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Banach Function Spaces and Interpolation Methods
I. The Abstract Theory

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Interpolation methods are introduced which have specific application in the function space setting. The methods are indexed by $(\rho; j)$ or $(\rho; k)$, where ρ is a rearrangement-invariant norm and j and k are natural modifications of the J- and K-functionals of Peetre. Theorems of interpolation, equivalence, stability, and duality are established under simple restrictions on the indices of ρ. Applications are given (in Part II) to the interpolation of weak-type operators and, in particular, to the Hilbert transform and the conjugate operator. In part III, the ρ-methods are used to establish generalized Hausdorff–Young estimates for the Fourier transform.

1. INTRODUCTION

The various interpolation methods for pairs of Banach spaces have found widespread applications in areas such as partial differential equations, approximation theory, and harmonic analysis. However, while these methods have their origins in the classical interpolation theorems of Riesz-Thorin and Marcinkiewicz, they have had little systematic application elsewhere in the function space setting. Thus, the existing interpolation theorems for Banach function spaces (cf. [7, 9, 13, 14, 22]) were all established by means of function-theoretic techniques.

In this series of papers we introduce the ρ-methods of interpolation and give various applications, some of which are outlined below. The methods are indexed by $(\rho; j)$ or $(\rho; k)$, where ρ is a rearrangement-invariant norm on $(0, \infty)$, j is an indefinite integral of Peetre's J-functional, and k is the derivative of the K-functional. The resulting theory is less general than that of the Ψ-methods proposed by Peetre [19, 20] but nevertheless appears to be adequate for function space purposes. Moreover, there is a gain in precision in the sense that the rather awkward conditions imposed on Ψ in [19–21] reduce in our case to simple restrictions on the Boyd indices [7] of ρ.

Our main results are listed below. Compatible couples of Banach spaces are denoted by (X_1, X_2), (Y_1, Y_2), etc., and ρ is a rearrangement-invariant (r.i.) norm whose indices (β, α) are unrestricted unless otherwise stated. To avoid repetition, we refer to the text of the later sections for the other notations not described here.

THEOREM A. *The space $(X_1, X_2)_{\rho;k}$ is intermediate between X_1 and X_2:*

$$X_1 \cap X_2 \subseteq (X_1, X_2)_{\rho;k} \subseteq X_1 + X_2 .$$

The same inclusions hold for the space $(X_1, X_2)_{\rho;j}$.

THEOREM B. *The spaces $(X_1, X_2)_{\rho;k}$ and $(Y_1, Y_2)_{\rho;k}$ are interpolation spaces with respect to (X_1, X_2) and (Y_1, Y_2). The "convexity inequality" for the various operator norms is*

$$M \leqslant M_2 h(M_2/M_1),$$

where h is the indicator function of ρ.

There is a similar theorem for the $(\rho; j)$ spaces.

THEOREM C (Equivalence Theorem). *If the indices of ρ satisfy $0 < \beta \leqslant \alpha < 1$, then*

$$(X_1, X_2)_{\rho;j} = (X_1, X_2)_{\rho;k} ,$$

with equivalent norms.

THEOREM D (Stability Theorem). *Let (X_1, X_2) be a compatible couple, and let Y_i, $i = 1, 2$, be intermediate spaces of X_1 and X_2 of class $\mathscr{H}(\theta_i; X_1, X_2)$, where $0 \leqslant \theta_1 < \theta_2 \leqslant 1$.*

(a) *Suppose ν is a* r.i. *norm whose indices satisfy $0 < \beta_\nu \leqslant \alpha_\nu < 1$. Then there is a* r.i. *norm ρ whose indices are given by*

$$\beta_\rho = \beta_\nu(1 - \theta_1) + (1 - \beta_\nu)(1 - \theta_2); \qquad \alpha_\rho = \alpha_\nu(1 - \theta_1) + (1 - \alpha_\nu)(1 - \theta_2) \tag{1.1}$$

such that $(X_1, X_2)_{\rho;k} = (Y_1, Y_2)_{\nu;k}$, with equivalent norms.

(b) *Suppose ρ is a* r.i. *norm whose indices satisfy $1 - \theta_1 < \beta_\rho \leqslant \alpha_\rho < 1 - \theta_2$. Then there is a* r.i. *norm ν whose indices are given by* (1.1) *such that $(X_1, X_2)_{\rho;k} = (Y_1, Y_2)_{\nu;k}$, with equivalent norms.*

THEOREM E (Duality Theorem). *Suppose (X_1, X_2) is a conjugate couple, and let ρ be an absolutely continuous* r.i. *norm whose indices satisfy* $0 < \beta \leqslant \alpha < 1$. *Then*

$$(X_1, X_2)^*_{\rho;j} = (X_2^*, X_1^*)_{\rho';k} = (X_2^*, X_1^*)_{\rho';j} = (X_1, X_2)^*_{\rho;k},$$

with equivalent norms, where ρ' is the associate norm of ρ. If, in addition, X_1, X_2 and ρ are reflexive, then so are $(X_1, X_2)_{\rho;j}$ and $(X_1, X_2)_{\rho;k}$.

Calderón's theorem [9] for the pair (L^1, L^∞) follows directly from Theorem B.

THEOREM F (Calderón). *Let ρ be an arbitrary* r.i. *norm. If a linear operator T is bounded on L^1 (into itself) and on L^∞, then T is bounded on L^ρ.*

We remark that in the proofs of these theorems, only the most elementary properties of the indices are used, and nowhere do we need to draw on existing interpolation theorems. This is quite crucial of course since we intend using the ρ-methods to derive interpolation theorems for r.i. spaces.

In Part II of the paper (cf. [3]), we consider the interpolation of weak-type operators; we indicate also some partial results concerning strong-type interpolation (cf. [14]). A r.i. space L^ρ has the weak interpolation property with respect to L^p and L^q if every linear operator of weak-types (p, p) and (q, q) is bounded on L^ρ. A necessary and sufficient condition for this is that the indices (β, α) of L^ρ satisfy $q^{-1} < \beta \leqslant \alpha < p^{-1}$, at least when $1 \leqslant p < q < \infty$ and the underlying measure space is a subspace of the real line or the integers (cf. [7]). The case $q = \infty$ is exceptional in that the restriction on β is removed altogether (we might expect to have $\beta > 0$). The difficulty arises because there is no satisfactory notion of an operator of weak-type (∞, ∞). It is therefore conventionally agreed that weak- and strong-type (∞, ∞) shall mean the same thing.

However, in practice, one deals not with operators of a single weak-type but rather with operators that are simultaneously of two distinct weak-types. Given that point of view, we define in a natural way the class $W(p, \infty)$ of operators of simultaneous weak-types (p, p) and (∞, ∞). The corresponding classes $W(p, q)$, $1 \leqslant p < q < \infty$, correspond to the usual definitions. Closely related are the classes $AW(p, q)$ of operators of averaged weak-types (p, p) and (q, q). Boyd's theorem now reads: A necessary and sufficient condition that each operator of class $\Omega(p, q)$ (or $AW(p, q)$) be bounded on L^ρ is that $q^{-1} < \beta \leqslant \alpha < p^{-1}$, where $1 \leqslant p < q \leqslant \infty$.

Both the Hilbert transform (on the line) and the conjugate operator (on the circle) are of class $AW(1, \infty)$ so, by interpolation, they are bounded on every r.i. space L^ρ whose indices satisfy $0 < \beta \leqslant \alpha < 1$ (cf. [6, 10]).

In the third part of the paper [4], the ρ-methods are used to derive the generalized Hausdorff–Young theorem established by the author in [2] (we also remove the restriction that the indices be equal). A similar application leads to estimates for the Fourier coefficients of functions of class $L(\log^+ L)^p$, $p > 0$. The special case $p = 1$ reduces to the classical theorem of Hardy and Littlewood [11, Theorem 1].

2. Rearrangement-Invariant Norms

Let $\mathscr{M}^+$ (resp. $\mathscr{M}$) denote the class of nonnegative (resp. complex) Lebesgue measurable functions on the half-line $R^+ = (0, \infty)$, and denote the decreasing rearrangement of a function f in $\mathscr{M}^+$ or $\mathscr{M}$ by f^* (cf. [7, 9]). We shall write $\langle f, g\rangle$ for the inner product $\int_0^\infty f(t)\, g(t)\, dt$. The characteristic function of the interval $(0, t)$, where $t > 0$, will be denoted by χ_t, and when $t = 1$ we shall write χ for χ_1.

A rearrangement-invariant norm (r.i. norm) is a functional ρ: $\mathscr{M}^+ \to [0, \infty]$ which, for all $f, f_n, g \in \mathscr{M}^+$ and all scalars $t, \lambda > 0$, has the following properties (cf. [2, 7, 15, 16, 17]):

$$\rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}; \qquad \rho(\lambda f) = \lambda\rho(f);$$
$$\rho(f + g) \leqslant \rho(f) + \rho(g); \tag{2.1}$$

$$f \leqslant g \text{ a.e.} \Rightarrow \rho(f) \leqslant \rho(g) \quad \text{(monotonicity)}; \tag{2.2}$$

$$\rho(f) = \rho(f^*) \quad \text{(rearrangement-invariance)}; \tag{2.3}$$

$$\rho(\chi_t) < \infty; \qquad \langle f, \chi_t\rangle \leqslant A_t\rho(f), \tag{2.4}$$

where A_t is a constant depending on t but not on f;

$$f_n \uparrow f \text{ a.e.} \Rightarrow \rho(f_n) \uparrow \rho(f) \quad \text{(Fatou property)}. \tag{2.5}$$

The rearrangement-invariant space L^ρ consists of all (classes of) functions in $\mathscr{M}$ for which $\rho(|f|) < \infty$. It is a Banach space under the norm $\|f\| = \rho(|f|)$. We shall write $\rho(f)$ for $\rho(|f|)$.

