{a}\right[$.

Here $C_\nu^q(\Lambda(g))$ denotes the multifractal q-box dimension of $\Lambda(g)$ w.r.t. ν (c.f. Section 2.7), and D_ν^q denotes the generalized Rényi dimension of ν. We note that the result in (vii) is a slight extension of the result in [Ra]. Rand [Ra] proves that $\dim \Lambda(\alpha) = \tau^*(\alpha)$, whereas we in addition show that $\dim \Lambda(\alpha) = \operatorname{Dim} \Lambda(\alpha)$, i.e. $\Lambda(\alpha)$ is a fractal in the sense of Taylor [Tay1, Tay2].

Note that Theorem 5.1 and Theorem 6.1 show that graph directed self-similar measures in $\mathbb{R}^d$ with totally disconnected support and "cookie-cutter" measures on $\mathbb{R}$ are Taylor multifractal measures (c.f. (1.6)).

2.7. *Multifractal Box Dimensions*

We begin by recalling the definition of the upper and lower box-dimension. Let $E \subseteq \mathbb{R}^d$ and $M_\delta(E)$ denote the smallest number of sets of diameter at most δ which can cover E. Then the lower and upper box-dimension of E respectively are defined as

$$\underline{C}(E) = \liminf_{\delta \searrow 0} \frac{\log M_\delta(E)}{-\log \delta}$$

$$\bar{C}(E) = \limsup_{\delta \searrow 0} \frac{\log M_\delta(E)}{-\log \delta}.$$

If $\bar{C}(E) = \underline{C}(E)$ we refer to the common value as the box-dimension and denote it by $C(E)$. If $N_\delta(E)$ denotes the largest number of disjoint balls of radius δ with centres in E then

$$\underline{C}(E) = \liminf_{\delta \searrow 0} \frac{\log N_\delta(E)}{-\log \delta}, \qquad \bar{C}(E) = \limsup_{\delta \searrow 0} \frac{\log N_\delta(E)}{-\log \delta}$$

by [Fa2, p. 41]. The reader is referred to [Fa2] for more information about box-dimensions.

We will now define multifractal box-dimensions. Let $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $q \in \mathbb{R}$. For $E \subseteq \mathbb{R}^d$ and $\delta > 0$ write

$$S_{\mu,\delta}^q(E) = \sup \left\{ \sum_i \mu(B(x_i, \delta))^q \mid (B(x_i, \delta))_{i \in \mathbb{N}} \text{ is a centered packing of } E \right\}.$$

The upper respectively lower multifractal q-box dimension $\overline{C}_\mu^q(E)$ and $\underline{C}_\mu^q(E)$ of E (with respect to the measure μ) is defined by

$$\overline{C}_\mu^q(E) = \limsup_{\delta \searrow 0} \frac{\log S_{\mu,\delta}^q(E)}{-\log \delta}$$

$$\underline{C}_\mu^q(E) = \liminf_{\delta \searrow 0} \frac{\log S_{\mu,\delta}^q(E)}{-\log \delta}.$$

If $\overline{C}_\mu^q(E) = \underline{C}_\mu^q(E)$ we refer to the common value as the q-box dimension of E (with respect to the measure μ) and denote it by $C_\mu^q(E)$. A somewhat similar definition appears in [Fa2, p. 225] and [Str]. Also observe that

$$\underline{C}_\mu^0(E) = \underline{C}(E), \qquad \overline{C}_\mu^0(E) = \overline{C}(E),$$

There is another equally natural way to define q-box dimensions. For $q \in \mathbb{R}$ and $\delta > 0$ write

$$T_{\mu,\delta}^q(E) = \inf\left\{\sum_i \mu(B(x_i,\delta))^q \mid (B(x_i,\delta))_i \text{ is a centered covering of } E\right\}$$

and set

$$\overline{L}_\mu^q(E) = \limsup_{\delta \searrow 0} \frac{\log T_{\mu,\delta}^q(E)}{-\log \delta}$$

$$\underline{L}_\mu^q(E) = \liminf_{\delta \searrow 0} \frac{\log T_{\mu,\delta}^q(E)}{-\log \delta}.$$

The next results summarize the most important inequalities between $\underline{C}_\mu^q$, $\overline{C}_\mu^q$, $\underline{L}_\mu^q$, $\overline{L}_\mu^q$ and Δ_μ^q.

PROPOSITION 2.19. *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $E \subseteq \mathbb{R}^d$. Then*

(i) $\dim_\mu^q(E) \leqslant \underline{L}_\mu^q(E) = \underline{C}_\mu^q(E)$ *for* $q \leqslant 0$.

(ii) $\overline{L}_\mu^q(E) = \overline{C}_\mu^q(E) = \Delta_\mu^q(E)$ *for* $q \leqslant 0$.

PROPOSITION 2.20. *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $E \subseteq \mathbb{R}^d$. Then*

(i) $\underline{L}_\mu^q(E) \leqslant \underline{C}_\mu^q(E)$ *for* $0 < q$.

(ii) $\overline{L}_\mu^q(E) \leqslant \overline{C}_\mu^q(E) \leqslant \Delta_\mu^q(E)$ *for* $0 < q$.

PROPOSITION 2.21. *Let $E \subseteq \mathbb{R}^d$ and $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$. Then*

(i) $\dim_\mu^q(E) \leqslant \underline{L}_\mu^q(E)$ *for* $0 < q$.

PROPOSITION 2.22.. *Let* $E \subseteq \mathbb{R}^d$ *and* $\mu \in \mathscr{P}_1(\mathbb{R}^d, E)$. *Then*

(i) $\dim_\mu^q(E) \leqslant \underline{L}_\mu^q(E) = \underline{C}_\mu^q(E)$ *for* $0 < q$.

(ii) $\bar{L}_\mu^q(E) = \bar{C}_\mu^q(E) = \Delta_\mu^q(E)$ *for* $0 < q$.

By combining Theorem 2.17, Proposition 2.19 and Proposition 2.22 we get the following corollary.

COROLLARY 2.23. *If* $\mu \in \mathscr{P}_1(\mathbb{R}^d)$ *then*

$$f_\mu(\alpha) \leqslant \inf_q (\alpha q + \underline{C}_\mu^q(\operatorname{supp} \mu)) \qquad \alpha \in]\underline{a}, \bar{a}[$$

It is known that the inequality in the previous corollary can be replaced by equality in certain cases, cf. [Ra, Ca, Ed, Lo1]. However, Corollary 2.23 shows that $f_\mu(\alpha)$ always is majorized by $\inf_q(q\alpha + \underline{C}_\mu^q(\operatorname{supp} \mu))$ for $\mu \in \mathscr{P}_1(\mathbb{R}^d)$.

2.8. *Generalized Rényi Dimensions*

In 1983 Hentschel & Procaccia [He], Grassberger & Procaccia [Gr1] and Grassberger [Gr2] proposed a multifractal formalism parallel to (but independent of) the $f(\alpha)$ formalism introduced by Halsey et al. [Ha]. Hentschel & Procaccia [He] and Grassberger & Procaccia [Gr1] introduced a one-parameter family of numbers $(D_q)_{q \in \mathbb{R}}$ based on some generalized entropies due to Rényi [Re1, Re2]. Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. For $q \in \mathbb{R}$ and $\delta > 0$ write

$$h_\delta^q(\mu) = \frac{1}{q-1} \sup \left\{ \log \sum_i \mu B(x_i, \delta)^q \,\middle|\, (B(x_i, \delta))_i \text{ is a centered packing of } \operatorname{supp} \mu \right\}$$

$$= \frac{1}{q-1} \log(S_{\mu, \delta}^q(\operatorname{supp} \mu)) \qquad \text{for} \quad q \neq 1,$$

and

$$h_\delta^1(\mu) = \inf \left\{ \sum_i \mu(E_i) \log \mu(E_i) \mid (E_i)_i \text{ is countable Borel partition of } \operatorname{supp} \mu,\ \operatorname{diam} E_i \leqslant \delta \right\}$$

The numbers $h_\delta^q(\mu)$ are intimately connected with generalized Rényi entropies [Re1, Re2]: Let $p = (p_1, ..., p_n)$ be a probability vector (i.e. $p_i \geqslant 0$

and $\sum p_i = 1$) and $\alpha \in \mathbb{R}\setminus\{1\}$, then the α Rényi entropy $I_\alpha(p)$ of p is defined by

$$I_\alpha(p) = \frac{1}{1-\alpha} \log_2 \left(\sum_i p_i^\alpha \right).$$

The generalized entropies I_α where introduced by A. Rényi [Re1] in 1960 in an attempt to characterize the class of mean value functions which induce additive entropy functions. The reader is referred to [Ac] for more information about this question and Rényi entropies in general.

Following Hentschel & Procaccia [He, formula (3.13)] we define the q Rényi dimensions $\underline{D}_\mu^q$ and $\bar{D}_\mu^q$ of μ by

$$\underline{D}_\mu^q = \liminf_{\delta \searrow 0} \frac{h_\delta^q(\mu)}{-\log \delta}$$

$$\bar{D}_\mu^q = \limsup_{\delta \searrow 0} \frac{h_\delta^q(\mu)}{-\log \delta}$$

(in [He] all limits are assumed to exist and Hentschel et al. therefore only consider $D_\mu^q = \lim_{\delta \searrow 0} h_\delta^q(\mu)/-\log \delta$). A parallel development of q Rényi dimensions using integrals was also suggested in [He, formula (3.14)]. For $r > 0$ and $q \in \mathbb{R}\setminus\{0\}$ write

$$I_{\mu,r}^q = \frac{1}{q} \log \left(\int_{\operatorname{supp}\mu} \mu(B(x,r))^q \, d\mu(x) \right) \qquad \text{for} \quad q \neq 0$$

$$I_{\mu,r}^0 = \int_{\operatorname{supp}\mu} \log \mu(B(x,r)) \, d\mu(x) \qquad \text{for} \quad q = 0$$

and

$$\bar{I}_\mu^q = \limsup_{r \searrow 0} \frac{I_{\mu,r}^q}{-\log r}$$

$$\underline{I}_\mu^q = \liminf_{r \searrow 0} \frac{I_{\mu,r}^q}{-\log r}.$$

Observe that

$$I_{\mu,r}^q = \log \|\mu(B(\cdot, r))\|_q,$$

where $\|\cdot\|_q$ denotes the usual q-norm; such norms are usually only defined for $q>0$ but we will also allow $q<0$. The numbers $\bar{I}_\mu^q$ and $\underline{I}_\mu^q$ have been studied by Cutler [Cu3] who investigated the relation between $\bar{I}_\mu^q$ and $\int \bar{\alpha}_\mu(x)\,d\mu(x)$ (and $\underline{I}_\mu^q$ and $\int \bar{\alpha}_\mu(x)\,d\mu(x)$). Our main result states that $(q-1)\,\underline{D}_\mu^q \vee (q-1)\,\bar{D}_\mu^q$ and $(q-1)\,\underline{I}_\mu^{q-1} \vee (q-1)\,\bar{I}_\mu^{q-1}$ are equal to $\Delta_\mu^q(\operatorname{supp}\mu)$.

THEOREM 2.24. *Let $\mu\in\mathscr{P}_1(\mathbb{R}^d)$. Then the following holds*

(i) $\Delta_\mu^q(\operatorname{supp}\mu)=(q-1)\,\underline{D}_\mu^q \vee (q-1)\,\bar{D}_\mu^q$.

(ii) $\Delta_\mu^q(\operatorname{supp}\mu)=(q-1)\,\underline{I}_\mu^{q-1} \vee (q-1)\,\bar{I}_\mu^{q-1}$.

It follows from Theorem 2.24 that our multifractal formalism contains the generalized Rényi dimensions $\underline{D}_\mu^q$ and $\bar{D}_\mu^q$, and $\underline{I}_\mu^{q-1}$ and $\bar{I}_\mu^{q-1}$ in a very natural way.

3. SOME EXAMPLES

Before we turn toward the proofs of the results stated in Section 2 we consider four examples in order to illustrate the concepts that we have introduced.

3.1.

EXAMPLE 1. Let $\varnothing\neq X\subseteq\mathbb{R}^d$ be a bounded Borel set such that $\lambda_d(X)>0$ (λ_d denotes the Lebesgue measure in $\mathbb{R}^d$). Let $f\in\mathscr{L}_1(X)$ and assume $\gamma:=\inf_x f(x)>0$ and $\Gamma:=\sup_x f(x)<\infty$. Let $\lambda_d \mid X$ denote the restriction of λ_d to X and put $\mu=(\int f\,d\lambda_d)^{-1} f d(\lambda_d \mid X)$. Then

$$\dim_\mu^q(X)=\operatorname{Dim}_\mu^q(X)=\Delta_\mu^q(X)=d-dq \qquad \text{for} \quad 0\leqslant q \tag{3.1}$$

$$\dim_\mu^q(X)\geqslant d-dq \qquad \text{for} \quad q\in\mathbb{R}. \tag{3.2}$$

If in addition

$$0<\liminf_{r\to 0}\frac{\lambda_d(B(x,r)\cap X)}{\lambda_d(B(x,r))} \qquad \text{for all} \quad x\in X, \tag{3.3}$$

then

$$\dim_\mu^q(X)=\operatorname{Dim}_\mu^q(X)=d-dq \qquad \text{for} \quad q<0. \tag{3.4}$$

We will now prove (3.1), (3.2) and (3.4). Write $\Omega=\lambda_d(B(0,1))$ and $I=\int f\,d\lambda_d$.

Claim 1. $d-dq \leqslant \dim_\mu^q(X)$ for $q \in \mathbb{R}$. Proof of Claim 1: Put

$$X_m = \left\{ x \in X \,\middle|\, \frac{1}{2} \leqslant \frac{\lambda_d(B(x,r) \cap X)}{\lambda_d(B(x,r))} \leqslant \frac{3}{2} \text{ for } 0 < r < \frac{1}{m} \right\}, \qquad m \in \mathbb{N}.$$

It follows from Lebesgues differentiation theorem that $\lambda_d(\bigcup_m X_m) = \lambda_d(X) > 0$, and we may thus choose $N \in \mathbb{N}$ such that $\lambda_d(X_N) \geqslant \frac{1}{2}\lambda_d(X)$. Let $(B(x_i, r_i))_{i \in \mathbb{N}}$ be a centred $1/N$-covering of X_N. Then

$$\begin{aligned}
&\sum_i \mu(B(x_i, r_i))^q (2r_i)^{d-dq} \\
&= 2^{d(1-q)} I^{-q} \sum_i \left(\int_{B(x_i, r_i) \cap X} f \, d\lambda_d \right)^q (r_i^d)^{1-q} \\
&\geqslant (2^d/\Omega)^{(1-q)} (\gamma^q \wedge \Gamma^q) I^{-q} \sum_i \lambda_d(B(x_i, r_i) \cap X)^q \lambda_d(B(x_i, r_i))^{1-q} \\
&\geqslant c_0 ((\tfrac{1}{2})^q \wedge (\tfrac{3}{2})^q) \sum_i \lambda_d(B(x_i, r_i)) \\
&\geqslant c_1 \lambda_d \left(\bigcup_i B(x_i, r_i) \right) \geqslant c_1 \lambda_d(X_N) \geqslant c_1 \tfrac{1}{2} \lambda_d(X)
\end{aligned}$$

where

$$c_0 = (2^d/\Omega)^{1-q} (\gamma^q \wedge \Gamma^q) I^{-q} \qquad \text{and} \qquad c_1 = c_0((\tfrac{1}{2})^q \wedge (\tfrac{3}{2})^q).$$

Hence

$$\begin{aligned}
\mathscr{H}_\mu^{q,d-dq}(X) &\geqslant \mathscr{H}_\mu^{q,d-dq}(X_N) \geqslant \bar{\mathscr{H}}_\mu^{q,d-dq}(X_N) \\
&\geqslant \bar{\mathscr{H}}_{\mu,1/N}^{q,d-dq}(X_N) \geqslant c_1 \tfrac{1}{2} \lambda_d(X) > 0
\end{aligned}$$

i.e. $d - dq \leqslant \dim_\mu^q(X)$.

Claim 2. $\Delta_\mu^q(X) \leqslant d - dq$ for $0 \leqslant q$. Proof of Claim 2: Suppose $q \geqslant 0$, $\delta \leqslant 1$ and $(B(x_i, r_i))_i$ is a centered δ-packing of X. Then

$$\begin{aligned}
\bar{\mathscr{P}}_{\mu,\delta}^{q,d-dq}(X) &\leqslant \sum_i \mu(B(x_i, r_i))^q (2r_i)^{d-dq} \\
&= 2^{d(1-q)} I^{-q} \sum_i \left(\int_{B(x_i, r_i) \cap X} f \, d\lambda_d \right)^q (r_i^d)^{1-q} \\
&\leqslant 2^{d(1-q)} (\Gamma/I)^q \sum_i (\lambda_d(B(x_i, r_i)))^q (r_i^d)^{1-q} \\
&= c_2 \sum_i \lambda_d(B(x_i, r_i)) = c_2 \lambda_d \left(\bigcup_i B(x_i, r_i) \right) \leqslant c_2 \lambda_d(B(X, 1)),
\end{aligned}$$

where $B(X, 1) = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, X) \leqslant 1\}$ and $c_2 = (2^d/\Omega)^{(1-q)} (\Gamma/I)^q$. Letting $\delta \searrow 0$ now yields $\overline{\mathscr{P}}_{\mu}^{q,\, d-dq}(X) \leqslant c_2 \lambda_d(B(X, 1)) < \infty$, i.e. $\Delta_{\mu}^q(X) \leqslant d - dq$.

Claim 3. If (3.3) is satisfied then $\operatorname{Dim}_{\mu}^q(X) \leqslant d - dq$ for $q < 0$. Proof of Claim 3: Let $q < 0$ and put

$$X_m = \left\{ x \in X \;\middle|\; \frac{1}{m} < \frac{\lambda_d(B(x, r) \cap X)}{\lambda_d B(x, r)} \text{ for } 0 < r < \frac{1}{m} \right\}, \qquad m \in \mathbb{N}.$$

Let $m \in \mathbb{N}$, $0 < \delta \leqslant 1$ and $(B(x_i, r_i))_i$ be a centered δ-covering of X_m. By calculuations similar to those in the proof of Claim 2, we obtain

$$\begin{aligned}
\overline{\mathscr{P}}_{\mu,\delta}^{q,\, d-dq}(X_m) &\leqslant \sum_i \mu(B(x_i, r_i))^q (2r_i)^{d-dq} \\
&= (2^d/\Omega)^{1-q} (\gamma/I)^q \sum_i \left(\frac{\lambda_d(B(x_i, r_i) \cap X)}{\lambda_d(B(x_i, r_i))} \right)^q \lambda_d(B(x_i, r_i)) \\
&\leqslant c_3 \left(\frac{1}{m}\right)^q \sum_i \lambda_d(B(x_i, r_i)) = c_3 \left(\frac{1}{m}\right)^q \lambda_d\left(\bigcup_i B(x_i, r_i) \right) \\
&\leqslant c_3 \left(\frac{1}{m}\right)^q \lambda_d(B(X, 1)),
\end{aligned}$$

where $c_3 = (2^d/\Omega)^{1-q} (\gamma/I)^q$.

Letting $\delta \searrow 0$ now yields

$$\mathscr{P}_{\mu}^{q,\, d-dq}(X_m) \leqslant \overline{\mathscr{P}}_{\mu}^{q,\, d-dq}(X_m) \leqslant c_3 \left(\frac{1}{m}\right)^q \lambda_d(B(X, 1)),$$

whence $\operatorname{Dim}_{\mu}^q(X_m) \leqslant d - dq$ for all $m \in \mathbb{N}$. It follows from (3.3) that $X = \bigcup_m X_m$ whence $\operatorname{Dim}_{\mu}^q(X) = \sup_m \operatorname{Dim}_{\mu}^q(X_m) \leqslant d - dq$.

Formulas (3.1), (3.2) and (3.4) follow immediately from claims 1–3. Observe that the proof of Claim 3 shows that if

$$0 < \inf_{x \in X} \liminf_{r \searrow 0} \frac{\lambda_d(B(x, r) \cap X)}{\lambda_d(B(x, r))} \tag{3.5}$$

then

$$\dim_{\mu}^q(X) = \operatorname{Dim}_{\mu}^q(X) = \Delta_{\mu}^q(X) = d - dq. \tag{3.6}$$

Our next example shows that if the hypothesis $\gamma > 0$ or $\Gamma < \infty$ is omitted then the conclusions in (3.1) and (3.6) are false.

3.2

EXAMPLE 2. Let $a \in]-1, \infty[$ and define $f_a: \mathbb{R} \to \mathbb{R}$ by $f_a(x) = x^a$ for $x \in]0, 1[$ and $f(x) = 0$ otherwise. Put $\mu = (a+1) f_a \, dx \in \mathscr{P}(\mathbb{R})$. Clearly $\operatorname{supp}(\mu) = [0, 1] := I$. Then $b(q) = B(q) = -(a+1)q \vee (1-q)$.

Proof. We will only prove the statement for $b(q)$. The other statement is verified in the same way. We claim that

$$\dim_\mu^q(\{0\}) = -(a+1)q. \tag{3.7}$$

Indeed, let $\varepsilon, \delta > 0$ and $(B_i = B(x_i, r_i))_i$ be a centered δ-covering of $\{0\}$. Then

$$\sum_i \mu(B_i)^q (2r_i)^{-(a+1)q-\varepsilon} = (a+1) 2^{-(a+1)q-\varepsilon} \sum_i r_i^{-\varepsilon}$$
$$\geqslant (a+1) 2^{-(a+1)q-\varepsilon} \delta^{-\varepsilon}.$$

Hence $\overline{\mathscr{H}}_{\mu,\delta}^{q,-(a+1)q-\varepsilon}(\{0\}) \geqslant (a+1) 2^{-(a+1)q-\varepsilon} \delta^{-\varepsilon}$ for all $\delta > 0$, whence $\mathscr{H}_\mu^{q,-(a+1)q-\varepsilon}(\{0\}) = \infty$, and so $\dim_\mu^q(\{0\}) \geqslant -(a+1)q - \varepsilon$ for all $\varepsilon > 0$, i.e.

$$\dim_\mu^q(\{0\}) \geqslant -(a+1)q.$$

For all $\delta \in]0, 1[$ and $\eta > 0$,

$$\overline{\mathscr{H}}_{\mu,\delta}^{q,-(a+1)q+\eta}(\{0\}) \leqslant \mu(]-\delta, \delta[)^q (2\delta)^{-(a+1)q+\eta} = 2^{-(a+1)q+\eta} \delta^\eta$$

whence $\overline{\mathscr{H}}_\mu^{q,-(a+1)q+\eta}(\{0\}) \leqslant 0$ and so $\mathscr{H}_\mu^{q,-(a+1)q+\eta}(\{0\}) \leqslant 0$, i.e. $\dim_\mu^q(\{0\}) \leqslant -(a+1)q + \eta$ for all $\eta > 0$. This proves (3.7). It follows from (3.7) and Example 1 that

$$\dim_\mu^q(I) = \dim_\mu^q\left(\{0\} \cup \bigcup_{n=1}^{\infty} \left]\frac{1}{n}, 1\right]\right) = \dim_\mu^q(\{0\}) \vee \bigvee_{n=1}^{\infty} \dim_\mu^q\left(\left]\frac{1}{n}, 1\right]\right)$$
$$= -(a+1)q \vee (1-q)$$

(since $\dim_\mu^q(]1/n, 1]) = 1 - q$ by Example 1). ∎

This example was investigated in Hasley et al. [Ha] in a very heuristic way for $a \in]-1, 0[$. The case $a = -\frac{1}{2}$ has also been studied by Collet [Co2].

3.3.

EXAMPLE 3. We will now study a discrete measure μ on $\mathbb{R}$ with support equal to $[0, 1]$. Let $r \in]0, \frac{1}{2}[$ and define $\mu \in \mathscr{P}(\mathbb{R})$ by

$$\mu = a \sum_{n=1}^{\infty} \sum_{p=1,3,\dots,2^n-1} r^n \delta_{p/2^n}, \qquad a = \frac{1-2r}{r}.$$

The multifractal spectrum of this measure has been studied in a recent paper by Aversa and Bandt [Av].

Define the map $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(q) = \begin{cases} 1 + q\dfrac{\log r}{\log 2} & \text{for } \ q \leqslant -\dfrac{\log 2}{\log r} \\ 0 & \text{for } \ -\dfrac{\log 2}{\log r} \leqslant q \end{cases}$$

We will now prove that

$$b_\mu = B_\mu = \varphi. \tag{3.8}$$

The proof of (3.8) will be divided into several lemmas. For $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, 2^n - 1\}$ write

$$E_{nk} = \left[\frac{k}{2^n}; \frac{k+1}{2^n}\right].$$

LEMMA 3.1. *Let* $\varepsilon > 0$. *Then*

$$S(\varepsilon) := \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \mu(E_{nk})^q \left(2\frac{1}{2^{n+1}}\right)^{\varphi(q)+\varepsilon} < \infty.$$

Proof. Fix $n \in \mathbb{N}$. Then clearly

$$\begin{aligned} S_n := \sum_{k=0}^{2^n-1} \mu(E_{nk})^q \left(2\frac{1}{2^{n+1}}\right)^{\varphi(q)+\varepsilon} &= \left(\frac{1-2r}{r}\right)^q 2\left(\frac{r^{n+1}}{1-2r} + r^n\right)^q \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon} \\ &+ \left(\frac{1-2r}{r}\right)^q \sum_{j=1}^{n-1} 2^{n-j} \left(\frac{r^{n+1}}{1-2r} + r^n + r^{n-j}\right)^q \left(\frac{1}{2n}\right)^{\varphi(q)+\varepsilon}. \end{aligned} \tag{3.9}$$

If $0 \leqslant q \leqslant 1$ then

$$\left(\frac{r^{n+1}}{1-2r} + r^n + r^{n-j}\right)^q \leqslant \left(\frac{r^{n+1}}{1-2r} + r^n\right)^q + r^{q(n-j)}. \tag{3.10}$$

If $q \leqslant 0$ or $1 \leqslant q$ then Jensen's inequality implies that

$$\left(\frac{r^{n+1}}{1-2r}+r^n+r^{n-j}\right)^q \leqslant 2^{q-1}\left(\left(\frac{r^{n+1}}{1-2r}+r^n\right)^q+r^{q(n-j)}\right). \tag{3.11}$$

It follows from (3.9), (3.10) and (3.11) that

$$\begin{aligned} S_n &\leqslant \left(\frac{1-2r}{r}\right)^q \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon} \left[2\left(\frac{r}{1-2r}+1\right)^q r^{nq} + (1 \vee 2^{q-1}) \right. \\ &\quad \left. \times \left(\frac{r}{1-2r}+1\right)^q r^{nq} \sum_{j=1}^{n-1} 2^{n-j} + (1 \vee 2^{q-1}) \sum_{j=1}^{n-1} 2^{n-j} r^{q(n-j)}\right] \\ &\leqslant \begin{cases} \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon} \left[c_1(2r^q)^n + c_2 \frac{1-(2r^q)^{n-1}}{1-2r^q}\right] & \text{for } q \neq -\frac{\log 2}{\log r} \\ \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon} [c_1 + c_2 n] & \text{for } q = -\frac{\log 2}{\log r} \end{cases} \end{aligned} \tag{3.12}$$

where $c_1, c_2 > 0$ are suitable constants. If $q < -(\log 2/\log r)$ then

$$\frac{1-(2r^q)^{n-1}}{1-2r^q} \leqslant c_3 (2r^q)^n, \tag{3.13}$$

where c_3 is a constant. If $-(\log 2/\log r) < q$ then

$$\frac{1-(2r^q)^{n-1}}{1-2r^q} \leqslant (1-2r^q)^{-1} := c_4. \tag{3.14}$$

It follows from (3.12), (3.13) and (3.14) that

$$S_n \leqslant \begin{cases} 2^{-n(\varphi(q)+\varepsilon)}[c_5(2r^q)^n + c_6] & \text{for } q \neq -\frac{\log 2}{\log r} \\ 2^{-n(\varphi(q)+\varepsilon)}[c_1 + c_2 n] & \text{for } q = -\frac{\log 2}{\log r}, \end{cases}$$

where $c_5, c_6 > 0$ are suitable constants.

For $q \neq -(\log 2/\log r)$ this implies that

$$S(\varepsilon) = \sum_n S_n \leqslant c_5 \sum_n (2^{1-\varphi(q)-\varepsilon} r^q)^n + c_6 \sum_n (2^{-\varphi(q)-\varepsilon})^n < \infty,$$

since $2^{1-\varphi(q)-\varepsilon}r^q,\ 2^{-\varphi(q)-\varepsilon} \in [0, 1[$. For $q = -(\log 2/\log r)$ we have

$$S(\varepsilon) = \sum_n S_n \leqslant c_1 \sum_n 2^{-n\varepsilon} + c_2 \sum_n n 2^{-n\varepsilon} < \infty. \quad \blacksquare$$

LEMMA 3.2. $B_\mu(q) \leqslant \varphi(q)$ *for* $q \leqslant 0$.

Proof. Let $\delta \in]0, 1[$ and $(B(x_i, r_i))_i$ be a centered δ-packing of $[0, 1]$. For each i choose $n_i \in \mathbb{N}$ such that

$$\frac{1}{2^{n_i+1}} \leqslant \frac{r_i}{2} \leqslant \frac{1}{2^{n_i}},$$

and pick $k_i \in \{0, ..., 2^{n_i} - 1\}$ such that $E_{n_i k_i} \subseteq B(x_i, r_i)$. Since $\mu(E_{n_i k_i}) \geqslant r^{n_i+1}/(1-2r)$,

$$\begin{aligned}\sum_i \mu(B(x_i, r_i))^q (2r_i)^{\varphi(q)} &\leqslant a^q \sum_i \left(\frac{r^{n_i+1}}{1-2r}\right)^q \left(\frac{1}{2^{n_i}}\right)^{\varphi(q)} 4^{\varphi(q)} \\ &\leqslant c \sum_i (r^q 2^{-\varphi(q)})^{n_i} \\ &\leqslant c \sum_i (r^q 2^{-\varphi(q)})^{-\log r_i/\log 2} \\ &\leqslant c \sum_i r_i \leqslant c(1 + 2\delta),\end{aligned}$$

where $c = 4^{\varphi(q)}(ra/1 - 2r)^q$, whence

$$\mathscr{P}_\mu^{q, \varphi(q)}([0, 1]) \leqslant \overline{\mathscr{P}}_\mu^{q, \varphi(q)}([0, 1]) \leqslant \overline{\mathscr{P}}_{\mu, \delta}^{q, \varphi(q)}([0, 1]) \leqslant c(1 + 2\delta) < \infty,$$

i.e. $B_\mu(q) \leqslant \varphi(q)$. $\blacksquare$

LEMMA 3.3. $B_\mu(q) \leqslant \varphi(q)$ *for* $0 \leqslant q \leqslant 1$.

Proof. Let $\varepsilon > 0$, $\delta \in]0, \frac{1}{4}[$ and $(B(x_i, r_i))_i$ be a centered δ-packing of $[0, 1]$. Put

$$I_n = \left\{ i \,\middle|\, \frac{1}{2^{n+1}} < 2r_i \leqslant \frac{1}{2^n} \right\}, \qquad n \in \mathbb{N}.$$

Now fix $n \in \mathbb{N}$ and $i \in I_n$. We may clearly choose $j(i) \in \{0, ..., 2^n - 1\}$ such that

$$B(x_i, r_i) \subseteq E_{n, j(i)} \cup E_{n, j(i)+1},$$

whence (since $(x + y)^q \leqslant x^q + y^q$ for $0 < q \leqslant 1$ and $x, y \geqslant 0$)

$$\mu(B(x_i, r_i))^q \leqslant \mu(E_{n, j(i)} \cup E_{n, j(i)+1})^q \leqslant \mu(E_{n, j(i)})^q + \mu(E_{n, j(i)+1})^q.$$

However, each $E_{n,j}$ will intersect at most three balls $B(x_i, r_i)$ with $i \in I_n$, i.e.: for each $n \in \mathbb{N}$ there are at most three integers $i_1, i_2, i_3 \in I_n$ such that $j(i_1) = j(i_2) = j(i_3)$. Hence

$$\sum_{i \in I_n} \mu(B(x_i, r_i))^q (2r_i)^{\varphi(q)+\varepsilon} \leqslant \sum_{i \in I_n} \mu(E_{n,j(i)})^q \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon}$$
$$+ \sum_{i \in I_n} \mu(E_{n,j(i)+1})^q \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon}$$
$$\leqslant 2\left(3 \sum_{j=1}^{2^n-1} \mu(E_{nj})^q \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon}\right) \quad (3.15)$$

It follows immediately from (3.15) and Lemma 3.1 that

$$\sum_i \mu(B(x_i, r_i))^q (2r_i)^{\varphi(q)+\varepsilon} = \sum_{n=1}^{\infty} \left(\sum_{i \in I_n} \mu(B(x_i, r_i))^q (2r_i)^{\varphi(q)+\varepsilon} \right)$$
$$\leqslant 6 \sum_{n=1}^{\infty} \sum_{j=1}^{2^n-1} \mu(E_{nj})^q \left(\frac{1}{2^n}\right)^{\varphi(q)+\varepsilon} = 6S(\varepsilon) < \infty,$$

whence

$$\mathscr{P}_\mu^{q,\varphi(q)+\varepsilon}([0,1]) \leqslant \overline{\mathscr{P}}_\mu^{q,\varphi(q)+\varepsilon}([0,1]) \leqslant \overline{\mathscr{P}}_{\mu,\delta}^{q,\varphi(q)+\varepsilon}([0,1]) \leqslant S(\varepsilon) < \infty,$$

i.e. $B_\mu(q) \leqslant \varphi(q) + \varepsilon$ for all $\varepsilon > 0$. ∎

LEMMA 3.4. *$B_\mu(q) = \varphi(q) = 0$ for $-(\log 2/\log r) \leqslant q \leqslant 1$.*

Proof. Lemma 3.3 shows that $B_\mu(q) \leqslant \varphi(q) = 0$ and Proposition 2.10 yields $B_\mu(q) \geqslant 0 = \varphi(q)$. ∎

LEMMA 3.5. *$B_\mu(q) \leqslant \varphi(q)$ for $1 \leqslant q$.*

Proof. Obvious since $B_\mu(q) \leqslant 0 = \varphi(q)$ for $1 \leqslant q$. ∎

LEMMA 3.6. *$B_\mu(0) = \varphi(0) = 1$.*

Proof. Obvious since $B_\mu(0) = \mathrm{Dim}_\mu^0([0,1]) = \mathrm{Dim}([0,1]) = 1$. ∎

By combining Lemma 3.2-Lemma 3.6 and recalling that B_μ is convex (cf. Proposition 2.9) we get $B_\mu = \varphi$.

