Discrete Mathematics 41 (1982) 123–130
North-Holland Publishing Company

NONCONVEX VERTICES OF POLYHEDRAL 2-MANIFOLDS

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Received 6 May 1980
Revised 12 September 1980 and 24 September 1981

1. Introduction

A *polyhedral 2-manifold* is a 2-manifold that is the union of convex polygons, called its *facets*, such that the intersection of any two facets is either empty, a vertex of each facet, or an edge of each facet. Polyhedral 2-manifolds may be viewed as generalizations of 3-dimensional convex polytopes. One property that convex polytopes have is that each vertex is *convex*, that is, there is a plane that intersects the set of facets that meet the vertex such that the intersection is the boundary of a convex polygon.

Any polyhedral 2-manifold of genus greater than or equal to 1 must have a nonconvex vertex. This was first mentioned in print by Altshuler [1]; however, it probably has been known before because every such manifold must have a saddle point, and a saddle point is nonconvex.

In her thesis, J. Simutis mentions the possibility that every toroidal polytope (i.e., polyhedral 2-manifold of genus 1) has at least six nonconvex vertices [3]. In this paper we construct a toroidal polytope with only five nonconvex vertices and prove that every toroidal polytope has at least four nonconvex vertices.

It might seem that the number of nonconvex vertices that a polyhedral 2-manifold must have increases with the genus; however, we show that this is not so. We construct polyhedral 2-manifolds of every positive genus that have at most seven nonconvex vertices.

2. Definitions

A 2-cell complex C is a collection of convex k-dimensional polytopes $-1 \leq k \leq 2$, called the *faces* of C, such that

(i) the intersection of any two faces of C is a face of both faces,

(ii) any face of a face of C is a face of C.

* Research supported by NSF Grant MCS76-07466.

0012-365X/82/0000–0000/$02.75

A polyhedral 2-manifold is therefore a 2-cell complex whose union is a 2-manifold. A 2-*pseudomanifold* is a simplicial 2-complex that is connected and has the property that every edge belongs to exactly two 2-simplices. A *polyhedral 2-pseudomanifold* is a 2-cell complex whose union is homeomorphic to a 2-pseudomanifold.

The *star* of a vertex of any of these types of manifolds is the union of the set of faces meeting the vertex. The *link* of the vertex is the relative boundary of the star. In a 2-manifold, the star of a vertex is always a cell and the link is always a simple circuit.

Lemma 1. *If the link of each vertex of a polyhedral 2-pseudomanifold is a simple circuit, then it is a polyhedral 2-manifold.*

Proof. We must show that every point has a neighborhood homeomorphic to a cell. Clearly all points in a 2-pseudomanifold, except possibly the vertices, will have such a neighborhood. Since the link of each vertex is a simple circuit, the star must be a closed cell; thus, the vertices also have the desired neighborhoods.

As stated above, a vertex is *convex* if there is a plane that intersects its star in the boundary of a convex polygon. The star of a vertex generates a cone with the vertex as the vertex of the cone. The cone consists of the union of the rays from the vertex through each of the points of its link. If the cones associated with two different vertices are congruent, we shall say that the vertices are *congruent*. Clearly, if one of two congruent vertices is convex, then so is the other.

For a given polyhedral 2-manifold, a *general direction* is one such that each plane perpendicular to that direction contains at most one vertex. A *general plane* is one that is perpendicular to a general direction. For a given direction u, the *index* of a vertex with respect to u as 1 minus one half the number of times that the link of the vertex intersects the plane through the vertex perpendicular to u. Thus, a relative maximum or minimum will have index 1. A vertex that is not a maximum, a minimum or a saddle point will have index 0. A saddle point will have negative index. A convex vertex will have index either 0 or 1. A theorem of Banchoff [2] states that the sum of the indices of the vertices of a polyhedral 2-manifold (in which the facets are all triangles) is the Euler characteristic of the manifold. If we have a polyhedral 2-manifold that has facets that are not triangles, we may add diagonals across facets to change the facets to triangles without changing any of the indices; thus, Banchoff's theorem applies to all polyhedral 2-manifolds.

We shall say that a vertex is a saddle point *with respect to a plane* if it is a saddle point (i.e., of negative index) with respect to a direction perpendicular to that plane.