The associate norm ρ' defined for $g \in \mathscr{M}^+$ by

$$\rho'(g) = \sup\{\langle f^*, g^*\rangle : \rho(f) \leqslant 1\} \tag{2.6}$$

is again a r.i. norm, and the second associate norm $\rho'' = (\rho')'$ coincides with ρ [15, 17]. Thus ρ has the representation

$$\rho(f) = \sup\{\langle f^*, g^*\rangle : \rho'(g) \leqslant 1\}, \tag{2.7}$$

from which the important Hölder inequality

$$|\langle f, g\rangle| \leqslant \langle |f|, |g|\rangle \leqslant \langle f^*, g^*\rangle \leqslant \rho(f)\,\rho'(g) \tag{2.8}$$

follows.

The smallest and the largest of the r.i. spaces are respectively $L^1 \cap L^\infty$ and $L^1 + L^\infty$, this in the sense that the continuous embeddings (i.e., the identity is continuous)

$$L^1 \cap L^\infty \subseteq L^\rho \subseteq L^1 + L^\infty \tag{2.9}$$

hold for any r.i. space L^ρ. Indeed, the norm on $L^1 + L^\infty$ is given by

$$\|f\|_{L^1+L^\infty} = \int_0^1 f^*(t)\,dt = \langle f^*, \chi\rangle, \quad f \in L^1 + L^\infty \tag{2.10}$$

(cf. [8, p. 184]). Hence, by Hölder's inequality (2.8),

$$\|f\|_{L^1+L^\infty} \leqslant \rho'(\chi)\,\rho(f), \quad f \in L^\rho. \tag{2.11}$$

Passing to the associate spaces and noting that $L^1 \cap L^\infty$ and $L^1 + L^\infty$ are mutually associate [18], we derive from (2.11) the inequality

$$\rho(f) \leqslant \rho(\chi)\,\|f\|_{L^1\cap L^\infty}, \qquad f \in L^1 \cap L^\infty. \tag{2.12}$$

But, by (2.4), both $\rho(\chi)$ and $\rho'(\chi)$ are finite so the inclusions (2.9) follow at once from (2.11) and (2.12).

A r.i. norm ρ is absolutely continuous if it has the property that $\rho(f_n) \downarrow 0$ whenever $\{f_n\}$ is a sequence of functions in L^ρ such that $f_n \downarrow 0$ a.e. [15, p. 14]. The associate space $L^{\rho'}$ is (isometrically isomorphic to) a closed subspace of the dual space $(L^\rho)^*$ of L^ρ, with equality when ρ is absolutely continuous.

THEOREM 2.1 [15, Theorem 3]. *The associate space $L^{\rho'}$ is isometrically isomorphic to the dual space $(L^\rho)^*$ if and only if ρ is absolutely continuous. In particular, L^ρ is reflexive if and only if both ρ and ρ' are absolutely continuous.*

An important property of r.i. spaces, and one that we shall use frequently, is as follows.

THEOREM 2.2 [16, Theorem 11.7]. *Let* ρ *be a* r.i. *norm. If, for all* $t > 0$,

$$\int_0^t f^*(s)\, ds \leqslant \int_0^t g^*(s)\, ds,$$

then $\rho(f) \leqslant \rho(g)$.

If f^{**} denotes the maximal average

$$f^{**}(t) = t^{-1} \int_0^t f^*(s)\, ds, \qquad 0 < t < \infty,$$

then Theorem 2.2 asserts that

$$f^{**} \leqslant g^{**} \Rightarrow \rho(f) \leqslant \rho(g), \tag{2.13}$$

which can be regarded as a generalization of the monotonicity property (2.2).

For each $s > 0$, the dilation operator E_s is defined on $L^1 + L^\infty$ by

$$(E_s f)(t) = f(st), \qquad 0 < t < \infty. \tag{2.14}$$

THEOREM 2.3. *Let* ρ *be a* r.i. *norm, and let* $s > 0$. *Then* E_s *is a bounded linear operator on* L^ρ *into itself with norm* $h(s) \equiv h_\rho(s) \leqslant$ $\max(1, s^{-1})$.

Proof. If $s > 1$, then $(E_s f)^* = E_s f^* \leqslant f^*$. Hence, by the monotonicity property (2.2), we have $\rho(E_s f) \leqslant \rho(f)$, i.e., $h(s) \leqslant 1$.

On the other hand, when $s > 1$, we have $t = s^{-1} < 1$ and $\langle E_s f^*, g^* \rangle = t \langle f^*, E_t g^* \rangle \leqslant s^{-1} \langle f^*, g^* \rangle$. Hence, from (2.7), $\rho(E_s f) \leqslant s^{-1} \rho(f)$, i.e., $h(s) \leqslant s^{-1}$.

The function $s \to h_\rho(s)$ is called the indicator function of ρ. Clearly

$$\rho(E_s f) \leqslant h_\rho(s)\, \rho(f), \qquad f \in L^\rho, \quad s > 0. \tag{2.15}$$

3. INTERSECTION, SUM, AND RELATIVE COMPLETION

A pair (X_1, X_2) of Banach spaces is called a compatible couple if there is a Hausdorff topological vector space $\mathscr{X}$ and continuous embeddings $X_1 \subseteq \mathscr{X}$, $X_2 \subseteq \mathscr{X}$. In this case, both the intersection $X_1 \cap X_2$ and the sum $X_1 + X_2$ are Banach spaces under the respective norms

$$\|f\|_{X_1 \cap X_2} = \max(\|f\|_1, \|f\|_2), \qquad f \in X_1 \cap X_2, \tag{3.1}$$

and

$$\|f\|_{X_1 + X_2} = \inf_{f = f_1 + f_2} (\|f_1\|_1 + \|f_2\|_2), \qquad f \in X_1 + X_2 \tag{3.2}$$

(cf. [8, p. 165]).

A Banach space X is intermediate between X_1 and X_2 if $X_1 \cap X_2 \subseteq X \subseteq X_1 + X_2$. For any such intermediate space X, we denote by X^0 the closure in X of $X_1 \cap X_2$, and by $\bar{X}$ the closure of X in $X_1 + X_2$. Thus $X^0 \subseteq X \subseteq \bar{X}$. The following elementary identities were noted by Berens [5]:

$$\bar{X}_1 = X_1 + X_2^0; \qquad \bar{X}_2 = X_1^0 + X_2;$$
$$\bar{X}_1 \cap \bar{X}_2 = X_1^0 + X_2^0 = \overline{X_1 \cap X_2}. \tag{3.3}$$

A compatible couple (X_1, X_2) in which $X_1^0 = X_1$ and $X_2^0 = X_2$ will be called a conjugate couple. In this case, $X_1 \cap X_2$ is dense in both X_1 and X_2 so the dual spaces X_1^* and X_2^* can be embedded in a canonical way in the dual of $X_1 \cap X_2$. In particular, (X_1^*, X_2^*) is again a compatible couple.

THEOREM 3.1 [1, Theorem 8.III]. *Let (X_1, X_2) be a conjugate couple. Then*

$$(X_1 \cap X_2)^* \cong X_1^* + X_2^*; \qquad (X_1 + X_2)^* \cong X_1^* \cap X_2^*,$$

where "$\cong$" denotes "isometrically isomorphic."

Let X be an intermediate space between X_1 and X_2, and let $B_X(R) = \{f \in X: \|f\|_X \leqslant R\}$ be the closed ball in X of radius $R > 0$. The relative completion of X in $X_1 + X_2$ consists of the union $\tilde{X} = \bigcup_{R>0} \bar{B}_X(R)$, where $\bar{B}_X(R)$ is the closure of $B_X(R)$ in $X_1 + X_2$. It is a Banach space under the norm $\|f\|_{\tilde{X}} = \inf\{R > 0: f \in \bar{B}_X(R)\}$ [1, Sect. 10].

Note that an element $f \in X_1 + X_2$ belongs to $\tilde{X}$ if and only if it is the limit in $X_1 + X_2$ of a sequence of elements $f_n \in X$ for which $\sup_n \|f_n\|_X < \infty$.

We shall need the identity [1, Theorem 10. VIII]:

$$(X_1 \cap X_2)^\sim = \tilde{X}_1 \cap \tilde{X}_2. \tag{3.4}$$

A compatible couple (X_1, X_2) for which $\tilde{X}_1 = X_1$ and $\tilde{X}_2 = X_2$ will be called relatively complete. This is the case when, for instance, both X_1 and X_2 are reflexive [1, Corollary 10. VI].

4. THE $(\rho; j)$ INTERPOLATION METHOD

When (X_1, X_2) is a compatible couple, the J-functional of Peetre is defined on $X_1 \cap X_2$ for each $t > 0$ by

$$J(t; f) \equiv J(t; f; X_1, X_2) = \max(\|f\|_1, t\|f\|_2), \qquad f \in X_1 \cap X_2 \tag{4.1}$$

(cf. [8, p. 166]). Note that for any $f \in X_1 \cap X_2$,

$$J(t; f; X_2, X_1) = t J(t^{-1}; f; X_1, X_2). \tag{4.2}$$

It is clear that for each fixed $f \in X_1 \cap X_2$, $J(t;f)$ is a continuous, piecewise linear, and convex function of $t > 0$. Comparing (3.1) and (4.1), we see that $J(1;f) = \|f\|_{X_1 \cap X_2}$; in fact, for each $t > 0$,

$$\min(1, t) \|f\|_{X_1 \cap X_2} \leqslant J(t; f) \leqslant \max(1, t) \|f\|_{X_1 \cap X_2}. \tag{4.3}$$

Thus $\{J(t; \cdot): 0 < t < \infty\}$ is a family of equivalent norms on $X_1 \cap X_2$. For any $f \in X_1 \cap X_2$, one derives from (3.2) the elementary inequality

$$\|f\|_{X_1 + X_2} \leqslant \min(1, t^{-1}) J(t; f). \tag{4.4}$$

Suppose the element $f \in X_1 + X_2$ is representable in the form

$$f = \int_0^\infty u(t)\, dt/t, \tag{4.5}$$

where $u = u(t)$: $(0, \infty) \to X_1 \cap X_2$ is strongly measurable and the integral converges in $X_1 + X_2$, i.e., $\int_0^\infty \| u(t)\|_{X_1+X_2}\, dt/t$ is finite (cf. [8, pp. 166–169]). For such a representation u of f, define the j-functional $j(s; u)$ (finite or infinite) by

$$j(s; u) = \int_s^\infty t^{-1} J(t; u(t))\, dt/t, \qquad 0 < s < \infty. \tag{4.6}$$

When ρ is a r.i. norm, the space $(X_1, X_2)_{\rho;j}$ consists of all elements $f \in X_1 + X_2$ that have a representation u of the form (4.5) for which $\rho(j(s; u)) \equiv \rho(j(\cdot; u)) < \infty$. It is a Banach space for the norm

$$\|f\|_{\rho;j} = \inf\{\rho(j(s; u))\}, \tag{4.7}$$

where the infimum is taken over all representations u of f.