We will prove that $b_\mu = \varphi$.

LEMMA 3.7. *$b_\mu(q) = \varphi(q)$ for $0 \leqslant q \leqslant -(\log 2/\log r)$.*

Proof. We have $b_\mu(q) \leqslant B_\mu(q) = \varphi(q)$ and it is thus sufficient to prove that $b_\mu(q) \geqslant \varphi(q)$. Let $0 \leqslant q \leqslant -(\log 2/\log r)$, $\delta > 0$ and

$(B_i = B(x_i, r_i))$ be a centered δ-covering of $[0, 1]$. Put $I_n = \{i \mid 1/2^{n+1} < 2r_i \leqslant 1/2^n\}$ for $n \in \mathbb{N}$. Now fix $i \in I_n$. Since $1/2^{n+2} < 2r_i$ we may choose $j(i) \in \{0, ..., 2^{n+2-1}\}$ such that

$$E_{n+2, j(i)} \subseteq B_i$$

i.e.

$$\mu(B_i)^q \geqslant \mu(E_{n+2, j(i)})^q \geqslant \left(\frac{r^{n+3}}{1-2r}\right)^q = cr^{nq},$$

where $c = (r^3/1-2r)^q$. Hence (since $2^{-\varphi(q)} r^q = 2^{-1}$)

$$\begin{aligned}\sum_i \mu(B_i)^q (2r_i)^{\varphi(q)} &= \sum_n \sum_{i \in I_n} \mu(B_i)^q (2r_i)^{\varphi(q)} \geqslant \sum_n \operatorname{card}(I_n)\, cr^{nq} \left(\frac{1}{2^{n+1}}\right)^{\varphi(q)} \\ &= c2^{-\varphi(q)} \sum_n (2^{-\varphi(q)} r^q)^n \operatorname{card}(I_n) \\ &= c2^{-\varphi(q)} \sum_n 2^{-n} \operatorname{card}(I_n) \\ &\geqslant c2^{-\varphi(q)} \sum_n \left(\sum_{i \in I_n} 2r_i\right) \geqslant c2^{-\varphi(q)}\end{aligned}$$

and so

$$\mathscr{H}_\mu^{q, \varphi(q)}([0, 1]) \geqslant \bar{\mathscr{H}}_\mu^{q, \varphi(q)}([0, 1]) \geqslant \bar{\mathscr{H}}_{\mu, \delta}^{q, \varphi(q)}([0, 1]) \geqslant c2^{-\varphi(q)} > 0,$$

which implies that $b_\mu(q) \geqslant \varphi(q)$. ▮

LEMMA 3.8. *$b_\mu(q) = \varphi(q)$ for $-(\log 2/\log r) \leqslant q \leqslant 1$.*

Proof. By equation (1.5), $0 \leqslant b_\mu(q) \leqslant B_\mu(q) = \varphi(q) = 0$. ▮

LEMMA 3.9. *$b_\mu(q) = \varphi(q)$ for $q < 0$.*

Proof. Assume that there exists $q < 0$ such that $b_\mu(q) \neq \varphi(q)$. Then $b_\mu(q) < \varphi(q)$ since $b_\mu \leqslant B_\mu = \varphi$. Put $p = -(\log 2/\log r)$, $\alpha = q/(q-p) \in]0, 1[$ and observe that $\alpha p + (1-\alpha) q = 0$. Now $B_\mu(p) = b_\mu(p) = 0$ and $b_\mu(\alpha p + (1-\alpha) q) = b_\mu(0) = B_\mu(0) = 1$, and Proposition 2.10 therefore implies that

$$\begin{aligned}1 = b_\mu(\alpha p + (1-\alpha) q) &\leqslant \alpha B_\mu(p) + (1-\alpha) b_\mu(q) = (1-\alpha) b_\mu(q) < (1-\alpha) \varphi(q) \\ &= \left(1 - \frac{q}{q-p}\right)\left(1 - \frac{q}{p}\right) = 1,\end{aligned}$$

which is a contradiction. ▮

LEMMA 3.10. *$b_\mu(q)=\varphi(q)$ for $1<q$.*

Proof. The proof is similar to the proof of Lemma 3.9. ∎

3.4.

EXAMPLE 4. We will now construct a Borel probability measure $\mu\in\mathscr{P}_1(\mathbb{R})$ such that:

(1) The measure μ is not a Taylor multifractal measure (cf. (1.6)), in fact

$$b_\mu(q)\neq B_\mu(q)\qquad\text{for}\quad q\in\mathbb{R}\backslash\{1\}.$$

(2) The function b_μ is not convex.

Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence in $]0,\infty[$ such that

$$a_0=1,\qquad 2a_{n+1}<a_n\qquad\text{for}\quad n\in\mathbb{N}_0,\qquad d<D,\tag{3.16}$$

where

$$d:=\liminf_n\frac{n\log 2}{-\log a_n},\qquad D:=\limsup_n\frac{n\log 2}{-\log a_n}.$$

For each $n\in\mathbb{N}_0$ we construct a family $\mathscr{I}_n=(I_{n1},\ldots,I_{n,2^n})$ of closed disjoint intervals of $[0,1]$ such that $\operatorname{diam}(I_{ni})=a_n$ for $i=1,\ldots,2^n$. We will construct $\mathscr{I}_n$ by induction on n.

The Case $n=0$. Put $I_{01}=[0,1]$ and $\mathscr{I}_0=(I_{01})$.

The Case $n\in\mathbb{N}$. Suppose we have constructed $\mathscr{I}_n=(I_{n1},\ldots,I_{n,2^n})$. Fix $i\in\{1,\ldots,2^n\}$. Then I_{ni} is a closed subinterval of $[0,1]$ and $\operatorname{diam}I_{ni}=a_n$. It follows from (3.16) that we can choose two disjoint closed subintervals I and J of I_{ni} such that $\operatorname{diam}I=a_{n+1}=\operatorname{diam}J$ and I lies to the left of J. Now put $I_{n+1,2i-1}=I$ and $I_{n+1,2i}=J$. This completes the construction of $\mathscr{I}_{n+1}$.

Now, put

$$E=\bigcap_n\bigcup_i I_{ni}.$$

The set E is called a symmetrical perfect set, cf. [Ka]. Symmetrical perfect sets have been studied by e.g. Kahane & Salem [Ka] and Marion [Mar].

Now define a Borel probability measure μ on E such that

$$\mu(I_{ni})=2^{-n}\qquad\text{for all}\quad n\in\mathbb{N},\qquad i=1,\ldots,2^n.$$

It is clear that $\operatorname{supp}\mu = E$. We will now prove that

$$b_\mu(q) \leqslant d(1-q) \wedge D(1-q) \tag{3.17}$$

$$B_\mu(q) \geqslant d(1-q) \vee D(1-q) \tag{3.18}$$

Properties (1) and (2) follow immediately from (3.17) and (3.18) since $d < D$ and $b_\mu(1) = B_\mu(1) = 0$. The proof of (3.17) and (3.18) is divided into four lemmas.

LEMMA 3.11. *$b_\mu(q) \leqslant d(1-q)$ for $q < 1$.*

Proof. Let $\varepsilon > 0$ and $F \subseteq E$. Since

$$\liminf_n \frac{n \log 2}{-\log a_n} < d + \frac{\varepsilon}{1-q},$$

there exists a subsequence $(n_k)_k$ of integers such that

$$\frac{n_k \log 2}{-\log a_{n_k}} < d + \frac{\varepsilon}{1-q}.$$

And so (since $0 < 1-q$)

$$2^{n_k(1-q)} a_{n_k}^{d(1-q)+\varepsilon} \leqslant 1 \qquad \text{for all } k. \tag{3.19}$$

Now fix $k \in \mathbb{N}$ and let $I(k) = \{ i \mid I_{n_k, i} \cap F \neq \varnothing \}$. For each $i \in I(k)$ choose $x_i \in I_{n_k, i} \cap F$ and observe that $B(x_i, a_{n_k})$ can at most intersect 3 different members of $\mathscr{I}_{n_k}$, i.e.

$$I_{n_k, i} \cap E \subseteq B(x_i, a_{n_k}) \cap E \subseteq I_{n_k, i-1} \cup I_{n_k, i} \cup I_{n_k, i+1}.$$

Hence

$$\mu(B(x_i, a_{n_k}))^q \leqslant (1 \vee 3^q)\, \mu(I_{n_k, i})^q = (1 \vee 3^q)\, 2^{-n_k q}. \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\begin{aligned}
\mathscr{H}^{q, d(1-q)+\varepsilon}_{\mu, a_{n_k}}(F) &\leqslant \sum_{i \in I(k)} \mu(B(x_i, a_{n_k}))^q\, (2a_{n_k})^{d(1-q)+\varepsilon} \\
&\leqslant 2^{d(1-q)+\varepsilon}(1 \vee 3^q) \sum_{i=1}^{2^{n_k}} 2^{-n_k q} a_{n_k}^{d(1-q)+\varepsilon} \\
&= c 2^{n_k} 2^{-n_k q} a_{n_k}^{d(1-q)+\varepsilon} \leqslant c < \infty,
\end{aligned}$$

where $c := 2^{d(1-q)+\varepsilon}(1 \vee 3^q)$. Letting $k \to \infty$ now yields

$$\overline{\mathscr{H}}_{\mu}^{q, d(1-q)+\varepsilon}(F) \leqslant c$$

for all $F \subseteq E$. Hence

$$\mathscr{H}_{\mu}^{q, d(1-q)+\varepsilon}(E) \leqslant c$$

i.e. $b_\mu(q) = \dim_\mu^q(E) \leqslant d(1-q) + \varepsilon$ for all $\varepsilon > 0$. ∎

LEMMA 3.12. $b_\mu(q) \leqslant D(1-q)$ *for* $1 < q$.

Proof. Let $\varepsilon > 0$ and $F \subseteq E$. Since

$$D + \frac{\varepsilon}{1-q} < \limsup_n \frac{n \log 2}{-\log a_n},$$

there exists a subsequence $(n_k)_k$ of integers such that

$$D + \frac{\varepsilon}{1-q} < \frac{n_k \log 2}{-\log a_{n_k}}.$$

And so (since $1 - q < 0$)

$$2^{n_k(1-q)} a_{n_k}^{D(1-q)+\varepsilon} \leqslant 1 \qquad \text{for all} \quad k.$$

Now proceed as in the proof of Lemma 3.11. ∎

LEMMA 3.13. $B_\mu(q) \geqslant D(1-q)$ *for* $q < 1$.

Proof. Let $\varepsilon > 0$ and $E \subseteq \bigcup_i E_i$. Since

$$D - \frac{\varepsilon}{1-q} < \limsup_n \frac{n \log 2}{-\log a_n},$$

there exists a sequence $(n_k)_k$ of integers such that

$$D - \frac{\varepsilon}{1-q} < \frac{n_k \log 2}{-\log a_{n_k}} \qquad \text{for} \quad k \in \mathbb{N}.$$

And so (since $0 < 1 - q$)

$$1 \leqslant 2^{n_k(1-q)} a_{n_k}^{D(1-q)-\varepsilon} \qquad \text{for} \quad k \in \mathbb{N}. \tag{3.21}$$

Now fix $i, k \in \mathbb{N}$ and let $I(k, i) = \{j \mid I_{n_k, j} \cap E_i \neq \emptyset\}$. For each $j \in I(k, i)$ choose $x_j \in I_{n_k, j} \cap E_i$, and observe that $B(x_j, a_{n_k})$ can intersect at most 3 different members of $\mathcal{I}_{n_k}$, i.e.

$$I_{n_k, j} \cap E \subseteq B(x_j, a_{n_k}) \cap E \subseteq I_{n_k, j-1} \cup I_{n_k, j} \cup I_{n_k, j+1}.$$

Hence

$$\mu(B(x_j, a_{n_k}))^q \geq (1 \wedge 3^q)\, \mu(I_{n_k, i})^q = (1 \wedge 3^q)\, 2^{-n_k q} \tag{3.22}$$

and:

$$\left.\begin{array}{l}\text{the family } (B(x_j, a_{n_k}))_{j \in I(k, i)} \text{ can be divided into} \\ 3 \text{ disjoint centered } a_{n_k}\text{-packings of } E_i\end{array}\right\}, \tag{3.23}$$

Also

$$\mu(E_i) \leq \frac{\operatorname{card} I(k, i)}{2^{n_k}}. \tag{3.24}$$

It follows from (3.21) through (3.24) that

$$\begin{aligned}
3\overline{\mathcal{P}}^{q, D(1-q)-\varepsilon}_{\mu, a_{n_k}}(E_i) &\geq \sum_{j \in I(k, i)} \mu(B(x_j, a_{n_k}))^q (2a_{n_k})^{D(1-q)-\varepsilon} \\
&\geq 2^{D(1-q)-\varepsilon}(1 \wedge 3^q) \sum_{j \in I(k, i)} 2^{-n_k q} a_{n_k}^{D(1-q)-\varepsilon} \\
&= c \operatorname{card}(I(k, i))\, 2^{-n_k q} a_{n_k}^{D(1-q)-\varepsilon} \\
&\geq c\mu(E_i)\, 2^{n_k} 2^{-n_k q} a_{n_k}^{D(1-q)-\varepsilon} \geq c\mu(E_i),
\end{aligned}$$

where $c = 2^{D(1-q)-\varepsilon}(1 \wedge 3^q)$. Letting $k \to \infty$ now yields

$$\overline{\mathcal{P}}^{q, D(1-q)-\varepsilon}_{\mu}(E_i) \geq c\mu(E_i)/3.$$

Hence

$$\sum_i \overline{\mathcal{P}}^{q, D(1-q)-\varepsilon}_{\mu}(E_i) \geq (c/3) \sum_i \mu(E_i) \geq (c/3)\, \mu\left(\bigcup_i E_i\right) \geq (c/3)\, \mu(E) = c/3,$$

which implies that

$$\mathcal{P}^{q, D(1-q)-\varepsilon}_{\mu}(E) \geq c/3 > 0,$$

i.e. $B_\mu(q) = \operatorname{Dim}^q_\mu(E) \geq D(1-q) - \varepsilon$ for all $\varepsilon > 0$. ∎

LEMMA 3.14. $B_\mu(q) \geq d(1-q)$ *for* $1 < q$.

Proof. Let $\varepsilon > 0$ and $E \subseteq \bigcup_i E_i$. Since

$$\liminf_n \frac{n \log 2}{-\log a_n} < d - \frac{\varepsilon}{1-q},$$

there exists a sequence $(n_k)_k$ of integers such that

$$\frac{n_k \log 2}{-\log a_{n_k}} < d - \frac{\varepsilon}{1-q} \qquad \text{for} \quad k \in \mathbb{N}.$$

And so (since $1-q<0$)

$$2^{n_k(1-q)} a_{n_k}^{d(1-q)-\varepsilon} \geq 1.$$

Now proceed as in the proof of Lemma 3.13. ∎

We can in fact with some additional work prove that equality holds in (3.17) and (3.18). We also note that the equalities in (3.17) and (3.18) for $q=0$, i.e.

$$\dim(E) = b_\mu(0) = d, \qquad \operatorname{Dim}(E) = B_\mu(0) = D,$$

are well known, cf. e.g. [Tr, pp. 66–67].

Finally we prove that $\mu \in \mathscr{P}_1(\mathbb{R})$. Let $x \in E$ and $0 < r < \frac{1}{2}$. We claim that

$$\frac{\mu B(x, 2r)}{\mu B(x, r)} \leq 12. \tag{3.25}$$

Pick $n \in \mathbb{N}$ satisfying $a_n < 2r \leq a_{n-1}$ and choose an integer i such that $x \in I_{n-1,i}$. Since $\operatorname{diam}(I_{n-1,i}) = a_{n-1} \geq 2r$,

$$B(x, 2r) \cap E \subseteq I_{n-1,i-1} \cup I_{n-1,i} \cup I_{n-1,i+1}. \tag{3.26}$$

Also choose an integer j such that $x \in I_{n+1,j}$ and observe that

$$I_{n+1,j} \subseteq B(x, r). \tag{3.27}$$

Indeed, if $y \in I_{n+1,j}$ then $|x-y| \leq \operatorname{diam} I_{n+1,j} = a_{n+1} < \frac{1}{2} a_n < r$. It follows from (3.26) and (3.27) that

$$\frac{\mu B(x, 2r)}{\mu B(x, r)} \leq \frac{\mu(I_{n-1,i-1} \cup I_{n-1,i+1} \cup I_{n-1,i+1})}{\mu(I_{n+1,j})} = \frac{3 \cdot 2^{-(n-1)}}{2^{-(n+1)}} = 12$$

which proves (3.25). Inequality (3.25) clearly implies that $\mu \in \mathscr{P}_1(\mathbb{R})$.

4. Proofs

4.1. *Technical Lemmas*

We begin by stating two covering lemmas which we will apply later.

THEOREM 4.1. *Let $\mathscr{B}$ be a family of closed balls contained in a bounded subset of $\mathbb{R}^d$. Then there exists a countable or finite subfamily $(B(x_i, r_i))_i$ of $\mathscr{B}$ such that*

(i) *$(B(x_i, r_i))_i$ is a pairwise disjoint family.*

(ii) *$(\bigcup_{B \in \mathscr{B}} B) \setminus (\bigcup_{i=1}^{k} B(x_i, r_i)) \subseteq \bigcup_{i=k+1}^{\infty} B(x_i, 5r_i)$ for all k.*

Proof. Cf. [Fa, Lemma 1.9]. ▮

We now state Besicovitch covering theorem

THEOREM 4.2 (Besicovitch Covering Theorem). *Let $d \in \mathbb{N}$. Then there exists an integer $\zeta \in \mathbb{N}$ which satisfies the following: Let $A \subseteq \mathbb{R}^d$ and for each $x \in A$ fix a number $r_x > 0$ such that $\sup_{x \in A} r_x < \infty$. Then there exist ζ countable or finite subfamilies $\mathscr{B}_1, ..., \mathscr{B}_\zeta$ of $\{B(x, r_x) \mid x \in A\}$ such that*

(i) *$A \subseteq \bigcup_i \bigcup_{B \in \mathscr{B}_i} B$*

(ii) *$\mathscr{B}_i$ is a family of disjoint sets.*

Proof. Cf. [Gu, p. 5]. ▮

The next lemma investigates the scaling properties of $\mathscr{H}_\mu^{q,t}$ and $\mathscr{P}_\mu^{q,t}$.

LEMMA 4.3. *Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ and $T: \mathbb{R}^d \to \mathbb{R}^d$ be a similarity map with ratio r such that $T(\operatorname{supp} \mu) \subseteq \operatorname{supp} \nu$. For $q \in \mathbb{R}$ write*

$$\underline{J}_{\mu,\nu}^{q}(T) = \liminf_{\rho \searrow 0} \inf_{x \in \operatorname{supp} \mu} \left(\frac{\nu T(B(x, \rho))}{\mu B(x, \rho)} \right)^q,$$

$$\bar{J}_{\mu,\nu}^{q}(T) = \limsup_{\rho \searrow 0} \sup_{x \in \operatorname{supp} \mu} \left(\frac{\nu T(B(x, \rho))}{\mu B(x, \rho)} \right)^q.$$

If $q, t \in \mathbb{R}$ and $E \subseteq \operatorname{supp} \mu$ then

(i)

$$\underline{J}_{\mu,\nu}^{q}(T)\, r^t \mathscr{H}_\mu^{q,t}(E) \leqslant \mathscr{H}_\nu^{q,t}(TE) \leqslant \bar{J}_{\mu,\nu}^{q}(T)\, r^t \mathscr{H}_\mu^{q,t}(E).$$

(ii)

$$\underline{J}_{\mu,\nu}^{q}(T)\, r^t \mathscr{P}_\mu^{q,t}(E) \leqslant \mathscr{P}_\nu^{q,t}(TE) \leqslant \bar{J}_{\mu,\nu}^{q}(T)\, r^t \mathscr{P}_\mu^{q,t}(E).$$

Proof. (i) Let $\varepsilon > 0$, $0 < \delta \leqslant \varepsilon$ and $F \subseteq \operatorname{supp} \mu$. Let $(B(x_i, r_i))_i$ be a centered δ-covering of F. Observe that $(B(Tx_i, rr_i))_i$ is a centered $r\delta$-covering of TF. Hence

$$\bar{\mathscr{H}}^{q,t}_{\nu, r\delta}(TF) \leqslant \sum_i \nu(B(Tx_i, rr_i))^q (2rr_i)^t = \sum_i \nu(TB(x_i, r_i))^q (2rr_i)^t$$

$$\leqslant \sup_{\rho < \varepsilon} \sup_{x \in \operatorname{supp} \mu} \left(\frac{\nu(TB(x, \rho))}{\mu B(x, \rho)} \right)^q r^t \sum_i \mu(B(x_i, r_i))^q (2r_i)^t,$$

whence

$$\bar{\mathscr{H}}^{q,t}_{\nu, r\delta}(TF) \leqslant \sup_{\rho < \varepsilon} \sup_{x \in \operatorname{supp} \mu} \left(\frac{\nu(TB(x, \rho))}{\mu B(x, \rho)} \right)^q r^t \bar{\mathscr{H}}^{q,t}_{\mu, \delta}(F),$$

for all $\varepsilon > 0$ and $0 < \delta \leqslant \varepsilon$. By first letting $\delta \searrow 0$ and then letting $\varepsilon \searrow 0$ this inequality yields $\bar{\mathscr{H}}^{q,t}_{\nu}(TF) \leqslant \bar{J}^q_{\mu, \nu}(T)\, r^t \bar{\mathscr{H}}^{q,t}_{\mu}(F)$ for all $F \subseteq \operatorname{supp} \mu$, which clearly implies that

$$\mathscr{H}^{q,t}_{\nu}(TE) \leqslant \bar{J}^q_{\mu\nu}(T)\, r^t \mathscr{H}^{q,t}_{\mu}(E),$$

for all $E \subseteq \operatorname{supp} \mu$ since T is one-to-one. Similarly we may prove that

$$\underline{J}^q_{\mu, \nu}(T)\, r^t \mathscr{H}^{q,t}_{\mu}(TE) \leqslant \mathscr{H}^{q,t}_{\nu}(TE)$$

for $E \subseteq \operatorname{supp} \mu$.

(ii) Similar to the proof of (i). ∎

4.2. *Proofs of the Results in Section* 2.2

Proof of Proposition 2.2. It is obvious that $\mathscr{H}^{q,t}_{\mu}$ is monotone, $\mathscr{H}^{q,t}_{\mu}(\varnothing) = 0$ and that $\mathscr{H}^{q,t}_{\mu}(E \cup F) = \mathscr{H}^{q,t}_{\mu}(E) + \mathscr{H}^{q,t}_{\mu}(F)$ whenever the distance between E and F is positive. It is thus sufficient to prove that $\mathscr{H}^{q,t}_{\mu}$ is countably additive. Let $(F_n)_n$ be a sequence of subsets of X. Let $\delta, \varepsilon > 0$ and choose a centered δ-covering $(B_{ni} := B(x_{ni}, r_{ni}))_{i \in \mathbb{N}}$ of F_n such that

$$\sum_i \mu(B_{ni})^q (2r_{ni})^t \leqslant \bar{\mathscr{H}}^{q,t}_{\mu, \delta}(F_n) + \frac{\varepsilon}{2^n} \leqslant \bar{\mathscr{H}}^{q,t}_{\mu}(F_n) + \frac{\varepsilon}{2^n} \leqslant \mathscr{H}^{q,t}_{\mu}(F_n) + \frac{\varepsilon}{2^n}.$$

Since $(B_{ni})_{n,i}$ is a centered δ-covering of $\bigcup_n F_n$,

$$\bar{\mathscr{H}}^{q,t}_{\mu, \delta}\left(\bigcup_n F_n \right) \leqslant \sum_n \sum_i \mu(B_{ni})^q (2r_{ni})^t \leqslant \sum_n \mathscr{H}^{q,t}_{\mu}(F_n) + \varepsilon.$$

Letting $\delta \searrow 0$ yields, since $\varepsilon > 0$ was arbitrary,

$$\bar{\mathscr{H}}_{\mu}^{q,t}\left(\bigcup_n F_n\right) \leqslant \sum_n \mathscr{H}_{\mu}^{q,t}(F_n) \qquad \text{for all} \quad F_1, F_2, \ldots \subseteq X. \tag{4.1}$$

Let $E_1, E_2, \ldots \subseteq X$, $\eta > 0$ and choose $F \subseteq \bigcup_n E_n$ such that $\mathscr{H}_{\mu}^{q,t}(\bigcup_n E_n) - \eta \leqslant \bar{\mathscr{H}}_{\mu}^{q,t}(F)$. Then (4.1) implies that

$$\begin{aligned}\mathscr{H}_{\mu}^{q,t}\left(\bigcup_n E_n\right) &\leqslant \bar{\mathscr{H}}_{\mu}^{q,t}(F) + \eta = \bar{\mathscr{H}}_{\mu}^{q,t}\left(\bigcup_n (F \cap E_n)\right) + \eta \\ &\leqslant \sum_n \mathscr{H}_{\mu}^{q,t}(F \cap E_n) + \eta \leqslant \sum \mathscr{H}_{\mu}^{q,t}(E_n) + \eta\end{aligned}$$

for all $\eta > 0$. ∎

Proof of Proposition 2.3. It follows immediately from [Mu, Theorem 11.3] that $\mathscr{P}_{\mu}^{q,t}$ is an outer measure. It is obvious that $\mathscr{P}_{\mu}^{q,t}(E \cup F) = \bar{\mathscr{P}}_{\mu}^{q,t}(E) + \bar{\mathscr{P}}_{\mu}^{q,t}(F)$ whenever $E, F \subseteq X$ and $\operatorname{dist}(E, F) > 0$. This clearly implies that $\mathscr{P}_{\mu}^{q,t}$ is additive on sets that are separated by a positive distance, and thus a metric outer measure. ∎

Proof of Proposition 2.4. (i) Obvious.

(iii) Let $E \subseteq \mathbb{R}^d$. For $m \in \mathbb{N}$ write

$$E_m = \left\{ x \in E \;\middle|\; \frac{\mu B(x, 5r)}{\mu B(x, r)} < m \text{ for } 0 < r < \frac{1}{m} \right\},$$

where we put $a/0 = 1$ for $a \geqslant 0$. Fix $m \in \mathbb{N}$ and let $F \subseteq E_m$. We will now prove that

$$\bar{\mathscr{H}}_{\mu}^{q,t}(F) \leqslant \bar{\mathscr{P}}_{\mu}^{q,t}(F).$$

We may clearly assume that $\mathscr{P}_{\mu}^{q,t}(F) < \infty$. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that

$$\bar{\mathscr{H}}_{\mu}^{q,t}(F) - \frac{\varepsilon}{3} \leqslant \bar{\mathscr{H}}_{\mu,\delta}^{q,t}(F) \qquad \text{for} \quad \delta \leqslant \delta_1.$$

Next choose $\delta_2 > 0$ such that

$$\bar{\mathscr{P}}_{\mu,\delta}^{q,t}(F) \leqslant \bar{\mathscr{P}}_{\mu}^{q,t}(F) + \frac{\varepsilon}{3} \qquad \text{for} \quad \delta \leqslant \delta_2.$$

Let $\mathscr{V} = \{B(x, r) \mid x \in F,\ r < \delta_1/5 \wedge \delta_2 \wedge 1/m\}$. Then $\mathscr{V}$ is a Vitali covering of F and we may thus apply [Fa, Lemma 1.9] to choose a countable (or finite) subfamily $(B(x_i, x_i) := B_i)_i \subseteq \mathscr{V}$ such that $B_i \cap B_j = \varnothing$ for $i \neq j$ and

$$F \setminus \bigcup_{i=1}^{k} B_i \subseteq \bigcup_{i=k+1}^{\infty} B(x_i, 5r_i) \qquad \text{for all} \quad k.$$

Since $x_i \in E_m$ and $r_i < 1/m$,

$$\sum_i \mu(B(x_i, 5r_i))^q (10r_i)^t \leqslant 5^t \sum_i (m\mu(B(x_i, r_i)))^q (2r_i)^t$$

$$\leqslant m^q 5^t \bar{\mathscr{P}}^{q,t}_{\mu,\delta_2}(F) \leqslant m^q 5^t \left(\bar{\mathscr{P}}^{q,t}_{\mu}(F) + \frac{\varepsilon}{3} \right) < \infty.$$

We may thus choose $K \in \mathbb{N}$ such that

$$\sum_{i=K+1}^{\infty} \mu(B(x_i, 5r_i))^q (10r_i)^t \leqslant \frac{\varepsilon}{3}.$$

Hence

$$\bar{\mathscr{H}}^{q,t}_{\mu}(F) \leqslant \bar{\mathscr{H}}^{q,t}_{\mu,\delta_1}(F) + \frac{\varepsilon}{3}$$

$$\leqslant \sum_{i=1}^{K} \mu(B_i)^q (2r_i)^t + \sum_{i=K+1}^{\infty} \mu(B(x_i, 5r_i))^q (10r_i)^t + \frac{\varepsilon}{3}$$

$$\leqslant \sum_i \mu(B_i)^q (2r_i)^t + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$\leqslant \bar{\mathscr{P}}^{q,t}_{\mu,\delta_2}(F) + \frac{2\varepsilon}{3} \leqslant \bar{\mathscr{P}}^{q,t}_{\mu}(F) + \varepsilon$$

for all $\varepsilon > 0$. This yields

$$\bar{\mathscr{H}}^{q,t}_{\mu}(F) \leqslant \bar{\mathscr{P}}^{q,t}_{\mu}(F) \qquad \text{for all} \quad F \subseteq E_m. \tag{4.2}$$

Let $E_m \subseteq \bigcup_i F_i$. Then (4.2) implies that

$$\mathscr{H}^{q,t}_{\mu}(E_m) = \mathscr{H}^{q,t}_{\mu}\left(\bigcup_i (F_i \cap E_n) \right) \leqslant \sum_i \mathscr{H}^{q,t}_{\mu}(F_i \cap E_m)$$

$$\leqslant \sum_i \sup_{F \subseteq F_i \cap E_m} \bar{\mathscr{H}}^{q,t}_{\mu}(F) \leqslant \sum_i \sup_{F \subseteq F_i \cap E_m} \bar{\mathscr{P}}^{q,t}_{\mu}(F)$$

$$\leqslant \sum_i \bar{\mathscr{P}}^{q,t}_{\mu}(F_i),$$

whence

$$\mathscr{H}_{\mu}^{q,t}(E_m) \leqslant \mathscr{P}_{\mu}^{q,t}(E_m) \qquad \text{for all} \quad m \in \mathbb{N}.$$

This completes the proof since $E_m \nearrow E$ and $\mathscr{P}_{\mu}^{q,t}$ is regular.

(ii) This follows by an argument similar to the proof of (iii). However, since $q \leqslant 0$, $\mu(B(x, 5r))^q \leqslant \mu(B(x, r))^q$ for all x and we need not assume that $\mu \in \mathscr{P}_0(\mathbb{R}^d)$.