One method of construction that we shall use is formation of convex hulls. The convex hull of a collection of sets $S_1, \ldots, S_n$ will be denoted by $\mathrm{con}(S_1, \ldots, S_n)$. If we have two faces F_1 and F_2 of a polyhedral 2-manifold that meet on an edge e,

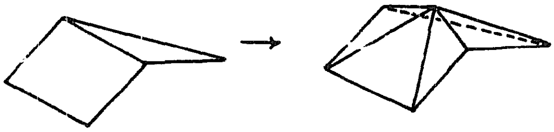

Fig. 1.

then by *capping* the edge that they meet on, we mean replacing F_1 and F_2 by polygons formed in the following way. Take a point p very near the middle of e, but not in the convex hull of the two facets. For each edge of F_1 or F_2, other than e, take the convex hull of p with that edge. These are the polygons that we replace F_1 and F_2 with (Fig. 1).

3. The main results

Theorem 1. *For each $i>0$, there exists a polyhedral 2-manifold M_i, of genus i, with at most seven non-convex vertices.*

Proof. We begin by describing the intersections, A_i, B_i and C_i of M_i with the planes $z=0$, $z=1$, and $z=-1$, respectively. The set A_i consists of $2i+1$ triangles with a common vertex V as shown in Fig. 2. We label the innermost triangle T_1, the next innermost T_2, etc. The righthand edge of T_j will be called a_j. The upper and left edges of T_j will be called b_j and c_j, respectively. The set A_i is situated in the xy-plane in E^3 such that the origin is inside T_1 and the triangles are symmetric about the x axis. The vertex belonging to c_j and b_j will be denoted r_j.

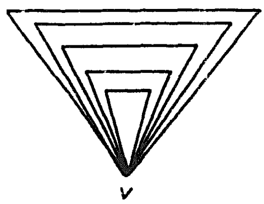

Fig. 2.

Now we will perform certain modifications on A_i to produce the set B_i. We translate each of the triangles $T_1, \ldots, T_{2i-1}$ such that the bottom vertex of each T_j is in the relative interior of a_{j+1}, for $1 \leq j \leq 2i-1$, as in Fig. 3.

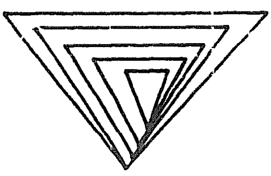

Fig. 3.

These translations can be made in such a way that the origin is inside the translate of T_1.

We now extend the top edge of the translate of T_1 a small distance to the right, and we add a segment from this new endpoint parallel to a_1, as shown in Fig. 4.

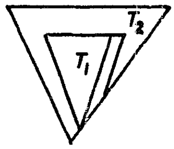

Fig. 4.

We extend each of the translates of the edges c_j, $1 \leq j \leq 2i-1$, until they meet the translates of a_{j+2} (see Fig. 5).

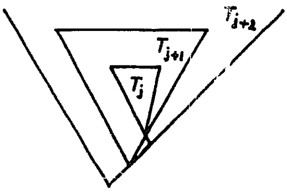

Fig. 5.

From this configuration we can extract a closed polygonal curve K_i as shown in Fig. 6.

We multiply the set K_i by a constant α to be determined later, and translate the resulting set one unit upward. This is the set B_i. On the curve K_i each edge corresponds to a parallel edge in A_i. If x is an edge in A_i then we denote the corresponding edge in B_i by x'.

The set C_i is the reflection of B_i through the origin. Because of the symmetry of the triangles in A_i, each edge of C_i is parallel to a unique edge of A_i. If x is an edge of A_i, we denote the corresponding parallel edge in C_i by x''.

The manifold M_i contains as faces, all polygons of the form con(x, x') and con(x, x''), for all edges x of A_i. It also contains all polygons of the form con(p, e) and con(q, g) where $p=(0,0,1+\varepsilon)$, $q=(0,0,-1,-\varepsilon)$, e is an edge of B_i and g is an edge of C_i (ε is a constant to be determined later).

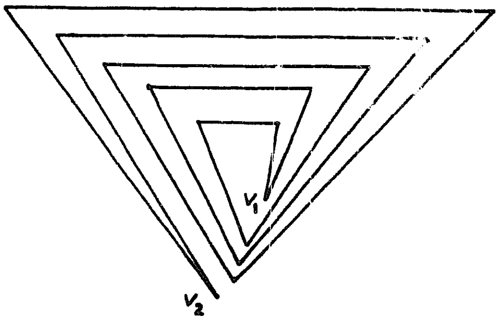

Fig. 6.

We will now examine the convexity of the vertices and we shall see how to choose the constants α and ε so that there are at most seven nonconvex vertices. It is important to observe here that $\mathrm{con}(T_j, b_j', c_j')$ and $\mathrm{con}(T_j, a_j'', b_j'')$ are convex polytopes and that their union is a convex polytope P_j provided α is small enough. The vertex r_j is congruent to a vertex of P_j and is therefore convex. By symmetry, the vertices lying in both a_j and b_j are also convex.