THEOREM 4.1. *Let (X_1, X_2) be a compatible couple. Then for any* r.i. *norm ρ, the space $(X_1, X_2)_{\rho;j}$ is intermediate between X_1 and X_2.*

Proof. We show first that $X_1 \cap X_2 \subseteq (X_1, X_2)_{\rho;j}$. For each $n = 1, 2, \ldots$, let $\varphi_n(t)$ be a continuous function with support in $[a_n, b_n]$, where $a_n \uparrow 1$, $b_n \downarrow 1$ as $n \to \infty$, such that $\int_0^\infty \varphi_n(t)\, dt/t = 1$. For each $f \in X_1 \cap X_2$, set $u_n(t) = f\varphi_n(t)$. Then u_n represents f and $J(t; u_n(t)) = \varphi_n(t) J(t; f)$. Hence $j(s; u_n) = 0$ if $s > b_n$ and for all s,

$$j(s; u_n) = \int_s^\infty t^{-1} J(t; u_n(t))\, dt/t \leqslant a_n^{-1} J(a_n; f).$$

Thus $j(s; u) \leqslant \chi_{b_n}(s)\, a_n^{-1} J(a_n; f)$ so using (4.7) we have $\|f\|_{\rho;j} \leqslant \rho(\chi_{b_n})\, a_n^{-1} J(a_n; f)$. Letting $n \to \infty$, we obtain

$$\|f\|_{\rho;j} \leqslant \rho(\chi)\, \|f\|_{X_1 \cap X_2}, \qquad f \in X_1 \cap X_2. \tag{4.8}$$

But $\rho(\chi) < \infty$ by (2.4) so this establishes the continuous embedding $X_1 \cap X_2 \subseteq (X_1, X_2)_{\rho;j}$.

To see that $(X_1, X_2)_{\rho;j} \subseteq X_1 + X_2$, let $f \in (X_1, X_2)_{\rho;j}$ and let u be any representation of f. From (4.4) we have

$$\|f\|_{X_1+X_2} \leqslant \int_0^\infty \|u(t)\|_{X_1+X_2}\, dt/t \leqslant \int_0^\infty \min(1, t^{-1})\, t^{-1} J(t; u(t))\, dt,$$

and a computation shows that the last integral is just $\int_0^1 j(s; u)\, ds$. Hence, by Hölder's inequality (2.8), $\|f\|_{X_1+X_2} \leqslant \rho'(\chi)\, \rho(j(s; u))$, so passing to the infimum we have

$$\|f\|_{X_1+X_2} \leqslant \rho'(\chi)\, \|f\|_{\rho;j}, \qquad f \in (X_1, X_2)_{\rho;j}. \tag{4.9}$$

Once again $\rho'(\chi)$ is finite by (2.4) and the inclusion $(X_1, X_2)_{\rho;j} \subseteq X_1 + X_2$ follows.

Before turning to the interpolation theorem for the $(\rho; j)$ methods we need some terminology [8, p. 179]. Let $(X_1, X_2) \subseteq \mathscr{X}$ and $(Y_1, Y_2) \subseteq \mathscr{Y}$ be compatible couples and let X be intermediate for the first couple, Y for the second. Denote by $\mathscr{B}(X_1, X_2; Y_1, Y_2)$ the class of linear operators T: $X_1 + X_2 \to Y_1 + Y_2$ whose restriction to X_i is bounded from X_i to Y_i with norm M_i, $i = 1, 2$. We say that X and Y are interpolation spaces (with respect to (X_1, X_2) and (Y_1, Y_2)) if each operator $T \in \mathscr{B}(X_1, X_2; Y_1, Y_2)$ restricts to a bounded linear operator on X into Y (we denote its norm by M).

THEOREM 4.2 (The $(\rho; j)$ Interpolation Theorem). *Let (X_1, X_2) and (Y_1, Y_2) be two compatible couples and let ρ be an arbitrary* r.i. *norm. Then $(X_1, X_2)_{\rho;j}$ and $(Y_1, Y_2)_{\rho;j}$ are interpolation spaces with respect to (X_1, X_2) and (Y_1, Y_2). Furthermore, if $T \in \mathscr{B}(X_1, X_2; Y_1, Y_2)$, then*

$$M \leqslant M_2 h(M_2/M_1), \tag{4.10}$$

where h is the indicator function of ρ.

Proof. Let $f \in (X_1, X_2)_{\rho;j}$, and let u be any representation of f. If $T \in \mathscr{B}(X_1, X_2; Y_1, Y_2)$, then $Tf = \int_0^\infty (Tu)(t)\, dt/t$, where $(Tu)(t) = T(u(t)) \in Y_1 \cap Y_2$. Thus

$$\|Tf\|_{\rho;j} \leqslant \rho(j(s; Tu)) = \rho\left(\int_s^\infty t^{-1} J(t; (Tu)(t))\, dt/t\right). \tag{4.11}$$

But $J(t; (Tu)(t)) \leqslant M_1 J(\lambda t; u(t)) = M_1 J(\lambda t; v(\lambda t))$, where $\lambda = M_2/M_1$ and $v(t) = u(\lambda^{-1}t)$. Hence

$$\int_s^\infty t^{-1} J(t; (Tu)(t))\, dt/t \leqslant M_1 \int_s^\infty t^{-1} J(\lambda t; v(\lambda t))\, dt/t = M_2 j(\lambda s; v).$$

It follows from (4.11) and (2.15) that

$$\| Tf \|_{\rho;j} \leqslant M_2 \rho(j(\lambda s; v)) \leqslant M_2 h(\lambda)\, \rho(j(s; v)).$$

Observing that $\int_0^\infty u(t)\, dt/t = \int_0^\infty v(t)\, dt/t = f$, we can take the infimum over v to obtain $\|Tf\|_{\rho;j} \leqslant M_2 h(M_2/M_1) \| f \|_{\rho;j}$, and from this the "convexity inequality" (4.10) follows.

In order to set up the duality theory (Sect. 11), it is essential to have $X_1 \cap X_2$ dense in $(X_1, X_2)_{\rho;j}$ (cf. the discussion in Sect. 3). The next theorem shows that this is the case whenever ρ is absolutely continuous.

THEOREM 4.3 (The Density Theorem). *Let (X_1, X_2) be a compatible couple, and let ρ be a* r.i. *norm. If ρ is absolutely continuous, then $X_1 \cap X_2$ is dense in $(X_1, X_2)_{\rho;j}$.*

Proof. Let $f \in (X_1, X_2)_{\rho;j}$. As in [8, Proposition 3.2.8], there is a strongly continuous function $v = v(t)$ on $(0, \infty)$ into $X_1 \cap X_2$ which represents f and satisfies $\rho(j(s; v)) < \infty$. For each ϵ, $0 < \epsilon < 1$, set $f_\epsilon = \int_0^\infty v_\epsilon(t)\, dt/t = \int_\epsilon^{1/\epsilon} v(t)\, dt/t$. The strong continuity of v ensures that $f_\epsilon \in X_1 \cap X_2$, for each ϵ. Moreover, $f - f_\epsilon = \int_0^\infty (v(t) - v_\epsilon(t))\, dt/t$ so

$$\| f - f_\epsilon \|_{\rho;j} \leqslant \rho(j(s; v - v_\epsilon)) \leqslant \rho(j(s; v)) < \infty. \tag{4.12}$$

Thus each of the functions $s \to j(s; v - v_\epsilon)$, $0 < \epsilon < 1$, belongs to L^ρ and they decrease monotonically to 0 as $\epsilon \to 0$. Since ρ is absolutely continuous, we find that $\rho(j(s; v - v_\epsilon)) \to 0$ as $\epsilon \to 0$. It follows from (4.12) that $f_\epsilon \to f$ in $(X_1, X_2)_{\rho;j}$, as $\epsilon \to 0$. This completes the proof.

5. THE $(\rho; k)$ INTERPOLATION METHOD

When (X_1, X_2) is a compatible couple, the K-functional of Peetre is defined on $X_1 + X_2$ for each $t > 0$ by

$$K(t; f) \equiv K(t; f; X_1, X_2) = \inf_{f = f_1 + f_2} (\| f_1 \|_1 + t \| f_2 \|_2), \quad f \in X_1 + X_2 \tag{5.1}$$

(cf. [8, p. 167]). Note that

$$K(t; f; X_2; X_1) = tK(t^{-1}; f; X_1, X_2), \qquad f \in X_1 + X_2. \tag{5.2}$$

Comparing (5.1) and (3.2), we see that $K(1;f) = \|f\|_{X_1+X_2}$, and in fact for each $t > 0$,

$$\min(1, t)\|f\|_{X_1+X_2} \leqslant K(t;f) \leqslant \max(1, t)\|f\|_{X_1+X_2}, \qquad f \in X_1 + X_2. \quad (5.3)$$

Thus $\{K(t; \cdot): 0 < t < \infty\}$ is a family of equivalent norms on $X_1 + X_2$. Generalizing (4.4) we have

$$K(t;f) \leqslant \min(1, t/s)\, J(s;f), \qquad f \in X_1 \cap X_2. \quad (5.4)$$

For each $f \in X_1 + X_2$, $K(t; f)$ is a continuous, increasing, and concave function of $t > 0$ [8, p. 167] whereas $t^{-1}K(t;f)$ is continuous and decreasing. It follows that $K(t;f)$ has a unique integral representation of the form

$$K(t;f) - K(0+;f) = \int_0^t k(s;f)\, ds, \qquad f \in X_1 + X_2, \quad (5.5)$$

where $k(s;f) = k(s;f; X_1, X_2)$ is nonnegative, right continuous, and decreasing for $s > 0$. We shall restrict attention to those f in $X_1 + X_2$ for which $K(0+;f) \equiv \lim_{t\to 0} K(t;f) = 0$. As the next theorem shows, this is not always the case.