(iv) Let $\zeta \in \mathbb{N}$ be the integer that appears in Besicovitch covering theorem. We first prove that

$$\bar{\mathscr{H}}_{\mu}^{q,t}(F) \leqslant \zeta \bar{\mathscr{P}}_{\mu}^{q,t}(F) \tag{4.3}$$

for all $F \subseteq \mathbb{R}^d$. Let $\delta > 0$ and write $\mathscr{V} = \{B(x, r) \mid x \in F,\ 0 < r < \delta\}$. It follows from Besicovitch covering theorem that there exist ζ countable (or finite) subfamilies $(B(x_{ij}, r_{ij}))_j$, $i = 1, ..., \zeta$ of $\mathscr{V}$ such that $(B(x_{ij}, r_{ij}))_{i,j}$ is a centered δ-covering of F and $(B(x_{ij}, r_{ij}))_j$ is a centered δ-packing of F for each i. Hence

$$\begin{aligned}\bar{\mathscr{H}}_{\mu,\delta}^{q,t}(F) &\leqslant \sum_{i=1}^{\zeta} \sum_j \mu(B(x_{ij}, r_{ij}))^q (2r_{ij})^t \leqslant \sum_{i=1}^{\zeta} \bar{\mathscr{P}}_{\mu,\delta}^{q,t}(F) \\ &= \zeta \bar{\mathscr{P}}_{\mu,\delta}^{q,t}(F).\end{aligned}$$

Letting $\delta \searrow 0$ yields (4.3). Let $E \subseteq \mathbb{R}^d$ and $E \subseteq \bigcup_i E_i$. Then (4.3) implies that

$$\begin{aligned}\mathscr{H}_{\mu}^{q,t}(E) &= \mathscr{H}_{\mu}^{q,t}\left(\bigcup_i (E \cap E_i)\right) \leqslant \sum_i \mathscr{H}_{\mu}^{q,t}(E \cap E_i) \\ &= \sum_i \sup_{F \subseteq E \cap E_i} \bar{\mathscr{H}}_{\mu}^{q,t}(F) \leqslant \zeta \sum_i \sup_{F \subseteq E \cap E_i} \bar{\mathscr{P}}_{\mu}^{q,t}(F) \\ &\leqslant \zeta \sum_i \bar{\mathscr{P}}_{\mu}^{q,t}(E_i),\end{aligned}$$

whence $\mathscr{H}_{\mu}^{q,t}(E) \leqslant \zeta \mathscr{P}_{\mu}^{q,t}(E)$. ∎

4.3. *Proofs of the Results in Section* 2.3

Proof of Proposition 2.5. The statements are true for $q = 0$ by (1.2) so we may assume that $q \neq 0$.

(i) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in \bar{X}^{\alpha} \;\middle|\; \frac{\log \mu B(x, r)}{\log r} \leqslant \alpha + \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Let $m \in \mathbb{N}$ and $0 < \eta < 1/m$. Let $(B_i := B(x_i, r_i))_{i \in \mathbb{N}}$ be a centered η-covering of T_m. Then clearly

$$\begin{aligned}&\frac{\log \mu(B(x_i, r_i))}{\log r_i} \leqslant \alpha + \frac{\delta}{q}\\ &\Downarrow\\ &\mu(B(x_i, r_i)) \geqslant r_i^{\alpha + (\delta/q)}\\ &\Downarrow\\ &\mu(B(x_i, r_i))^q \geqslant r_i^{\alpha q + \delta}\\ &\Downarrow\\ &\mu(B(x_i, r_i))^q (2r_i)^t \geqslant 2^t r_i^{\alpha q + \delta + t}.\end{aligned}$$

Hence

$$\begin{aligned}\mathscr{H}_\eta^{q\alpha + t + \delta}(T_m) &\leqslant \sum_i \operatorname{diam} B(x_i, r_i)^{q\alpha + t + \delta} \leqslant 2^{q\alpha + t + \delta} \sum_i r_i^{\alpha q + \delta + t}\\ &\leqslant 2^{q\alpha + \delta} \sum_i \mu(B(x_i, r_i))^q (2r_i)^t,\end{aligned}$$

whence

$$\mathscr{H}_\eta^{q\alpha + t + \delta}(T_m) \leqslant 2^{q\alpha + \delta} \bar{\mathscr{H}}_{\mu, \eta}^{q, t}(T_m) \qquad \text{for} \quad \eta < \frac{1}{m}.$$

Letting $\eta \searrow 0$ now yields

$$\mathscr{H}^{q\alpha + t + \delta}(T_m) \leqslant 2^{q\alpha + \delta} \bar{\mathscr{H}}_\mu^{q, t}(T_m) \leqslant 2^{q\alpha + \delta} \mathscr{H}_\mu^{q, t}(T_m) \qquad \text{for} \quad m \in \mathbb{N}.$$

Clearly $T_m \nearrow \bar{X}^\alpha$, whence

$$\mathscr{H}^{q\alpha + t + \delta}(\bar{X}^\alpha) = \sup_m \mathscr{H}^{q\alpha + t + \delta}(T_m) \leqslant 2^{q\alpha + \delta} \sup_m \mathscr{H}_\mu^{q, t}(T_m) \leqslant 2^{q\alpha + \delta} \mathscr{H}_\mu^{q, t}(\bar{X}^\alpha).$$

(ii) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in \underline{X}_\alpha \mid \alpha + \frac{\delta}{q} \leqslant \frac{\log \mu B(x, r)}{\log r} \text{ for } 0 < r < \frac{1}{m} \right\},$$

and proceed as in case (i).

(iii) Follows immediately from (i) and (ii).

(iv) We first prove that $\dim(\underline{X}^\alpha) \leqslant \alpha q + B(q)$ for $0 < q$. It is sufficient to prove that

$$\dim(\underline{X}^\alpha) \leqslant \alpha q + t + \delta$$

for $t > B(q)$ and $\delta > 0$. Fix $t > B(q)$ and $\delta > 0$. Since $\mathscr{P}_\mu^{q,t}(\underline{X}^\alpha) = 0$ we may choose a covering $(E_i)_{i \in \mathbb{N}}$ of $\underline{X}^\alpha$ such that $\sum_i \overline{\mathscr{P}}_\mu^{q,t}(E_i) < 1$. Let $I = \{i \in \mathbb{N} \mid \underline{X}^\alpha \cap E_i \neq \varnothing\}$. Since $\underline{X}^\alpha = \bigcup_{i \in I} (\underline{X}^\alpha \cap E_i)$,

$$\dim(\underline{X}^\alpha) = \sup_{i \in I} \dim(\underline{X}^\alpha \cap E_i),$$

and it is thus sufficient to prove that

$$\dim(\underline{X}^\alpha \cap E_i) \leqslant \alpha q + t + \delta$$

for each $i \in I$. Now fix $i \in I$. Then

$$\lim_{\eta \searrow 0} \overline{\mathscr{P}}_{\mu,\eta}^{q,t}(E_i) = \overline{\mathscr{P}}_\mu^{q,t}(E_i) < 1,$$

and we may choose an integer $N \in \mathbb{N}$ such that

$$\overline{\mathscr{P}}_{\mu,1/n}^{q,t}(E_i) < 1 \qquad \text{for} \quad n \geqslant N. \tag{4.4}$$

Let $x \in \underline{X}^\alpha$. Then $\liminf_{r \searrow 0} \log \mu B(x,r)/\log r \leqslant \alpha < \alpha + (\delta/q)$ and we can thus choose a sequence $(r_n(x))_{n \in \mathbb{N}}$ such that $0 < r_n(x) < 1/n$ and

$$\frac{\log \mu B(x, r_n(x))}{\log r_n(x)} \leqslant \alpha + \frac{\delta}{q}.$$

Hence

$$\begin{aligned}
&\mu B(x, r_n(x)) \geqslant r_n(x)^{\alpha + (\delta/q)} \\
&\Downarrow \\
&\mu(B(x, r_n(x)))^q \geqslant (r_n(x)^{\alpha + (\delta/q)})^q = r_n(x)^{\alpha q + \delta} \qquad (4.5) \\
&\Downarrow \\
&\mu(B(x, r_n(x)))^q\,(2r_n(x))^t \geqslant 2^t r_n(x)^{\alpha q + t + \delta}.
\end{aligned}$$

Put $\mathscr{V}_n = \{B(x, r_k(x)) \mid x \in \underline{X}^\alpha \cap E_i,\ k \geqslant n\}$ for $n \geqslant N$. The family $\mathscr{V}_n$ is clearly a Vitali covering of $\underline{X}^\alpha \cap E_i$, and we can thus (cf. [Fa1, Theorem 1.10]) choose a countable, disjoint subfamily $(V_{nj} := B(x_{nj}, r_{nj}))_{j\in\mathbb{N}} \subseteq \mathscr{V}_n$ such that either

$$\sum_j (\operatorname{diam} V_{nj})^{q\alpha+t+\delta} = \infty$$

or

$$\mathscr{H}^{q\alpha+t+\delta}\left((\underline{X}^\alpha \cap E_i)\setminus \bigcup_j V_{nj}\right) = 0.$$

However, (4.4) and (4.5) imply that

$$\begin{aligned}\sum_j (\operatorname{diam} V_{nj})^{q\alpha+t+\delta} &\leqslant 2^{q\alpha+t+\delta}\sum_j r_{nj}^{q\alpha+t+\delta} \leqslant 2^{q\alpha+\delta}\sum_j \mu(B(x_{nj}, r_{nj}))^q (2r_{nj})^t\\ &\leqslant 2^{q\alpha+\delta}\overline{\mathscr{P}}^{q,t}_{\mu,1/n}(\underline{X}^\alpha \cap E_i) \leqslant 2^{q\alpha+\delta}\overline{\mathscr{P}}^{q,t}_{\mu,1/n}(E_i)\\ &\leqslant 2^{q\alpha+\delta} < \infty, \end{aligned} \tag{4.6}$$

whence

$$\mathscr{H}^{q\alpha+t+\delta}\left((\underline{X}\cap E_i)\setminus \bigcup_j V_{nj}\right) = 0 \qquad \text{for} \quad n \geqslant N. \tag{4.7}$$

Put $V = \bigcap_{n\geqslant N}\bigcup_j V_{nj}$. Then (4.7) implies that $\mathscr{H}^{q\alpha+t+\delta}((\underline{X}^\alpha \cap E_i)\setminus V) = 0$, i.e. $\dim((\underline{X}^\alpha \cap E_i)\setminus V) \leqslant \alpha q + t + \delta$. It follows from (4.6) that

$$\mathscr{H}^{q\alpha+t+\delta}(V) = \sup_{n\geqslant N} \mathscr{H}^{q\alpha+t+\delta}_{1/n}(V) \leqslant \sup_{n\geqslant N}\sum_j (\operatorname{diam} V_{nj})^{q\alpha+t+\delta} \leqslant 2^{q\alpha+\delta},$$

whence $\dim V \leqslant \alpha q + t + \delta$. Hence

$$\dim(\underline{X}^\alpha \cap E_i) \leqslant \max(\dim((\underline{X}^\alpha \cap E_i)\setminus V), \dim V) \leqslant q\alpha + t + \delta$$

for $i \in I$.

Mutatis mutandis $\dim(\overline{X}_\alpha) \leqslant \alpha q + B(q)$ for $q < 0$. ∎

Proof of Proposition 2.6. The statements are true for $q = 0$ by (1.2) so we may assume that $q \neq 0$.

(i) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in \overline{X}^\alpha \;\middle|\; \frac{\log \mu B(x, r)}{\log r} \leqslant \alpha + \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m}\right\}.$$

Fix $m \in \mathbb{N}$, $E \subseteq T_m$ and $0 < \eta < 1/m$. Let $(B_i := B(x_i, r_i))_{i \in \mathbb{N}}$ be a centered η-packing of E. Then clearly

$$\frac{\log \mu B(x_i, r_i)}{\log r_i} \leqslant \alpha + \frac{\delta}{q}$$
$$\Downarrow$$
$$\mu B(x_i, r_i) \geqslant r_i^{\alpha + (\delta/q)}$$
$$\Downarrow$$
$$\mu(B(x_i, r_i))^q \geqslant r_i^{\alpha q + \delta}$$
$$\Downarrow$$
$$(2r_i)^t \mu(B(x_i, r_i))^q \geqslant 2^t r_i^{\alpha q + t + \delta}.$$

Hence

$$\sum_i (2r_i)^{\alpha q + t + \delta} \leqslant 2^{\alpha q + \delta} \sum_i \mu(B_i)^q (2r_i)^t \leqslant 2^{\alpha q + \delta} \bar{\mathscr{P}}^{q,t}_{\mu,\eta}(E),$$

whence

$$\bar{\mathscr{P}}^{\alpha q + t + \delta}_{\eta}(E) \leqslant 2^{\alpha q + \delta} \bar{\mathscr{P}}^{q,t}_{\mu,\eta}(E).$$

Letting $n \searrow 0$ now yields

$$\bar{\mathscr{P}}^{\alpha q + t + \delta}(E) \leqslant 2^{\alpha q + \delta} \bar{\mathscr{P}}^{q,t}_{\mu}(E) \qquad \text{for} \quad E \subseteq T_m. \tag{4.8}$$

Now let $T_m \subseteq \bigcup_i E_i$. Then (4.8) implies that

$$\begin{aligned}\mathscr{P}^{\alpha q + t + \delta}(T_m) &= \mathscr{P}^{\alpha q + t + \delta}\left(\bigcup_i (T_m \cap E_i)\right) \leqslant \sum_i \mathscr{P}^{\alpha q + t + \delta}(T_m \cap E_i) \\ &\leqslant \sum_i \bar{\mathscr{P}}^{\alpha q + t + \delta}(T_m \cap E_i) \leqslant 2^{\alpha q + \delta} \sum_i \bar{\mathscr{P}}^{q,t}_{\mu}(T_m \cap E_i) \\ &\leqslant 2^{\alpha q + \delta} \sum_i \bar{\mathscr{P}}^{q,t}_{\mu}(E_i),\end{aligned}$$

whence

$$\mathscr{P}^{\alpha q + t + \delta}(T_m) \leqslant 2^{\alpha q + \delta} \mathscr{P}^{q,t}_{\mu}(T_m)$$

for all m. Since $\bar{X}^{\alpha} = \bigcup_m T_m$ this completes the proof.

(ii) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in \underline{X}_\alpha \;\middle|\; \alpha + \frac{\delta}{q} \leqslant \frac{\log \mu B(x, r)}{\log r} \text{ for } 0 < r < \frac{1}{m} \right\},$$

and proceed as before.

(iii) Follows immediately from (i) and (ii). ∎

Proof of Proposition 2.7. The statements are true for $q = 0$ by (1.2), so we may assume that $q \neq 0$.

(i) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in A \;\middle|\; \frac{\log \mu B(x, r)}{\log r} \leqslant \alpha - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Let $m \in \mathbb{N}$, $E \subseteq T_m$ and $0 < \eta < 1/m$. Let $(E_i)_{i \in \mathbb{N}}$ be a covering of E with $r_i := \operatorname{diam} E_i < \eta$ for all i. Put $I = \{ i \mid E_i \cap E \neq \emptyset \}$ and choose $x_i \in E_i \cap E$. Then $(B(x_i, r_i))_i$ is a centered η-covering of E, with

$$\frac{\log \mu B(x_i, r_i)}{\log r} \leqslant \alpha - \frac{\delta}{q}$$

$$\Downarrow$$

$$\mu B(x_i, r_i) \geqslant r_i^{\alpha - (\delta/q)}$$

$$\Downarrow$$

$$\mu(B(x_i, r_i))^q \leqslant r_i^{\alpha q - \delta}$$

$$\Downarrow$$

$$\mu(B(x_i, r_i))^q (2r_i)^t \leqslant 2^t r_i^{\alpha q + t - \delta}.$$

Hence

$$\overline{\mathscr{H}}^{q,t}_{\mu,\eta}(E) \leqslant \sum_{i \in I} \mu(B(x_i, r_i))^q (2r_i)^t \leqslant 2^t \sum_{i \in I} r_i^{\alpha q + t - \delta}$$
$$\leqslant 2^t \sum_i (\operatorname{diam} E_i)^{\alpha q + t - \delta},$$

whence

$$\overline{\mathscr{H}}^{q,t}_{\mu,\eta}(E) \leqslant 2^t \mathscr{H}^{\alpha q + t - \delta}_\eta(E) \qquad \text{for} \quad \eta < \frac{1}{m}.$$

Letting $\eta \searrow 0$ now yields

$$\overline{\mathscr{H}}^{q,t}_{\mu}(E) \leqslant 2^t \mathscr{H}^{\alpha q + t - \delta}(E) \leqslant 2^t \mathscr{H}^{\alpha q + t - \delta}(T_m)$$

for all $E \subseteq T_m$, whence

$$\mathscr{H}_\mu^{q,t}(T_m) \leqslant 2^t \mathscr{H}^{\alpha q + t - \delta}(T_m) \qquad \text{for} \quad m \in \mathbb{N}.$$

Since $\bigcup_m T_m = A$ this completes the proof.

(ii) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in A \mid \alpha - \frac{\delta}{q} \leqslant \frac{\log \mu B(x, r)}{\log r} \text{ for } 0 < r < \frac{1}{m} \right\},$$

and proceed as in (i). ∎

Proof of Proposition 2.8. The statements are true for $q = 0$ by (1.2) so we may assume that $q \neq 0$.

(i) For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in A \;\middle|\; \frac{\log \mu B(x, r)}{\log r} \leqslant \alpha - \frac{\delta}{q} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Let $m \in \mathbb{N}$, $E \subset T_m$ and $0 < \eta < 1/m$. Let $B(x_i, r_i)_{i \in \mathbb{N}}$ be a centered δ-packing of E. Then

$$\frac{\log \mu B(x_i, r_i)}{\log r_i} \leqslant \alpha - \frac{\delta}{q}$$

$$\Downarrow$$

$$\mu B(x_i, r_i) \geqslant r_i^{\alpha - (\delta/q)}$$

$$\Downarrow$$

$$\mu(B(x_i, r_i))^q \leqslant r_i^{\alpha q - \delta}$$

$$\Downarrow$$

$$\mu(B(x_i, r_i))^q (2r_i)^t \leqslant 2^t r_i^{\alpha q + t - \delta},$$

whence

$$\begin{aligned} \sum \mu(B(x_i, r_i))^q (2r_i)^t &\leqslant 2^t \sum_i r_i^{\alpha q + t - \delta} = 2^{-\alpha q + \delta} \sum_i (2r_i)^{\alpha q + t - \delta} \\ &\leqslant 2^{-\alpha q + \delta} \overline{\mathscr{P}}_\eta^{\alpha q + t - \delta}(E), \end{aligned}$$

and so $\overline{\mathscr{P}}_{\mu,\eta}^{q,t}(E) \leqslant 2^{-\alpha q + \delta} \overline{\mathscr{P}}_\eta^{\alpha q + t - \delta}(E)$. Letting $\eta \searrow 0$ now yields

$$\overline{\mathscr{P}}_\mu^{q,t}(E) \leqslant 2^{-\alpha q + \delta} \overline{\mathscr{P}}^{\alpha q + t - \delta}(E) \tag{4.9}$$

for all $E \subseteq T_m$. Let $(E_i)_i$ be a covering of T_m. Then we get by (4.9),

$$\mathscr{P}_\mu^{q,t}(T_m) \leqslant \mathscr{P}_\mu^{q,t}\left(\bigcup_i (T_m \cap E_i)\right) \leqslant \sum_i \mathscr{P}_\mu^{q,t}(T_m \cap E_i)$$

$$\leqslant \sum_i \bar{\mathscr{P}}_\mu^{q,t}(T_m \cap E_i) \leqslant 2^{-\alpha q+\delta} \sum_i \bar{\mathscr{P}}^{\alpha q+t-\delta}(T_m \cap E_i)$$

$$\leqslant 2^{-\alpha q+t} \sum_i \bar{\mathscr{P}}^{\alpha q+t-\delta}(E_i),$$

and so

$$\mathscr{P}_\mu^{q,t}(T_m) \leqslant 2^{-\alpha q+t} \mathscr{P}^{\alpha q+t-\delta}(T_m).$$

Since $\bigcup_m T_m = A$ this completes the proof.

(ii) For $m \in \mathbb{N}$ write

$$T_m = \left\{x \in A \mid \alpha - \frac{\delta}{q} \leqslant \frac{\log \mu B(x,r)}{\log r} \text{ for } 0 < r < \frac{1}{m}\right\}$$

and proceed as in (i). ∎

LEMMA 4.4. *If X is a metric space, $\mu \in \mathscr{P}(X)$ and $\alpha \geqslant 0$, then*

(i) $\underline{X}^\alpha = \varnothing$ *for* $\alpha < \underline{A}$.

(ii) $\underline{X}_\alpha = \varnothing$ *for* $\bar{a} < \alpha$.

(iii) $\bar{X}_\alpha = \varnothing$ *for* $\bar{A} < \alpha$.

(iv) $\bar{X}^\alpha = \varnothing$ *for* $\alpha < \underline{a}$.

Proof. (i) Suppose $\alpha < \underline{A}$ and $x \in \underline{X}^\alpha$. Since $\alpha < \underline{A}$ there exist real numbers ε, $q_0 > 0$ such that $\alpha + \varepsilon < -(B(q_0)/q_0)$, i.e. $-q_0(\alpha+\varepsilon) > B(q_0)$. Put $t = -q_0(\alpha + \varepsilon)$. Since $x \in \underline{X}^\alpha$,

$$\liminf_{r \searrow 0} \frac{\log \mu B(x,r)}{\log r} \leqslant \alpha < \alpha + \varepsilon.$$

We can thus choose a sequence $(r_n)_n$ such that $r_n \searrow 0$, $0 < r_n < 1/n$ and

$$\begin{gathered} \frac{\log \mu B(x,r_n)}{\log r_n} < \alpha + \varepsilon \\ \Downarrow \\ \mu(B(x,r_n)) \geqslant r_n^{\alpha+\varepsilon} \\ \Downarrow \\ \mu(B(x,r_n))^{q_0}(2r_n)^t \geqslant 2^t r_n^{q_0(\alpha+\varepsilon)+t} = 2^t \end{gathered} \tag{4.10}$$

here the last equality in (4.10) is due to the fact that $q_0(\alpha+\varepsilon)+t=0$. Hence

$$\bar{\mathscr{P}}^{q_0,t}_{\mu,1/n}(\{x\})\geqslant\mu(B(x,r_n))^{q_0}(2r_n)^t\geqslant 2^t \qquad \text{for all} \quad n\in\mathbb{N},$$

whence $\bar{\mathscr{P}}^{q_0,t}_{\mu}(\{x\})\geqslant 2^t$. This clearly implies that $\mathscr{P}^{q_0,t}_{\mu}(\{x\})\geqslant 2^t$ whence $-q_0(\alpha+\varepsilon)=t\leqslant \mathrm{Dim}^{q_0}_{\mu}(\{x\})\leqslant B(q_0)$ contradicting the fact that $-q_0(\alpha+\varepsilon)>B(q_0)$.

(ii) Suppose $\bar{a}<\alpha$ and $x\in\underline{X}_\alpha$. Since $\bar{a}<\alpha$ there exist real numbers $\varepsilon>0$, $q_0<0$ such that $\alpha-\varepsilon>-(b(q_0)/q_0)$, i.e. $-q_0(\alpha-\varepsilon)>b(q_0)$. Put $t=-q_0(\alpha-\varepsilon)$. Since $x\in\underline{X}_\alpha$

$$\alpha-\varepsilon<\alpha\leqslant\liminf_{r\searrow 0}\frac{\log\mu B(x,r)}{\log r}.$$

We can thus choose $r_0>0$ such that all $0<r<r_0$ satisfies

$$\begin{aligned}
&\alpha-\varepsilon<\frac{\log\mu B(x,r)}{\log r}\\
&\Downarrow\\
&r^{\alpha-\varepsilon}\geqslant\mu B(x,r)\\
&\Downarrow\\
&r^{q_0(\alpha-\varepsilon)}\leqslant\mu(B(x,r))^{q_0} \qquad (\text{because } q_0<0)\\
&\Downarrow\\
&2^t\leqslant 2^t=2^t r^{q_0(\alpha-\varepsilon)+t}\leqslant\mu(B(x,r))^{q_0}(2r)^t.
\end{aligned} \tag{4.11}$$

Here the first equality in (4.11) is due to the fact that $q_0(\alpha-\varepsilon)+t=0$. Hence

$$\mathscr{H}^{q_0,t}_{\mu}(\{x\})\geqslant\bar{\mathscr{H}}^{q_0,t}_{\mu}(\{x\})\geqslant\bar{\mathscr{H}}^{q_0,t}_{\mu,r_0}(\{x\})\geqslant 2^t. \tag{4.12}$$

It follows from (4.12) that

$$-q_0(\alpha-\varepsilon)=t\leqslant\dim^{q_0}_{\mu}(\{x\})\leqslant b(q_0)$$

contradicting the fact that $-q_0(\alpha-\varepsilon)>b(q_0)$.

(iii–iv) The proofs of (iii) and (iv) are similar to those of (i) and (ii). ∎

4.4. *Proofs of Results in Section* 2.4

Proof of Proposition 2.10. (i) Obvious since $x \to a^x$ is decreasing for $0 < a \leqslant 1$.

(ii) Follows immediately from (i).

(iii) Let $\varepsilon, \delta > 0$ and $E \subseteq X$. For all centered $(\varepsilon \wedge \delta)$-packings $(B_i = B(x_i, \varepsilon_i))_{i \in \mathbb{N}}$ of E,

$$\begin{aligned}\sum_i \mu(B_i)^{\alpha p + (1-\alpha) q}(2\varepsilon_i)^{\alpha t + (1-\alpha) s} &= \sum_i (\mu(B_i)^p (2\varepsilon_i)^t)^\alpha (\mu(B_i)^q (2\varepsilon_i)^s)^{1-\alpha}\\ &\leqslant \left(\sum_i \mu(B_i)^p (2\varepsilon_i)^t\right)^\alpha \left(\sum_i \mu(B_i)^q (2\varepsilon_i)^s\right)^{1-\alpha}\\ &\leqslant (\overline{\mathcal{P}}^{p,t}_{\mu,\varepsilon}(E))^\alpha (\overline{\mathcal{P}}^{q,s}_{\mu,\delta}(E))^{1-\alpha}\end{aligned}$$

where we have used Hölders inequality. Hence

$$\begin{aligned}\overline{\mathcal{P}}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s}_\mu(E) &\leqslant \overline{\mathcal{P}}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s}_{\mu, \varepsilon \wedge \delta}(E)\\ &\leqslant (\overline{\mathcal{P}}^{p,t}_{\mu,\varepsilon}(E))^\alpha (\overline{\mathcal{P}}^{q,s}_{\mu,\delta}(E))^{1-\alpha} \qquad \text{for all} \quad \varepsilon, \delta > 0,\end{aligned}$$

whence

$$\overline{\mathcal{P}}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s}_\mu(E) \leqslant (\overline{\mathcal{P}}^{p,t}_\mu(E))^\alpha (\overline{\mathcal{P}}^{q,s}_\mu(E))^{1-\alpha}.$$

(iv) Let $\varepsilon > 0$. Then (by (iii))

$$\begin{aligned}&\overline{\mathcal{P}}^{\alpha p + (1-\alpha) q, \alpha \Delta^p_\mu(E) + (1-\alpha) \Delta^q_\mu(E) + \varepsilon}(E)\\ &\quad\leqslant (\overline{\mathcal{P}}^{p, \Delta^p_\mu(E) + \varepsilon}(E))^\alpha (\overline{\mathcal{P}}^{q, \Delta^q_\mu(E) + \varepsilon}(E))^{1-\alpha} = 0 \cdot 0 = 0,\end{aligned}$$

i.e.

$$\Delta^{\alpha p + (1-\alpha) q}_\mu(E) \leqslant \alpha \Delta^p_\mu(E) + (1-\alpha) \Delta^q_\mu(E) + \varepsilon$$

which proves the assertion since $\varepsilon > 0$ was arbitrary.

(v) The proofs are similar to (i).

(vi) Write $B = B_{\mu, E}$. It is obvious that B decreasing. We will now prove that B is convex. Let $p, q \in \mathbb{R}$, $\alpha \in [0, 1]$ and $\varepsilon > 0$. Write $B(p) = t$ and $B(q) = s$. Clearly

$$\mathcal{P}^{q, s+\varepsilon}_\mu(E) = 0 = \mathcal{P}^{p, t+\varepsilon}_\mu(E).$$

We can thus choose coverings $(H_i)_{i\in\mathbb{N}}$ and $(K_i)_{i\in\mathbb{N}}$ of E such that

$$\sum_i \bar{\mathscr{P}}_\mu^{p,t+\varepsilon}(H_i)\leqslant 1, \qquad \sum_i \bar{\mathscr{P}}_\mu^{q,s+\varepsilon}(K_i)\leqslant 1.$$

For $n\in\mathbb{N}$ write $E_n=\bigcup_{i,j=1}^n (H_i\cap K_j)$. Fix $n\in\mathbb{N}$. Then clearly (by (iii))

$$\begin{aligned}
&\mathscr{P}_\mu^{\alpha p+(1-\alpha)q,\,\alpha t+(1-\alpha)s+\varepsilon}(E_n)\\
&=\mathscr{P}_\mu^{\alpha p+(1-\alpha)q,\,\alpha(t+\varepsilon)+(1-\alpha)(s+\varepsilon)}\left(\bigcup_{i,j=1}^n (H_i\cap K_j)\right)\\
&\leqslant \sum_{i,j=1}^n \mathscr{P}_\mu^{\alpha p+(1-\alpha)q,\,\alpha(t+\varepsilon)+(1-\alpha)(s+\varepsilon)}(H_i\cap K_j)\\
&\leqslant \sum_{i,j=1}^n \bar{\mathscr{P}}_\mu^{\alpha p+(1-\alpha)q,\,\alpha(t+\varepsilon)+(1-\alpha)(s+\varepsilon)}(H_i\cap K_j)\\
&\leqslant \sum_{i,j=1}^n (\bar{\mathscr{P}}_\mu^{p,t+\varepsilon}(H_i\cap K_j))^\alpha\,(\bar{\mathscr{P}}_\mu^{q,s+\varepsilon}(H_i\cap K_j))^{1-\alpha} \qquad \text{(by (iv))}\\
&\leqslant \left(\sum_{i,j=1}^n \bar{\mathscr{P}}_\mu^{p,t+\varepsilon}(H_i\cap K_j)\right)^\alpha \left(\sum_{i,j=1}^n \bar{\mathscr{P}}_\mu^{q,s+\varepsilon}(H_i\cap K_j)\right)^{1-\alpha}\\
&\qquad \text{(by Hölder)}\\
&\leqslant \left(\sum_{i=1}^n\sum_{j=1}^n \bar{\mathscr{P}}_\mu^{p,t+\varepsilon}(H_i)\right)^\alpha \left(\sum_{j=1}^n\sum_{i=1}^n \bar{\mathscr{P}}_\mu^{q,s+\varepsilon}(K_j)\right)^{1-\alpha}\\
&=\left(n\sum_{i=1}^n \bar{\mathscr{P}}_\mu^{p,t+\varepsilon}(H_i)\right)^\alpha \left(n\sum_{j=1}^n \bar{\mathscr{P}}_\mu^{q,s+\varepsilon}(K_j)\right)^{1-\alpha}\\
&\leqslant n^\alpha n^{1-\alpha}=n<\infty.
\end{aligned}$$

Hence $\operatorname{Dim}_\mu^{\alpha p+(1-\alpha)q}(E_n)\leqslant \alpha t+(1-\alpha)s+\varepsilon$ for all $n\in\mathbb{N}$. Since $E\subseteq\bigcup_n E_n$ this implies that

$$\begin{aligned}
B(\alpha p+(1-\alpha)q)&=\operatorname{Dim}_\mu^{\alpha p+(1-\alpha)q}(E)\leqslant \operatorname{Dim}_\mu^{\alpha p+(1-\alpha)q}\left(\bigcup_n E_n\right)\\
&=\sup_n \operatorname{Dim}_\mu^{\alpha p+(1-\alpha)q}(E_n)\leqslant \alpha B(p)+(1-\alpha)B(q)+\varepsilon
\end{aligned}$$

which proves convexity of B since $\varepsilon>0$ was arbitrary.

