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Deficient Bernstein Polynomials

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In [1] it was shown that the Bernstein polynomials of certain piecewise linear functions are deficient, in a sense soon to be made precise. The proof given there was highly computational and failed to illuminate the cause of the deficiency. In this note we give a much simplified proof, which also yields a fuller understanding of the phenomenon. We then apply the method used to obtain a partial converse of the theorem in [1].

The result referred to is as follows. Denote by P_n the set of algebraic polynomials of degree $\leqslant n$. For $f \in C[0, 1]$, the Bernstein polynomial of degree n of f is defined by $B_n(f; x) = B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$.

THEOREM A [1]. *Let f be a piecewise linear function having (possible) changes of slope only at the points i/m, $i = 1, 2, ..., m-1$. Then, for all $n \geqslant 1$, $B_{mn+1} \in P_{mn}$ and $B_{mn+1}(x) \equiv B_{mn}(x)$.*

Proof. We rely upon the following formula of Averbach (see [3, p. 306]):

$$\frac{B_n(x) - B_{n+1}(x)}{(1-x)^{n+1}} = \sum_{k=1}^{n} \left\{ \binom{n}{k} f\left(\frac{k}{n}\right) + \binom{n}{k-1} f\left(\frac{k-1}{n}\right) - \binom{n+1}{k} f\left(\frac{k}{n+1}\right) \right\} z^k, \tag{1}$$

where $z = x/(1-x)$. The term in brackets is equal to

$$\frac{n!}{k!\,(n-k)!} f\left(\frac{k}{n}\right) + \frac{n!}{(k-1)!\,(n+1-k)!} f\left(\frac{k-1}{n}\right) - \frac{(n+1)!}{k!\,(n+1-k)!} f\left(\frac{k}{n+1}\right)$$

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$$=\frac{n!}{(k-1)!\,(n-k)!}\left\{\frac{1}{k}f\left(\frac{k}{n}\right)+\frac{1}{n+1-k}f\left(\frac{k-1}{n}\right)\right.$$
$$\left.-\frac{n+1}{k(n+1-k)}f\left(\frac{k}{n+1}\right)\right\}. \tag{2}$$

On the other hand, the second-order divided difference of f based on the points $(k-1)/n$, $k/(n+1)$, and k/n satisfies

$$f\left[\frac{k-1}{n},\frac{k}{n+1},\frac{k}{n}\right]=n^2(n+1)\left\{\frac{1}{k}f\left(\frac{k}{n}\right)+\frac{1}{n+1-k}f\left(\frac{k-1}{n}\right)\right.$$
$$\left.-\frac{n+1}{k(n+1-k)}f\left(\frac{k}{n+1}\right)\right\}. \tag{3}$$

Comparing (2) and (3) we see that the coefficient of z^k in (1) is a positive multiple of $f[(k-1)/n, k/(n+1), k/n]$. Hence,

$$\frac{B_n(x)-B_{n+1}(x)}{(1-x)^{n+1}}=\sum_{k=1}^{n} b_{n,k}f\left[\frac{k-1}{n},\frac{k}{n+1},\frac{k}{n}\right]z^k. \tag{4}$$

Replacing n by mn we obtain

$$\frac{B_{mn}(x)-B_{mn+1}(x)}{(1-x)^{mn+1}}=\sum_{k=1}^{mn} b_{mn,k}f\left[\frac{k-1}{mn},\frac{k}{mn+1},\frac{k}{mn}\right]z^k. \tag{5}$$

Since f is *linear* in each of the intervals $[(k-1)/mn, k/mn]$, each of the divided differences in (5) is zero, so that $B_{mn+1}(x)\equiv B_{mn}(x)$, and the proof is complete.

In [4] it was conjectured that the converse of Theorem A is true; namely that the functions in that theorem are the only ones which satisfy $B_{mn+1}(x)\equiv B_{mn}(x)$, $n=1, 2, 3, \ldots$. We now make use of (5) to obtain partial confirmation of this conjecture.