THEOREM 5.1 [5, p. 8]. *If $f \in X_1 + X_2$, then*

(a) $\lim_{t\to 0} K(t;f) = 0$ *if and only if* $f \in X_1{}^0 + X_2$;

(b) $\lim_{t\to\infty} t^{-1}K(t;f) = 0$ *if and only if* $f \in X_1 + X_2{}^0$;

(c) $\lim_{t\to 0} K(t;f) = \lim_{t\to\infty} t^{-1}K(t;f) = 0$ *if and only if* $f \in X_1{}^0 + X_2{}^0$.

Thus, using part (a) of the theorem, we have from (5.5)

$$K(t;f) = \int_0^t k(s;f)\, ds, \qquad f \in X_1{}^0 + X_2. \quad (5.6)$$

When (X_1, X_2) is a compatible couple and ρ a r.i. norm, we denote by $(X_1, X_2)_{\rho;k}$ the space of elements $f \in X_1{}^0 + X_2$ for which

$$\|f\|_{\rho;k} \equiv \rho(k(t;f)) \quad (5.7)$$

is finite. It is a Banach space under the norm given by (5.7).

Recall that χ is the characteristic function of the interval (0, 1) so its decreasing rearrangement χ^* is just χ. Furthermore, the average χ^{**} is given by $\chi^{**}(t) = \min(1, t^{-1})$, $0 < t < \infty$.

THEOREM 5.2. *Let (X_1, X_2) be a compatible couple. Then for each* r.i. *norm ρ, the space $(X_1, X_2)_{\rho;k}$ is intermediate between $X_1{}^0$ and X_2 (hence intermediate between X_1 and X_2).*

Proof. For any $f \in (X_1, X_2)_{\rho;k}$, we have from (5.6)

$$\|f\|_{X_1+X_2} = K(1;f) = \int_0^1 k(s;f)\, ds = \langle k(s;f), \chi(s)\rangle.$$

Hence, by Hölder's inequality (2.8)

$$\|f\|_{X_1+X_2} \leqslant \rho'(\chi)\, \rho(k(s;f)) = \rho'(\chi)\, \|f\|_{\rho;k}, \qquad f \in (X_1, X_2)_{\rho;k}, \quad (5.8)$$

and so $(X_1, X_2)_{\rho;k} \subseteq X_1{}^0 + X_2$.

On the other hand, any f belonging to $X_1 \cap X_2$ lies in $X_1{}^0 + X_2$ and hence satisfies (5.6). From (2.12) we have

$$\|f\|_{\rho;k} = \rho(k(t;f)) \leqslant \rho(\chi) \max\left\{\int_0^\infty k(t;f)\, dt, \sup_t k(t;f)\right\}.$$

But $\int_0^\infty k(t;f)\, dt = \lim_{t\to\infty} K(t;f) \leqslant \|f\|_1$ by (5.6); moreover, since $k(t;f)$ is decreasing, $\sup_t k(t;f) = \lim_{t\to 0} t^{-1}K(t;f) \leqslant \|f\|_2$. Hence

$$\|f\|_{\rho;k} \leqslant \rho(\chi) \max\{\|f\|_1, \|f\|_2\} = \rho(\chi)\, \|f\|_{X_1\cap X_2}, \qquad f \in X_1 \cap X_2, \quad (5.9)$$

which shows that $X_1 \cap X_2 \subseteq (X_1, X_2)_{\rho;k}$.

It is not immediately obvious that $\|\cdot\|_{\rho;k}$ satisfies the triangle inequality. However, if $f, g \in (X_1, X_2)_{\rho;k}$, then by the triangle inequality for $K(t;\cdot)$ we have

$$\int_0^t k(s;f+g)\, ds \leqslant \int_0^t \{k(s;f) + k(s;g)\}\, ds, \qquad 0 < t < \infty.$$

It follows from Theorem 2.2 that $\rho(k(s;f+g)) \leqslant \rho(k(s;f) + k(s;g))$, so using the triangle inequality for ρ we have finally $\|f+g\|_{\rho;k} \leqslant \|f\|_{\rho;k} + \|g\|_{\rho;k}$.

THEOREM 5.3 (The $(\rho; k)$ Interpolation Theorem). *Let (X_1, X_2) and (Y_1, Y_2) be two compatible couples, and let ρ be a* r.i. *norm. Then $(X_1, X_2)_{\rho;k}$ and $(Y_1, Y_2)_{\rho;k}$ are interpolation spaces with respect to (X_1, X_2) and (Y_1, Y_2). Furthermore, if $T \in \mathscr{B}(X_1, X_2; Y_1, Y_2)$, then*

$$M \leqslant M_2 h(M_2/M_1),$$

where h is the indicator function of ρ.

Proof. We note first that if $f \in X_1^0 + X_2$, then $Tf \in Y_1^0 + Y_2$. Hence, for any $f \in (X_1, X_2)_{\rho;k} \subseteq X_1^0 + X_2$, an identity of the form (5.6) holds for both f and Tf. In that case, the inequality $K(t; Tf) \leqslant M_1 K(\lambda t; f)$ (cf. [8, p. 180]), where $\lambda = M_2/M_1$, reduces to

$$\int_0^t k(s; Tf)\, ds \leqslant \int_0^t M_2 k(\lambda s; f)\, ds, \qquad 0 < t < \infty.$$

Hence, by Theorem 2.2, $\rho(k(s; Tf)) \leqslant \rho(M_2 k(\lambda t; f))$, so using (2.15) we find that

$$\| Tf \|_{\rho;k} = \rho(k(s; Tf)) \leqslant M_2 h(\lambda)\, \rho(k(t; f)) = M_2 h(\lambda)\, \| f \|_{\rho;k} .$$

This completes the proof.

6. Calderón's Theorem

We consider the Lebesgue spaces L^1 and L^∞ over a σ-finite measure space of the kind considered in [7, 9], i.e., nonatomic with finite or infinite measure, or completely atomic with atoms of equal positive measure. If L^{ρ_0} is a r.i. space on $\mathscr{M}$, then there is a r.i. norm ρ on $(0, \infty)$ such that $\rho_0(f) = \rho(f^*)$ (cf. [7, 16]). Moreover, for such measure spaces, the K-functional is given by [8, p. 184]

$$K(t; f; L^1, L^\infty) = \int_0^t f^*(s)\, ds, \qquad 0 < t < \infty, \tag{6.1}$$

so $k(t; f)$ is just $f^*(t)$. The norm on $(L^1, L^\infty)_{\rho;k}$ is therefore

$$\| f \|_{\rho;k} = \rho(k(t; f)) = \rho(f^*) = \rho_0(f).$$

Hence

$$(L^1, L^\infty)_{\rho;k} = L^{\rho_0}. \tag{6.2}$$

It follows from (6.2) and the interpolation theorem (Theorem 5.3) that any operator $T \in \mathscr{B}(L^1, L^\infty; L^1, L^\infty)$ is bounded on L^{ρ_0}, which is precisely Calderón's theorem (Theorem F).

7. Indices of r.i. Spaces

We interrupt our exposition of the abstract theory at this point so as to include a brief discussion of the Boyd indices of r.i. spaces.

In Section 2 we saw that the indicator function h of a r.i. norm ρ is finite everywhere on $(0, \infty)$. Moreover, the identity $E_{st} = E_s E_t$ gives rise to the inequality

$$h(st) \leqslant h(s)\, h(t), \qquad 0 < s, t < \infty, \tag{7.1}$$

i.e., h is submultiplicative on $(0, \infty)$.

Boyd [7] defined the indices of ρ by

$$\alpha \equiv \alpha_\rho = \inf_{0<s<1} \frac{-\log h(s)}{\log s}; \qquad \beta \equiv \beta_\rho = \sup_{1<s<\infty} \frac{-\log h(s)}{\log s}, \tag{7.2}$$

and showed that, as a consequence of the property (7.1),

$$\alpha = \lim_{s\to 0} \frac{-\log h(s)}{\log s}; \qquad \beta = \lim_{s\to\infty} \frac{-\log h(s)}{\log s}. \tag{7.3}$$

The indices satisfy

$$0 \leqslant \beta \leqslant \alpha \leqslant 1, \tag{7.4}$$

and the indices α', β' of the associate norm ρ' are given by

$$\alpha' = 1 - \beta; \qquad \beta' = 1 - \alpha. \tag{7.5}$$

We shall make frequent use of the next two lemmas. They combine a number of results to be found in [6, 7] whose elementary proofs we omit.

LEMMA 7.1 (Boyd). *Let ρ be a* r.i. *norm with indicator function h and upper index α. The following statements are equivalent.*

(i) $\alpha < 1$;
(ii) $\int_0^1 h(s)\, ds < \infty$;
(iii) $sh(s) \to 0$, *as* $s \to 0$;
(iv) $sh(s) < 1$, *for some* $s < 1$;
(v) $h(s) \leqslant Ks^{-\gamma}$, *for all* $s < 1$, *where* K *and* γ *are positive constants and* $\gamma < 1$.

LEMMA 7.2 (Boyd). *Let ρ be a* r.i. *norm with indicator function h and lower index β. The following statements are equivalent.*

(i) $\beta > 0$;
(ii) $\int_1^\infty h(s)\, ds/s < \infty$;
(iii) $h(s) \to 0$, *as* $s \to \infty$;
(iv) $h(s) < 1$, *for some* $s > 1$;
(v) $h(s) \leqslant Ks^{-\gamma}$, *for all* $s > 1$, *where* K *and* γ *are positive constants and* $\gamma < 1$.

In the next lemma, χ is, as usual, the characteristic function of (0, 1) so $\chi^{**}(t) = \min(1, t^{-1})$.

LEMMA 7.3. *Let* ρ *be a* r.i. *norm with upper index* $\alpha < 1$. *Then* $\rho(\chi^{**}) < \infty$.