(vii)–(viii) Similar to (i) and (ii). ∎

Proof of Proposition 2.11. (ii) Let $t = B_{\mu,E}(p)$ and $s = b_{\mu,E}(q)$. We must now prove that

$$\dim_\mu^{\alpha p+(1-\alpha)q}(E) \leqslant \alpha t + (1-\alpha)s + \varepsilon$$

for all $\varepsilon > 0$. Fix $\varepsilon > 0$. For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in E \,\middle|\, \frac{\mu B(x, 5r)}{\mu B(x, r)} < m \text{ for } 0 < r < \frac{1}{m} \right\}$$

and observe that $T_m \nearrow E$. It is thus sufficient to prove that

$$\mathscr{H}_\mu^{\alpha p+(1-\alpha)q,\, \alpha t+(1-\alpha)s+\varepsilon}(T_m) < \infty$$

for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $F \subseteq T_m$. Let (F_i) be a covering of F and $\delta > 0$. Fix $i \in \mathbb{N}$ and choose $0 < \delta_i$ such that

$$\bar{\mathscr{P}}_{\mu,\delta_i}^{p,t+\varepsilon}(F_i) \leqslant \bar{\mathscr{P}}_\mu^{p,t+\varepsilon}(F_i) + \frac{1}{2^i}.$$

Clearly $\dim_\mu^q(F_i \cap F) \leqslant b_{\mu,E}(q) = s < s + \varepsilon$ whence $\mathscr{H}_\mu^{q,s+\varepsilon}(F_i \cap F) = 0$, i.e. $\bar{\mathscr{H}}_{\mu,\eta}^{q,s+\varepsilon}(F_i \cap F) = 0$ for all $\eta > 0$, and we can thus choose a centered $(\delta/5 \wedge 1/m \wedge \delta_i)$-covering $(B(x_{ij}, r_{ij}))_{j \in I_i}$ of $F_i \cap F$ such that

$$\sum_j \mu(B(x_{ij}, r_{ij}))^q (2r_{ij})^{s+\varepsilon} \leqslant \frac{1}{2^i}.$$

We may now apply [Fa1, Lemma 1.9] to choose a subset J_i of I_i such that

$$\bigcup_{j \in I_i} B(x_{ij}, r_{ij}) \subseteq \bigcup_{j \in J_i} B(x_{ij}, 5r_{ij})$$

$$B(x_{ij}, r_{ij}) \cap B(x_{ik}, r_{ik}) = \varnothing \qquad \text{for } j, k \in J_i \text{ and } j \neq k.$$

Since $\{B(x_{ij}, 5r_{ij}) \mid j \in J_i\}$ is a centered δ-covering of $F_i \cap F$ and $\{B(x_{ij}, r_{ij}) \mid j \in J_i\}$ is a centered δ_i-packing of F_i we get

$$\begin{aligned}
&\bar{\mathscr{H}}_{\mu,\delta}^{\alpha p+(1-\alpha)q,\, \alpha t+(1-\alpha)s+\varepsilon}(F) \\
&\quad \leqslant \sum_i \sum_{j \in J_i} \mu(B(x_{ij}, 5r_{ij}))^{\alpha p+(1-\alpha)q} (2 \cdot 5r_{ij})^{\alpha t+(1-\alpha)s+\varepsilon} \\
&\quad \leqslant 5^{\alpha t+(1-\alpha)s+\varepsilon} \sum_i \sum_{j \in J_i} m^{\alpha p+(1-\alpha)q} \\
&\qquad \times \mu(B(x_{ij}, r_{ij}))^{\alpha p+(1-\alpha)q} (2r_{ij})^{\alpha t+(1-\alpha)s+\varepsilon}
\end{aligned}$$

$$= c \sum_i \sum_{j \in J_i} (\mu(B(x_{ij}, r_{ij}))^p (2r_{ij})^{t+\varepsilon})^\alpha (\mu(B(x_{ij}, r_{ij}))^q (2r_{ij})^{s+\varepsilon})^{1-\alpha}$$

$$\leqslant c \left(\sum_i \sum_{j \in J_i} \mu(B(x_{ij}, r_{ij}))^p (2r_{ij})^{t+\varepsilon} \right)^\alpha \left(\sum_i \sum_{j \in J_i} \mu(B(x_{ij}, r_{ij}))^q (2r_{ij})^{s+\varepsilon} \right)^{1-\alpha}$$

$$\leqslant c \left(\sum_i \overline{\mathscr{P}}^{p, t+\varepsilon}_{\mu, \delta_i}(F_i) \right)^\alpha \left(\sum_i \frac{1}{2^i} \right)^{1-\alpha}$$

$$\leqslant c \left(\sum_i \left(\overline{\mathscr{P}}^{p, t+\varepsilon}_{\mu}(F_i) + \frac{1}{2^i} \right) \right)^\alpha$$

$$\leqslant c \left(\sum_i \overline{\mathscr{P}}^{p, t+\varepsilon}_{\mu}(F_i) + 1 \right)^\alpha$$

for $\delta > 0$, where $c = 5^{\alpha t + (1-\alpha) s + \varepsilon} m^{\alpha p + (1-\alpha) q}$. Letting $\delta \searrow 0$ now yields

$$\overline{\mathscr{H}}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s + \varepsilon}_{\mu}(F) \leqslant c \left(\sum_i \overline{\mathscr{P}}^{p, t+\varepsilon}_{\mu}(F_i) + 1 \right)^\alpha$$

for all coverings $(F_i)_i$ of F. Hence

$$\overline{\mathscr{H}}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s + \varepsilon}_{\mu}(F) \leqslant c(\mathscr{P}^{p, t+\varepsilon}_{\mu}(F) + 1)^\alpha$$

for all $F \subset T_m$, which in turn implies that

$$\begin{aligned} \mathscr{H}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s + \varepsilon}_{\mu}(T_m) &= \sup_{E \subseteq T_m} \overline{\mathscr{H}}^{\alpha p + (1-\alpha) q, \alpha t + (1-\alpha) s + \varepsilon}_{\mu}(E) \\ &\leqslant \sup_{E \subseteq T_m} c(\mathscr{P}^{p, t+\varepsilon}_{\mu}(E) + 1)^\alpha \\ &\leqslant c(\mathscr{P}^{p, t+\varepsilon}_{\mu}(T_m) + 1)^\alpha = c(0+1)^\alpha = c < \infty. \end{aligned}$$

(i) The proof of (i) is similar to the proof of (ii). ∎

Proof of Proposition 2.12. (i) Since $B_\mu(1) = 0$,

$$\alpha = \alpha \cdot 1 + B_\mu(1) \geqslant B^*_\mu(\alpha) \geqslant b^*_\mu(\alpha) \qquad \text{for} \quad \alpha \geqslant 0.$$

(ii) It is sufficient to prove that

$$\alpha(1-q) - \varepsilon \leqslant b_\mu(q)$$

for all $\varepsilon > 0$ and $q \in \mathbb{R}$. Now fix $\varepsilon > 0$ and $q \in \mathbb{R}$ and choose $\eta, \gamma > 0$ such that

$$(1 + |q|)\,\eta + \gamma < \varepsilon.$$

Write

$$T = \{x \in \operatorname{supp} \mu \mid \alpha_\mu(x) = \alpha\}$$

and

$$T_m = \left\{x \in T \mid \alpha - \eta < \frac{\log \mu B(x, r)}{\log r} < \alpha + \eta \text{ for } 0 < r < \frac{1}{m}\right\}$$

for $m \in \mathbb{N}$. Since $T_m \nearrow T$ and $\mu(T) > 0$ we may choose $M \in \mathbb{N}$ such that $\mu(T_M) > 0$. It follows immediately from Corollary 2.9 that $\dim(T_M) \geqslant \alpha - \eta$ whence

$$\mathscr{H}^{\alpha - \eta - \gamma}(T_M) = \infty \tag{4.13}$$

Observe that if $x \in T_M$ and $0 < r < 1/M$ then

$$\alpha - \eta < \frac{\log \mu B(x, r)}{\log r} < \alpha + \eta,$$

and a small computation now yields

$$\mu(B(x, r))^q \geqslant r^{\alpha q} r^{\eta |q|} \tag{4.14}$$

Let $0 < \delta < 1/M$ and $(B(x_i, r_i))_i$ be a centered δ-covering of T_M. Then (4.14) implies that

$$\begin{aligned}
\sum_i \mu(B(x_i, r_i))^q (2r_i)^{\alpha(1-q)-\varepsilon} &\geqslant \sum_i r_i^{\alpha q + \eta |q|} (2r_i)^{\alpha(1-q)-\varepsilon} \\
&= 2^{\alpha(1-q)-\varepsilon} \sum_i r_i^{\alpha + \eta |q| - \varepsilon} \geqslant 2^{\alpha(1-q)-\varepsilon} \sum_i r_i^{\alpha - \eta - \gamma} \\
&= c \sum_i (2r_i)^{\alpha - \eta - \gamma} = c \sum_i (\operatorname{diam} B(x_i, r_i))^{\alpha - \eta - \gamma} \\
&\geqslant c \mathscr{H}^{\alpha - \eta - \gamma}_{2\delta}(T_M),
\end{aligned}$$

where $c = 2^{-\alpha q - \varepsilon + \eta + \gamma}$. Hence

$$\overline{\mathscr{H}}^{q, \alpha(1-q)-\varepsilon}_{\mu, \delta}(T_M) \geqslant c \mathscr{H}^{\alpha - \eta - \gamma}_{2\delta}(T_M).$$

Letting $\delta \searrow 0$ now yields

$$\mathscr{H}_\mu^{q,\alpha(1-q)-\varepsilon}(T_M) \geqslant \bar{\mathscr{H}}_\mu^{q,\alpha(1-q)-\varepsilon}(T_M) \geqslant c\mathscr{H}^{\alpha-\eta-\gamma}(T_M) = \infty$$

from which we infer that $\alpha(1-q)-\varepsilon \leqslant \dim_\mu^q(T_M) \leqslant b_\mu(q)$. ▮

Proof of Proposition 2.13. (i) Let $q < p$. We must now prove that

$$B(p) + \underline{A}(p-q) \leqslant B(q) = \operatorname{Dim}_\mu^q(\operatorname{supp})\,\mu).$$

It is sufficient to prove that

$$\mathscr{P}_\mu^{q,\,B(p)+(\underline{A}-\eta)(p-q)-\varepsilon}(\operatorname{supp}\mu) > 0$$

for all $\eta, \varepsilon > 0$. Fix $\eta, \varepsilon > 0$. Since $\underline{A} \in I'_+$ there exists $z > 0$ such that $\underline{A} - \eta \leqslant -B(z)/z < \underline{A}$. For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in \underline{X}_{\underline{A}} \;\middle|\; -\frac{B(z)}{z} < \frac{\log \mu B(x,r)}{\log r} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Clearly

$$\begin{aligned}\mathscr{P}_\mu^{q,\,B(p)+(\underline{A}-\eta)(p-q)-\varepsilon}(\operatorname{supp}\mu) &\geqslant \mathscr{P}^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}\left(\bigcup_m T_m\right)\\ &= \sup_m \mathscr{P}^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}(T_m).\end{aligned}$$

It is thus sufficient to prove that there exists an integer $M \in \mathbb{N}$ such that

$$\mathscr{P}_\mu^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}(T_M) > 0. \tag{4.15}$$

Observe that

$$\operatorname{supp}\mu = \underline{X}_{\underline{A}} \cup \bigcup_n \underline{X}^{\underline{A}-1/n} = \underline{X}_{\underline{A}} = \bigcup_m T_m,$$

since $\underline{X}^\alpha = \varnothing$ for $\alpha < \underline{A}$ by Lemma 4.4. Hence

$$\infty = \mathscr{P}_\mu^{p,\,B(p)-\varepsilon/2}(\operatorname{supp}\mu) = \sup_m \mathscr{P}_\mu^{p,B(p)-\varepsilon/2}(T_m),$$

and we may thus choose $M \in \mathbb{N}$ satisfying

$$\nu := \mathscr{P}_\mu^{p,\,B(p)-\varepsilon/2}(T_M) > 0.$$

We claim that M satisfies (4.15).

First observe that if $x \in T_M$ and $0 < r < 1/M$ then

$$
\begin{aligned}
&-\frac{B(z)}{z} < \frac{\log \mu B(x,r)}{\log r} \\
&\Downarrow \\
&r^{-B(z)/z} \geqslant \mu B(x,r) \\
&\Downarrow \\
&r^{-B(z)} \geqslant \mu(B(x,r))^{z} \qquad (4.16)\\
&\Downarrow \\
&\mu(B(x,r))^{z}\,(2r)^{B(z)+(\varepsilon/2)(z/(p-q))} \leqslant 2^{B(z)+(\varepsilon/2)(z/(p-q))} r^{(\varepsilon/2)(z/(p-q))} \\
&\qquad \leqslant 2^{B(z)+(\varepsilon/2)(z/(p-q))} \left(\frac{1}{M}\right)^{(\varepsilon/2)(z/(p-q))} := c.
\end{aligned}
$$

Let $E \subseteq T_M$, $0 < \delta < 1/M$ and $(B_i = B(x_i, r_i))_i$ be a centered δ-packing of E. Then

$$
\begin{aligned}
\overline{\mathscr{P}}^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}_{\mu,\delta}(E) &\geqslant \sum_i \mu(B_i)^q\,(2r_i)^{B(p)-(B(z)/z)(p-q)-\varepsilon} \\
&= \sum_i \left(\mu(B_i)^p\,(2r_i)^{B(p)-(\varepsilon/2)}\right) \\
&\qquad \times \left(\mu(B_i)^z\,(2r_i)^{B(z)+(\varepsilon/2)(z/(p-q))}\right)^{(q-p)/z} \\
&\geqslant c^{(q-p)/z} \sum_i \mu(B_i)^p\,(2r_i)^{B(p)-(\varepsilon/2)} \\
&\qquad \left[\text{by (4.16) since } \frac{q-p}{z} < 0\right]
\end{aligned}
$$

Hence

$$
\overline{\mathscr{P}}^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}_{\mu,\delta}(E) \geqslant c^{(q-p)/z}\,\overline{\mathscr{P}}^{p,\,B(p)-(\varepsilon/2)}_{\mu,\delta}(E),
$$

and letting $\delta \searrow 0$ yields

$$
\overline{\mathscr{P}}^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}_{\mu}(E) \geqslant c^{(q-p)/z}\,\overline{\mathscr{P}}^{p,\,B(p)-(\varepsilon/2)}_{\mu}(E) \qquad \text{for} \quad E \subseteq T_M. \quad (4.17)
$$

Now, if $T_M \subseteq \bigcup_i E_i$ then (4.17) implies that

$$\begin{aligned}0 < \nu c^{(q-p)/z} &= \mathscr{P}_\mu^{p,\,B(p)-(\varepsilon/z)}(T_M)\, c^{(q-p)/z}\\ &= c^{(q-p)/z} \mathscr{P}_\mu^{p,\,B(p)-(\varepsilon/z)}\left(\bigcup_i (T_M \cap E_i)\right)\\ &\leqslant c^{(q-p)/z} \sum_i \mathscr{P}_\mu^{p,\,B(p)-(\varepsilon/z)}(T_M \cap E_i) \leqslant c^{(q-p)/z} \sum_i \overline{\mathscr{P}}_\mu^{p,\,B(p)-(\varepsilon/z)}(T_M \cap E_i)\\ &\leqslant \sum_i \overline{\mathscr{P}}_\mu^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}(T_M \cap E_i) \leqslant \sum_i \overline{\mathscr{P}}_\mu^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}(E_i),\end{aligned}$$

which shows that

$$0 < \nu c^{(q-p)/z} \leqslant \mathscr{P}^{q,\,B(p)-(B(z)/z)(p-q)-\varepsilon}(T_M).$$

By monotonicity $\underline{E} := \lim_{q \to \infty} (B(q) + \underline{A}q)$ exists. Also $B(q) + \underline{A}q$ is non-negative for $q > 0$ by the definition of $\underline{A}$, whence $\underline{E} \geqslant \underline{0}$.

(ii) Follows easily from the fact that B is decreasing.

(iii) Let $q < p$. We must now prove that

$$B(q) - \bar{A}(p-q) \leqslant B(p) = \mathrm{Dim}_\mu^p(\operatorname{supp} \mu).$$

It is thus sufficient to prove that

$$\mathscr{P}_\mu^{p,\,B(q)-(\bar{A}+\eta)(p-q)-\varepsilon}(\operatorname{supp} \mu) > 0$$

for all $\eta, \varepsilon > 0$. Since $\bar{A} \in I'_-$ there exists $z < 0$ such that $\bar{A} < -B(z)/z \leqslant \bar{A} + \eta$. For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in \bar{X}^{\bar{A}} \,\middle|\, \frac{\log \mu B(x,r)}{\log r} < -\frac{B(z)}{z} \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Now use the fact that $\operatorname{supp} \mu = \bar{X}^{\bar{A}} \cup \bigcup_n \bar{X}_{\bar{A}+1/n} = \bar{X}^{\bar{A}} = \bigcup_m T_m$ (since $\bar{X}_\alpha = \varnothing$ for $\bar{A} < \alpha$ by Lemma 4.8) and proceed as in case i.

(iv) Follows easily from the fact that B is decreasing. ∎

Note that Proposition 2.13 does not hold if B_μ, $\underline{A}$, $\bar{A}$, I_+, and I_- are replaced by $b = b_\mu$, $\underline{a}$, $\bar{a}$, $J_+ := \{-b(q)/q \mid q > 0\}$ and $J_- := \{-b(q)/q \mid q < 0\}$ respectively. Indeed, let μ be the measure from Example 4 in Section 3. Then $\bar{a} = d$ and $\underline{a} = D$. Also

$$b_\mu(q) + \underline{a}q = \begin{cases} (D-d)\,q + d & q \leqslant 1 \\ D & 1 \leqslant q \end{cases}$$

which is not decreasing and

$$b_\mu(q) + \bar{a}q = \begin{cases} d & q \leqslant 1 \\ -(D-d)\,q + D & 1 \leqslant q \end{cases}$$

which is not increasing.

4.5. *Proofs of the Results in Section* 2.5

Proof of Theorem 2.14. (i) Let $a := \inf_{x \in E} \bar{d}_\mu^{q,t}(x, \nu)$. The assertion is obvious for $a = 0$ so we may assume that $a > 0$. For $m \in \mathbb{N}$ write

$$E_m = \left\{ x \in E \,\middle|\, \frac{\mu B(x, 5r)}{\mu B(x, r)} < m \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Now observe that $E_m \nearrow E$. It is thus sufficient to prove that

$$\mathscr{H}_\mu^{q,t}(E_m)(a - \eta) \leqslant \nu(E) + \varepsilon$$

for all $m \in \mathbb{N}$ and $\eta, \varepsilon > 0$. Fix $m \in \mathbb{N}$, $\varepsilon > 0$ and $0 < \eta < a$. By inner regularity it is sufficient to prove that

$$\mathscr{H}_\mu^{q,t}(F)(a - \eta) \leqslant \nu(E) + \varepsilon$$

for all closed subsets F of E_m. Let F be a closed subset of E_m. By definition $\mathscr{H}_\mu^{q,t}(F) = \sup_{H \subseteq F} \bar{\mathscr{H}}_\mu^{q,t}(H)$ and it is thus sufficient to prove that

$$\bar{\mathscr{H}}_\mu^{q,t}(H)(a - \eta) \leqslant \nu(E) + \varepsilon$$

for $H \subseteq F$. Now fix $H \subseteq F$. For $\delta > 0$ write $B(F, \delta) = \{x \in \mathbb{R}^d \mid \operatorname{dist}(F, x) \leqslant \delta\}$. Since F is closed, $B(F, \delta) \searrow F$ as $\delta \searrow 0$, and we can therefore choose $\delta_0 > 0$ such that

$$\nu(B(F, \delta)) \leqslant \nu(F) + \frac{\varepsilon}{3} \qquad \text{for} \quad \delta < \delta_0.$$

Since $\bar{\mathscr{H}}_\mu^{q,t}(H) \leqslant \mathscr{H}_\mu^{q,t}(H) < \infty$ it is possible to choose $\delta < \delta_0$ satisfying

$$\bar{\mathscr{H}}_\mu^{q,t}(H) - \frac{\varepsilon}{3(a - \eta)} \leqslant \bar{\mathscr{H}}_{\mu,\delta}^{q,t}(H). \tag{4.18}$$

Put $\mathscr{V} = \{B(x, r) \mid x \in H,\ 5r < \delta,\ \nu(B(x, r)) \geqslant (a - \eta)\, \mu(B(x, r))^q\, (2r)^t\}$. It follows from [Fa2, Lemma 1.9] that there exists a countable, disjoint subfamily $(B_i = B(x_i, r_i))_i \subseteq \mathscr{V}$ such that

$$H \setminus \bigcup_{i=1}^{k} B_i \subseteq \bigcup_{i > k} B(x_i, 5r_i), \tag{4.19}$$

for all $k \in \mathbb{N}$. Observe that

$$\sum_i \mu(B(x_i, 5r_i))^q (2 \cdot 5r_i)^t \leqslant 5^t m \sum_i \mu(B(x_i, r_i))^q (2r_i)^t$$
$$\leqslant 5^t m (a-\eta)^{-1} \sum_i \nu(B(x_i, r_i))$$
$$\leqslant 5^t m (a-\eta)^{-1} \nu\left(\bigcup_i B(x_i, r_i)\right) < \infty.$$

We may thus choose $N \in \mathbb{N}$ such that

$$\sum_{i>N} \mu(B(x_i, 5r_i))^q (2 \cdot 5r_i)^t < \frac{\varepsilon}{3} (a-\eta)^{-1}.$$

It follows from (4.18) and (4.19) that

$$\bar{\mathscr{H}}_\mu^{q,t}(H)(a-\eta) \leqslant (a-\eta)\, \bar{\mathscr{H}}_{\mu,\delta}^{q,t}(H) + \frac{\varepsilon}{3}$$
$$\leqslant (a-\eta)\left(\sum_i \mu(B_i)^q (2r_i)^t + \sum_{i>N} \mu(B(x_i, 5r_i))^q (2 \cdot 5r_i)^t\right) + \frac{\varepsilon}{3}$$
$$\leqslant \sum_i \nu(B(x_i, r_i)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$
$$\leqslant \nu\left(\bigcup_i B(x_i, r_i)\right) + \frac{2\varepsilon}{3} \leqslant \nu(B(F, \delta)) + \frac{2\varepsilon}{3}$$
$$\leqslant \nu(F) + \varepsilon \leqslant \nu(E) + \varepsilon.$$

(ii) Put $a := \sup_{x \in E} \bar{d}_\mu^{q,t}(x, \nu)$. It is sufficient to prove that

$$\nu(E) \leqslant \mathscr{H}_\mu^{q,t}(E)(a+\eta) + \varepsilon$$

for all $\varepsilon, \eta > 0$. Fix $\varepsilon, \eta > 0$ and write $E_m = \{x \in E \mid \nu(B(x, r)) \leqslant (a+\eta)\, \mu(B(x, r))^q (2r)^t \text{ for } 0 < r < 1/m\}$, $m \in \mathbb{N}$, and observe that $E_m \nearrow E$. It is therefore sufficient to prove that

$$\nu(E_m) \leqslant \mathscr{H}_\mu^{q,t}(E)(a+\eta) + \varepsilon$$

for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. Since $\bar{\mathscr{H}}_\mu^{q,t}(E_m) \leqslant \mathscr{H}_\mu^{q,t}(E_m) < \infty$ we may choose a centered $1/m$-covering $(B(x_i, r_i))_i$ of E_m such that

$$\sum \mu(B(x_i, r_i))^q (2r_i)^t \leqslant \bar{\mathscr{H}}_{\mu, 1/m}^{q,t}(E_m) + \frac{\varepsilon}{a+\eta}.$$

Hence

$$\nu(E_m) \leqslant \sum_i \nu(B(x_i, r_i)) \leqslant (a+\eta) \sum_i \mu(B(x_i, r_i))^q (2r_i)^t$$
$$\leqslant \mathscr{H}_\mu^{q,t}(E_m)(a+\eta) + \varepsilon. \quad ∎$$

Proof of Theorem 2.15. Proof of $\mathscr{P}_\mu^{q,t}(E) \inf_{x \in E} \underline{d}_\mu^{q,t}(x, \nu) \leqslant \nu(E)$: Let $a := \inf_{x \in E} \underline{d}_\mu^{q,t}(x, \nu)$. The assertion is obvious for $a = 0$ so we may assume that $a > 0$. It is sufficient to prove that

$$\mathscr{P}_\mu^{q,t}(E)(a-\eta) \leqslant \nu(E) + \varepsilon$$

for η, $\varepsilon > 0$. Fix $\varepsilon > 0$ and $0 < \eta < a$. By inner regularity it is sufficient to prove that

$$\mathscr{P}_\mu^{q,t}(F)(a-\eta) \leqslant \nu(E) + \varepsilon$$

for all closed subsets F of E. Let F be a closed subset of E. For $\delta > 0$ write $B(F, \delta) = \{x \in \mathbb{R}^d \mid \operatorname{dist}(F, x) \leqslant \delta\}$. Since F is closed, $B(F, \delta) \searrow F$ for $\delta \searrow 0$, and we can therefore choose $\delta_0 > 0$ such that

$$\nu(B(F, \delta)) \leqslant \nu(F) + \varepsilon \qquad \text{for} \quad 0 < \delta < \delta_0.$$

For $m \in \mathbb{N}$ write

$$F_m = \left\{ x \in F \mid \nu(B(x, r)) \geqslant (a-\eta)\, \mu(B(x, r))^q (2r)^t \text{ for } 0 < r < \frac{1}{m} \right\}.$$

Fix $m \in \mathbb{N}$ and $0 < \delta < 1/m \wedge \delta_0$. Let $(B(x_i, r_i))_i$ be a centered δ-packing of F_m. Then

$$(a-\eta) \sum_i \mu(B(x_i, r_i))^q (2r_i)^t \leqslant \sum_i \nu(B(x_i, r_i)) = \nu\left(\bigcup_i B(x_i, r_i)\right)$$
$$\leqslant \nu(B(F, \delta)) \leqslant \nu(F) + \varepsilon \leqslant \nu(E) + \varepsilon.$$

Hence

$$(a-\eta)\, \mathscr{P}_\mu^{q,t}(F_m) \leqslant (a-\eta)\, \overline{\mathscr{P}}_\mu^{q,t}(F_m) \leqslant (a-\eta)\, \overline{\mathscr{P}}_{\mu,\delta}^{q,t}(F_m) \leqslant \nu(E) + \varepsilon.$$

Clearly $F_m \nearrow F$, whence

$$(a-\eta)\, \mathscr{P}_\mu^{q,t}(F) \leqslant \nu(E) + \varepsilon.$$

Proof of $\nu(E) \leqslant \mathscr{P}_\mu^{q,t}(E) \sup_{x\in E} \underline{d}_\mu^{q,t}(x, \nu)$: Let $a := \sup_{x\in E} \underline{d}_\mu^{q,t}(x, \nu)$. It is sufficient to prove that

$$\nu(F) \leqslant a\bar{\mathscr{P}}_\mu^{q,t}(F) \qquad \text{for all} \quad F \subseteq E. \tag{4.20}$$

Indeed, assume that (4.20) is satisfied and let $(E_i)_{i\in\mathbb{N}}$ be a cover of E. Then

$$\nu(E) = \nu\left(\bigcup_i (E \cap E_i)\right) \leqslant \sum_i \nu(E \cap E_i) \leqslant a \sum_i \bar{\mathscr{P}}_\mu^{q,t}(E \cap E_i) \leqslant a \sum_i \bar{\mathscr{P}}_\mu^{q,t}(E_i),$$

whence

$$\nu(E) \leqslant a\mathscr{P}_\mu^{q,t}(E).$$

Fix $F \subseteq E$. In order to prove (4.20) it is clearly enough to prove that

$$\nu(F) \leqslant (a+\eta)\, \bar{\mathscr{P}}_\mu^{q,t}(F) + \varepsilon$$

for all η, $\varepsilon > 0$. Let $\eta, \varepsilon > 0$. Choose $\delta > 0$ such that

$$\bar{\mathscr{P}}_{\mu,\delta}^{q,t}(F) \leqslant \bar{\mathscr{P}}_\mu^{q,t}(F) + \frac{\varepsilon}{a+\eta}.$$

Put

$$\mathscr{V} = \{B(x, r) \mid x \in F,\, r < \delta,\, \nu(B(x, r)) \leqslant (a+\eta)\, \mu(B(x, r))^q\, (2r)^t\}.$$

It follows from Vitalis covering theorem (cf. [Gu]) that there exists a δ-packing $(B(x_i, r_i))_i \subseteq \mathscr{V}$ of F satisfying

$$\nu\left(F \setminus \bigcup_i B(x_i, r_i)\right) = 0.$$

Hence

$$\begin{aligned}\nu(F) &= \nu\left(F \cap \bigcup_i B(x_i, r_i)\right) \leqslant \sum_i \nu(B(x_i, r_i))\\ &\leqslant (a+\eta) \sum \mu(B(x_i, r_i))^q\, (2r_i)^t \leqslant (a+\eta)\, \bar{\mathscr{P}}_{\mu,\delta}^{q,t}(F)\\ &\leqslant (a+\eta)\, \bar{\mathscr{P}}_\mu^{q,t}(F) + \varepsilon. \quad \blacksquare\end{aligned}$$

COROLLARY 4.5. *If* $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$ *and* $\mathscr{H}_\mu^{q,t}(E) < \infty$ *then*

$$\bar{d}_\mu^{q,t}(x, \mathscr{H}_\mu^{q,t} \mid E) = 1 \qquad \textit{for} \quad \mathscr{H}_\mu^{q,t}\textit{-a.a.} \quad x \in E.$$

Proof. "$\leqslant$" Put $\nu = \mathscr{H}^{q,t}_{\mu}|E$, $F = \{x \in E \mid \bar{d}^{q,t}_{\mu}(x, \nu) > 1\}$ and $F_m = \{x \in E \mid \bar{d}^{q,t}_{\mu}(x, \nu) \geqslant 1 + 1/m\}$. Then theorem 2.14(i) implies that

$$\mathscr{H}^{q,t}_{\mu}(F_m)\left(1 + \frac{1}{m}\right) \leqslant \nu(F_m) = \mathscr{H}^{q,t}_{\mu}(F_m),$$

whence $\mathscr{H}^{q,t}_{\mu}(F_m) = 0$. Since $F = \bigcup_m F_m$ this completes the proof.

"$\geqslant$" Put $\nu = \mathscr{H}^{q,t}_{\mu}|E$, $F = \{x \in E \mid \bar{d}^{q,t}_{\mu}(x, \nu) < 1\}$ and $F_m = \{x \in E \mid \bar{d}^{q,t}_{\mu}(x, \nu) \leqslant 1 - 1/m\}$. Now apply Theorem 2.14(ii) and proceed as in the previous part of the proof. ∎

COROLLARY 4.6. *If* $\mathscr{P}^{q,t}_{\mu}(E) < \infty$ *then*

$$\underline{d}^{q,t}_{\mu}(x, \mathscr{P}^{q,t}_{\mu}|E) = 1 \qquad \textit{for} \quad \mathscr{P}^{q,t}_{\mu}\textit{-a.a.} \quad x \in E.$$

Proof. The proof is similar to the proof of the previous corollary. ∎

Proof of Corollary 2.16. (i)$\Rightarrow$(ii). Clearly

$$\mathscr{H}^{q,t}_{\mu}(F) = \mathscr{P}^{q,t}_{\mu}(F) \qquad \text{for} \quad F \subseteq E, \tag{4.21}$$

(indeed, if $F \subseteq E$ then $\mathscr{H}^{q,t}_{\mu}(F) + \mathscr{H}^{q,t}_{\mu}(E \backslash F) = \mathscr{H}^{q,t}_{\mu}(E) = \mathscr{P}^{q,t}_{\mu}(E) = \mathscr{P}^{q,t}_{\mu}(F) + \mathscr{P}^{q,t}_{\mu}(E \backslash F)$, and the inequality $\mathscr{H}^{q,t}_{\mu} \leqslant \mathscr{P}^{q,t}_{\mu}$ now yields (4.21)). By Corollary 4.5 and (4.21),

$$\bar{d}^{q,t}_{\mu}(x, \mathscr{H}^{q,t}_{\mu}|E) = 1 \qquad \text{for} \quad \mathscr{P}^{q,t}_{\mu}\text{-a.a.} \quad x \in E. \tag{4.22}$$

Now put $\nu = \mathscr{H}^{q,t}_{\mu}|E$, $F = \{x \in E \mid \underline{d}^{q,t}_{\mu}(x, \nu) < 1\}$ and $F_m = \{x \in E \mid \underline{d}^{q,t}_{\mu}(x, \nu) \leqslant 1 - 1/m\}$. Then Theorem 2.15 and (4.22) imply that

$$\begin{aligned}\mathscr{P}^{q,t}_{\mu}(F_m) &= \mathscr{H}^{q,t}_{\mu}(F_m) \qquad \text{(by (4.21))}\\ &= \nu(F_m) \leqslant \mathscr{P}^{q,t}_{\mu}(F_m)\left(1 - \frac{1}{m}\right),\end{aligned}$$

whence $\mathscr{P}^{q,t}_{\mu}(F_m) = 0$. Since $F = \bigcup_m F_m$ this shows that $\mathscr{P}^{q,t}_{\mu}(F) = 0$, i.e.

$$1 \leqslant \underline{d}^{q,t}_{\mu}(x, \mathscr{H}^{q,t}_{\mu}|E) \qquad \text{for} \quad \mathscr{P}^{q,t}_{\mu}\text{-a.a.} \quad x \in E. \tag{4.23}$$

The statement in (ii) now follows from (4.22) and (4.23).

(ii)$\Rightarrow$(i) Put $F = \{x \in E \mid d^{q,t}_{\mu}(x, \mathscr{H}^{q,t}_{\mu}|E) = 1\}$ and $\nu = \mathscr{H}^{q,t}_{\mu}|E$. It follows from Theorem 2.15 and (ii) that

$$\begin{aligned}\mathscr{P}^{q,t}_{\mu}(E) &= \mathscr{P}^{q,t}_{\mu}(F) \qquad \text{(by (ii))}\\ &\leqslant \nu(F) = \mathscr{H}^{q,t}_{\mu}(F) \leqslant \mathscr{H}^{q,t}_{\mu}(E) \leqslant \mathscr{P}^{q,t}_{\mu}(E).\end{aligned}$$

(i) $\Rightarrow$ (iii) The proof is very similar to the proof of (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (i) The proof is very similar to the proof of (ii) $\Rightarrow$ (i). ∎

4.6. *Proofs of the Results in Section* 2.7

Proof of Propositions 2.19–2.22. The proofs of Proposition 2.19 through Proposition 2.22 follows from the next eight claims which we will prove below.

Claim 1. $\underline{L}^q_\mu(E) \leqslant \underline{C}^q_\mu(E)$, $\overline{L}^q_\mu(E) \leqslant \overline{C}^q_\mu(E)$ for $q \in \mathbb{R}$.