Let P_j' be the polytope obtained by taking the convex hull of P_j and a point $P=(0,0,1+\varepsilon)$ where $\varepsilon>0$ and is sufficiently small that the faces of P_j' meeting P are just triangles, namely the convex hulls of P with edges of the top triangular face of P_j'. The vertices in B_i that belong to both b_j' and c_j' are congruent to vertices of P_j' and are thus convex. A similar argument shows that the vertices belonging to both a_j' and b_j' are convex. By symmetry, the corresponding vertices in C_i are also convex.

Let $Q_j=\mathrm{con}(c_j, c_j', a_{j+2}, a_{j-2}', P)$. The vertices in B_i other than v_1 and v_2 that are the intersections of c_j' and a_j' (see Fig. 6) are congruent to vertices of the corresponding Q_j and are thus convex. The corresponding vertices in C_i are therefore also convex.

This leaves only seven vertices, p, q, V, and two vertices each, in B_i and C_i, that have not been shown to be convex. Although it is not necessary for the proof of this theorem, it can be shown that they are indeed not convex.

Our next task is to prove that M_i is a manifold. It is clearly a pseudomanifold and since each vertex except possibly V, has a neighborhood homeomorphic to a cell, we can show that M_i is a manifold by showing that the link of V is a simple circuit.

This we prove by induction on i. The reader may check that for $i=1$, the link is a simple circuit. We shall assume that the link is a circuit in M_{i-1} and prove it for M_i.

The cross section B_i is essentially what you get if you separate the two edges meeting v_2 in B_{i-1} (see Fig. 6) and add the edges a_{2i}', a_{2i+1}', b_{2i}', b_{2i+1}', c_{2i}' and c_{2i+1}'. Having observed this, let us see how we would modify the link of V in M_{i-1} to get the link in M_i. In M_{i-1} the link uses the edges c_{2i-1}' and c_{2i-2}'. In M_i these two edges do not meet at a vertex. This sequence of two edges is replaced by the sequence c_{2i-1}', a_{2i+1}', an edge from B_i to A_i, an edge from A_i to C_i, c_{2i+1}'', c_{2i}'', an edge from C_i to A_i, an edge from A_i to B_i, a_{2i}', c_{2i-2}'. The net effect is that one simple path in the link has been replaced by another simple path. Similarly, two consecutive edges in C_{i-1} will be replaced by a longer simple path. These changes do not change the fact that the link is a circuit.

We can determine the genus of M_i by examining the index of the saddle point V. If we take a plane very close to the xy-plane, then with respect to this general direction there is only one saddle point, V. There is one maximum (using the upward direction as positive) and one minimum. The index of V is $1-\frac{1}{2}(4i+2)=-2i$. By Banchoff's Theorem we conclude that the Euler characteristic of M_i is $2-2i$. the genus is therefore i.

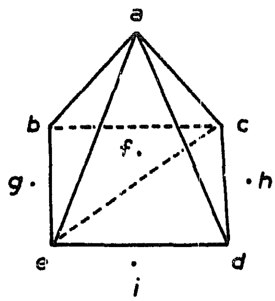

Fig. 7.

For manifolds of genus 1, we can do a little better. We will construct a torus with only five nonconvex vertices. The author conjectures that for the torus, this is the best possible.

Theorem 2. *There exists a toroidal polytope T with only five nonconvex vertices.*

Proof. Let a, b, c, d, and e be the vertices of a triangular bipyramid P as shown in Fig. 7. Let f be a point not in P but very close to the centroid of the face *aed*. Let X be the set consisting of the triangles *ebf*, *bfc*, *fcd*, *cde*, and *deb*.

To the set X we add the tetrahedra *abef*, *acbf*, and *acdf*. Next we cap the edges *be*, *cd*, and *de* (in that order) calling the three new vertices g, h, and i, respectively. The torus T is the boundary of this solid in E^3 (see Fig. 8).