Proof. The lemma is an immediate consequence of the theorems of Shimogaki [22] or Boyd [7] but we prefer to give a direct proof. We write

$$\chi^{**}(t) = t^{-1} \int_0^t \chi^*(s)\, ds = \int_0^1 \chi(st)\, ds = \int_0^1 (E_s\chi)(t)\, ds,$$

so by (2.15)

$$\rho(\chi^{**}) \leqslant \int_0^1 \rho(E_s\chi)\, ds \leqslant \rho(\chi) \left(\int_0^1 h(s)\, ds\right).$$

But $\alpha < 1$, so Lemma 7.1 shows that the integral is finite, and $\rho(\chi)$ is finite because of (2.4). Hence $\rho(\chi^{**}) < \infty$.

Note that the converse is false: if $L^\rho = L^1 + L^\infty$, then $\rho(\chi^{**}) < \infty$ but $\alpha = 1$ (and $\beta = 0$).

8. THE $(\rho; J)$ AND $(\rho; K)$ METHODS

We next give a brief description of the $(\rho; J)$ and $(\rho; K)$ methods which are more closely related to the $(\theta, q; J)$ and $(\theta, q; K)$ methods of Peetre [8, Chap. 3]. Some additional assumptions on ρ are required but then the theory is quite similar to that already developed for the $(\rho; j)$ and $(\rho; k)$ methods. We shall therefore keep the proofs to a minimum.

As before, (X_1, X_2) denotes a compatible couple and ρ is a r.i. norm on $(0, \infty)$. An element $f \in X_1 + X_2$ belongs to the space $(X_1, X_2)_{\rho;J}$ if it has a representation $f = \int_0^\infty u(t)\, dt/t$ of the form (4.5) for which $\rho(t^{-1}J(t; u(t))) < \infty$. For each $f \in (X_1, X_2)_{\rho;J}$, we set

$$\|f\|_{\rho;J} = \inf \left\{\rho(t^{-1}J(t; u(t))) : f = \int_0^\infty u(t)\, dt/t\right\}. \tag{8.1}$$

THEOREM 8.1. *The space* $(X_1, X_2)_{\rho;J}$ *is a Banach space under the norm* (8.1) *in which* $X_1 \cap X_2$ *is continuously embedded. If* ρ *also satisfies*

$$\rho'(\chi^{**}) < \infty, \tag{8.2}$$

then $(X_1, X_2)_{\rho;J}$ *is intermediate between* X_1 *and* X_2. *In this case, the interpolation theorem* (*Theorem* 4.2) *holds with* $(\rho; j)$ *replaced by* $(\rho; J)$.

COROLLARY 8.2. *If $\beta > 0$, then $(X_1, X_2)_{\rho;J}$ is intermediate between X_1 and X_2, and the interpolation theorem holds.*

Proof. If $\beta > 0$, then by (7.5), $\alpha' < 1$. Hence, by Lemma 7.3, $\rho'(\chi^{**}) < \infty$, and the statement follows from Theorem 8.1.

THEOREM 8.3. *If $\beta > 0$, then $(X_1, X_2)_{\rho;J} \subseteq (X_1, X_2)_{\rho;j}$.*

The space $(X_1, X_2)_{\theta,q;J}$ of Peetre [8, 19, 20] is defined in much the same way but by means of the norm

$$\|f\|_{\theta,q;J} = \inf\left\{\int_0^\infty (t^{-\theta}J(t; u(t)))^q \, dt/t\right\}^{1/q}. \tag{8.3}$$

It is intermediate between X_1 and X_2 if $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, and in the extreme cases $q = 1$, $\theta = 0$ or 1; the interpolation theorem holds for $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$ [8, Chap. 3].

Berens [5] has characterized the extreme spaces as follows:

$$(X_1, X_2)_{0,1;J} = X_1^0; \qquad (X_1, X_2)_{1,1;J} = X_2^0. \tag{8.4}$$

These characterizations enable us to identify the space $(X_1, X_2)_{\rho;j}$ for the norms ρ^1, ρ^∞, $\rho^{1\cap\infty}$, and $\rho^{1+\infty}$ which are the respective norms on the spaces L^1, L^∞, $L^1 \cap L^\infty$, and $L^1 + L^\infty$.

THEOREM 8.4. *Let (X_1, X_2) be a compatible couple.*

(a) *If $\rho = \rho^1$, then $(X_1, X_2)_{\rho;j} = X_1^0$;*
(b) *if $\rho = \rho^\infty$, then $(X_1, X_2)_{\rho;j} = X_2^0$;*
(c) *if $\rho = \rho^{1\cap\infty}$, then $(X_1, X_2)_{\rho;j} = X_1^0 \cap X_2^0 = X_1 \cap X_2$;*
(d) *if $\rho = \rho^{1+\infty}$, then $(X_1, X_2)_{\rho;j} = X_1^0 + X_2^0 = \overline{X_1 \cap X_2}$.*

Proof. We prove only part (a). When $\rho = \rho^1$ and $f \in (X_1, X_2)_{\rho;j}$ with representation $f = \int_0^\infty u(t)\, dt/t$, we have

$$\rho(j(s; u)) = \int_0^\infty ds \int_s^\infty t^{-1}J(t; u(t))\, dt/t = \int_0^\infty t^{-1}J(t; u(t))\, dt.$$

Hence by (8.3) and (8.4), $(X_1, X_2)_{\rho;j} = (X_1, X_2)_{0,1;J} = X_1^0$.

Since $L^1 \cap L^\infty$ and $L^1 + L^\infty$ are respectively the smallest and the largest of the r.i. spaces (cf. (2.9)), one interpretation of the last theorem is that the $(\rho; j)$ spaces depend only on the closures in X_1 and X_2 of the intersection $X_1 \cap X_2$.

COROLLARY 8.5. *For each* r.i. *norm* ρ, *the space* $(X_1, X_2)_{\rho;j}$ *is intermediate between* X_1^0 *and* X_2^0.

We turn now to the K-methods. The space $(X_1, X_2)_{\rho;K}$ consists of those $f \in X_1 + X_2$ for which the norm

$$\|f\|_{\rho;K} = \rho(t^{-1}K(t;f)) \tag{8.5}$$

is finite.

THEOREM 8.6. *For any* r.i. *norm* ρ, *there is the embedding*

$$(X_1, X_2)_{\rho;K} \subsetneq (X_1, X_2)_{\rho;k}. \tag{8.6}$$

Proof. If $f \in (X_1, X_2)_{\rho;K}$, then $\rho(t^{-1}K(t;f)) < \infty$. Let $g(t) = t^{-1} \lim_{s\to 0} K(s;f)$. Since $K(t;f)$ is increasing, we have from (2.10) and Hölder's inequality (2.8)

$$\|g\|_{L^1+L^\infty} \leqslant \rho'(\chi)\,\rho(g) \leqslant \rho'(\chi)\,\rho(t^{-1}K(t;f)) < \infty.$$

But $g(t)$ is a constant multiple of $1/t$, so if its $(L^1 + L^\infty)$-norm is finite, it must be identically zero. Hence $\lim_{s\to 0} K(s;f) = 0$, so by Theorem 5.1(a), $f \in X_1^0 + X_2$. It follows then from (5.6) that $k(t;f) \leqslant t^{-1}K(t;f)$ for all $t > 0$. Hence

$$\|f\|_{\rho;k} = \rho(k(t;f)) \leqslant \rho(t^{-1}K(t;f)) = \|f\|_{\rho;K},$$

and this completes the proof.

THEOREM 8.7. *For each* r.i. *norm* ρ, $(X_1, X_2)_{\rho;K}$ *is a Banach space continuously embedded in* $X_1^0 + X_2$. *If* ρ *also satisfies*

$$\rho(\chi^{**}) < \infty, \tag{8.7}$$

then $(X_1, X_2)_{\rho;K}$ *is intermediate between* X_1^0 *and* X_2. *In this case, the interpolation theorem (Theorem* 5.3) *holds with* $(\rho; k)$ *replaced by* $(\rho; K)$.

COROLLARY 8.8. *If* $\alpha < 1$, *then the space* $(X_1, X_2)_{\rho;K}$ *is intermediate between* X_1 *and* X_2, *and the interpolation theorem holds.*

THEOREM 8.9. *If* $\alpha < 1$, *then* $(X_1, X_2)_{\rho;k} \subsetneq (X_1, X_2)_{\rho;K}$.

Proof. For each $f \in (X_1, X_2)_{\rho;k}$ we write

$$t^{-1}K(t;f) = t^{-1}\int_0^t k(s;f)\,ds = \int_0^1 k(st;f)\,ds.$$

Hence (cf. proof of Lemma 7.3)

$$\rho(t^{-1}K(t;f)) \leqslant \rho(k(t;f)) \left(\int_0^1 h(s)\,ds\right).$$

By Lemma 7.1, the integral is finite so $\|f\|_{\rho;K} \leqslant c\,\|f\|_{\rho;k}$, as desired.

The space $(X_1, X_2)_{\theta,q;K}$ is defined in a similar way but by means of the norm

$$\|f\|_{\theta,q;K} = \left\{\int_0^\infty (t^{-\theta}K(t;f))^q\,dt/t\right\}^{1/q}. \tag{8.8}$$

It is intermediate between X_1 and X_2 for $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, and in the extreme cases $q = \infty$, $\theta = 0$ or 1. The interpolation theorem is valid for $0 < \theta < 1$ [8, p. 180]. Berens [5] has characterized the extreme cases in terms of the relative completion (cf. Sect. 3). Thus

$$(X_1, X_2)_{0,\infty;K} = \tilde{X}_1\,; \qquad (X_1, X_2)_{1,\infty;K} = \tilde{X}_2\,.$$

THEOREM 8.10. *Let (X_1, X_2) be a compatible couple.*

(a) *If $\rho = \rho^1$, then $(X_1, X_2)_{\rho;k} = (X_1{}^0)^\sim$;*
(b) *if $\rho = \rho^\infty$, then $(X_1, X_2)_{\rho;k} = \tilde{X}_2$;*
(c) *if $\rho = \rho^{1\cap\infty}$, then $(X_1, X_2)_{\rho;k} = (X_1{}^0 \cap X_2)^\sim = (X_1 \cap X_2)^\sim$;*
(d) *if $\rho = \rho^{1+\infty}$, then $(X_1, X_2)_{\rho;k} = (X_1{}^0)^\sim + X_2 = X_1{}^0 + X_2$.*

Proof. (a) If $f \in (X_1, X_2)_{\rho;k}$, where $\rho = \rho^1$, then from (8.8)

$$\rho(k(t;f)) = \int_0^\infty k(t;f)\,dt = \sup_t K(t;f) = \|f\|_{1,\infty;K}\,.$$

Hence, by (8.9)

$$(X_1, X_2)_{\rho;k} = (X_1, X_2)_{1,\infty;K} \cap (X_1{}^0 + X_2) = \tilde{X}_1 \cap (X_1{}^0 + X_2) = (X_1{}^0)^\sim.$$

The proof of part (b) is similar. Combining parts (a), (b), and using the identity (3.4), we find

$$(X_1, X_2)_{\rho;k} = (X_1{}^0)^\sim \cap \tilde{X}_2 = (X_1{}^0 \cap X_2)^\sim = (X_1 \cap X_2)^\sim,$$

where $\rho = \rho^{1\cap\infty}$. This establishes part (c).