Claim 2. $\underline{L}^q_\mu(E) \geqslant \underline{C}^q_\mu(E)$, $\overline{L}^q_\mu(E) \geqslant \overline{C}^q_\mu(E)$ for $0 < q$ and $\mu \in \mathscr{P}_1(\mathbb{R}^d, E)$.

Claim 3. $\underline{L}^q_\mu(E) \geqslant \underline{C}^q_\mu(E)$, $\overline{L}^q_\mu(E) \geqslant \overline{C}^q_\mu(E)$ for $q \leqslant 0$.

Claim 4. $\overline{C}^q_\mu(E) \leqslant \Delta^q_\mu(E)$ for $q \in \mathbb{R}$.

Claim 5. $\overline{C}^q_\mu(E) > \Delta^q_\mu(E)$ for $0 < q$ and $\mu \in \mathscr{P}_1(\mathbb{R}^d, E)$.

Claim 6. $\overline{C}^q_\mu(E) \geqslant \Delta^q_\mu(E)$ for $q \leqslant 0$.

Claim 7. $\dim^q_\mu(E) \leqslant \underline{L}^q_\mu(E)$ for $0 < q$ and $\mu \in \mathscr{P}_0(\mathbb{R}^d, E)$.

Claim 8. $\dim^q_\mu(E) \leqslant \underline{L}^q_\mu(E)$ for $q \leqslant 0$.

Proof of Claim 1. Let ζ be the integer that appears in Besicovitch covering theorem (i.e. Theorem 2.2). Let $\delta > 0$ and put $\mathscr{V} = \{B(x, \delta) \mid x \in E\}$. It follows from Theorem 2.2 that there exists ζ coutable (or finite) subfamilies $(B(x_{ij}, \delta))_j$, $i = 1, ..., \zeta$ of $\mathscr{V}$ such that $(B(x_{ij}, \delta))_{ij}$ is a cover of E and $(B(x_{ij}, \delta))_j$ is a packing of E for $i = 1, ..., \zeta$. Hence

$$T^q_{\mu,\delta}(E) \leqslant \sum_i \sum_j \mu(B(x_{ij}, \delta))^q \leqslant \sum_{i=1}^{\zeta} S^q_{\mu,\delta}(E) = \zeta S^q_{\mu,\delta}(E).$$

Taking logarithms and letting $\delta \searrow 0$ yields Claim 1.

Proof of Claim 2. Since $\mu \in \mathscr{P}_1(\mathbb{R}^d, E)$, we may choose $A \in]0, \infty[$ and $r_0 > 0$ such that

$$\sup_{x \in E} \frac{\mu B(x, 3r)}{\mu B(x, r)} \leqslant A \qquad \text{for} \quad 0 < r < r_0.$$

Fix $0 < \delta < r_0$. Let $(B(x_i, \delta))_i$ be a centered packing of E and $(B(y_i, \delta/2))_i$ a centered covering of E. For each $i \in \mathbb{N}$ choose an integer $k(i)$ such that $x_i \in B(y_{k(i)}, \delta/2)$ and observe that

$$i \neq j \Rightarrow k(i) \neq k(j)$$

Hence

$$\begin{aligned}\sum_i \mu(B(x_i,\delta))^q &= \sum_i \left(\frac{\mu B(x_i,\delta)}{\mu B(y_{k(i)},\delta/2)}\right)^q \mu\left(B\left(y_{k(i)},\frac{\delta}{2}\right)\right)^q \\ &\leqslant \sum_i \left(\frac{\mu B(y_{k(i)},3\delta/2)}{\mu B(y_{k(i)},\delta/2)}\right)^q \mu\left(B\left(y_{k(i)},\frac{\delta}{2}\right)\right)^q \\ &\leqslant A^q \sum_i \left(B\left(y_{k(i)},\frac{\delta}{2}\right)\right)^q \\ &\leqslant A^q \sum_j \mu\left(B\left(y_j,\frac{\delta}{2}\right)\right)^q \qquad \text{(by (2.4))}\end{aligned}$$

whence $S^q_{\mu\delta}(E) \leqslant A^q T^q_{\mu,(\delta/2)}(E)$ for $0<\delta<r_0$. Taking logarithms and letting $\delta \searrow 0$ yields the desired results.

Proof of Claim 3. Similar to the proof of Claim 2.

Proof of Claim 4. Put $t := \Delta^q_\mu(E)$. Let $\varepsilon>0$. We may choose $0<\delta_\varepsilon<1$ such that

$$\bar{\mathscr{P}}^{q,t+\varepsilon}_{\mu,\delta}(E)<1 \qquad \text{for} \quad 0<\delta<\delta_\varepsilon.$$

Fix $0<\delta<\delta_\varepsilon$ and let $(B(x_i,\delta))_i$ be a centered packing of E. Then

$$\begin{aligned}\sum_i \mu(B(x_i,\delta))^q &= (2\delta)^{-(t+\varepsilon)} \sum_i \mu(B(x_i,\delta))^q (2\delta)^{t+\varepsilon} \\ &\leqslant (2\delta)^{-(t+\varepsilon)} \bar{\mathscr{P}}^{q,t+\varepsilon}_{\mu,\delta}(E) \leqslant (2\delta)^{-(t+\varepsilon)},\end{aligned}$$

whence $S^q_{\mu,\delta}(E) \leqslant (2\delta)^{-(t+\varepsilon)}$. Taking logarithms then yields

$$\frac{\log S^q_{\mu,\delta}(E)}{-\log\delta} \leqslant \frac{(t+\varepsilon)\log 2}{\log\delta} + (t+\varepsilon)$$

for $0<\delta<\delta_\varepsilon$. Letting $\delta \searrow 0$ now yields $\bar{C}^q_\mu(E) \leqslant t+\varepsilon$ which completes the proof since $\varepsilon>0$ was arbitrary.

Proof of Claim 5. Put $t := \Delta^q_\mu(E)$. Since $\mu \in \mathscr{P}_1(\mathbb{R}^d, E)$, we may choose $A \in \,]0,\infty[$ and $1>r_0>0$ such that

$$\sup_{x\in E} \frac{\mu B(x,2r)}{\mu B(x,r)} < A \qquad \text{for} \quad 0<r<r_0.$$

Now fix $\varepsilon > 0$ and write $a := \log(A(1-2^{-\varepsilon/2})^{-1} 2^{t-(\varepsilon/2)}(1 \vee (\frac{1}{2})^{t-(\varepsilon/2)}) 2^{t-\varepsilon})$. In order to prove Claim 5 it is sufficient to prove that

$$\forall \delta_0 \in]0, r_0[: \exists \delta \in]0, \delta_0[: t - \varepsilon - \frac{a}{\log \delta} \leqslant \frac{\log S^q_{\mu,\delta}(E)}{-\log \delta}.$$

Let $\delta_0 \in]0, 1[$. Since $\infty = \overline{\mathscr{P}}^{q, t-(\varepsilon/2)}_{\mu}(E) \leqslant \overline{\mathscr{P}}^{q, t-(\varepsilon/2)}_{\mu, \delta_0}(E)$ there exists a centered δ_0-packing $(B_i = B(x_i, r_i))_i$ of E such that

$$\sum_i \mu(B(x_i, r_i))^q (2r_i)^{t-(\varepsilon/2)} > 1.$$

For $n \in \mathbb{N}$ write

$$I_n = \left\{ i \,\middle|\, \frac{\delta_0}{2^{n+1}} \leqslant r_i < \frac{\delta_0}{2^n} \right\}$$

$$\mu_n = \sum_{i \in I_n} \mu(B(x_i, r_i))^q.$$

Clearly

$$\begin{aligned}
1 &< \sum_i \mu(B(x_i, r_i))^q (2r_i)^{t-(\varepsilon/2)} \\
&\leqslant 2^{t-(\varepsilon/2)} \left(1 \vee \left(\frac{1}{2}\right)^{t-(\varepsilon/2)}\right) \sum_n \mu_n \left(\frac{\delta_0}{2^n}\right)^{t-(\varepsilon/2)} \\
&\leqslant c_1 \sum_n \mu_n \left(\frac{\delta_0}{2^n}\right)^{t-\varepsilon} \left(\frac{\delta_0}{2^n}\right)^{\varepsilon/2} \\
&\leqslant c_1 \left(\sup_m \mu_m \left(\frac{\delta_0}{2^m}\right)^{t-\varepsilon}\right) \sum_n \left(\frac{\delta_0}{2^n}\right)^{\varepsilon/2} \\
&\leqslant c_2 \sup_m \mu_m \left(\frac{\delta_0}{2^m}\right)^{t-\varepsilon},
\end{aligned}$$

where $c_2 = 2^{t-(\varepsilon/2)}(1 \vee (\frac{1}{2})^{t-(\varepsilon/2)})(1-2^{-\varepsilon/2})^{-1}$, and we may thus choose $N \in \mathbb{N}$ such that

$$1 < c_2 \mu_N \left(\frac{\delta_0}{2^N}\right)^{t-\varepsilon}.$$

Now put $\delta := \delta_0/2^{N+1}$. Then $\delta \in]0, \delta_0[$ and $(B(x_i, \delta))_{i \in I_N}$ is a centered packing of E (because $\delta := \delta_0/2^{N+1} < r_i$ for $i \in I_N$), whence

$$
\begin{aligned}
S^q_{\mu,\delta}(E) &\geqslant \sum_{i \in I_N} \mu(B(x_i, \delta))^q \\
&\geqslant \sum_{i \in I_N} \left(\frac{\mu B(x_i, \delta_0/2^{N+1})}{\mu B(x_i, \delta_0/2^N)}\right)^q \mu(B(x_i, r_i))^q \\
&\geqslant A^{-1} \sum_{i \in I_N} \mu(B(x_i, r_i))^q = A^{-1}\mu_N \\
&> A^{-1}c_2^{-1}\left(\frac{\delta_0}{2^N}\right)^{-(t-\varepsilon)} = A^{-1}c_2^{-1}2^{-(t-\varepsilon)}\delta^{t-\varepsilon}.
\end{aligned}
$$

Taking logarithms now yields

$$\frac{\log S^q_{\mu,\delta}(E)}{-\log \delta} \geqslant -\frac{a}{\log \delta} + t - \varepsilon.$$

Proof of Claim 6. Similar to the proof of Claim 5.

Proof of Claim 7. For $m \in \mathbb{N}$ write

$$T_m = \left\{ x \in E \,\middle|\, \frac{\mu B(x, 3r)}{\mu B(x, r)} < m \text{ for } 0 < m < \frac{1}{m} \right\}.$$

Since $\bigcup_m T_m = E$, $\dim^q_\mu(E) = \sup_m \dim^q_\mu(T_m)$ and it is thus sufficient to prove that

$$\dim^q_\mu(T_m) \leqslant \underline{L}^q_\mu(E)$$

for all $m \in \mathbb{N}$. Now fix $m \in \mathbb{N}$. We must now prove that

$$\dim^q_\mu(T_m) \leqslant t$$

for all $\underline{L}^q_\mu(E) < t$. Let $\underline{L}^q_\mu(E) < t$. We must now prove that $\mathscr{H}^{q,t}_\mu(T_m) < \infty$ i.e.

$$\sup_{T \subseteq T_m} \bar{\mathscr{H}}^{q,t}_\mu(T) < \infty.$$

Fix $T \subseteq T_m$. Since $t > \underline{L}^{q,t}_\mu(E) = \liminf_{\delta \searrow 0} \log T^q_{\mu,\delta}(E)/-\log \delta$ there exists a sequence $(\delta_n)_n$ such that $\delta_n \searrow 0$, $\delta_n \in]0, 1[$ and

$$t > \frac{\log T^q_{\mu,\delta_n}(E)}{-\log \delta_n} \qquad \text{for} \quad n \in \mathbb{N}.$$

Hence: For $n \in \mathbb{N}$ then there exists a centered covering $(B(x_{ni}, \delta_n))$ of E satisfying

$$\delta_n^{-t} > \sum_i \mu(B(x_{ni}, \delta_n)).$$

Let $n \in \mathbb{N}$ and put $I = \{i \mid B(x_{ni}, \delta_n) \cap T \neq \varnothing\}$. For $i \in I$ choose $y_i \in B(x_{ni}, \delta_n) \cap T$. Then $(B(y_i, 2\delta_n))_{i \in I}$ is a centered $2\delta_n$-covering of T, whence

$$\begin{aligned}
\bar{\mathscr{H}}^{q,t}_{\mu, 2\delta_n}(T) &\leqslant \sum_{i \in I} \mu(B(y_i, 2\delta_n))^q (4\delta)^t \\
&= 4^t \sum_{i \in I} \left(\frac{\mu(B(y_i, 2\delta_n))}{\mu B(x_{ni}, \delta_n)} \right)^q \mu(B(x_{ni}, \delta_n))^q \delta_n^t \\
&\leqslant 4^t \sum_{i \in I} \left(\frac{\mu B(x_{ni}, 3\delta_n)}{\mu B(x_{ni}, \delta_n)} \right)^q \mu(B(x_{ni}, \delta_n))^q \delta_n^t \\
&\leqslant 4^t m^q \sum_{i \in I} \mu(B(x_{ni}, \delta_n))^q \delta_n^t \leqslant 4^t m^q.
\end{aligned}$$

Letting $n \to \infty$ gives $\bar{\mathscr{H}}^{q,t}_\mu(T) \leqslant 4^t m^q$ for $T \subseteq T_m$, whence $\mathscr{H}^{q,t}_\mu(T_m) \leqslant 4^t m^q$ and the proof is complete.

Proof of Claim 8. Similar to the proof of Claim 7. ∎

4.7. *Proofs of the Results in Section* 2.8

We begin by proving two small lemmas.

LEMMAS 4.7. *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. Then*

(i)

$$q\underline{I}^q_\mu \wedge q\bar{I}^q_\mu \leqslant \underline{L}^{q+1}_\mu(\operatorname{supp} \mu) \quad \textit{for} \quad q < 0$$
$$q\underline{I}^q_\mu \vee q\bar{I}^q_\mu \leqslant \bar{L}^{q+1}_\mu(\operatorname{supp} \mu) \quad \textit{for} \quad q < 0$$

(ii)

$$\underline{C}^{q+1}_\mu(\operatorname{supp} \mu) \leqslant q\underline{I}^q_\mu \wedge q\bar{I}^q_\mu \quad \textit{for} \quad 0 < q$$
$$\bar{C}^{q+1}_\mu(\operatorname{supp} \mu) \leqslant q\underline{I}^q_\mu \vee q\bar{I}^q_\mu \quad \textit{for} \quad 0 < q$$

(iii)

$$\underline{C}^{q+1}_\mu(\operatorname{supp} \mu) = q\underline{I}^q_\mu \wedge q\bar{I}^q_\mu \quad \textit{for} \quad q \in \mathbb{R} \quad \textit{and} \quad \mu \in \mathscr{P}_1(\mathbb{R}^d)$$
$$\bar{C}^{q+1}_\mu(\operatorname{supp} \mu) = q\underline{I}^q_\mu \vee q\bar{I}^q_\mu \quad \textit{for} \quad q \in \mathbb{R} \quad \textit{and} \quad \mu \in \mathscr{P}_1(\mathbb{R}^d).$$

Proof. The proof follows from Proposition 2.19 and Proposition 2.20 and the next four claims which we will prove below.

Claim 1. $\underline{C}^{q+1}_\mu(\operatorname{supp} \mu) \leqslant q\underline{I}^q_\mu \wedge q\bar{I}^q_\mu$, $\bar{C}^{q+1}_\mu(\operatorname{supp} \mu) \leqslant q\underline{I}^q_\mu \vee q\bar{I}^q_\mu$ for $q < 0$ and $\mu \in \mathscr{P}_1(\mathbb{R}^d)$.

Claim 2. $\underline{C}^{q+1}_\mu(\operatorname{supp} \mu) \leqslant q\underline{I}^q_\mu \wedge q\bar{I}^q_\mu$, $\bar{C}^{q+1}_\mu(\operatorname{supp} \mu) \leqslant q\underline{I}^q_\mu \vee q\bar{I}^q_\mu$ for $0 < q$.

Claim 3. $q\underline{I}^q_\mu \wedge q\overline{I}^q_\mu \leqslant \underline{L}^{q+1}_\mu(\operatorname{supp}\mu)$, $q\underline{I}^q_\mu \vee q\overline{I}^q_\mu \leqslant \overline{L}^{q+1}_\mu(\operatorname{supp}\mu)$ for $0<q$ and $\mu \in \mathscr{P}_1(\mathbb{R}^d)$.

Claim 4. $q\underline{I}^q_\mu \wedge q\overline{I}^q_\mu \leqslant \underline{L}^{q+1}_\mu(\operatorname{supp}\mu)$, $q\underline{I}^q_\mu \vee q\overline{I}^q_\mu \leqslant \overline{L}^{q+1}_\mu(\operatorname{supp}\mu)$ for $q<0$.

Proof of Claim 1. Since $\mu \in \mathscr{P}_1(\mathbb{R}^d)$ we may choose $A \in \,]0, \infty[$ and $r_0 > 0$ such that

$$\sup_{x \in \operatorname{supp}\mu} \frac{\mu B(x, 3r)}{\mu B(x, r)} < A \qquad \text{for} \quad 0<r<r_0.$$

Let $0<\delta<r_0$ and $(B(x_i, \delta))_i$ be a centered packing of $\operatorname{supp}\mu$. Then clearly

$$\begin{aligned}
\sum_i \mu(B(x_i, \delta))^{q+1} &= \sum_i \mu(B(x_i, \delta))^q \mu(B(x_i, \delta)) \\
&= \sum_i \mu(B(x_i, \delta))^q \int_{B(x_i, \delta)} d\mu(z) \\
&\leqslant \sum_i \int_{B(x_i, \delta)} \left(\frac{\mu B(x_i, \delta)}{\mu B(z, 2\delta)}\right)^q \mu(B(z, 2\delta))^q \, d\mu(z) \\
&\leqslant \sum_i \int_{B(x_i, \delta)} \left(\frac{\mu B(x_i, \delta)}{\mu B(x_i, 3\delta)}\right)^q \mu(B(z, 2\delta))^q \, d\mu(z) \\
&\leqslant A^{-q} \sum_i \int_{B(x_i, \delta)} \mu(B(z, 2\delta))^q \, d\mu(z) \\
&= A^{-q} \int_{\bigcup_i B(x_i, \delta)} \mu(B(z, 2\delta))^q \, d\mu(z) \\
&\leqslant A^{-q} \int_{\operatorname{supp}\mu} \mu(B(z, 2\delta))^q \, d\mu(z),
\end{aligned}$$

whence $\log(S^{q+1}_{\mu,\delta}(\operatorname{supp}\mu)) \leqslant -q\log(A) + qI^q_{\mu,2\delta}$. Letting $\delta \searrow 0$ yields Claim 1.

Proof of Claim 2. Similar to Claim 1.

Proof of Claim 3. Since $\mu \in \mathscr{P}_1(\mathbb{R}^d)$ we may choose $A \in \,]0, \infty[$ and $r_0 > 0$ such that

$$\sup_{x \in \operatorname{supp}\mu} \frac{\mu B(x, 3r)}{\mu B(x, r)} < A \qquad \text{for} \quad 0<r<r_0.$$

Let $0<\delta<r_0$ and $(B(x_i,\delta))_i$ be a centered covering of $\operatorname{supp}\mu$. Then

$$\begin{aligned}
&\int_{\operatorname{supp}\mu} \mu(B(x,2\delta))^q\,d\mu(x))\\
&\quad\leqslant \sum_i \int_{B(x_i,\delta)} \mu(B(x,2\delta))^q\,d\mu(x)\\
&\quad\leqslant \sum_i \int_{B(x_i,\delta)} \left(\frac{\mu B(x,2\delta)}{\mu B(x_i,\delta)}\right)^q \mu(B(x_i,\delta))^q\,d\mu(x)\\
&\quad\leqslant \sum_i \int_{B(x_i,\delta)} \left(\frac{\mu B(x_i,3\delta)}{\mu B(x_i,\delta)}\right)^q \mu(B(x_i,\delta))^q\,d\mu(x)\\
&\quad\leqslant A^q \sum_i \int_{B(x_i,\delta)} \mu(B(x_i,\delta))^q\,d\mu(x)\\
&\quad= A^q \sum_i \mu(B(x_i,\delta))^{q+1},
\end{aligned}$$

whence $qI^q_{\mu,2\delta}\leqslant q\log(A)+\log(T^{q+1}_{\mu,\delta}(\operatorname{supp}\mu))$. Letting $\delta\searrow 0$ yields Claim 3.

Proof of Claim 4. Similar to Claim 3. ∎

LEMMA 4.8. *Let* $\mu\in\mathscr{P}(\mathbb{R}^d)$. *Then the following hold*

(i) $(q-1)\,\underline{D}^q_\mu \vee (q-1)\,\overline{D}^q_\mu=\overline{C}^q_\mu(\operatorname{supp}\mu)$

(ii) $(q-1)\,\underline{D}^q_\mu \wedge (q-1)\,\overline{D}^q_\mu=\underline{C}^q_\mu(\operatorname{supp}\mu)$.

Proof. The proof follows immediately from the definitions. ∎

Proof of Theorem 2.24. Follows immediately from Proposition 2.19, Lemma 4.7 and Lemma 4.8. ∎

4.8. *Proofs of the Results in Section* 2.6

We will first prove Theorem 2.17.

Proof of Theorem 2.17. Theorem 2.17 follows immediately from Proposition 2.5 through 2.8 and Lemma 4.4. ∎

We will now prove Theorem 2.18. The proof is based on Lemma 4.9 and Theorem 4.10. Lemma 4.9 is a small lemma concerning Legendre transforms. The lemma is believed to be known. However, we have not been able to find any references, so we include the proof for sake of completeness. Theorem 4.10 is a large deviation result inspired by some theorem in [El1, El2].

LEMMA 4.9. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be decreasing and convex, $t \in \mathbb{R}$ and $\varepsilon > 0$.*

(i) *If $f'_-(t) \leqslant \alpha$ then there exists $\delta > 0$ such that*

$$\varepsilon + f^*(-\alpha) + \alpha(t+\delta) - f(t+\delta) > 0.$$

(ii) *If $\alpha \leqslant f'_+(t)$ then there exists $\delta > 0$ such that*

$$\varepsilon + f^*(-\alpha) + \alpha(t-\delta) - f(t-\delta) > 0.$$

Proof. (i) We divide the proof into two cases.

Case 1: $\varepsilon + f^*(-\alpha) + \alpha t > f(t)$. Since f is convex and therefore continuous there exists $\delta > 0$ such that

$$\varepsilon + f^*(-\alpha) + \alpha t > -\alpha\delta + f(t+\delta).$$

Case 2: $f(t) \geqslant \varepsilon + f^*(-\alpha) + \alpha t$. We have $\varepsilon + f^*(-\alpha) > \inf_x(-x\alpha + f(x))$ and we can therefore choose $x \in \mathbb{R}$ such that $\varepsilon + f^*(-\alpha) > -x\alpha + f(x)$. Now put $\delta = x - t$. Then clearly

$$\varepsilon + f^*(-\alpha) + \alpha t > -x\alpha + f(x) + \alpha t = -\alpha\delta + f(t+\delta).$$

Also $\delta > 0$. Otherwise $0 \geqslant \delta = x - t$, i.e. $t \geqslant x$ whence $\alpha(t-x) \geqslant f'_-(t)(t-x) \geqslant f(t) - f(x)$ and so $\varepsilon + f^*(-\alpha) + \alpha t > (-\alpha x + f(x)) + \alpha t = \alpha(t-x) + f(x) \geqslant f(t)$ which is a contradiction.

(ii) The proof is similar to the proof of case (i). ∎

THEOREM 4.10. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(W_n)_n$ a sequence of negative random variables defined on Ω and $(a_n)_n$ a sequence of positive real numbers such that $a_n \to \infty$ as $n \to \infty$. Define $c_n\colon \mathbb{R} \to \overline{\mathbb{R}}$ by*

$$c_n(t) = \frac{1}{a_n} \log \mathbb{E}(\exp(tW_n)),$$

where $\mathbb{E}$ denotes expectation w.r.t. P. Assume

(1) *Each function $c_n(t)$ is finite for all $t \in \mathbb{R}$.*

(2) *$c(t) := \lim_n c_n(t)$ exists and is finite for all $t \in \mathbb{R}$.*

Then the following hold

(i) *The function c is decreasing and convex.*

(ii) *If $t \in \mathbb{R}$ and $c'_-(t) \leqslant c'_+(t) < \alpha$ then*

$$\limsup_{n \to \infty} \frac{1}{a_n} \log(e^{-a_n c(t)} \mathbb{E}(\exp(tW_n)\, 1_{\{(W_n/a_n) \geqslant \alpha\}})) < 0.$$

(iii) *If* $\sum_n e^{-\varepsilon a_n} < \infty$ *for all* $\varepsilon > 0$ *then*

$$\limsup_n \frac{W_n}{a_n} \leqslant c'_+(0) \qquad P\text{-a.s.}$$

(iv) *If* $t \in \mathbb{R}$ *and* $\alpha < c'_-(t) \leqslant c'_+(t)$ *then*

$$\limsup_{n \to \infty} \frac{1}{a_n} \log(e^{-a_n c(t)} \mathbb{E}(\exp(tW_n)\, 1_{\{(W_n/a_n) \leqslant \alpha\}})) < 0$$

(v) *If* $\sum_n e^{-\varepsilon a_n} < \infty$ *for all* $\varepsilon > 0$ *then*

$$c'_-(0) \leqslant \liminf_n \frac{W_n}{a_n} \qquad P\text{-a.s.}$$

Proof. (i) Obvious.

(ii) We claim that

$$\varepsilon := c(t) - (c^*(-\alpha) + \alpha t) > 0.$$

Otherwise $c(t) \leqslant c^*(-\alpha) + \alpha t \leqslant (c(t) - \alpha t) + \alpha t = c(t)$, i.e. $c^*(-\alpha) = c(t) - \alpha t$ whence $\alpha \in \partial c(t) = [c'_-(t), c'_+(t)]$ (cf. e.g. [El2, Theorem VI.5.3]; here ∂c denotes the subdifferential of c), contradicting the fact that $\alpha > c'_+(t)$. It follows from Lemma 4.9 that there exists $\delta > 0$ such that

$$\varepsilon + c^*(-\alpha) + \alpha(t + \delta) - c(t + \delta) > 0. \tag{4.24}$$

Hence

$$\begin{aligned}
&\frac{1}{a_n} \log (e^{-a_n c(t)} \mathbb{E}(\exp(t\, W_n)\, 1_{\{(W_n/a_n) \geqslant \alpha\}}))\\
&\quad = \frac{1}{a_n} \log \left(e^{-a_n c(t)} \int_{\{(W_n/a_n) \geqslant \alpha\}} e^{tW_n}\, dP \right)\\
&\quad = \frac{1}{a_n} \log \left(e^{-a_n(\varepsilon + c^*(-\alpha) + \alpha t) - a_n \alpha \delta} \int_{\{(W_n/a_n) \geqslant \alpha\}} e^{tW_n + a_n \alpha \delta}\, dP \right)\\
&\quad \leqslant \frac{1}{a_n} \log \left(e^{-a_n(\varepsilon + c^*(-\alpha) + \alpha(t+\delta))} \int_{\{(W_n/a_n) \geqslant \alpha\}} e^{tW_n + \delta W_n}\, dP \right)\\
&\quad \leqslant \frac{1}{a_n} \log(e^{-a_n(\varepsilon + c^*(-\alpha) + \alpha(t+\delta))} \mathbb{E}(\exp((t+\delta)\, W_n)))\\
&\quad = \frac{1}{a_n} \log(e^{-a_n(\varepsilon + c^*(-\alpha) + \alpha(t+\delta)) + a_n c_n(t+\delta)})\\
&\quad = -(\varepsilon + c^*(-\alpha) + \alpha(t+\delta) - c_n(t+\delta)).
\end{aligned} \tag{4.25}$$

The desired result follow from (4.24) and (4.25) since $c_n(t+\delta)\to c(t+\delta)$ as $n\to\infty$.

(iii) For $n, m\in\mathbb{N}$ write

$$T_{nm}=\left\{\frac{W_n}{a_n}\geqslant c'_+(0)+\frac{1}{m}\right\}.$$

Now fix $m\in\mathbb{N}$. It follows from (ii) that there exists a number $\varepsilon>0$ and an integer $N\in\mathbb{N}$ such that

$$\frac{1}{a_n}\log(e^{-a_n c(0)}\mathbb{E}(\exp(0W_n)\,1_{T_{nm}}))\leqslant-\varepsilon$$

for $n\geqslant N$. Hence (since $c(0)=0$) $P(T_{nm})=\mathbb{E}(\exp(0W_n)\,1_{T_{nm}})\leqslant e^{a_n c(0)}e^{-a_n\varepsilon}=e^{-a_n\varepsilon}$ for $n\geqslant N$, i.e.

$$\begin{aligned}\sum_n P\left(\frac{W_n}{a_n}\geqslant c'_+(0)+\frac{1}{m}\right)&=\sum_n P(T_{nm})\\&=\sum_{n<N}P(T_{nm})+\sum_{N\leqslant n}P(T_{nm})\\&\leqslant\sum_{n<N}P(T_{nm})+\sum_{N\leqslant n}e^{-a_n\varepsilon}<\infty.\end{aligned}$$

Borel-Cantelli's lemma therefore implies that

$$P\left(\frac{W_n}{a_n}>c'_+(0)+\frac{1}{m}\ n\text{-i.o.}\right)=0\qquad\text{for all}\quad m\in\mathbb{N},$$

whence

$$P\left(\limsup_n\frac{W_n}{a_n}\leqslant c'_+(0)\right)=P\left(\Omega\setminus\bigcup_m\left\{\frac{W_n}{a_n}\geqslant c'_+(0)+\frac{1}{m}\ n\text{-i.o.}\right\}\right)=1.$$

(iv and v) The proofs of (iv) and (v) are similar to the proofs of (ii) and (iii). ∎

If $x\in\operatorname{supp}\mu$ and $(r_n)_n$ is a sequence in $]0,1[$ such that $r_n\to0$ then we write

$$\underline{\alpha}_\mu(x,r_n)=\liminf_n\frac{\log\mu B(x,r_n)}{\log r_n},$$

$$\bar{\alpha}_\mu(x,r_n)=\limsup_n\frac{\log\mu B(x,r_n)}{\log r_n}.$$

If $\underline{\alpha}_\mu(x, r_n)$ and $\bar{\alpha}_\mu(x, r_n)$ coincide we denote the common value by $\alpha_\mu(x, r_n)$. We are now ready to prove Theorem 2.18.

Proof of Theorem 2.18. (i) Write $r_n := r_{q,n}$, $t := t_q$, $\varphi := \varphi_q$, $\nu := \nu_q$, $\underline{K} := \underline{K}_q$, $\bar{K} := \bar{K}_q$ and

$$M := \{x \in \operatorname{supp} \mu \mid -c'_+(0) \leqslant \underline{\alpha}_\mu(x, r_n) \leqslant \bar{\alpha}_\mu(x, r_n) \leqslant -c'_-(0)\}.$$

First, observe that

$$t = b_\mu(q) = B_\mu(q) = \Lambda_\mu(q). \tag{4.26}$$

Indeed let $\varepsilon > 0$. Now choose $0 < \delta_\varepsilon < (r_q \wedge 1)$ such that $|\varphi(r)/\log r| < \varepsilon/2$ for $0 < r < \delta_\varepsilon$. If $0 < \delta < \delta_\varepsilon$ and $(B(x_i, r_i))_i$ is a centered δ-packing of $\operatorname{supp} \mu$ then

$$\begin{aligned} \sum_i \mu(B(x_i, r_i))^q (2r_i)^{t+\varepsilon} &\leqslant (2^\varepsilon/\underline{K}) \sup_i (r_i^\varepsilon e^{-\varphi(r_i)}) \sum_i \nu(B(x_i, r_i)) \\ &\leqslant (2^\varepsilon/\underline{K})\, \delta^{\varepsilon/2} \end{aligned}$$

(since $r_i^\varepsilon e^{-\varphi(r_i)} \leqslant r_i^\varepsilon e^{|(\varepsilon/2) \log r_i|} = r_i^{\varepsilon/2} \leqslant \delta^{\varepsilon/2}$), whence $\bar{\mathscr{P}}^{q, t+\varepsilon}_{\mu, \delta}(\operatorname{supp} \mu) \leqslant (2^\varepsilon/\underline{K})\, \delta^{\varepsilon/2}$ for $0 < \delta \leqslant \delta_\varepsilon$. Letting $\delta \searrow 0$ now yields $\bar{\mathscr{P}}^{q, t+\varepsilon}_\mu(\operatorname{supp} \mu) = 0$, i.e. $\Lambda_\mu(q) \leqslant t + \varepsilon$ for $\varepsilon > 0$. Mutatis mutandis $t \leqslant b_\mu(q)$.