THEOREM 1. *Let $f\in C[0,1]$ and suppose that $f\in C^2((i-1)/m, i/m)$, $i=1, 2, \ldots, m$. If $B_{mn+1}(x)\equiv B_{mn}(x)$, $n=1, 2, 3, \ldots$, then f is piecewise linear on $[0,1]$, with (possible) changes of slope only at the points i/m, $i=1, 2, \ldots, m-1$.*

Proof. If $B_{mn+1}(x)\equiv B_{mn}(x)$, then, by (5) we have

$$f\left[\frac{k-1}{mn},\frac{k}{mn+1},\frac{k}{mn}\right]=0,\quad k=1, 2, \ldots, mn;\ n=1, 2, 3, \ldots\quad (\text{since } b_{mn,k}>0).$$

Now $f \in C^2$ on each of the intervals $((i-1)/m, i/m)$, $i=1, 2, ..., m$. Hence, for any triple $((k-1)/mn, k/(mn+1), k/mn)$ lying in $((i-1)/m, i/m)$, we have $f[(k-1)/mn, k/(mn+1), k/mn] = f''(\theta)/2$ for some $\theta \in ((k-1)/mn, k/mn)$ (see [2, p. 249]). The triples of the form $((k-1)/mn, k/(mn+1), k/mn)$ are dense in $((i-1)/m, i/m)$. Thus, f'' has a dense set of zeros on $((i-1)/m, i/m)$. The continuity of f'' now yields $f'' \equiv 0$ on each such interval. As a result, f is linear on $((i-1)/m, i/m)$, $i=1, 2, ..., m$.

Another condition which can be used in place of $f \in C^2$ is convexity.

THEOREM 2. *If $f \in C[0, 1]$ is convex (or concave) on $((i-1)/m, i/m)$, $i=1, 2, ..., m$, and satisfies $B_{mn+1}(x) \equiv B_{mn}(x)$, $n=1, 2, 3, ...,$ then f is as in the conclusion of Theorem* 1.

Proof. Consider the triple $((i-1)/m, i/(m+1), i/m)$. Since $f[(i-1)/m, i/(m+1), i/m] = 0$, f is linear on these three points. But this, together with the convexity of f (or its concavity), guarantees that f is linear on $((i-1)/m, i/m)$, $i=1, 2, ..., m$.

We now show that the only piecewise linear functions which satisfy $B_{mn+1}(x) \equiv B_{mn}(x)$, $n=1, 2, 3, ...,$ are those of Theorem A.

THEOREM 3. *If f is piecewise linear on $[0, 1]$ and $B_{mn+1}(x) \equiv B_{mn}(x)$, $n=1, 2, 3, ...,$ then the knots of f can occur only at i/m, $i=1, 2, ..., m-1$.*

Proof. Suppose f has a knot at $x_0 \in ((i-1)/m, i/m)$. Then, for some $\varepsilon > 0$, f is linear in $(x_0 - \varepsilon, x_0)$ with slope s_1, and linear in $(x_0, x_0 + \varepsilon)$ with a different slope, s_2. Now there exists *some* triple $((j-1)/km, j/(km+1), j/km)$, with $(j-1)/km \in (x_0 - \varepsilon, x_0)$ and $j/km \in (x_0, x_0 + \varepsilon)$. We know that $f[(j-1)/km, j/(km+1), j/km]$ must equal 0. But, if $s_1 \neq s_2$, then this divided difference is not 0. Hence, there can be no knots in $((i-1)/m, i/m)$, $i=1, 2, ..., m$.

Remarks. 1. It is possible to weaken the hypothesis of Theorem 2 and merely require that f be *piecewise* convex on $((i-1)/m, i/m)$, $i=1, 2, ..., m$. Indeed, suppose f is convex (or concave) on $(a, b) \subset ((i-1)/m, i/m)$. By a modification of the proof of Theorem 2, we can show that f must be linear on (a, b). Hence, if f is piecewise convex on $[(i-1)/m, i/m]$, $i=1, 2, ..., m$, then f is actually *piecewise linear* on each of these intervals, and the result follows from Theorem 3.

2. Theorems 1–3 and extensive numerical calculations done with J. A. Roulier strengthen our belief that the full converse of Theorem A holds.

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