Finally, if $\rho = \rho^{1+\infty}$, then for any $f \in (X_1, X_2)_{\rho;k}$,

$$\|f\|_{\rho;k} = \rho(k(t;f)) = \int_0^1 k(t;f)\,dt = K(1;f) = \|f\|_{X_1+X_2}\,.$$

Hence $(X_1, X_2)_{\rho;k} = (X_1 + X_2) \cap (X_1{}^0 + X_2) = X_1{}^0 + X_2 = (X_1{}^0)^\sim + X_2{}^\sim$.

9. The Equivalence Theorem

Our objective in this section is to show the equivalence of all four methods (j, J, k and K) whenever the indices of ρ lie strictly between 0 and 1.

THEOREM 9.1. *Let* (X_1, X_2) *be a compatible couple, and let* ρ *be a* r.i. *norm. Then* $(X_1, X_2)_{\rho;j} \subseteq (X_1, X_2)_{\rho;k}$.

Proof. Let $f \in (X_1, X_2)_{\rho;j}$ and suppose f has a representation $f = \int_0^\infty u(t)\, dt/t$. Then from (5.4)

$$K(t;f) \leqslant \int_0^\infty K(t; u(s))\, ds/s \leqslant \int_0^\infty \min(1, s^{-1}t)\, J(s; u(s))\, ds/s$$

$$= \int_0^t J(s; u(s))\, ds/s + t \int_t^\infty s^{-1} J(s; u(s))\, ds/s.$$

The first integral is $\int_0^t dx \int_x^t s^{-1} J(s; u(s))\, ds/s$, and the second can be written in the form $\int_0^t dx \int_t^\infty s^{-1} J(s; u(s))\, ds/s$, so their sum is $\int_0^t dx \int_x^\infty s^{-1} J(s; u(s))\, ds/s = \int_0^t j(x; u)\, dx$. Hence

$$K(t;f) \leqslant \int_0^t j(x; u)\, dx, \qquad 0 < t < \infty. \tag{9.1}$$

Now by Corollary 8.5, $f \in (X_1, X_2)_{\rho;j} \subseteq X_1{}^0 + X_2{}^0$, so we can use (5.6) to rewrite (9.1) in the form

$$\int_0^t k(x;f)\, dx \leqslant \int_0^t j(x; u)\, dx, \qquad 0 < t < \infty. \tag{9.2}$$

Hence, by Theorem 2.2, $\rho(k(x;f)) \leqslant \rho(j(x; u))$. Taking the infimum over all representations u, we obtain

$$\|f\|_{\rho;k} = \rho(k(x;f)) \leqslant \|f\|_{\rho;j}, \tag{9.3}$$

and this completes the proof.

THEOREM 9.2. *If* $\beta > 0$, *then* $(X_1, X_2)_{\rho;K} \subseteq (X_1, X_2)_{\rho;J}$.

Proof. Each f in $(X_1, X_2)_{\rho;K}$ belongs also to $(X_1, X_2)_{\rho;k}$ (Theorem 8.6) and hence to $X_1{}^0 + X_2$. By Theorem 5.1, we have therefore

$$\lim_{t\to 0} K(t;f) = 0. \tag{9.4}$$

On the other hand, using (5.6), we have for each $t > 0$,

$$t^{-1}K(t;f) = \int_0^1 k(st;f)\,ds = \int_0^1 E_t\{k(s;f)\}\,ds.$$

Therefore, by Hölder's inequality (2.8) and (2.15),

$$t^{-1}K(t;f) \leqslant \rho'(\chi)\,\rho(E_t\{k(s;f)\}) \leqslant \rho'(\chi)\,h_\rho(t)\,\rho(k(s;f)).$$

But $\beta > 0$ so $h_\rho(t) \to 0$ as $t \to \infty$ by Lemma 7.2. Hence

$$\lim_{t\to\infty} t^{-1}K(t;f) = 0. \tag{9.5}$$

The estimates (9.4) and (9.5) enable us to find a representation $f = \int_0^\infty u(t)\,dt/t$ of f for which $J(t;u(t)) \leqslant 4eK(t;f)$ [8, Lemma 3.2.10]. Hence

$$\|f\|_{\rho;J} \leqslant \rho(t^{-1}J(t;u(t))) \leqslant 4e\rho(t^{-1}K(t;f)) = 4e\,\|f\|_{\rho;K}.$$

This completes the proof.

THEOREM 9.3 (The Equivalence Theorem). *Let* (X_1, X_2) *be a compatible couple, and let* ρ *be a* r.i. *norm whose indices satisfy* $0 < \beta \leqslant \alpha < 1$. *Then*

$$(X_1, X_2)_{\rho;j} = (X_1, X_2)_{\rho;J} = (X_1, X_2)_{\rho;K} = (X_1, X_2)_{\rho;k}, \tag{9.6}$$

with equivalent norms.

Proof. The continuous embeddings $J \to j \to k \to K \to j$ follow from Theorems 8.3, 9.1, 8.9, and 9.2, respectively.

When ρ is the L^{pq}-norm

$$\rho(f) = \left\{\int_0^\infty [t^{1/p}f^{**}(t)]^q\,dt/t\right\}^{1/q}, \tag{9.7}$$

the ρ-methods reduce to Peetre's (θ, q)-methods.

THEOREM 9.4. *If* ρ *is the* L^{pq}-*norm,* $1 < p < \infty \leqslant q \leqslant \infty$ *then Theorem* 9.3 *holds and the spaces in* (9.6) *are equivalent to*

$$(X_1, X_2)_{\theta,q;J} = (X_1, X_2)_{\theta,q;K}, \qquad \theta = 1 - 1/p.$$

Proof. The L^{pq}-norm has indices $\alpha = \beta = p^{-1}$ so $1 < p < \infty$ implies $0 < \beta = \alpha < 1$. Hence Theorem 9.3 applies. But $(X_1, X_2)_{\rho;K} = (X_1, X_2)_{\theta,q;K}$ (cf. (8.5), (8.8)) and $(X_1, X_2)_{\rho;J} = (X_1, X_2)_{\theta,q;J}$ (cf. (8.1), (8.3)), if $\theta = 1 - 1/p$.

10. The Stability Theorem

Before presenting the theorem of stability (or reiteration), we need two preliminary results concerning the construction of r.i. norms.

Theorem 10.1. *Let a and b be any numbers satisfying* $0 \leqslant b < a \leqslant 1$, *and let* $c = (a - b)^{-1}$. *Let* ν *be a* r.i. *norm whose indices satisfy*

$$0 < \beta_\nu \leqslant \alpha_\nu < 1. \tag{10.1}$$

Then the functional ρ *defined by*

$$\rho(f(t)) = \nu(t^{bc} f^*(t^c)) \tag{10.2}$$

is a r.i. *norm whose indices are given by*

$$\beta_\rho = \beta_\nu a + (1 - \beta_\nu) b; \qquad \alpha_\rho = \alpha_\nu a + (1 - \alpha_\nu) b. \tag{10.3}$$

In particular,

$$b < \beta_\rho \leqslant \alpha_\rho < a. \tag{10.4}$$

Proof. We need to verify that properties (2.1)–(2.5) hold for the functional ρ defined by (10.2). However, most of the proof is routine so we shall only establish the triangle inequality for ρ.

If $f, g \in \mathscr{M}^+$, then for any $s > 0$, we have $\int_0^s (f + g)^*(u)\, du \leqslant \int_0^s (f^* + g^*)(u)\, du$. Hence [16, Sect. 5], $\int_0^s u^{a-1}(f + g)^*(u)\, du \leqslant \int_0^s u^{a-1}(f^* + g^*)(u)\, du$, since u^{a-1} is nonincreasing in u. A change of variable leads to the inequality

$$\int_0^s t^{bc}(f + g)^* (t^c)\, dt \leqslant \int_0^s t^{bc}(f^* + g^*)(t^c)\, dt, \qquad s > 0.$$

Hence by Theorem 2.2 (applied to ν) and (10.2), we find that $\rho(f+g) \leqslant \rho(f) + \rho(g)$, as required.

Now let us compute the indices of ρ. For any $s > 0$ and any $f \in L^\rho$,

$$\rho((E_s f)(t)) = \nu(t^{bc} f^*(st^c)) = \nu(s^{-b}(s^{a-b}t)^{bc} f^*((s^{a-b}t)^c))$$
$$= s^{-b} \nu(E_{s^{a-b}}\{t^{bc} f^*(t^c)\}).$$

Applying (2.15) (for ν), we have

$$\rho(E_s f) \leqslant s^{-b} h_\nu(s^{a-b})\, \nu(t^{bc} f^*(t^c)) = s^{-b} h_\nu(s^{a-b})\, \rho(f).$$

Hence, again by (2.15) (for ρ),

$$h_\rho(s) \leqslant s^{-b} h_\nu(s^{a-b}), \qquad s > 0. \tag{10.5}$$

To obtain the reverse inequality, we solve (10.2) for ν:

$$\nu(g(t)) \equiv \nu(g^*(t)) = \rho(t^{-b}g^*(t^{a-b})), \tag{10.6}$$

and apply the same argument as above but this time with the roles of ν and ρ interchanged. The resulting inequality is

$$h_\nu(s) \leqslant s^{bc}h_\rho(s^c), \qquad s > 0,$$

which together with (10.5) gives

$$h_\rho(s) = s^{-b}h_\nu(s^{a-b}), \qquad s > 0. \tag{10.7}$$

The desired identities (10.3) now follow easily from (10.7) by taking the appropriate limits as in (7.3). This completes the proof.