Next observe that

$$\underline{\alpha}_\nu(x) = \underline{\alpha}_\nu(x, r_n), \qquad \bar{\alpha}_\nu(x) = \bar{\alpha}_\nu(x, r_n) \tag{4.27}$$

for $x \in \operatorname{supp} \nu$. Indeed if $n \in \mathbb{N}$ and $r_{n+1} \leqslant r < r_n$ then

$$\frac{\log r_n}{\log r_{n+1}} \frac{\log \mu B(x, r_n)}{\log r_n} \leqslant \frac{\log \mu B(x, r)}{\log r} \leqslant \frac{\log r_{n+1}}{\log r_n} \frac{\log \mu B(x, r_{n+1})}{\log r_{n+1}}. \tag{4.28}$$

Equation (4.27) follows from (4.28) since $\log r_{n+1}/\log r_n \to 1$ as $n \to \infty$ by assumption. In a similar way we obtain

$$\underline{\alpha}_\mu(x) = \underline{\alpha}_\mu(x, r_n), \qquad \bar{\alpha}_\mu(x) = \bar{\alpha}_\mu(x, r_n)$$

for $x \in \operatorname{supp} \mu$, whence

$$M = \underline{X}_{-c'_+(0)} \cap \bar{X}^{-c'_-(0)} \tag{4.29}$$

Consider the probability space $(\Omega, \mathscr{F}, P) = (\operatorname{supp} \mu, \mathscr{B}(\operatorname{supp} \mu), \nu)$. Define random variables W_n on Ω by $W_n(x) = \log \mu B(x, r_n)$ and put $a_n = -\log r_n$. It follows immediately from Theorem 4.10 that $-c'_+(0) \leqslant \underline{\alpha}_\mu(x, r_n) \leqslant \bar{\alpha}_\mu(x, r_n) \leqslant -c'_-(0)$ for ν almost all x, i.e.

$$\nu(M) = 1. \tag{4.30}$$

Also, if $x \in \operatorname{supp}\mu$ and $0 < r < (r_q \wedge 1)$ then

$$\frac{\log \nu B(x,r)}{\log r} \geqslant \frac{\log \bar{K}}{\log r} + q\frac{\log \mu B(x,r)}{\log r} + t\frac{\log(2r)}{\log r} + \frac{\varphi(r)}{\log r},$$

whence

$$\underline{\alpha}_\nu(x) = \underline{\alpha}_\nu(x, r_n) \geqslant \begin{cases} q(-c'_-(0)) + t & \text{for } q \leqslant 0 \\ q(-c'_+(0)) + t & \text{for } 0 \leqslant q \end{cases} \tag{4.31}$$

for $x \in M$. Now the result follows from (4.26), (4.29), (4.30), (4.31) and Corollary 2.9.

(ii) Follows immediately from (i).

(iii) We will first prove that

$$B_\mu(p+q) - B_\mu(q) = c_q(p) \tag{4.32}$$

for $p, q \in \mathbb{R}$. It follows from (4.26) that $t_q = b_\mu(q) = B_\mu(q) = \Lambda_\mu(q)$ for all $q \in \mathbb{R}$, i.e.

$$b_\mu = B_\mu = \Lambda_\mu. \tag{4.33}$$

It also follows by arguments very similar to the proof of Lemma 4.7 that

$$c_q(p) = \lim_n \frac{1}{-\log r_{q,n}} \log\left(\int_{\operatorname{supp}\mu} \mu(B(x, r_{q,n}))^{p+q-1}\, d\mu(x)\right) - t_q. \tag{4.34}$$

Arguments similar to the proof of (4.27) yield

$$\begin{aligned}\lim_n \frac{1}{-\log r_{q,n}} &\log\left(\int_{\operatorname{supp}\mu} \mu(B(x, r_{q,n}))^{p+q-1}\, d\mu(x)\right) \\ &= \lim_{r \searrow 0} \frac{1}{-\log r} \log\left(\int_{\operatorname{supp}\mu} \mu(B(x, r))^{p+q-1}\, d\mu(x)\right) \\ &= (p+q-1)\, I_\mu^{p+q-1}.\end{aligned} \tag{4.35}$$

It follows from (4.33)–(4.35), Lemma 4.7 and Proposition 2.22, that $B_\mu(p+q) = \Lambda_\mu(p+q) = \bar{C}_\mu^{p+q}(\operatorname{supp}\mu) = (p+q-1)\, I_\mu^{p+q-1} = c_q(p) + t_q$ which proves equation (4.32).

We will now prove (iii). Fix $q \in \operatorname{dom} B'_\mu$. Define random variables $W_{q,n}\colon \operatorname{supp}\mu \to \mathbb{R}$ for $n \in \mathbb{N}$ by $W_{q,n}(x) = \log \mu(B(x, r_{q,n}))$ and put $a_{q,n} = -\log r_{q,n}$. It follows from (4.32) that c_q is differentiable at 0 with $c'_q(0) = B'_\mu(q)$ and [El2, Theorem II.4.3] therefore implies that

$$\frac{W_{q,n}}{a_{q,n}} \xrightarrow{\exp} c_q(0) \qquad \text{w.r.t. } \nu_q \quad \text{as} \quad n \to \infty$$

(where $\xrightarrow{\exp}$ denotes exponential convergence, cf. [El2, p. 48]), and so by [El2, Theorem II.4.4] (since $\sum_n \exp(-a_{q,n}\varepsilon) < \infty$ for all $\varepsilon > 0$)

$$\begin{aligned}\alpha_\mu(x) &= \alpha_\mu(x, r_{q,n}) \qquad \text{(by (4.27))}\\ &= \lim_n \frac{W_{q,n}(x)}{-a_{q,n}} = -c_q'(0) = -B_\mu'(q) \qquad \text{for} \quad \nu_q\text{-a.a. } x.\end{aligned}$$

(iv) Let $q \in \operatorname{dom} B_\mu'$. It follows from (4.32) that c_q is differentiable at 0 with $c_q'(0) = B_\mu'(q)$, and (i) therefore implies that

$$f_\mu(-B_\mu'(q)) = b_\mu^*(-B_\mu'(q)) = B_\mu^*(-B_\mu'(q))$$

which proves (iv). ▮

5. Multifractal Analysis of Graph Directed Self-Similar Mearures

In this section we prove that the upper bounds in Theorem 2.17 are the exact values of $f_\mu(\alpha) = \dim(\underline{X}_\alpha \cap \overline{X}^\alpha)$ and $F_\mu(\alpha) = \operatorname{Dim}(\underline{X}_\alpha \cap \overline{X}^\alpha)$ (and not just upper bounds) if μ is a graph directed self-similar measure in $\mathbb{R}^d$ with totally disconnected support. Self-similar sets and measures were first introduced by Moran [Mo] in 1946 and later by Hutchinson [Hu] in 1981. Self-similar sets and measures were subsequently generalized to graph directed self-similar sets and measures by Bandt [Ban], Barnsely et al. [Bar], Mauldin & Williams [Mau] and others. Recently a textbook [Edg] by C. Edgar on graph directed self-similar sets and measures has appeared. A rigorous analysis of the multifractal decomposition of graph directed measures in $\mathbb{R}^d$ with totally disconnected supports has appeared in two recent papers by Cawley & Mauldin [Ca] and Edgar & Mauldin [Ed].

5.1. *Mauldin–Williams Graphs*

Let (V, E) be a finite directed multigraph. The set V is the set of vertices and E is the set of edges. For $u, v \in V$ let E_{uv} denote the set of edges from u to v and write $E_u = \bigcup_{v \in V} E_{uv}$. A path in the graph is a finite string $e_1 e_2 \cdots e_n$ of edges such that the terminal vertex of the edge e_i is the initial vertex of the next edge e_{i+1} and an infinite path in the graph is an infinite string $e_1 e_2 \cdots$ of edges such that $e_1 \cdots e_n$ is a path for all $n \in \mathbb{N}$. For $e \in E$ let $i(e)$ and $t(e)$ denote the initial and terminal vertex of e respectively. For $u, v \in V$ and $n \in \mathbb{N}$ write

$$E_{uv}^{(n)} = \{e_1 \cdots e_n \text{ is a path such that } i(e_1) = u \text{ and } \tau(e_n) = v\}$$

$$E_{uv}^{(*)} = \bigcup_{n \in \mathbb{N}} E_{uv}^{(n)}$$

$$E_u^{(n)} = \bigcup_{v \in V} E_{uv}^{(n)}, \qquad E_u^{(*)} = \bigcup_{v \in V} E_{uv}^{(*)}$$

$$E^{(n)} = \bigcup_{u \in V} E_u^{(n)}, \qquad E^{(*)} = \bigcup_{u \in V} E_u^{(*)}$$

$$E_u^{\mathbb{N}} = \{e_1 e_2 \cdots \text{ is an infinite path such that } i(e_1) = u\}$$

$$E^{\mathbb{N}} = \bigcup_{u \in V} E_u^{\mathbb{N}}$$

If $\alpha = \alpha_1 \cdots \alpha_n$, $\beta = \beta_1 \cdots \beta_m \in E^{(*)}$ are paths and the terminal vertex $t(\alpha_n)$ of α is equal to the initial vertex $i(\beta_1)$ of β then we write $\alpha\beta = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m$. If $\alpha = \alpha_1 \cdots \alpha_n \in E^{(n)}$ and $k \in \{1, ..., n\}$ then we write $\alpha | k = \alpha_1 \cdots \alpha_k$. Similarly, if $\alpha = \alpha_1 \cdots \alpha_n \in E^{(*)}$ is a path, $\omega = \omega_1 \omega_2 \cdots \in E^{\mathbb{N}}$ is an infinite path with $t(\alpha_n) = i(\omega_1)$ and $m \in \mathbb{N}$ is an integer then write $\alpha\omega = \alpha_1 \cdots \alpha_n \omega_1 \omega_2 \cdots$ and $\omega | m = \omega_1 \cdots \omega_m$. For $\alpha = \alpha_1 \cdots \alpha_n \in E^{(n)}$ put $|\alpha| = n$. If $\alpha = \alpha_1 \cdots \alpha_n \in E^{(n)}$, $\beta = \beta_1 \cdots \beta_m \in E^{(m)}$ with $|\alpha| \leqslant |\beta|$ and $\alpha_1 = \beta_1, ..., \alpha_n = \beta_n$ then we write $\alpha \leqslant \beta$. Similarly, if $\alpha = \alpha_1 \cdots \alpha_n \in E^{(*)}$ is a path and $\omega = \omega_1 \omega_2 \cdots \in E^{\mathbb{N}}$ is an infinite path with $\alpha_1 = \omega_1, ..., \alpha_n = \omega_n$ we write $\alpha \leqslant \omega$. Finally, if $\alpha \in E^{(n)}$ and $\omega \in E^{\mathbb{N}}$ then we will always write $\alpha = \alpha_1 \cdots \alpha_n$ or $\alpha = \alpha(1) \cdots \alpha(n)$ and $\omega = \omega_1 \omega_2 \cdots$ or $\omega = \omega(1)\, \omega(2) \cdots$.

A list $(V, E, (r_e)_{e \in E}, (T_e)_{e \in E})$ where

(1) (V, E) is a finite directed multigraph.

(2) $r_e \in \,]0, 1[$ for all $e \in E$.

(3) $T_e \colon \mathbb{R}^d \to \mathbb{R}^d$ is a similarity map with similarity ratio r_e

is called a Mauldin–Williams graph (MW graph), cf. [Mau] and [Edg]. If $\alpha = \alpha_1 \cdots \alpha_n \in E^{(*)}$ then write $T_\alpha = T_{\alpha_1} \circ \cdots \circ T_{\alpha_n}$ and $r_\alpha = r_{\alpha_1} \cdots r_{\alpha_n}$.

Let $G = (V, E(r_e)_e, (T_e)_e)$ be a MW graph. It follows from [Mau] (cf. also [Edg]) that there exists a unique list $(K_u)_{u \in V}$ of non-empty compact sets such that

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} T_e(K_v) \qquad \text{for all} \quad u \in V,$$

in fact

$$K_u = \bigcap_{n \in \mathbb{N}} \bigcup_{\alpha \in E_u^{(n)}} K_\alpha,$$

where we have written $K_\alpha = T_\alpha K_v$ for $\alpha \in E_{uv}^{(n)}$. The sets K_u, $u \in V$, are called the invariant self-similar sets associated with G. Put

$$\Delta := \min\{\operatorname{dist}(T_e(K_v), T_\varepsilon(K_w)) \mid u, v, w \in V, \\ e \in E_{uv}, \varepsilon \in E_{uw}, \varepsilon \in E_{uw}, e \neq \varepsilon\}. \tag{5.1}$$

It is well known that K_u is totally disconnected for all $u \in V$ if and only if $\Delta > 0$.

5.2. *The Code Space*

Let $G = (V, E, (r_e)_e, (T_e)_e)$ be a MW graph. We will use the "code space" $E^{\mathbb{N}}$ in our investigations of self-similar sets and measures. Let $u \in V$ and $\omega \in E_u^{\mathbb{N}}$. Since $(T_{\omega|n}(K_{\tau(\omega_n)}))_n$ is a decreasing sequence of non-empty compact sets such that $\operatorname{diam}(T_{\omega|n}(K_{\omega_n})) \searrow 0$, $\bigcap_n T_{\omega|n}(K_{t(\omega_n)})$ is a singleton. Now define "the code map"

$$\pi_u : E_u^{\mathbb{N}} \to \mathbb{R}^d$$

by

$$\{\pi_u(\omega)\} = \bigcap_n T_{\omega|n}(K_{t(\omega_n)}).$$

It is readily seen that

$$\pi_u(E_u^{\mathbb{N}}) = K_u$$

and

$$K_u = \bigcup_{\omega \in E_u^{\mathbb{N}}} \bigcap_n T_{\omega|n}(K_{t(\omega_n)}) = \bigcap_n \bigcup_{\alpha \in E_u^{(n)}} T_\alpha(K_{t(\alpha_n)}).$$

Finally, if $\alpha \in E_u^{(n)}$ write $[\alpha] = \{\omega \in E_u^{\mathbb{N}} \mid \omega | n = \alpha\}$.

5.3. *Self Similar Measures*

A MW graph with probabilities is a list $G = (V, E, (r_e)_{e \in E}, (T_e)_{e \in E}, (p_e)_e)$ where

(1) $(V, E, (r_e)_e, T_e)_e)$ is a MW graph

(2) $p_e \in]0, 1[$ for $e \in E$ and

$$\sum_{v \in V} \sum_{e \in E_{uv}} p_e = 1 \qquad \text{for all } u.$$

For $\alpha = \alpha_1 \cdots \alpha_n \in E^{(n)}$ write $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_n}$. Then clearly $p_\alpha = \sum_{e \in E_v} p_{\alpha e}$ for all $\alpha \in E_{uv}^{(*)}$. Therefore, for each $u \in V$, there exists a unique Borel probability measure $\hat{\mu}_u$ on $E_u^{\mathbb{N}}$ (equipped with product topology) such that

$$\hat{\mu}_u([\alpha]) = p_\alpha \qquad \text{for} \quad \alpha \in E_u^{(*)}. \tag{5.2}$$

Let

$$\mu_u = \hat{\mu}_u \circ \pi_u^{-1}. \tag{5.3}$$

It follows from [Hu] that supp $\mu_u = K_u$. The measures μ_u are called the graph directed self-similar measure associated with G.

5.4 *Statement of Main Result*

Fix a MW graph with probabilities $G = (E, V, (r_e)_e, (T_e)_e, (p_e)_e)$. Let $(K_u)_u$ be the invariant self-similar sets associated with G, and let $\mu_u = \hat{\mu}_u \circ \pi_u^{-1}$ be the graph directed self-similar measures associated with G (cf. (5.2) and (5.3)). Assume that (E, V) is strongly connected, (i.e. $E_{uv}^{(*)} \neq \emptyset$ for $u, v \in V$), and that card $E_{uv} \geqslant 2$ for all $u, v \in V$ with $E_{uv} \neq \emptyset$.

For each $q, t \in \mathbb{R}$ we define a square matrix $A(q, t)$ indexed by V such that the entry $A_{uv}(q, t)$ in the uth row and the vth column is

$$A_{uv}(q, t) = \sum_{e \in E_{uv}} p_e^q r_e^t.$$

Let $\Phi(q, t)$ denote the spectral radius of $A(q, t)$. It follows from [Ed] that for each $q \in \mathbb{R}$ there exists a unique $\beta(q)$ such that

$$\Phi(q, \beta(q)) = 1.$$

It is proved in [Ed] that β is a real analytic map. Put $\alpha = -\beta'$ and write

$$K_u(a) = \{x \in K_u \mid \alpha_\mu(x) = a\}$$

for $a \geqslant 0$. We now state our main result concerning the multifractality of graph directed self-similar measures.

THEOREM 5.1. *Assume $\Delta > 0$. Then*

(i) *For each $q \in \mathbb{R}$,*

$$0 < \mathscr{H}_{\mu_u}^{q, \beta(q)}(K_u(\alpha(q))) \leqslant \mathscr{P}_{\mu_u}^{q, \beta(q)}(K_u(\alpha(q))) \leqslant \overline{\mathscr{P}}_{\mu_u}^{q, \beta(q)}(K_u) < \infty.$$

(ii) *For each $q \in \mathbb{R}$ there exists a number $c_q \in]0, \infty[$ such that*

$$\mathscr{H}_{\mu_u}^{q, \beta(q)} \mid \operatorname{supp} \mu_u = c_q \mathscr{P}_{\mu_u}^{q, \beta(q)} \mid \operatorname{supp} \mu_u.$$

(iii) $\alpha_{\mu_u}(x) = \alpha(q)$ *for* $\mathscr{H}_{\mu_u}^{q, \beta(q)} \mid \operatorname{supp} \mu_u$*-a.a. x,*

$\alpha_{\mu_u}(x) = \alpha(q)$ *for* $\mathscr{P}_{\mu_u}^{q, \beta(q)} \mid \operatorname{supp} \mu_u$*-a.a. x.*

(iv) *If* $q, p \in \mathbb{R}$ *and* $\alpha(q) \neq \alpha(p)$ *then*

$$(\mathscr{H}^{q,\beta(q)}_{\mu_u} \mid \operatorname{supp} \mu_u) \perp (\mathscr{H}^{p,\beta(p)}_{\mu_u} \mid \operatorname{supp} \mu_u),$$
$$(\mathscr{P}^{q,\beta(q)}_{\mu_u} \mid \operatorname{supp} \mu_u) \perp (\mathscr{P}^{p,\beta(p)}_{\mu_u} \mid \operatorname{supp} \mu_u).$$

(v) *For each* $q \in \mathbb{R}$

$$b_{\mu_u}(q) = B_{\mu_u}(q) = \Delta^q_{\mu_u}(K_u) = C^q_{\mu_u}(K_u) = (1-q)\, D^q_{\mu_u} = \beta(q).$$

(vi) $\underline{a}_{\mu_u} = \underline{A}_{\mu_u} = \inf_{x \in K_u} \underline{\alpha}_{\mu_u}(x) := \underline{a}$, $\bar{a}_{\mu_u} = \bar{A}_{\mu_u} = \sup_{x \in K_u} \bar{\alpha}_{\mu_u}(x) := \bar{a}$.

(vii) $\dim K_u(\alpha) = \operatorname{Dim} K_u(\alpha) = b^*_{\mu_u}(\alpha) = B^*_{\mu_u}(\alpha) = \beta^*(\alpha)$ *for* $\alpha \in \,]\underline{a}, \bar{a}[$.

We note that the result in (ii) was first proved by Spear [Sp], in a slightly more general setting, for the case $q = 0$. We also note that the results in (v) and (vii) are minor extensions of the results in [Ca, Ed]. In [Ca] and [Ed] it is proved that $f_{\mu_u} = F_{\mu_u}$ (in a slightly more general setting, whereas we also prove that $f_{\mu_u} = F_{\mu_u} = (C^q_{\mu_u}(K_u))^* = ((1-q)\, D^q_{\mu_u})^*$.

Finally we note that a result very similar to the equation $\beta(q) = C^q_{\mu_u}(K_u) = (1-q)\, D^q_{\mu_u}$ has been proved in a recent paper by Strichartz [Str, Theorem 3.2] for the case $1 < q < \infty$.

It is an open problem whether the equations

$$f_{\mu_u} = \beta^*, \qquad F_{\mu_u} = \beta^*$$

hold in the case where the support of μ_u is not necessarily totally disconnected, cf. [Ca, p. 215] and [Ed, Section 5.3, Question (d)]. Cf. also Section 7.8 and Note Added in Proof (2) at the end of this paper.

5.5. *Proof of Main Result*

We begin by defining some auxiliary measures and proving some technical lemmas.

The matrix $A(q, \beta(q))$ is irreducible (because (V, E) is strongly connected) and has spectral radius 1. It therefore follows from Perron-Frobenius theorem (cf. e.g. [Se]) that there exist unique positive right and left eigenvectors $\rho = (\rho_v)_{v \in V}$, $\lambda = (\lambda_v)_{v \in V}$ such that

$$A(q, \beta(q))\, \rho = \rho \qquad \text{i.e.} \quad \sum_v \sum_{e \in E_{uv}} p_e^q r_e^{\beta(q)} \rho_v = \rho_u \qquad \text{for} \quad u \in V$$

$$\lambda A(q, \beta(q)) = \lambda \qquad \text{i.e.} \quad \sum_u \sum_{e \in E_{uv}} \lambda_u p_e^q r_e^{\beta(q)} = \lambda_v \qquad \text{for} \quad v \in V$$

$$1 = \|\rho\| = \sum_v \rho_v \qquad \rho_v > 0 \qquad \text{for} \quad v \in V$$

$$1 = \|\lambda\| = \sum_v \lambda_v \qquad \lambda_v > 0 \qquad \text{for} \quad v \in V.$$

Write $\underline{p} = \min_u p_u$, $\bar{p} = \max_u p_u$. Put

$$P_e = p_u^{-1} p_e^q r_e^{\beta(q)} p_v \qquad \text{for} \quad e \in E_{uv}$$

and write

$$P_\alpha = P_{\alpha_1} \cdots P_{\alpha_k} \qquad \text{for} \quad \alpha = \alpha_1 \cdots \alpha_k \in E^{(k)}.$$

Now observe that

$$\sum_{v \in V} \sum_{e \in E_{uv}} Pe = 1$$

$$\sum_{e \in E_v} P_{\alpha e} = P_\alpha \qquad \text{for} \quad \alpha \in E_{uv}^{(*)}.$$

This implies that there exists a unique Borel probability measure $\hat{\mu}_u^q$ on $E_u^{\mathbb{N}}$ such that

$$\hat{\mu}_u^q([\alpha]) = P_\alpha = p_u^{-1} p_\alpha^q r_\alpha^{\beta(q)} p_v \tag{5.4}$$

for $\alpha \in E_{uv}^{(*)}$. Put $\mu_u^q = \hat{\mu}_u^q \circ \pi_u^{-1}$. The auxiliary measures μ_u^q were also introduced by Edgar & Mauldin [Ed] in their multifractal analysis of graph directed self-similar measures.

The proof of Theorem 5.1 is based on the following five lemmas which we will prove below.

Lemma 5.2. *Assume that $\Delta > 0$ and let $0 < r < \Delta$ and $\alpha \geq 1$. Let $\omega \in E_u^{\mathbb{N}}$ and choose $k, l \in \mathbb{N}$ such that*

$$\max_v (\operatorname{diam} K_v)\, r_{\omega|k} < r \leq \max_v (\operatorname{diam} K_v)\, r_{\omega|k-1} \tag{5.5}$$

$$\Delta r_{\omega|l} < ar \leq \Delta r_{\omega|l-1}. \tag{5.6}$$

Then

$$0 \leq k - l \leq \varphi(a),$$

where

$$\varphi(t) := \frac{\log(\Delta/\max_v(\operatorname{diam} K_v)) + \log(t)}{-\log(\max_e r_e)} + 1 \qquad \textit{for} \quad t \geq 1.$$

Lemma 5.3. *If $\Delta > 0$ then $\mu_u \in \mathscr{P}_1(K_u)$.*

LEMMA 5.4. *If $\Delta > 0$ then there exists $\bar{K} \in]0, \infty[$ such that*

(i) $\overline{\mathscr{P}}^{q,\beta(q)}_{\mu_u}(K_u) \leqslant \bar{K}\mu^q_u(K_u)$.

(ii) $\mathscr{P}^{q,\beta(q)}_{\mu_u} \leqslant \bar{K}\mu^q_u$.

LEMMA 5.5. *If $\Delta > 0$ then there exists $\underline{K} \in]0, \infty[$ such that* $\underline{K}\mu^q_u \leqslant (\mathscr{H}^{q,\beta(q)}_{\mu_u} \mid \operatorname{supp} \mu_u)$.

LEMMA 5.6. *If $\Delta > 0$ then*

$$\underline{J}^q_{\mu_v, \mu_u}(T_e) = p^q_e = \bar{J}^q_{\mu_v, \mu_u}(T_e)$$

for $u, v \in V$ and $e \in E_{uv}$.

We will now show that Theorem 5.1 follows from Lemma 5.2 through Lemma 5.5

Proof of Theorem 5.1. It follows from [Ed, Lemma 4.1] that

$$\mu^q_u(K_u(\alpha(q))) = 1 \qquad \text{for} \quad u \in V \qquad \text{and} \qquad q \in \mathbb{R} \tag{5.7}$$

for $\Delta > 0$ (the proof of (5.7) is just a straightforward application of Birkhof's ergodic theorem).

(i, iii, iv) Follow from (5.7), Lemma 5.4 and Lemma 5.5.

(ii) We divide the proof into three steps.

Step 1. There exists $c_q \in]0, \infty[$ such that

$$\mathscr{H}^{q,\beta(q)}_{\mu_u}(K_u) = c_q \mathscr{P}^{q,\beta(q)}_{\mu_u}(K_u) \qquad \text{for} \quad u \in V.$$

Proof of Step 1. We have by Lemma 4.3 and Lemma 5.6,

$$\begin{aligned}
\sum_v A_{uv}(q, \beta(q))\, \mathscr{H}^{q,\beta(q)}_{\mu_u}(K_v) &= \sum_v \sum_{e \in E_{uv}} p^q_e r^{\beta(q)}_e \mathscr{H}^{q,\beta(q)}_{\mu_u}(K_v) \\
&= \sum_v \sum_{e \in E_{uv}} \mathscr{H}^{q,\beta(q)}_{\mu_v}(T_e K_v) \\
&= \mathscr{H}^{q,\beta(q)}_{\mu_u}\left(\bigcup_v \bigcup_{e \in E_{uv}} T_e K_v\right) \\
&= \mathscr{H}^{q,\beta(q)}_{\mu_u}(K_u),
\end{aligned}$$

i.e. $(\mathscr{H}^{q,\beta(q)}_{\mu_u}(K_u))_u$ is an eigenvector of $A(q, \beta(q))$ with eigenvalue 1. In a similar way we may prove that $(\mathscr{P}^{q,\beta(q)}_{\mu_u}(K_u))_u$ is an eigenvector of $A(q, \beta(q))$ with eigenvalue 1. Now, since $A(q, \beta(q))$ is irreducible and has spectral radius 1, Perron-Frobenius theorem [Se] implies that there exists $c_q \in]0, \infty[$ such that $(\mathscr{H}^{q,\beta(q)}_{\mu_u}(K_u))_u = c_q(\mathscr{P}^{q,\beta(q)}_{\mu_u}(K_u))_u$. This proves Step 1.

Step 2. If $u \in V$ and $\alpha \in E_u^{(*)}$ then

$$\mathscr{H}^{q,\beta(q)}_{\mu_u}(K_\alpha) = c_q \mathscr{P}^{q,\beta(q)}_{\mu_u}(K_\alpha).$$

Proof of Step 2. By Lemma 4.3, Lemma 5.6 and Step 1,

$$\begin{aligned}\mathscr{H}^{q,\beta(q)}_{\mu_u}(K_\alpha) &= \mathscr{H}^{q,\beta(q)}_{\mu_u}(T_{\alpha_1} \circ \cdots \circ T_{\alpha_{|\alpha|}} K_{t(\alpha_{|\alpha|})}) \\ &= p_\alpha^q r_\alpha^{\beta(q)} \mathscr{H}_{\mu_{t(\alpha_{|\alpha|})}}(K_{t(\alpha_{|\alpha|})}) \\ &= c_q p_\alpha^q r_\alpha^{\beta(q)} \mathscr{P}_{\mu_{t(\alpha_{|\alpha|})}}(K_{t(\alpha_{|\alpha|})}) \\ &= c_q \mathscr{P}^{q,\beta(q)}_{\mu_u}(K_\alpha),\end{aligned}$$

which proves Step 2.

Step 3. $\mathscr{H}^{q,\beta(q)}_{\mu_u} = c_q \mathscr{P}^{q,\beta(q)}_{\mu_u} \mid \operatorname{supp} \mu_u)$ for $u \in V$.

Proof of Step 3. By outer regularity, it suffices to prove that

$$\mathscr{H}^{q,\beta(q)}_{\mu_u}(G) = c_q \mathscr{P}^{q,\beta(q)}_{\mu_u}(G)$$

for all subsets G of $\operatorname{supp} \mu$ which are open relative to $\operatorname{supp} \mu$. Now let $G \subseteq \operatorname{supp} \mu$ be a subset which is open relative to $\operatorname{supp} \mu$. Let $A = \{\alpha \in E_u^{(*)} \mid K_\alpha \subseteq G\}$. Since G is open, $G = \bigcup_{\alpha \in A} K_\alpha$. Now, we need only cover G once: if $\alpha, \beta \in A$ and $[\alpha] \cap [\beta] \neq \varnothing$ then one of them is contained in the other, so we may discard the smaller one. So there is a set $A_0 \subseteq A$ such that

$$G = \bigcup_{\alpha \in A_0} K_\alpha,$$

and $[\alpha] \cap [\beta] = \varnothing$ for $\alpha, \beta \in A_0$, i.e. $(K_\alpha)_{\alpha \in A_0}$ is a disjoint family. Hence (by Step 2)

$$\begin{aligned}\mathscr{H}^{q,\beta(q)}_{\mu_u}(G) &= \mathscr{H}^{q,\beta(q)}_{\mu_u}\left(\bigcup_{\alpha \in A_0} K_\alpha\right) = \sum_{\alpha \in A_0} \mathscr{H}^{q,\beta(q)}_{\mu_u}(K_\alpha) \\ &= c_q \sum_{\alpha \in A_0} \mathscr{P}^{q,\beta(q)}_{\mu_u}(K_\alpha) = c_q \mathscr{P}^{q,\beta(q)}_{\mu_u}\left(\bigcup_{\alpha \in A_0} K_\alpha\right) \\ &= c_q \mathscr{P}^{q,\beta(q)}_{\mu_u}(G).\end{aligned}$$

(v) It follows from Lemma 5.4 and Lemma 5.5 that

$$0 < \underline{K}\mu_u^q(K_u(\alpha(q))) \leqslant \mathscr{H}_{\mu_u}^{q,\beta(q)}(K_u(\alpha(q))) \leqslant \mathscr{H}_{\mu_u}^{q,\beta(q)}(K_u)$$
$$\leqslant \mathscr{P}_{\mu_u}^{q,\beta(q)}(K_u) \leqslant \bar{\mathscr{P}}_{\mu_u}^{q,\beta(q)}(K_u) < \infty,$$

whence

$$\beta(q) = b_{\mu_u}(q) = B_{\mu_u}(q) = \Delta_{\mu_u}^q(\operatorname{supp}\mu)$$

(because $\operatorname{supp}\mu_u = K_u$). Finally Lemma 5.3 and Proposition 2.22 imply that $\Delta_{\mu_u}^q(\operatorname{supp}\mu) = C_{\mu_u}^q(\operatorname{supp}\mu_u)$.

(vi) It follows from (iv) that $\underline{a}_{\mu_u} = \underline{A}_{\mu_u}$ and $\bar{a}_{\mu_u} = \bar{A}_{\mu_u}$, and it is proved in [Ed, the proof of Proposition 3.3] that $\inf_{x \in K_u} \underline{\alpha}_{\mu_u}(x) = \underline{a}$ and $\bar{a} = \sup_{x \in K_u} \bar{a}_{\mu_u}(x)$ for $\Delta > 0$. It also follows from [Ed, Proposition 3.3] that

$$\underline{e} = \lim_{q \to \infty} (\beta(q) + \underline{a}q), \qquad \bar{e} = \lim_{q \to -\infty} (\beta(q) + \bar{a}q)$$

exist and $\underline{e}, \bar{e} \in [0, \infty[$. This clearly implies that $\underline{a}_{\mu_u} = \underline{a}$ and $\bar{a}_{\mu_u} = \bar{a}$.

(vii) "$\leqslant$" It follows immediately from (iv), (v), (vi) and Theorem 2.17 that

$$\dim K_u(\alpha) \leqslant \operatorname{Dim} K_u(\alpha) \leqslant B_{\mu_u}^*(\alpha) = b_{\mu_u}^*(\alpha).$$

"$\geqslant$" It follows from Proposition 2.7 (with $A = K_u(\alpha(q))$) that

$$0 < 2^{-\beta(q)} \mathscr{H}_{\mu_u}^{q,\beta(q)}(K_u(\alpha(q))) \leqslant \mathscr{H}^{\alpha(q)q + \beta(q) - \delta}(K_u(\alpha(q)))$$

for all $q \in \mathbb{R}$ and $\delta > 0$. Hence

$$\beta^*(\alpha(q)) \leqslant \alpha(q)\, q + \beta(q) \leqslant \dim(K_u(\alpha(q))) \leqslant \operatorname{Dim}(K_u(\alpha(q)))$$

for all $q \in \mathbb{R}$. This completes the proof since $\alpha(\mathbb{R}) =]\underline{a}, \bar{a}[$ by [Ed, Proposition 3.3]. ∎

We will now prove Lemma 5.2–Lemma 5.5. Put $D := \max_v(\operatorname{diam} K_v)$.