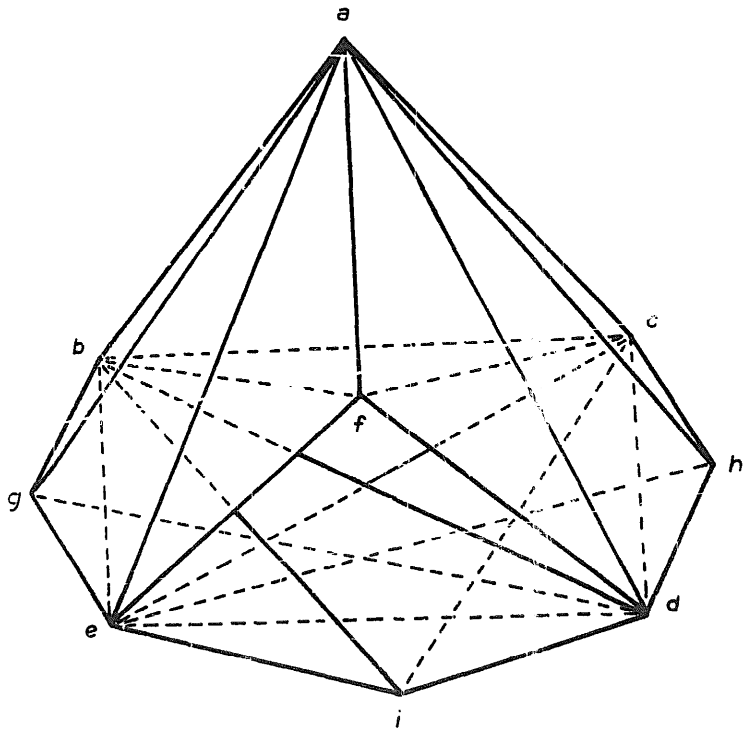

Fig. 8.

Table 1

Edges				Faces			Vertex v	link of v
ab	*bc*	*ce*	*dh*	*abc*	*aeg*	*cde*	a	*bc, ch, hd, df, fe, eg, gb*
bd	*bd*	*cf*	*di*	*abg*	*bcf*	*cdf*	b	*cf, fe, ei, id, dg, ga, ac*
ad	*be*	*ch*	*ef*	*ach*	*bdg*	*cei*	c	*de, ei, ih, ha, ab, bf, fd*
ae	*bf*	*ci*	*eg*	*adf*	*bdi*	*chi*	d	*eg, gb, bi, ih, ha, af, fc, ce*
af	*bg*	*de*	*ei*	*adh*	*bef*	*deg*	e	*fa, ag, gd, dc, ci, ib, bf*
ag	*bi*	*df*	*hi*	*aef*	*bei*	*dhi*	f	*ad, dc, cb, be, ea*
ah	*cd*	*dg*						

The vertex a is congruent to a vertex of P and is thus convex. Each of the vertices g, h, and i was added to the set when an edge belonging to two triangular faces was capped; thus, they are also convex. This leaves at most five vertices that can be nonconvex.

In Table 1 we list the edges and faces of T, and also the links of the vertices b, c, d, e, and f. The links of the other vertices are clearly simple circuits. Since T is a pseudomanifold (which can be checked from the table) in which all links are simple circuits, T is a manifold. It is a torus because its Euler characteristic is 0.

We now establish a lower bound on the minimum number of nonconvex vertices. In her thesis [3], J. Simutis showed that from a toroidal polytope with n nonconvex vertices, one can construct a simplicial (i.e., one with triangular faces) toroidal polytope with n nonconvex vertices. We remark here that her proof is independent of the genus of the manifold; thus we have

Lemma 2. *If there exists a polyhedral manifold of genus g, with exactly n nonconvex vertices, then there a simplicial polyhedral manifold of genus g with exactly n nonconvex vertices.*

Theorem 3. *Every polyhedral manifold of genus greater than or equal to 1 has at least four nonconvex vertices.*

Proof. We shall assume that the manifold is simplicial. Since it is simplicial we may move the vertices small distances without changing the convexity or nonconvexity of the vertices, thus we may also assume that the vertices are in general position.

We use another theorem of Simutis [3], that in a toroidal polytope (and by the same proof, in any polyhedral manifold) a saddle point must have a nonconvex vertex above and below it. If our manifold has at least two saddle points in some general direction, then there is a nonconvex vertex above the uppermost saddle point and a nonconvex vertex below the lowermost. Since saddle points are also nonconvex, we have at least four.

Suppose that in some general direction u there is only one saddle point, p. Since the vertices are in general position, for directions sufficiently close to u the index of p in that direction differs from the index in the direction u by at most 1. If p is the only saddle point in the direction u then by Banchoff's Theorem it has index at least -2, thus in directions sufficiently close to u, p will still be a saddle point.

Using a saddle point and two nonconvex vertices, one above and one below it, we have three nonconvex vertices, a, b, and c. Let S be a plane through a, b, and c, and let S' be a general plane near S, such that a is on S', and b and c are below it. If none of a, b or c is a saddle point with respect to S', then any saddle point in the manifold gives us a fourth nonconvex vertex. If one of the three vertices is the saddle point with respect to S', we choose a general plane S'' close to S', such that the saddle point with respect to S' is on S'' and the other two vertices are below it. Since the saddle point with respect to S' is still a saddle point with respect to S'', there is a nonconvex vertex above it. This gives our fourth nonconvex vertex.

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