In the next theorem we reverse the construction given above. That is, given a r.i. norm ρ satisfying (10.4), we define ν by means of (10.2) (or, equivalently, by (10.6)) and hope to show it has the appropriate properties. Unfortunately, the situation here is more complicated and we are unable to show that ν satisfies the triangle inequality. This is of little consequence however since we are able to construct a r.i. norm ν_1 which is equivalent to ν (i.e., the quasinormed space L^ν is in fact normable).

THEOREM 10.2. *Let a and b be numbers satisfying $0 \leqslant b < a \leqslant 1$, and let ρ be a* r.i. *norm whose indices satisfy*

$$b < \beta_\rho \leqslant \alpha_\rho < a. \tag{10.8}$$

Then the functional ν defined by

$$\nu(g(t)) = \rho(t^{-b}g^*(t^{a-b})) \tag{10.9}$$

is a r.i. *quasinorm (equivalent to a* r.i. *norm ν_1) whose indices are given by*

$$\beta_\nu = (\beta_\rho - b)/(a - b); \qquad \alpha_\nu = (\alpha_\rho - b)/(a - b). \tag{10.10}$$

In particular,

$$0 < \beta_\nu \leqslant \alpha_\nu < 1. \tag{10.11}$$

Proof. We show first that $\rho(\chi_s) < \infty$ for all $s > 0$ (cf. (2.4)). The indices of ρ lie strictly between b and a so there is a continuous embedding $X \equiv L^{1/b,\infty} \cap L^{1/a,\infty} \subseteq L^\rho$ (cf. [7]). Hence, for some constant K and all $f \in X$,

$$\rho(f) \leqslant K\|f\|_X = K \max(\sup_t t^b f^*(t), \sup_t t^a f^*(t)).$$

But $\nu(\chi_s(t)) = \rho(t^{-b}\chi_s(t^{a-b}))$ so we have

$$\nu(\chi_s(t)) \leqslant K \max(1, \sup_{0<t^{a-b}<s} t^{a-b}) = K \max(1, s) < \infty.$$

The remaining properties (2.1)–(2.5) are not difficult to establish, with the exception of the triangle inequality which we were unable to prove. However, if ν_1 is the functional defined by

$$\nu_1(g(t)) = \nu(g^{**}(t)) = \rho(t^{-b}g^{**}(t^{a-b})), \tag{10.12}$$

then it is clear that ν_1 is a norm (since $g \to g^{**}$ is subadditive). We sketch a proof that ν and ν_1 are equivalent. First, it is obvious that $\nu \leqslant \nu_1$ since $g^* \leqslant g^{**}$. In the other direction, the usual argument involving the dilation operators (cf. the proof of Lemma 7.3) leads to the inequality

$$\nu_1(g) \leqslant \left(\int_0^1 s^a h_\rho(s)\, ds/s\right) \nu(g).$$

The integral is finite because, by hypothesis, $\alpha_\rho < a$ (cf. Lemma 7.1 where we considered the case $a = 1$). Hence ν and ν_1 are equivalent.

The indices of ν (which, by equivalence, coincide with those of ν_1) are computed exactly as in the proof of the last theorem. We omit the details.

Let (X_1, X_2) be a compatible couple, and let θ be fixed, $0 \leqslant \theta \leqslant 1$. An intermediate space X of X_1 and X_2 is said to be of class $\mathscr{J}(\theta) = \mathscr{J}(\theta; X_1, X_2)$ if

$$\|f\|_X \leqslant At^{-\theta} J(t; f; X_1, X_2), \qquad f \in X_1 \cap X_2, \tag{10.13}$$

where A is a constant independent of f. Similarly, X belongs to the class $\mathscr{K}(\theta) = \mathscr{K}(\theta; X_1, X_2)$ if

$$K(t; f; X_1, X_2) \leqslant Bt^\theta \|f\|_X, \qquad f \in X. \tag{10.14}$$

If X belongs to both $\mathscr{J}(\theta)$ and $\mathscr{K}(\theta)$, we say X is of class $\mathscr{H}(\theta) = \mathscr{H}(\theta; X_1, X_2)$.

It is not difficult to see that X is of class $\mathscr{J}(\theta)$ if and only if $(X_1, X_2)_{\theta,1;J} \subseteq X$, and of class $\mathscr{K}(\theta)$ if and only if $X \subseteq (X_1, X_2)_{\theta,\infty;K}$ [8, p. 175].

THEOREM 10.3 (The Stability Theorem). *Let (X_1, X_2) be a compatible couple, and let Y_i, $i = 1, 2$, be intermediate spaces of X_1 and X_2 of class $\mathscr{H}(\theta_i; X_1, X_2)$, where $0 \leqslant \theta_1 < \theta_2 \leqslant 1$.*

(a) *Suppose* ν *is a* r.i. *norm whose indices satisfy* $0 < \beta_\nu \leqslant \alpha_\nu < 1$. *Then there is a* r.i. *norm* ρ *whose indices are given by*

$$\beta_\rho = \beta_\nu(1 - \theta_1) + (1 - \beta_\nu)(1 - \theta_2); \qquad \alpha_\rho = \alpha_\nu(1 - \theta_1) + (1 - \alpha_\nu)(1 - \theta_2) \tag{10.15}$$

such that

$$(X_1, X_2)_{\rho;k} = (Y_1, Y_2)_{\nu;k}, \tag{10.16}$$

with equivalent norms.

(b) *Suppose* ρ *is a* r.i. *norm whose indices satisfy* $1 - \theta_2 < \beta_\rho \leqslant \alpha_\rho < 1 - \theta_1$. *Then there is a* r.i. *norm* ν *whose indices are given by* (10.15), i.e.,

$$\beta_\nu = \frac{\beta_\rho - (1 - \theta_2)}{\theta_2 - \theta_1}; \qquad \alpha_\nu = \frac{\alpha_\rho - (1 - \theta_2)}{\theta_2 - \theta_1}, \tag{10.17}$$

such that

$$(Y_1, Y_2)_{\nu;k} = (X_1, X_2)_{\rho;k}, \tag{10.18}$$

with equivalent norms.

Proof. We remark that in all cases above, the indices of ρ and ν lie strictly between 0 and 1. Hence, by the equivalence theorem (Theorem 10.3), the k-spaces could equally well be replaced by any of the corresponding K, j, or J spaces. Thus to establish (10.16), it will suffice to show that there is a r.i. norm ρ with indices given by (10.15) such that

$$(Y_1, Y_2)_{\nu;K} \subseteq (X_1, X_2)_{\rho;K} \tag{10.19}$$

and

$$(X_1, X_2)_{\rho;J} \subseteq (Y_1, Y_2)_{\nu;J}. \tag{10.20}$$

Let $a = 1 - \theta_1$, $b = 1 - \theta_2$, $c = (a - b)^{-1} = (\theta_2 - \theta_1)^{-1}$. Then given ν, we construct ρ by means of the identity (10.2). By Theorem 10.1, ρ is a r.i. norm whose indices are given by (10.3) or, what is the same thing, by (10.15). Moreover, from (10.6) we have

$$\nu(g(t)) = \nu(g^*(t)) = \rho(t^{-b} g^*(t^{a-b})). \tag{10.21}$$

It will be convenient in what follows to write $K_X(t; x)$ instead of $K(t; x; X_1, X_2)$, and $K_Y(t; y)$ for $K(t; y; Y_1, Y_2)$.

Let $y \in (Y_1, Y_2)_{\nu;K}$, and suppose y has a decomposition $y = y_1 + y_2$ in $Y_1 + Y_2$. The spaces Y_i, $i = 1, 2$, are by assumption of class $\mathscr{H}(\theta_i; X_1, X_2)$, so by (10.14) there are constants B_1 and B_2 independent of y_1 and y_2 such that

$$K_X(t; y_i) \leqslant B_i t^{\theta_i} \| y_i \|_{Y_i}, \qquad i = 1, 2.$$

Hence

$$K_X(t;y) \leqslant K_X(t;y_1) + K_X(t;y_2) \leqslant B_1 t^{\theta_1}(\|y_1\|_{Y_1} + Bt^{\theta_2-\theta_1}\|y_2\|_{Y_2}),$$

where $B = B_2/B_1$. Taking the infimum over all such decompositions $y = y_1 + y_2$ of y, we obtain

$$K_X(t;y) \leqslant B_1 t^{\theta_1} K_Y(Bt^{\theta_2-\theta_1}; y).$$

Hence

$$\rho(t^{-1}K_X(t;y)) \leqslant C\rho(t^{\theta_1-1}K_Y(Bt^{\theta_2-\theta_1};y)) = C\rho(t^{-a}K_Y(Bt^{a-b};y))$$
$$= C\rho(t^{-b}\{t^{-(a-b)}K_Y(Bt^{a-b};y)\}).$$

Using (2.15), we can "remove" the constant B to get

$$\rho(t^{-1}K_X(t;y)) \leqslant D\rho(t^{-b}\varphi(t^{a-b})),$$

where $D = D(B_1, B_2, \theta_1, \theta_2, \rho)$, and $\varphi(t) = t^{-1}K_Y(t;y)$. Now φ is continuous and decreasing so $\varphi^* = \varphi$. Hence, by (10.21), the last inequality can be written as

$$\rho(t^{-1}K_X(t;y)) \leqslant D\nu(\varphi(t)) = D\nu(t^{-1}K_Y(t;y)).$$

It follows that $(Y_1, Y_2)_{\nu;K} \subseteq (X_1, X_2)_{\rho;K}$, i.e., (10.19) holds.