Proof of Lemma 5.2. Since $\Delta \leqslant \max_v(\operatorname{diam} K_v)$ and $\alpha \geqslant 1$, $k \geqslant l$. Since $k \geqslant l$ the right hand side of inequality (5.5) can be rewritten as

$$r \leqslant D r_{\omega|l} r_{\omega_{l+1}} \cdot \cdots \cdot r_{\omega_{k-1}}.$$

This inequality together with the lefthand side of inequality (5.6) implies that

$$a^{-1} \leqslant \frac{Dr_{\omega|l} \cdots r_{\omega_{k-1}}}{\Delta r_{\omega|l}} \leqslant (D/\Delta)(\max_e r_e)^{k-l-1}$$

which yields the desired result by taking logarithms. ∎

Proof of Lemma 5.3. Fix $a > 1$. Let $x = \pi_u(\omega)$, $\omega \in E_u^{\mathbb{N}}$ and $r > 0$. Now, choose integers $k, l \in \mathbb{N}$ such that

$$Dr_{\omega|k} < r \leqslant Dr_{\omega|k-1}$$

$$\Delta r_{\omega|l} < ar \leqslant \Delta r_{\omega|l-1},$$

and observe that $K_u \cap B(x, ar) \subseteq K_{\omega|l}$ and $K_{\omega|k} \subseteq B(x, r)$. Hence (by Lemma 5.2)

$$\frac{\mu_u B(x, ar)}{\mu_u B(x, r)} \leqslant \frac{\mu_u(K_{\omega|l})}{\mu_u(K_{\omega|k})} = \frac{p_{\omega|l}}{p_{\omega|k}} = \frac{1}{p_{\omega_{l+1}} \cdots p_{\omega_k}} \leqslant \frac{1}{(\min_e p_e)^{k-l}} \leqslant (\min_e r_e)^{-\varphi(a)}$$

whence

$$\limsup_{r \searrow 0} \left(\sup_{x \in K_u} \frac{\mu_u B(ar)}{\mu_u B(x, r)} \right) \leqslant (\min_e p_e)^{-\varphi(a)} < \infty. \quad ∎$$

Proof of Lemma 5.4. We divide the proof into two steps.

Step 1. There exists $\bar{K} \in\,]0, \infty[$ such that

$$\mathscr{P}_{\mu_u}^{q, \beta(q)}(K_\alpha) \leqslant \bar{\mathscr{P}}_{\mu_u}^{q, \beta(q)}(K_\alpha) \leqslant \bar{K} \mu_u^q(K_\alpha) \qquad \text{for} \quad \alpha \in E_u^{(*)}.$$

Proof of Step 1. Let $u \in V$, $\alpha \in E_u^{(*)}$ and $\varepsilon > 0$. The measure μ_u^q is finite and thus outer regular. We can therefore choose an open and bounded set G_ε such that $K_\alpha \subseteq G_\varepsilon$ and $\mu_u^q(G_\varepsilon \setminus K_\alpha) \leqslant \varepsilon$. Clearly $\delta_\varepsilon := \operatorname{dist}(K_\alpha, \mathbb{R}^d \setminus G_\varepsilon) > 0$. Let $0 < \delta < \Delta \wedge \operatorname{diam} K_\alpha \wedge \delta_\varepsilon$ and $(B(x_i, \varepsilon_i))_i$ be a centered δ-packing of K_α. For each i choose $\sigma_i \in [\alpha]$ such that $\pi_u(\sigma_i) = x_i$. Now, choose integers $k_i, l_i \neq \mathbb{N}$ such that

$$Dr_{\sigma_i|k_i} < \varepsilon_i \leqslant Dr_{\sigma_i|k_i-1}$$

$$\Delta r_{\sigma_i|l_i} < \varepsilon_i \leqslant \Delta r_{\sigma_i|l_i-1}.$$

Observe that $\sigma_i \in [\alpha]$ implies that $|\alpha| \leqslant k_i$. Also

$$K_u \cap B(x_i, \varepsilon_i) \subseteq K_{\sigma_i|l_i}$$

$$K_{\sigma_i|k_i} \subseteq B(x_i, \varepsilon_i).$$

Since $\varepsilon_i \leqslant Dr_{\sigma_i|k_i-1} \leqslant (D/\min_e r_e)\, r_{\sigma_i|k_i}$, $0 \leqslant \beta(q)$ implies that $(2\varepsilon_i)^{\beta(q)} \leqslant (2D/\min_e r_e)^{\beta(q)}\, r_{\sigma_i|k_i}^{\beta(q)}$. Since $Dr_{\sigma_i|k_i} < \varepsilon_i$, $\beta(q) < 0$ implies that $(2\varepsilon_i)^{\beta(q)} \leqslant (2D)^{\beta(q)}\, r_{\sigma_i|k_i}^{\beta(q)}$. In all cases

$$(2\varepsilon_i)^{\beta(q)} \leqslant \bar{K}_0\, r_{\sigma_i|k_i}^{\beta(q)} \qquad \text{for all } i, \tag{5.8}$$

where $\bar{K}_0 \in \,]0, \infty[$ is a suitable constant.

If $q \leqslant 0$ then

$$\mu_u(B(x_i, \varepsilon_i))^q \leqslant \mu_u(K_{\sigma_i|k_i})^q = p_{\sigma_i|k_i}^q.$$

If $0 < q$ then (by Lemma 5.2)

$$\begin{aligned} \mu_u(B(x_i, \varepsilon_i))^q &\leqslant \mu_u(K_{\sigma_i|l_i})^q = p_{\sigma_i|l_i}^q \\ &= \left(\frac{p_{\sigma_i|l_i}}{p_{\sigma_i|k_i}}\right)^q p_{\sigma_i|k_i}^q = \left(\frac{1}{p_{\sigma_i(l_i+1)} \cdots p_{\sigma_i(k_i)}}\right)^q p_{\sigma_i|k_i}^q \\ &\leqslant \frac{1}{(\min_e p_e^q)^{k_i - l_i}}\, p_{\sigma_i|k_i}^q \leqslant \frac{1}{(\min_e p_e^q)^{\varphi(1)}}\, p_{\sigma_i|k_i}^q \end{aligned}$$

In all cases

$$\mu_u(B(x_i, \varepsilon_i))^q \leqslant \bar{K}_1\, p_{\sigma_i|k_i}^q \qquad \text{for all } i, \tag{5.9}$$

where $\bar{K}_1 \in \,]0, \infty[$ is a suitable constant.

It follows from (5.8) and (5.9) that

$$\begin{aligned} \sum_i \mu(B(x_i, \varepsilon_i))^q (2\varepsilon_i)^{\beta(q)} &\leqslant \bar{K}_0 \bar{K}_1 \sum_i p_{\sigma_i|k_i}^q r_{\sigma_i|k_i}^{\beta(q)} \\ &\leqslant (\bar{p}/\underline{p})\, \bar{K}_0 \bar{K}_1 \sum_i p_{\sigma_i(1)}^{-1}\, p_{\sigma_i|k_i}^q r_{\sigma_i|k_i}^{\beta(q)} p_{\sigma_i(k_i)} \\ &= \bar{K} \sum_i \hat{\mu}_u^q([\sigma_i|k_i]) \leqslant \bar{K} \sum_i \mu_u^q(K_{\sigma_i|k_i}) \\ &\leqslant \bar{K} \sum_i \mu_u^q(B(x_i, \varepsilon_i)) = \bar{K} \mu_u^q\left(\bigcup_i B(x_i, \varepsilon_i)\right) \\ &\leqslant \bar{K} \mu_u^q(G_\varepsilon) \leqslant \bar{K}(\mu_u^q(K_\alpha) + \varepsilon), \end{aligned}$$

where $\bar{K} = (\bar{p}/\underline{p})\, \bar{K}_0 \bar{K}_1$. Hence

$$\bar{\mathscr{P}}_{\mu_u, \delta}^{q, \beta(q)}(K_\alpha) \leqslant \bar{K}(\mu_u^q(K_\alpha) + \varepsilon)$$

for $\varepsilon > 0$ and $0 < \delta < \Delta \wedge \operatorname{diam} K_\alpha \wedge \delta_\varepsilon$. This implies that

$$\overline{\mathscr{P}}^{q,\beta(q)}_{\mu_u}(K_\alpha) \leqslant \bar{K}\mu^q_u(K_\alpha),$$

which completes the proof of Step 1.

Step 2. There exists $\bar{K} \in]0, \infty[$ such that

$$(\mathscr{P}^{q,\beta(q)}_{\mu_u} \mid \operatorname{supp} \mu_u) \leqslant \bar{K}\mu^q_u.$$

Proof of Step 2. Let $G \subseteq K_u$ be an open subset of K_u and put $A = \{\alpha \in E_u^{(*)} \mid K_\alpha \subseteq G\}$. Since G is open $G = \bigcup_{\alpha \in A} K_\alpha$. Now we need only cover G once: if $\alpha, \beta \in A$ and $[\alpha] \cap [\beta] \neq \varnothing$, then one of them is contained in the other, so we may discard the smaller one. So there is a set $A_0 \subseteq A$ such that

$$G = \bigcup_{\alpha \in A_0} K_\alpha,$$

and $[\alpha] \cap [\beta] = \varnothing$ for $\alpha, \beta \in A_0$. Hence (by Step 1)

$$\begin{aligned} \mathscr{P}^{q,\beta(q)}_{\mu_u}(G) &= \mathscr{P}^{q,\beta(q)}_{\mu_u}\left(\bigcup_{\alpha \in A_0} K_\alpha\right) \leqslant \sum_{\alpha \in A_0} \mathscr{P}^{q,\beta(q)}_{\mu_u}(K_\alpha) \\ &\leqslant \bar{K} \sum_{\alpha \in A_0} \mu^q_u(K_\alpha) = \sum_{\alpha \in A_0} \hat{\mu}^q_u([\alpha]) \\ &= \bar{K}\hat{\mu}^q_u\left(\bigcup_{\alpha \in A_0} [\alpha]\right) \leqslant \bar{K}\hat{\mu}^q_u(\pi_u^{-1}(G)) = \bar{K}\mu^q_u(G) \qquad (5.10) \end{aligned}$$

for all open subsets $G \subseteq K_u$. Since $\mathscr{P}^{q,\beta(q)}_{\mu_u}$ and μ^q_u are finite Borel measures and thus outer regular, inequality (5.10) yields the desired results. ∎

Proof of Lemma 5.5. Let $E \subseteq K_u$ and $\delta < \Delta$. Let $(B_i = B(x_i, \varepsilon_i))_{i \in \mathbb{N}}$ be a centered δ-covering of E.

For each i choose $\sigma_i \in E_u^{\mathbb{N}}$ such that $x_i = \pi_u(\sigma_i)$. Next, choose integers $k_i, l_i \in \mathbb{N}$ satisfying

$$Dr_{\sigma_i \mid k_i} < \varepsilon_i \leqslant Dr_{\sigma_i \mid k_i - 1}$$

$$\Delta r_{\sigma_i \mid l_i} < \varepsilon_i \leqslant \Delta r_{\sigma_i \mid l_i - 1}$$

and observe that

$$K_u \cap B(x_i, \varepsilon_i) \subseteq K_{\sigma_i \mid l_i}$$

$$K_{\sigma_i \mid k_i} \subseteq B(x_i, \varepsilon_i).$$

Since $r_{\sigma_i|l_i} < \Delta^{-1}\varepsilon_i$, $0 \leqslant \beta(q)$ implies that

$$r_{\sigma_i|l_i}^{\beta(q)} \leqslant (2\Delta)^{-\beta(q)} (2\varepsilon_i)^{\beta(q)}.$$

Since $r_{\sigma_i|l_i} \geqslant (\min_e r_e)\, r_{\sigma_i|l_i-1} \geqslant (\min_e r_e)\, \Delta^{-1}\varepsilon_i$, $\beta(q) < 0$ implies that

$$r_{\sigma_i|l_i}^{\beta(q)} \leqslant ((2\Delta)^{-1} \min_e r_e)^{\beta(q)} (2\varepsilon_i)^{\beta(q)}.$$

We have in all cases

$$r_{\sigma_i|l_i}^{\beta(q)} \leqslant \underline{K}_0 (2\varepsilon_i)^{\beta(q)}, \tag{5.11}$$

where $\underline{K}_0 > 0$ is a suitable constant.

If $q \leqslant 0$ then

$$p^q_{\sigma_i|l_i} = \mu_u(K_{\sigma_i|l_i})^q \leqslant \mu_u(B(x_i, \varepsilon_i))^q.$$

If $0 < q$ then (by Lemma 5.2)

$$\begin{aligned} p^q_{\sigma_i|l_i} &= \mu_u(K_{\sigma_i|l_i})^q = \left(\frac{\mu_u(K_{\sigma_i|l_i})}{\mu_u(B(x_i, \varepsilon_i))}\right)^q \mu_u(B(x_i, \varepsilon_i))^q \\ &\leqslant \left(\frac{\mu_u(K_{\sigma_i|l_i})}{\mu_u(K_{\sigma_i|k_i})}\right)^q \mu_u(B(x_i, \varepsilon_i))^q = \left(\frac{p_{\sigma_i|l_i}}{p_{\sigma_i|k_i}}\right)^q \mu_u(B(x_i, \varepsilon_i))^q \\ &= \left(\frac{1}{p_{\sigma_i(l_i+1)} \cdots p_{\sigma_i(k_i)}}\right)^q \mu_u(B(x_i, \varepsilon_i))^q \\ &\leqslant \frac{1}{(\min_e p_e^q)^{k_i - l_i}} \mu_u(B(x_i, \varepsilon_i))^q \leqslant \frac{1}{(\min_e p_e^q)^{\varphi(1)}} \mu_u(B(x_i, \varepsilon_i))^q. \end{aligned}$$

We have in all cases

$$P^q_{\sigma_i|l_i} \leqslant \underline{K}_1 \mu_u(B(x_i, \varepsilon_i))^q \qquad \text{for all } i, \tag{5.12}$$

where $\underline{K}_1 \in \,]0, \infty[$ is a suitable constant.

It follows from (5.11) and (5.12) that

$$\begin{aligned} \mu_u^q(E) &\leqslant \sum_i \mu_u^q(B(x_i, \varepsilon_i)) \leqslant \sum_i \mu_u^q(K_{\sigma_i|l_i}) = \sum_i \rho^{-1}_{\sigma_i(1)} p^q_{\sigma_i|l_i} r^{\beta(q)}_{\sigma_i|l_i} \rho_{\sigma_i(l_i)} \\ &\leqslant (\bar{\rho}/\underline{\rho})\, \underline{K}_0 \underline{K}_1 \sum_i \mu_u(B(x_i, \varepsilon_i))^q (2\varepsilon_i)^{\beta(q)}, \end{aligned}$$

whence

$$\underline{K}\mu_u^q(E) \leqslant \mathscr{H}_{\mu_u,\delta}^{q,\beta(q)}(E) \leqslant \overline{\mathscr{H}}_{\mu_u}^{q,\beta(q)}(E) \leqslant \mathscr{H}_{\mu_u}^{q,\beta(q)}(E),$$

where $\underline{K} = (\underline{p}/\bar{p})(\underline{K}_0 \underline{K}_1)^{-1}$. ∎

Proof of Lemma 5.6. Let $0 < r < \Delta/(2 \max_\varepsilon r_\varepsilon)$ and $x \in K_v$. We claim that

$$\frac{\mu_u T_e(U(x,r))}{\mu_v U(x,r)} = p_e, \tag{5.13}$$

where $U(x,r)$ denotes the open ball with center x and radius r. Let $A = \{\alpha \in E_v^{(*)} \mid K_\alpha \subseteq U(x,r)\}$. Since $U(x,r)$ is open, $U(x,r) \cap K_v = \bigcup_{\alpha \in A} K_\alpha$. Now, we need only cover $U(x,r)$ once: if $\alpha, \beta \in A$ and $[\alpha] \cap [\beta] \neq \varnothing$, then one of them is contained in the other, so we may discard the smaller one. So there is a set $A_0 \subseteq A$ such that

$$U(x,r) \cap K_v = \bigcup_{\alpha \in A_0} K_\alpha, \tag{5.14}$$

and $[\alpha] \cap [\beta] = \varnothing$ for $\alpha, \beta \in A_0$, i.e. $(K_\alpha)_{\alpha \in A_0}$ is a disjoint family. Next observe that

$$T_e(U(x,r)) \cap K_u = T_e(U(x,r) \cap K_v). \tag{5.15}$$

Indeed, it is clear that $T_e(U(x,r)) \cap K_u \supseteq T_e(U(x,r) \cap K_v)$. Now let $y \in T_e(U(x,r)) \cap K_u$. We must now prove that $y \in T_e K_v$. Clearly

$$y \in T_e(U(x,r)) = U(T_e x, r_e r) \subseteq B(T_e K_v, r r_e) \subseteq B\left(T_e K_v, \frac{\Delta}{2}\right), \tag{5.16}$$

where $B(T_e K_v, \Delta/2) = \{z \in \mathbb{R}^d \mid \operatorname{dist}(T_e K_v, z) \leqslant \Delta/2\}$. Also $y \in K_u = \bigcup_{w \in V} \bigcup_{\varepsilon \in E_{uw}} T_\varepsilon K_w$, and we can thus choose $w \in V$ and $\varepsilon \in E_{uw}$ such that

$$y \in T_\varepsilon K_w. \tag{5.17}$$

However, since $B(T_e K_v, \Delta/2) \cap T_\varepsilon K_w = \varnothing$ for $(e,v) \neq (\varepsilon, w)$, (5.16) and (5.17) show that $\varepsilon = e$ and $w = v$, whence $y \in T_\varepsilon K_w = T_e K_v$. By (5.14) and (5.15),

$$\frac{\mu_u(T_c(U(x,r)))}{\mu_v(U(x,r))}=\frac{\mu_u(T_c(U(x,r))\cap K_u)}{\mu_v(U(x,r)\cap K_v)}=\frac{\mu_u(T_c(U(x,r)\cap K_v))}{\mu_v(U(x,r)\cap K_v)}$$

$$=\frac{\mu_u(T_c(\bigcup_{\alpha\in A_0}K_0))}{\mu_v(\bigcup_{\alpha\in A_0}K_\alpha)}=\frac{\mu_u(\bigcup_{\alpha\in A_0}T_cK_\alpha)}{\mu_v(\bigcup_{\alpha\in A_0}K_\alpha)}$$

$$=\frac{\mu_u(\bigcup_{\alpha\in A_0}K_{c\alpha})}{\mu_v(\bigcup_{\alpha\in A_0}K_\alpha)}=\frac{\sum_{\alpha\in A_0}p_{c\alpha}}{\sum_{\alpha\in A_0}p_\alpha}=\frac{\sum_{\alpha\in A_0}p_cp_\alpha}{\sum_{\alpha\in A_0}p_\alpha}=p_c,$$

which proves (5.13). The desired result follows immediately from (5.13). ∎

6. Multifractal Analysis of "Cookie-Cutter" Measures

In this section we prove that the upper bounds in Theorem 2.17 are the exact values of $f_\mu(\alpha)=\dim(\underline{X}_\alpha\cap\overline{X}^\alpha)$ and $F_\mu(\alpha)=\mathrm{Dim}(\underline{X}_\alpha\cap\overline{X}^\alpha)$ (and not just upper bounds) if μ is a "cookie-cutter" measure in $\mathbb{R}$. A rigorous analysis of the multifractal decomposition of "cookie-cutter" measures can also be found in a recent paper by Rand [Ra], (cf. also Bohr & Rand [Bo]).

6.1. *"Cookie-Cutter" Sets*

Let $I=[0,1]$ and $0<x_0<x_1<1$. Put $I_0=[0,x_0]$ and $I_1=[x_1,1]$. A "cookie-cutter" map is a map

$$g\colon I_0\cup I_1\to I$$

such that

(1) $g(I_0)=I=g(I_1)$.

(2) g is a $C^{1+\alpha}$ map for some $\alpha>0$ (i.e. g is a C^1 map and g' is α-Hölder continuous).

(3) $|g'(x)|>1$ for all $x\in I_0\cup I_1$.

The "cookie-cutter" set $\Lambda=\Lambda(g)$ associated with g is

$$\Lambda=\Lambda(g)=\{x\in I_0\cup I_1\mid\forall n\in\mathbb{N}_0\colon g^n(x)\in I_0\cup I_1\}.$$

Write $\Sigma^{(n)}=\{0,1\}^n$, $\Sigma^{(*)}=\bigcup_{n=0}^{\infty}\Sigma^{(n)}$ and $\sum^{\mathbb{N}}=\{0,1\}^{\mathbb{N}}$. If $\alpha\in\Sigma^{(n)}$ we will write $|\alpha|=n$. Also, if $\alpha\in\Sigma^{(n)}$ and $\omega\in\Sigma^{\mathbb{N}}$ we will always write $\alpha=(\alpha_1,...,\alpha_n)=(\alpha(1),...,\alpha(n))$ and $\omega=(\omega_1,\omega_2,...)=(\omega(1),\omega(2),...)$. Finally, if $\omega\in\sum^{\mathbb{N}}$ and $n\in\mathbb{N}$ then we put $\omega|n=(\omega_1,...,\omega_n)$. For $\alpha=(\alpha_0,...,\alpha_{n-1})\in\sum^{(n)}$ set

$$I_\alpha=\{x\in I_0\cup I_1\mid g^i(x)\in I_{\alpha_i}\text{ for }i=0,...,n-1\},$$

and observe that I_α is a closed interval and

$$\Lambda(g)=\bigcap_{n=0}^{\infty}\left(\bigcup_{\alpha\in\Sigma^{(n)}} I_\alpha\right).$$

Write $\gamma=\inf_{x\in I_0\cup I_1}|g'(x)|$ and $\Gamma=\sup_{x\in I_0\cup I_1}|g'(x)|$. It is easily seen that

$$(\Gamma^{-1})^{k-l}\operatorname{diam}(I_{\omega|l})\leqslant\operatorname{diam}(I_{\omega|k})\leqslant(\gamma^{-1})^{k-l}\operatorname{diam}(I_{\omega|l}) \tag{6.1}$$

for $\omega\in\Sigma^{\mathbb{N}}$ and $k, l\in\mathbb{N}_0$, with $k\geqslant l$.

The intersection $\bigcap_n I_{\omega|n}$ is clearly a singleton for each $\omega\in\Sigma^{\mathbb{N}}$. Now define $\pi:\Sigma^{\mathbb{N}}\to I$ by

$$\{\pi(\omega)\}=\bigcap_n I_{\omega|n}.$$

It is readily seen that $\pi(\Sigma^{\mathbb{N}})=\Lambda(g)$ and that π is a homeomorphism.

6.2. *"Cookie-Cutter" Measures*

Let $g: I_0\cup I_1\to I$ be a "cookie-cutter" map. If $h: I\to\mathbb{R}$ is a real valued map then write $S_n h(x)=\sum_{i=0}^{n-1}h(g^i(x))$ for $x\in I_0\cup I_1$. If $\varphi: I\to\mathbb{R}$ is a Hölder continuous function then we will denote the pressure of φ by $P(\varphi)$, the reader is referred to [Wa, Chapter 9] for a discussion of the pressure. Let $\varphi: I\to\mathbb{R}$ be a Hölder continuous function. The Gibbs state μ_φ of φ is the unique g-invariant Borel probability measure on $\Lambda(g)$ which statisfies the following:

there exist numbers $c_1, c_2\in\,]0,\infty[$ such that

$$c_1\leqslant\frac{\mu_\varphi(I_\alpha)}{e^{-nP(\varphi)+S_n\varphi(x)}}\leqslant c_2$$

for all $n\in\mathbb{N}$, $\alpha\in\Sigma^{(n)}$ and $x\in I_\alpha$.

The measure μ_φ is also called a "cookie-cutter" measure. Existence and uniqueness of Gibbs states are proved in Bowen [Bow] and Ruelle [Ru].

6.3. *Statement of Main Result*

Fix a "cookie-cutter" map $g: I_0\cup I_1\to I$ and a Hölder continuous function $\varphi: I\to\mathbb{R}$ and let $\nu=\mu_\varphi$ be the Gibbs state of φ. Let J_{g^n} denote the Jacobian derivative of g^n w.r.t. ν, i.e.

$$J_{g^n}(x)=\lim_{r\searrow 0}\frac{\nu(g^n B(x,r))}{\nu(B(x,r))}\qquad\text{for}\quad \nu\text{-a.a.}\quad x\in\Lambda(g).$$

The reader is referred to [Par, Chapter 10] for more information about Jacobian derivatives. Put

$$\varphi_{q,\tau} = -\tau \log(|g'|) - q \log(J_g)$$

for $q, \tau \in \mathbb{R}$. Then $\varphi_{q,\tau}$ is a Hölder continuous function on $\Lambda(g) = \Lambda$ (cf. [Ra]). Let $P(q, \tau) = P(\varphi_{q,\tau})$ denote the pressure of $\varphi_{q,\tau}$. It follows from [Ra, p. 534] that for each $q \in \mathbb{R}$ there exists a unique $\tau(q) \in \mathbb{R}$ such that

$$P(q, \tau(q)) = 0,$$

in fact, $q \to \tau(q)$ is real analytic. Write $\alpha = -\tau'$ and

$$\Lambda(a) = \{x \in \Lambda(g) \mid \alpha_\nu(x) = a\}.$$

We are now ready to state our main result on "cookie-cutters".

THEOREM 6.1. *The following assertions hold*

(i) $$0 < \mathscr{H}_\nu^{q,\tau(q)}(\Lambda(\alpha(q))) \leqslant \mathscr{P}_\nu^{q,\tau(q)}(\Lambda(\alpha(q))) \leqslant \overline{\mathscr{P}}_\nu^{q,\tau(q)}(\Lambda) < \infty.$$

(ii) *For each $q \in \mathbb{R}$ there exists a number $c_q \in]0, \infty[$ such that*

$$\mathscr{H}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu \leqslant \mathscr{P}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu \leqslant c_q \mathscr{H}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu.$$

(iii) $$\alpha_\nu(x) = \alpha(q) \quad \textit{for} \quad \mathscr{H}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu\text{-}a.a.\ x,$$

$$\alpha_\nu(x) = \alpha(q) \quad \textit{for} \quad \mathscr{P}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu\text{-}a.a.\ x.$$

(iv) *If $q, p \in \mathbb{R}$ and $\alpha(q) \neq \alpha(p)$ then*

$$(\mathscr{H}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu) \perp (\mathscr{H}_\nu^{p,\beta(p)} \mid \operatorname{supp} \nu),$$

$$(\mathscr{P}_\nu^{q,\beta(q)} \mid \operatorname{supp} \nu) \perp (\mathscr{P}_\nu^{p,\beta(p)} \mid \operatorname{supp} \nu).$$

(v) for each $q \in \mathbb{R}$,

$$b_\nu(q) = B_\nu(q) = \Delta_\nu^q(\Lambda) = C_\nu^q(\Lambda) = (1-q)\, D_\nu^q = \tau(q).$$

(vi) $\underline{a}_\nu = \underline{A}_\nu := \underline{a}$, $\bar{a}_\nu = \bar{A}_\nu := \bar{a}$.

(vii) $\dim \Lambda(\alpha) = \operatorname{Dim} \Lambda(\alpha) = b_\nu^*(\alpha) = B_\nu^*(\alpha) = \tau^*(\alpha)$ *for* $\alpha \in]\underline{a}, \bar{a}[$.

We note that the result in (vii) is a minor extension of the results in [Ra]. Rand [Ra] proves that $\dim \Lambda(\alpha) = \tau^*(\alpha)$, whereas we in addition show that $\dim \Lambda(\alpha) = \operatorname{Dim} \Lambda(\alpha)$, i.e. $\Lambda(\alpha)$ is a fractal in the sense of Taylor [Tay1, Tay2].

6.4. *Proof of Main Result*

We begin by defining some auxiliary measures. For $q \in \mathbb{R}$ let μ_q denote the Gibbs state of $\varphi_{q,\tau(q)}$. The proof of Theorem 6.1 is based on the following nine lemmas which we will prove below.

LEMMA 6.2 (The Principle of Bounded Variation). *Let $\psi: I_0 \cup I_1 \to \mathbb{R}$ be a Hölder continuous function of order a. Then there exists a number $C \in]0, \infty[$ such that*

$$|S_n \psi(x) - S_n \psi(y)| \leqslant C$$

for all $n \in \mathbb{N}$, $\alpha \in \Sigma^{(n)}$ and $x, y \in I_\alpha$.

LEMMA 6.3. *There exist numbers $\underline{c}, \bar{c} \in]0, \infty[$ such that*

$$\underline{c} \leqslant \operatorname{diam}(I_\alpha) \, |(g^n)'(x)| \leqslant \bar{c}$$

for all $n \in \mathbb{N}_0$, $\alpha \in \sum^{(n)}$ and $x \in I_\alpha$.

For $n \in \mathbb{N}_0$ and $\alpha \in \sum^{(n)}$ let J_α denote the "hole" in I_α, i.e.

$$I_\alpha = I_{\alpha 0} \cup J_\alpha \cup I_{\alpha 1},$$

$$I_\alpha \cap (I_{\alpha 0} \cup I_{\alpha 1})^\circ = \varnothing.$$

LEMMA 6.4. *There exists a number $c_0 \in]0, \infty[$ such that*

$$\operatorname{diam}(J_\alpha) \geqslant c_0 \operatorname{diam}(I_\alpha)$$

for $n \in \mathbb{N}_0$ and $\alpha \in \Sigma^{(n)}$.

LEMMA 6.5. $\nu \in \mathscr{P}_1(\Lambda(g))$.

LEMMA 6.6. *There exist numbers $\underline{k}, \bar{k} \in]0, \infty[$ such that*

$$\underline{k} \leqslant \nu(I_\alpha) \, J_{g^n}(x) \leqslant \bar{k}$$

for all $n \in \mathbb{N}$, $\alpha \in \Sigma^{(n)}$ and $x \in I_\alpha$.

LEMMA 6.7. *There exist numbers $\underline{K}, \bar{K} \in]0, \infty[$ such that*

$$\underline{K} \nu(I_\alpha)^q \operatorname{diam}(I_\alpha)^{\tau(q)} \leqslant \mu_q(I_\alpha) \leqslant \bar{K} \nu(I_\alpha)^q \operatorname{diam}(I_\alpha)^{\tau(q)}$$

for all $n \in \mathbb{N}_0$ and $\alpha \in \Sigma^{(n)}$.

LEMMA 6.8. *There exists a number* $\underline{C} \in]0, \infty[$ *such that*

$$\underline{C}\mu_q \leqslant (\mathscr{H}_\nu^{q, \tau(q)} \mid \operatorname{supp} \nu).$$

LEMMA 6.9. *There exists a number* $\bar{C} \in]0, \infty[$ *such that*

(i) $\bar{\mathscr{P}}_\nu^{q, \tau(q)}(\Lambda(g)) \leqslant \bar{C}\mu_q(\Lambda(g))$.

(ii) $(\mathscr{P}_\nu^{q, \tau(q)} \mid \operatorname{supp} \nu) \leqslant \bar{C}\mu_q$.

LEMMA 6.10. $\mu_q(\Lambda(\alpha(q))) = 1$.

Proof of Theorem 6.1. The proof follows from Lemma 6.2 through Lemma 6.10 and the arguments are similar to those in the proof of Theorem 5.1. ∎

Proof of Lemma 6.2. By uniform continuity, $\gamma := \inf_{x \in I_0 \cup I_1} |g'(x)| > 1$. It is easily seen by induction that

$$\operatorname{diam}(I_\alpha) \leqslant \gamma^{-m}$$

for all $m \in \mathbb{N}_0$ and $\alpha \in \Sigma^{(m)}$. Now let $n \in \mathbb{N}_0$, $\alpha \in \Sigma^{(n)}$ and $x, y \in I_\alpha$. For each $i \in \{0, ..., n-1\}$, $\{g^i(x), g^i(y)\} \subseteq I_{\alpha_i \cdots \alpha_{n-1}}$ whence

$$|g^i(x) - g^i(y)| \leqslant \gamma^{-(n-i)}.$$

Hence

$$|S_n\psi(x) - S_n\psi(y)| \leqslant \sum_{i=0}^{n-1} |\psi(g^i(x)) - \psi(g^i(y))| \leqslant \sum_{i=0}^{n-1} c\, |g^i(x) - g^i(y)|^a$$
$$\leqslant c \sum_{i=0}^{n-1} \gamma^{-(n-i)a} \leqslant c \sum_{i=0}^{\infty} \gamma^{-an} := C < \infty$$

for some $c \in]0, \infty[$. ∎

Proof of Lemma 6.3. Let $n \in \mathbb{N}_0$, $\alpha \in \Sigma^{(n)}$ and $x \in I_\alpha$. Choose $\underline{x}, \bar{x} \in I_\alpha$ such that

$$\inf_{y \in I_\alpha} |(g^n)'(y)| = |(g^n)'(\underline{x})| \qquad \text{and} \qquad \sup_{y \in I_\alpha} |(g^n)'(y)| = |(g^n)'(\bar{x})|.$$

Since g^n maps I_α homeomorphically onto I, the mean value theorem yields

$$|(g^n)'(\underline{x})| \operatorname{diam}(I_\alpha) = \inf_{y \in I_\alpha} |(g^n)'(y)| \operatorname{diam}(I_\alpha) \leqslant \operatorname{diam}(I)$$
$$\leqslant \sup_{y \in I_\alpha} |(g^n)'(y)| \operatorname{diam}(I_\alpha) = |(g^n)'(\bar{x})| \operatorname{diam}(I_\alpha).$$

Define $h: I_0 \cup I_1 \to \mathbb{R}$ by $h(x) = \log |g'(x)|$ and observe that h is Hölder continuous (because g' is Hölder continuous by assumption). It follows from the previous lemma that there exists a number $A > 0$ such that

$$|S_m h(y) - S_m h(z)| \leqslant A \tag{6.2}$$

for all $m \in \mathbb{N}_0$, $\beta \in \sum^{(m)}$ and $y, z \in I_\beta$. Since

$$S_n h(y) = \sum_{i=0}^{n-1} \log |g'(g^i(x))| = \log \left| \prod_{i=0}^{n-1} g'(g^i(y)) \right| = \log |(g^n)'(y)|,$$

equation (6.2) implies that

$$e^{-A} \leqslant \frac{|(g^n)'(\bar{x})|}{|(g^n)'(\underline{x})|} \leqslant e^A,$$

whence

$$\begin{aligned} \operatorname{diam}(I_\alpha)\, |(g^n)'(x)| &\leqslant \operatorname{diam}(I_\alpha)\, |(g^n)'(\bar{x})| \\ &\leqslant e^A \operatorname{diam}(I_\alpha)\, |(g^n)'(\underline{x})| \leqslant e^A \operatorname{diam}(I) = e^A. \end{aligned}$$

Similarly $\operatorname{diam}(I_\alpha)\, |(g^n)'(x)| \geqslant e^{-A}$. ∎

Proof of Lemma 6.4. It follows by an argument similar to the one given in Lemma 6.3 that there exist $\underline{k}, \bar{k} \in \,]0, \infty[$ such that

$$\underline{k} \leqslant \operatorname{diam}(J_\alpha)\, |(g^n)'(x)| \leqslant \bar{k}$$

for all $n \in \mathbb{N}_0$, $\alpha \in \Sigma^{(n)}$ and $x \in J_\alpha$. Let $n \in \mathbb{N}_0$, $\alpha \in \Sigma^{(n)}$ and $x \in J_\alpha \subseteq I_\alpha$. Then

$$\begin{aligned} \operatorname{diam}(J_\alpha) &= \operatorname{diam}(I_\alpha) \frac{\operatorname{diam}(J_\alpha)}{\operatorname{diam}(I_\alpha)} \geqslant \operatorname{diam}(I_\alpha) \frac{|(g^n)'(x)|^{-1} \underline{k}}{|(g^n)'(x)|^{-1} \bar{c}} \\ &= (\underline{k}/\bar{c}) \operatorname{diam}(I_\alpha). \end{aligned}$$ ∎

Proof of Lemma 6.5. Let $a > 1$ and c_0 be the constant that appears in Lemma 6.4. Let $x = \pi(\omega) \in \Lambda(g)$, $\omega \in \Sigma^{\mathbb{N}}$ and $r > 0$. Choose $l, k \in \mathbb{N}_0$ such that

$$\operatorname{diam}(I_{\omega|k+1}) \leqslant r < \operatorname{diam}(I_{\omega|k}) \tag{6.3}$$

$$c_0 \operatorname{diam}(I_{\omega|l+1}) \leqslant ar < c_0 \operatorname{diam}(I_{\omega|l}) \tag{6.4}$$

and observe that

$$I_{\omega|k+1} \subseteq B(x, r)$$

$$\Lambda(g) \cap B(x, ar) \subseteq I_{\omega|l+1}.$$

(Indeed, it is obvious that $I_{\omega|k+1} \subseteq B(x, r)$. Now let $y = \pi(\sigma) \in \Lambda(g) \cap B(x, ar)$ and assume $y \notin I_{\omega|l+1}$. Then there exists $j < l+1$ such that $\omega|j = \sigma|j$ and $\omega_{j+1} \neq \sigma_{j+1}$, whence

$$|y - x| \geqslant \operatorname{diam}(J_{\omega|j}) \geqslant c_0 \operatorname{diam}(I_{\omega|j}) \geqslant c_0 \operatorname{diam}(I_{\omega|l}) > ar,$$

which is a contradiction since $y \in B(x, ar)$.)

Since $a > 1$ and $c_0 \leqslant 1$, $k \geqslant l$. Equations (6.1), (6.3) and (6.4) therefore imply that

$$\frac{1}{a} = \frac{r}{ar} \leqslant \frac{\operatorname{diam}(I_{\omega|k})}{c_0 \operatorname{diam}(I_{\omega|l+1})} \leqslant \frac{(\gamma^{-1})^{k-l-1}}{c_0},$$

whence

$$k - l \leqslant 1 + \frac{\log(a/c_0)}{\log \gamma} := c(a).$$

Now the Gibbs state property implies that

$$\frac{\nu B(x, ar)}{\nu B(x, r)} \leqslant \frac{\nu(I_{\omega|l+1})}{\nu(I_{\omega|k+1})} \leqslant \frac{c_2 e^{-(l+1)P(\varphi) + S_{l+1}\varphi(x)}}{c_1 e^{-(k+1)P(\varphi) + S_{k+1}\varphi(x)}}$$

$$= \frac{c_2}{c_1} e^{(k-l)P(\varphi)} e^{-\sum_{i=l+1}^{k} \varphi(g^i(x))}$$

$$\leqslant \frac{c_2}{c_1} e^{|P(\varphi)|(k-l)} e^{(k-l)\|\varphi\|}$$

$$\leqslant \frac{c_2}{c_1} e^{(\|\varphi\| + |P(\varphi)|)c(a)} := k(a),$$

whence

$$\limsup_{r \searrow 0} \left(\sup_{x \in \Lambda(g)} \frac{\nu B(x, ar)}{\nu B(x, r)} \right) \leqslant k(a) < \infty. \quad \blacksquare$$

Proof of Lemma 6.6. It follows from the Gibbs state property that there exist numbers $0 < c_1 \leqslant c_2 < \infty$ such that

$$c_1 \leqslant \frac{\nu(I_\alpha)}{e^{-nP(\varphi) + S_n\varphi(x)}} \leqslant c_2 \tag{6.5}$$

for all $n \in \mathbb{N}$, $\alpha \in \Sigma^{(n)}$ and $x \in I_\alpha$. Fix $n \in \mathbb{N}$, $\alpha \in \Sigma^{(n)}$, $x \in I_\alpha \cap \Lambda(g)$ and $0 < r < c_0 \operatorname{diam}(I_\alpha)$, where c_0 is the constant that appears in Lemma 6.4. Write $x = \pi(\omega)$ for some $\omega \in [\alpha]$ and choose integers $k, l \in \mathbb{N}$ such that

$$\operatorname{diam}(I_{\omega|k+1}) \leqslant r < \operatorname{diam}(I_{\omega|k})$$

$$c_0 \operatorname{diam}(I_{\omega|l+1}) \leqslant r < c_0 \operatorname{diam}(I_{\omega|l}),$$

and observe that $l \geqslant n$ and

$$I_{\omega|k+1} \subseteq B(x, r), \qquad \Lambda(g) \cap B(x, r) \subseteq I_{\omega|l+1} \tag{6.6}$$

(cf. the proof of Lemma 6.5). It also follows from the proof of Lemma 6.5 that

$$0 \leqslant k - l \leqslant c < \infty,$$

where $c = c(1) := 1 - (\log(c_0)/\log \gamma)$. Since $x \in I_{\omega_0 \cdots \omega_k}$ and $g^n(x) \in I_{\omega_n \cdots \omega_l}$, (6.5) and (6.6) imply that

$$\begin{aligned}
\frac{\nu(g^n B(x, r))}{\nu(B(x, r))} &\leqslant \frac{\nu(g^n(I_{\omega|l+1}))}{\nu(I_{\omega|k+1})} = \frac{\nu(I_{\omega_n \cdots \omega_l})}{\nu(I_{\omega_0 \cdots \omega_k})} \\
&\leqslant \frac{c_2 e^{-(l-n+1)P(\varphi) + S_{l-n+1}\varphi(g^n(\pi(\omega)))}}{c_1 e^{-(k+1)P(\varphi) + S_{k+1}\varphi(\pi(\omega))}} \\
&\leqslant \frac{c_2}{c_1} e^{(k-l)|P(\varphi)|} e^{(k-l)\|\varphi\|} \frac{1}{e^{-nP(\varphi) + S_n\varphi(x)}} \\
&\leqslant \frac{c_2^2}{c_1} e^{c(|P(\varphi)| + \|\varphi\|)} \frac{1}{\nu(I_\alpha)}.
\end{aligned} \tag{6.7}$$

In a similar way we prove that

$$\frac{\nu(g^n B(x, r))}{\nu(B(x, r))} \geqslant \frac{c_1^2}{c_2} e^{-c(|P(\varphi)| + \|\varphi\|)} \frac{1}{\nu(I_\alpha)}. \tag{6.8}$$

It follows immediately from (6.7) and (6.8) that

$$\frac{c_1^2}{c_2} e^{-c(|P(\varphi)| + \|\varphi\|)} \frac{1}{\nu(I_\alpha)} \leqslant J_{g^n}(x) \leqslant \frac{c_2^2}{c_1} e^{c(|P(\varphi)| + \|\varphi\|)} \frac{1}{\nu(I_\alpha)}$$

for all $n \in \mathbb{N}$, $\alpha \in \Sigma^{(n)}$ and $x \in I_\alpha$. ∎

Proof of Lemma 6.7. Since $0 = P(q, \tau(q)) = P(\varphi_{q,\tau(q)})$, the Gibbs state property shows that there are constants $C_1, C_2 \in \,]0, \infty[$ such that

$$C_1 \leqslant \frac{\mu_q(I_\alpha)}{e^{S_n \varphi_{q,\tau(q)}(x)}} \leqslant C_2$$

for all $n \in \mathbb{N}_0$, $\alpha \in \Sigma^{(n)}$ and $x \in I_\alpha$. Fix $n \in \mathbb{N}_0$, $\alpha \in \Sigma^{(n)}$ and $x \in I_\alpha$. Then

$$\begin{aligned}\mu_q(I_\alpha) &\leqslant C_2 e^{S_n \varphi_{q,\tau(q)}(x)} \\ &= C_2 e^{-q\sum_{i=0}^{n-1} \log|g'(g^i(x))| - \tau(q) \sum_{i=0}^{n-1} \log(J_g(g^i(x)))} \\ &= C_2 e^{-q \log|(g^n)'(x)| - \tau(q) \log(\prod_{i=0}^{n-1} J_g(g^i(x)))} \\ &= C_2 \,|(g^n)'(x)|^{-q} \left(\prod_{i=0}^{n-1} J_g(g^i(x))\right)^{-\tau(q)}.\end{aligned}$$

It follows from [Par, Lemma 10.1] that $\prod_{i=0}^{n-1} J_g(g^i(x)) = J_{g^n}(x)$. Lemma 6.3 and 6.6 therefore imply that

$$\begin{aligned}\mu_q(I_\alpha) &\leqslant C_2 \,|(g^n)'(x)|^{-q} \,(J_{g^n}(x))^{-\tau(q)} \\ &\leqslant \bar{K} \operatorname{diam}(I_\alpha)^{\tau(q)} \,\nu(I_\alpha)^q,\end{aligned}$$

where $\bar{K} \in \,]0, \infty[$ is a suitable constant. In a similar way we may prove that

$$\underline{K} \operatorname{diam}(I_\alpha)^{\tau(q)} \,\nu(I_\alpha)^q \leqslant \mu_q(I_\alpha)$$

for some $\underline{K} \in \,]0, \infty[$. ∎

Proof of Lemma 6.8. Let $E \subseteq \Lambda(g)$ and $\delta > 0$. Let $(B_i = B(x_i, r_i))_{i \in \mathbb{N}}$ be a centered δ-covering of E. For each i choose $\omega_i \in \sum^{\mathbb{N}}$ such that $x_i = \pi(\omega_i)$. For each $i \in \mathbb{N}$ choose $k_i, l_i \in \mathbb{N}_0$ such that

$$\begin{aligned}\operatorname{diam}(I_{\omega_i|k_i+1}) &\leqslant r_i < \operatorname{diam}(I_{\omega_i|k_i}) \\ c_0 \operatorname{diam}(I_{\omega_i|l_i+1}) &\leqslant r_i < c_0 \operatorname{diam}(I_{\omega_i|l_i})\end{aligned}$$

(where c_0 is the number that appears in Lemma 6.4) and observe that

$$I_{\omega_i|k_i+1} \subseteq B(x_i, r_i), \qquad \Lambda(g) \cap B(x_i, r_i) \subseteq I_{\omega_i|l_i+1}.$$

Now clearly (by Lemma 6.7)

$$\begin{aligned}\mu_q(E) &\leqslant \sum_i \mu_q(B(x_i, r_i)) \leqslant \sum_i \mu_q(I_{\omega_i|l_i+1}) \\ &\leqslant \bar{K} \sum_i \nu(I_{\omega_i|l_i+1})^q \operatorname{diam}(I_{\omega_i|l_i+1})^{\tau(q)}.\end{aligned} \tag{6.9}$$

If $\tau(q)\geqslant 0$ then $\operatorname{diam}(I_{\omega_i|l_i+1})^{\tau(q)}\leqslant(2c_0)^{-\tau(q)}(2r_i)^{\tau(q)}$; if $\tau(q)<0$ then (6.1) implies that $\operatorname{diam}(I_{\omega_i|l_i+1})\geqslant\Gamma^{-1}\operatorname{diam}(I_{\omega_i|l_i})$ whence $2r_i\leqslant 2c_0\Gamma\operatorname{diam}(I_{\omega_i|l_i+1})$, i.e. $\operatorname{diam}(I_{\omega_i|l_i+1})^{\tau(q)}\leqslant(2c_0\Gamma)^{-\tau(q)}(2r_i)^{\tau(q)}$. In all cases

$$\operatorname{diam}(I_{\omega_i|l_i+1})^{\tau(q)}\leqslant k_1(2r_i)^{\tau(q)},\tag{6.10}$$

where k_1 is a suitable constant.

If $q<0$ then $A(g)\cap B(x_i,r_i)\subseteq I_{\omega_i|l_i+1}$ implies that

$$\nu(I_{\omega_i|l_i+1})^q\leqslant\nu(B(x_i,r_i))^q.\tag{6.11}$$

If $q\geqslant 0$ then $I_{\omega_i|l_i+1}\subseteq B(x_i,r_i/c_0)$ implies that

$$\nu(I_{\omega_i|l_i+1})^q\leqslant\left(\frac{\nu B(x_i,r_i/c_0)}{\nu B(x_i,r_i)}\right)^q\nu B(x_i,r_i)^q\leqslant M^q\nu(B(x_i,r_i))^q,\tag{6.12}$$

where

$$M:=\sup_{x\in A(g),\,r>0}\frac{\nu B(x,c_o^{-1}r)}{\nu B(x,r)}<\infty$$

(cf. Lemma 6.5). It follows from (6.10)–(6.12) that

$$\mu_q(E)\leqslant k_2\sum_i\nu(B(x_i,r_i))^q(2r_i)^{\tau(q)}$$

for a suitable constant k_2. Hence

$$\mu_q(E)\leqslant k_2\bar{\mathscr{H}}^{q,\tau(q)}_{\nu,\delta}(E)\leqslant\bar{\mathscr{H}}^{q,\tau(q)}_{\nu}(E)\leqslant k_2\mathscr{H}^{q,\tau(q)}_{\nu}(E).\ \blacksquare$$

Proof of Lemma 6.9. We divide the proof into two steps.

Step 1. There exists $\bar{C}\in\,]0,\infty[$ such that

$$\mathscr{P}^{q,\tau(q)}_{\nu}(I_\alpha)\leqslant\bar{\mathscr{P}}^{q,\tau(q)}_{\nu}(I_\alpha)\leqslant\bar{C}\mu_q(I_\alpha)\qquad\text{for}\quad\alpha\in\sum{}^{(*)}.$$

Proof of Step 1. Let $\alpha\in\sum^{(*)}$ and $\varepsilon>0$. Since μ_q is finite and therefore outer regular we may choose an open bounded set G_ε such that $I_\alpha\subseteq G_\varepsilon$ and $\mu_q(G_\varepsilon\backslash I_\alpha)\leqslant\varepsilon$. Clearly $\delta_\varepsilon:=\operatorname{dist}(I_\alpha,\mathbb{R}\backslash G_\varepsilon)>0$. Let $0<\delta<\delta_\varepsilon$ and $(B(x_i,r_i))_i$ be a centered δ-packing of I_α. For each $i\in\mathbb{N}$ choose $\omega_i\in[\alpha]$ such that $\pi(\omega_i)=x_i$ and integers $k_i,l_i\in\mathbb{N}$ such that

$$\operatorname{diam}(I_{\omega_i|k_i+1})\leqslant r_i<\operatorname{diam}(I_{\omega_i|k_i})$$
$$c_0\operatorname{diam}(I_{\omega_i|l_i+1})\leqslant r_i<c_0\operatorname{diam}(I_{\omega_i|l_i}),$$

where c_0 is the constant that appears in Lemma 6.4 and observe that

$$I_{\omega_i|k_i+1} \subseteq B(x_i, r_i), \qquad \Lambda(g) \cap B(x_i, r_i) \leqslant I_{\omega_i|l_i+1}.$$

If $\tau(q) \leqslant 0$ then

$$(2r_i)^{\tau(q)} \leqslant 2^{\tau(q)} \operatorname{diam}(I_{\omega_i|k_i+1})^{\tau(q)},$$

and if $0 < \tau(q)$ then (6.1) implies that

$$r_i < \operatorname{diam}(I_{\omega_i|k_i}) \leqslant \Gamma \operatorname{diam}(I_{\omega_i|k_i+1}).$$

whence

$$(2r_i)^{\tau(q)} \leqslant (2\Gamma)^{\tau(q)} \operatorname{diam}(I_{\omega_i|k_i+1})^{\tau(q)}.$$

We have in all cases

$$(2r_i)^{\tau(q)} \leqslant k_1 \operatorname{diam}(I_{\omega_i|k_i+1})^{\tau(q)}, \tag{6.13}$$

where k_1 is a suitable constant.

If $q \leqslant 0$ then

$$\nu(B(x_i, r_i))^q \leqslant \nu(I_{\omega_i|k_i+1})^q.$$

If $0 < q$ then the proof of Lemma 6.5 implies that

$$\begin{aligned} \nu(B(x_i, r_i))^q &\leqslant \left(\frac{\nu(I_{\omega_i|l_i+1})}{\nu(I_{\omega_i|k_i+1})}\right)^q \nu(I_{\omega_i|k_i+1})^q \\ &\leqslant \frac{c_2}{c_1} e^{(\|\varphi\| + P(\varphi))\, c(1)} \nu(I_{\omega_i|k_i+1})^q. \end{aligned}$$

In all cases

$$\nu(B(x_i, r_i))^q \leqslant k_2 \, \nu(I_{\omega_i|k_i+1})^q, \tag{6.14}$$

where k_2 is a suitable constant.

It follows from (6.13), (6.14) and Lemma 6.7 that

$$\begin{aligned} \sum_i \nu(B(x_i, r_i))^q (2r_i)^{\tau(q)} &\leqslant k_1 k_2 \sum_i \nu(I_{\omega_i|k_i+1})^q \operatorname{diam}(I_{\omega_i|k_i+1})^{\tau(q)} \\ &\leqslant k_1 k_2 \underline{K}^{-1} \sum_i \mu_q(I_{\omega_i|k_i+1}) \\ &\leqslant k_1 k_2 \underline{K}^{-1} \sum_i \mu_q(B(x_i, r_i)) \end{aligned}$$

$$= k_1 k_2 \underline{K}^{-1} \mu_q \Big(\bigcup_i B(x_i, r_i) \Big)$$
$$\leqslant k_1 k_2 \underline{K}^{-1} \mu_q (G_\varepsilon)$$
$$\leqslant k_1 k_2 \underline{K}^{-1} (\mu_q (I_\alpha) + \varepsilon).$$

Hence

$$\bar{\mathscr{P}}^{q, \tau(q)}_{\nu, \delta} (I_\alpha) \leqslant \bar{C} (\mu_q (I_\alpha) + \varepsilon)$$

for $\varepsilon > 0$ and $0 < \delta < \delta_\varepsilon$, which clearly implies that $\bar{\mathscr{P}}^{q, \tau(q)}_{\nu}(I_\alpha) \leqslant \bar{C} \mu_q (I_\alpha)$.

Step 2. There exists $\bar{C} \in \,]0, \infty[$ such that

$$\mathscr{P}^{q, \tau(q)}_{\nu} \leqslant \bar{C} \mu_q .$$

Proof of Step 2. The proof of step 2 is identical to the proof of Step 2 in Lemma 5.4. ∎

Proof of Lemma 6.10. For each $x \in \Lambda(g)$ let $\omega(x) = \pi^{-1}(x)$ (recall that π is a homeomorphism, in particular bijective). Since μ_q is an ergodic g-invariant measure, Birkhoff's ergodic theorem implies (because $J_{g^n}(x) = \prod_{i=0}^{n-1} J_g(g^i(x))$ for $x \in \Lambda(g)$ by [Par, Lemma 10.1]) that

$$\frac{1}{n} \log J_{g^n}(x) = \frac{1}{n} \log \Big(\prod_{i=0}^{n-1} J_g(g^i(x)) \Big) = \frac{1}{n} \sum_{i=0}^{n-1} \log J_g(g^i(x))$$
$$\to \int \log(J_g)\, d\mu_q \qquad \text{for} \quad \mu_q\text{-a.a.} \quad x \quad \text{as} \quad n \to \infty. \tag{6.15}$$

It follows from Lemma 6.6 that

$$\frac{1}{n} \log \underline{k} - \frac{1}{n} \log J_{g^n}(x) \leqslant \frac{1}{n} \log \nu(I_{\omega(x)|n})$$
$$\leqslant \frac{1}{n} \log \bar{k} - \frac{1}{n} \log J_{g^n}(x) \tag{6.16}$$

for all $n \in \mathbb{N}$ and $x \in \Lambda(g)$. By combining (6.15) and (6.16) we get

$$\frac{1}{n} \log \nu(I_{\omega(x)|n}) \to -\int \log(J_g)\, d\mu_q \qquad \text{for} \quad \mu_q\text{-a.a.} \quad x \quad \text{as} \quad n \to \infty. \tag{6.17}$$

It follows in the same way from Lemma 6.3 that

$$\frac{1}{n} \log \operatorname{diam}(I_{\omega(x)|n}) \to -\int \log |g'|\, d\mu_q \quad \text{for} \quad \mu_q\text{-a.a.} \quad x \quad \text{as} \quad n \to \infty. \tag{6.18}$$

Recall that $P(q, \tau) = P(\varphi_{q,\tau}) = P(-\tau \log |g'| - q \log(J_g))$. It follows from [Ru] that P is real-analytic and

$$\partial_1 P(q, \tau) = -\int \log(J_q)\, d\mu_{\varphi_{q,\tau}}, \qquad \partial_2 P(q, \tau) = -\int \log |g'|\, d\mu_{\varphi_{q,\tau}}$$

(where ∂_i denotes partial differentiation w.r.t. the ith variable.) Also, since $P(q, \tau(q)) = 0$,

$$\partial_1 P(q, \tau(q)) + \partial_2 P(q, \tau(q))\, \tau'(q) = 0$$

and so

$$\alpha(q) = -\tau'(q) = \frac{\partial_1 P(q, \tau(q))}{\partial_2 P(q, \tau(q))} = \frac{\int \log(J_g)\, d\mu_q}{\int \log |g'|\, d\mu_q}. \tag{6.19}$$

By putting (6.17), (6.18) and (6.19) together we get

$$\frac{\log \nu(I_{\omega(x)|n})}{\log \operatorname{diam}(I_{\omega(x)|n})} \to \frac{\int \log(J_g)\, d\mu_q}{\int \log |g'|\, d\mu_q} = \alpha(q)$$

$$\text{for } \mu_q\text{-a.a. } x \text{ as } n \to \infty. \tag{6.20}$$

It is readily seen that

$$\begin{gathered} \frac{\log \nu(I_{\omega(x)|n})}{\log \operatorname{diam}(I_{\omega(x)|n})} \to a \quad \text{as } n \to \infty \\ \Updownarrow \\ \frac{\log \nu B(x, r)}{\log r} \to a \quad \text{as } r \searrow 0. \end{gathered} \tag{6.21}$$

The desired conclusion now follows from (6.20) and (6.21). ∎

7. Remarks and Questions

Let X be a metric space and $\mu \in \mathscr{P}(X)$.

7.1. *Mutual Singularity of the Multifractal Hausdorff Measures*

Let $q, p \in \mathbb{R}$ and assume that $b := b_\mu$ is differentiable at q and p with $b'(q) \neq b'(p)$. It is then true that

$$(\mathscr{H}_\mu^{q, b(q)} \mid \operatorname{supp} \mu) \perp (\mathscr{H}_\mu^{p, b(p)} \mid \operatorname{supp} \mu)?$$

This satisfied for graph directed self-similar measures in $\mathbb{R}^d$ with totally disconnected support (cf. Theorem 5.1) and "cookie-cutter" measures (cf. Theorem 6.1).

7.2. *Mutual Singularity of the Multifractal Packing Measures*

Let $q, p \in \mathbb{R}$ and assume that $B := B_\mu$ is differentiable at q and p with $B'(q) \neq B'(p)$. It is then true that

$$(\mathscr{P}_\mu^{q, B(q)} \mid \operatorname{supp} \mu) \perp (\mathscr{P}_\mu^{p, B(p)} \mid \operatorname{supp} \mu)?$$

This is satisfied for graph directed self-similar measures in $\mathbb{R}^d$ with totally disconnected support (cf. Theorem 5.1) and "cookie-cutter" measures (cf. Theorem 6.1).

7.3. *Strict Monotonicity of $b_\mu(B_\mu)$*

Is $b_\mu(B_\mu)$ strictly decreasing for non-atomic μ?

7.4. *Fixed Points for b_μ and B_μ*

Is the converse of Proposition 2.12 true, i.e. if $\alpha \in [0, \infty[$ and

$$b_\mu^*(\alpha) = \alpha \qquad \text{or} \qquad B_\mu^*(\alpha) = \alpha$$

is it then true that

$$\mu(\{x \in \operatorname{supp} \mu \mid \alpha_\mu(x) = \alpha\}) > 0?$$

7.5. *The Relation between f_μ and $b_\mu(B_\mu)$*

Does there exist a measure $\mu \in \mathscr{P}(X)$ such that the support of f_μ contains a non-degenerate interval, μ is dimensional exact (i.e. there exists a number α such that $\alpha_\mu = \alpha$ μ-a.e.) and $f_\mu(x) < b_\mu^*(x)$ (or $f_\mu(x) < B_\mu^*(x)$) for all x in a non-degenerate interval contained in the support of f_μ?

7.6. *Multimeasures in the Sense of Kahane*

Kahane [Kah, p. 316] defines a multimeasure as follows. A multimeasure associated with a Borel probability measure μ on a metric space X is a family $(\mu_q)_{q \in \mathbb{R}}$ in $\mathscr{P}(X)$ satsfying the following three conditions:

(1) Normalization: there exist $\underline{c}, \bar{c} \in]0, \infty[$ such that

$$\underline{c}\mu_1 \leqslant \mu \leqslant \bar{c}\mu_1.$$

Remark. The normalization condition above is less restrictive than the normalization condition in [Kah] which requires that $\mu_1 = \mu$.

(2) Size of support:

$$\operatorname{supp} \mu_q = \operatorname{supp} \mu \qquad \text{for all} \quad q \in \mathbb{R}.$$

(3) The multifractal dimension exactness condition: There exists a family $(\alpha_q)_{q\in\mathbb{R}}$ of positive numbers such that

$$\alpha_\mu(x) = \alpha_q \qquad \text{for} \quad \mu_q\text{-a.a. } x,$$
$$\operatorname{Im} \alpha_\mu = \{\alpha_q \mid q \in \mathbb{R}\}.$$

Find conditions such that $(\mathscr{H}_\mu^{q,b_\mu(q)})_{q\in\mathbb{R}}$ (or $\mathscr{P}_\mu^{q,B_\mu(q)})_{q\in\mathbb{R}}$) is a multimeasure associated with μ. It follows from Theorem 5.1 and Theorem 6.1 that $(\mathscr{H}_\mu^{q,b_\mu(q)})_{q\in\mathbb{R}}$ and $(\mathscr{P}_\mu^{q,B_\mu(q)})_{q\in\mathbb{R}}$ are multimeasures for μ in the case where μ is a graph directed self-similar measure in $\mathbb{R}^d$ with totally disconnected support or a "cookie-cutter" measure on $\mathbb{R}$.

7.7. *Lower Bound for the Multifractal Spectrum*

Write $B := B_\mu$ and let $\alpha_+(q) := -B'_+(q)$ and $\alpha_-(q) := -B'_-(q)$ for $q \in \mathbb{R}$. Are the following inequalities satisfied

$$q\alpha_-(q) + B(q) \leqslant \operatorname{Dim}(\underline{X}_{\alpha_+(q)} \cap \overline{X}^{\alpha_-(q)}) \qquad \text{for} \quad q \leqslant 0$$
$$q\alpha_+(q) + B(q) \leqslant \operatorname{Dim}(\underline{X}_{\alpha_+(q)} \cap \overline{X}^{\alpha_-(q)}) \qquad \text{for} \quad 0 \leqslant q \quad ?$$

Theorem 2.18 gives a partial answer to this question.

7.8. *Arbitrary Graph Directed Self-Similar Measures*

Let $G = (E, V, (r_e)_e, (T_e)_e, (p_e)_e)$ be a MW graph with probabilities, cf. Section 5. Let $(K_u)_{u\in V}$ be the invariant self-similar sets associated with G, and let $(\mu_u)_{u\in V}$ be the graph directed self-similar measures associated with G. Finally, let β and α be the auxiliary functions introduced in Section 5.

If the support of μ_u is totally disconnected for all u then Theorem 5.1 shows that

$$f_{\mu_u} = \beta^* \tag{7.1}$$
$$F_{\mu_u} = \beta^* \tag{7.2}$$
$$0 < \mathscr{H}_{\alpha\mu_u}^{q,\beta(q)}(K_u(\alpha(q))) \leqslant \mathscr{P}_{\mu_u}^{q,\beta(q)}(K_u(\alpha(q))) \leqslant \overline{\mathscr{P}}_{\mu_u}^{q,\beta(q)}(K_u) < \infty \tag{7.3}$$

It is an open problem whether equations (7.1) and (7.2) hold in the case where the support of μ_u is not totally disconnected, cf. [Ca, p. 215] and [Ed, Section 5.3, Question (d)].

Are equations (7.1) through (7.3) satified in the case where the open set condition holds, i.e. if there exists a family of open, non-empty and bounded subsets $(U_u)_{u\in V}$ of $\mathbb{R}^d$ such that

(1) $T_e(U_v) \subseteq U_u$ for all $u, v \in V$ and $e \in E_{uv}$.

(2) $T_e(U_v) \cap T_\varepsilon(U_w) = \varnothing$ for all $u, v, w \in V$ with $v \neq w$ and $e \in E_{uv}$, $\varepsilon \in E_{uw}$. (See Note Added in Proof (2) at the end of this paper.)

7.9. *Concave Hulls of Multifractal Spectra*

If $f\colon \mathbb{R} \to \mathbb{R}$ is real valued function then $\hat{f}\colon \mathbb{R} \to [-\infty, \infty[$ denotes the concave hull of f. Is it true that

$$\hat{f}_\mu = b^*_\mu, \qquad \hat{F}_\mu = B^*_\mu?$$

The examples in Section 3 seem to indicate that this is the case.

Notes Added in Proof

(1) A substantial number of new results have been obtained since this paper was written (August 1992). Olsen [Ol1] has performed a multifractal analysis of random graph directed self-similar measures, and Arbeiter and Patzschke [AP] and Falconer [Fa3] have performed a multifractal analysis of random self-similar measures. Lau and Ngai [LN] have studied the multifractal structure of self-similar measures satisfying a very weak separation condition. Riedi [Ri], Olsen [Ol2, Ol3], and Schmeling and Siegmund-Schultze [SS] have studied various multifractal spectra of general self-affine measures. Finally we note that Riedi and Mandelbrot [RM] have studied the Hausdorff spectrum of self-similar measures generated by a countable number of similarities.

(2) It has recently been proven by Arbeiter and Patzschke [AP] that $f_\mu = F_\mu = \beta^*$ for random self-similar measures μ satisfying the open set condition.

(3) After this paper was accepted for publication, the author was informed by Professor S. J. Taylor and Professor J. Peyrière that the latter in [Pey] considered constructions related to (but less general than) the dimension functions b_μ, and that he in [BMP], in collaboration with G. Brown and G. Michon, has obtained results somewhat similar to parts of Theorem 2.18.

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