The proof of (10.20) is much the same (cf. [8, Proposition 3.2.19] so part (a) of the theorem is established.

To prove part (b), we make use of the fact that the constructions in Theorems 10.1 and 10.2 are mutually reciprocal. In other words, the identities (10.2) and (10.9) are obtained from one another simply by solving for one norm in terms of the other.

Thus, given ρ, we construct the quasinorm ν as in (10.9) (together with the equivalent norm ν_1). By equivalence, we have $(Y_1, Y_2)_{\nu;k} = (Y_1, Y_2)_{\nu_1;k}$. Now let ρ_1 be the r.i. norm constructed from ν_1 according to (10.2). Ovbiously ρ_1 is equivalent to ρ so $(X_1, X_2)_{\rho;k} = (X_1, X_2)_{\rho_1;k}$. But by part (a) of the theorem, $(X_1, X_2)_{\rho_1;k} = (Y_1, Y_2)_{\nu_1;k}$. Hence $(X_1, X_2)_{\rho;k} = (Y_1, Y_2)_{\nu;k}$, as required. This completes the proof.

11. The Duality Theorem

Throughout this final section (X_1, X_2) will denote a conjugate couple. Thus $X_1 \cap X_2$ is dense in X_1 and in X_2, or, in the terminology of Section 3, $X_1{}^0 = X_1$ and $X_2{}^0 = X_2$. In this case the duals of X_1

and X_2 can be regarded in a canonical way as subspaces of $(X_1 \cap X_2)^* = X_1^* + X_2^*$ (cf. Theorem 3.1), i.e., (X_1^*, X_2^*) is again a compatible couple (although not necessarily conjugate).

There is a natural duality between the J- and K-functionals exhibited by the identities

$$J(t; F; X_1^*, X_2^*) = \sup_{f \in X_1 + X_2} \frac{|(F, f)|}{K(t^{-1}; f; X_1, X_2)}, \quad F \in X_1^* \cap X_2^*; \quad (11.1)$$

$$K(t; F; X_1^*, X_2^*) = \sup_{f \in X_1 \cap X_2} \frac{|(F, f)|}{J(t^{-1}; f; X_1, X_2)}, \quad F \in X_1^* + X_2^*, \quad (11.2)$$

which follow from Theorem 3.1. This duality extends to the j- and k-functionals via the identity

$$\int_0^\infty t^{-1} J(t; f)\, t^{-1} K(t; g)\, dt = \int_0^\infty j(t; f)\, k(t; g)\, dt \quad (11.3)$$

which involves nothing more than an integration by parts.

Let X be an intermediate space of X_1 and X_2. In order that X^* be intermediate between X_1^* and X_2^*, it is necessary and sufficient that $X_1 \cap X_2$ be dense in X. Thus, if $X = (X_1, X_2)_{\rho;j}$, the density theorem (Theorem 4.3) will ensure that X^* is an intermediate space of X_1^* and X_2^* whenever ρ is absolutely continuous.

Lemma 11.1. *Let (X_1, X_2) be a conjugate couple, and let ρ be an absolutely continuous* r.i. *norm. Then*

$$(X_1, X_2)^*_{\rho;j} \subseteq (X_2^*, X_1^*)_{\rho';k}, \quad (11.4)$$

where ρ' is the associate norm of ρ. Moreover, the inclusion map has norm at most one.

Proof. Our proof is a modification of that given by Scherer [21, Theorem 3] for the Φ-methods developed in the context of approximation theory. In order to economize on notation, we shall write $K_{12}(t; f)$ for $K(t; f; X_1, X_2)$, $K_{21}(t; f)$ for $K(t; f; X_2, X_1)$, and $K^*_{12}(t; f)$ for $K(t; f; X_1^*, X_2^*)$, etc., with similar abbreviations for the J-, j-, and k-functionals.

Let $F \in (X_1, X_2)^*_{\rho;j}$, $F \neq 0$, and let λ satisfy $0 < \lambda < 1$. By (11.2), we can choose, for each $t > 0$, a nonzero element $g(t) \in X_1 \cap X_2$ such that

$$(F, g(t)) \geqslant \lambda K^*_{12}(t^{-1}; F)\, J_{12}(t; g(t)). \quad (11.5)$$

In fact, by the continuity of the J- and K-functionals, g can be chosen piecewise constant and hence strongly measurable in $X_1 \cap X_2$.

As φ ranges over the set $\mathscr{M}^+$ of all nonnegative measurable functions on $(0, \infty)$, so does $\psi(t) = t^{-1}J_{12}(t; \varphi(t)\, g(t))$. For each $\varphi \in \mathscr{M}^+$, let

$$\Phi(t) = \int_t^\infty \psi(s)\, ds/s = j_{12}(t; \varphi g), \tag{11.6}$$

and set $M = \{\varphi: \rho(\Phi) < \infty\}$, $M_0 = \{\Phi: \varphi \in M\} \subseteq L^\rho$. Then for any $\Psi \in L^{\rho'}$, we have from (2.6)

$$\rho'(\Psi) = \sup_{\Phi \in L^\rho}(\langle\Phi, \Psi\rangle/\rho(\Phi)) = \sup_{\Phi \in M_0}(\langle\Phi, \Psi\rangle/\rho(\Phi)). \tag{11.7}$$

The last identity follows from the fact that every nonnegative, continuously differentiable, and decreasing function Φ in L^ρ belongs to M_0 (take $\psi(s) = -s\Phi'(s)$ in (11.6)), and every nonnegative decreasing function in L^ρ can be approximated by such functions since ρ is absolutely continuous.

For any $\varphi \in M$, we have, just as in the proof of (9.1),

$$\int_0^\infty \| \varphi(t)\, g(t)\|_{X_1+X_2}\, dt/t = \int_0^\infty K_{12}(1; \varphi(t)\, g(t))\, dt/t \leqslant \int_0^1 j_{12}(t; \varphi g)\, dt$$

$$= \int_0^1 \Phi(t)\, dt \leqslant \rho'(\chi)\, \rho(\Phi) < \infty.$$

Hence, the integral $\int_0^\infty \varphi(t)\, g(t)\, dt/t$ converges in $X_1 + X_2$ to an element f_φ, say. But then f_φ belongs to $(X_1, X_2)_{\rho;j}$ because φg is a representation of f_φ for which $\rho(j_{12}(t; \varphi g)) = \rho(\Phi) < \infty$. Moreover, using the absolute continuity of ρ, we have $f_\varphi = \lim_{n\to\infty} \int_{1/n}^n \varphi(t) g(t)\, dt/t$ in $(X_1, X_2)_{\rho;j}$, so we can write $(F, f_\varphi) = \int_0^\infty (F, \varphi(t)\, g(t))\, dt/t$. Hence, from (11.5), (5.2), and (11.3), we have

$$(F, f_\varphi) \geqslant \lambda \int_0^\infty t^{-1} K_{21}^*(t; F)\, J_{12}(t; \varphi(t)\, g(t))\, dt/t$$

$$= \lambda \int_0^\infty k_{21}^*(t; F)\, j_{12}(t; \varphi g)\, dt;$$

equivalently, by (11.6),

$$(F, f_\varphi) \geqslant \lambda \int_0^\infty k_{21}^*(t; F)\, \Phi(t)\, dt. \tag{11.8}$$

Now φg represents f_φ so

$$\| f_\varphi \|_{\rho; j_{12}} \leqslant \rho(j_{12}(t; \varphi g)) = \rho(\Phi).$$

Together with (11.8) this yields the estimate

$$\frac{(F, f_\varphi)}{\| f_\varphi \|_{\rho; j_{12}}} \geqslant \frac{\int_0^\infty k_{21}^*(t; F)\, \Phi(t)\, dt}{\rho(\Phi)}.$$

Taking the supremum over all $\varphi \in M$ (equivalently, all $\Phi \in M_0$), we obtain via (11.7)

$$\sup_{\varphi \in M} \frac{(F, f_\varphi)}{\| f_\varphi \|_{\rho; j_{12}}} \geqslant \lambda \rho'(k_{21}^*(t; F)).$$

Finally, letting $\lambda \to 1$, this leads to the inequality

$$\| F \|_{(X_1 X_2)^*_{\rho; j}} \geqslant \| F \|_{(X_2^*, X_1^*)_{\rho'; k}}.$$

The normal embedding (11.4) is therefore established and the proof is complete.

In order to establish the reverse inclusion to (11.4), we impose one additional (and somewhat artificial) restriction on ρ (cf. (11.9)).

LEMMA 11.2. *Let (X_1, X_2) be a conjugate couple, and let ρ be an absolutely continuous* r.i. *norm. Suppose in addition that*

$$X_1^* \cap X_2^* \text{ is dense in } (X_2^*, X_1^*)_{\rho'; k}. \tag{11.9}$$

Then

$$(X_2^*, X_1^*)_{\rho' k} \subseteq (X_1, X_2)^*_{\rho; j}, \tag{11.10}$$

and the inclusion mapping has norm at most one.

Proof. Let $F \in X_1^* \cap X_2^* = (X_1 + X_2)^*$, and let $f \in X_1 \cap X_2$. Suppose that both F and f are nonzero. When $\lambda > 1$ is fixed, we can find a representation $f = \int_0^\infty u(t)\, dt/t$ of f such that

$$\rho(j_{12}(x; u)) \leqslant \lambda \| f \|_{\rho; j}. \tag{11.11}$$

Furthermore, for each $t > 0$, there is a decomposition $F = F_1(t) + F_2(t)$ of F in $X_1^* + X_2^*$ such that $\| F_1(t) \|_{X_1^*} + \| F_2(t) \|_{X_2^*} \leqslant \lambda K_{12}^*(t; F)$. Then for each $t > 0$ we have

$$\begin{aligned} |(F, u(t))| &\leqslant \| F_1(t) \|_{X_1^*} \| u(t) \|_{X_1} + \| F_2(t) \|_{X_2^*} \| u(t) \|_{X_2} \\ &\leqslant \lambda K_{12}^*(t^{-1}; F)\, J_{12}(t; u(t)). \end{aligned}$$

Now F is continuous on $X_1 + X_2$ so $(F, f) = \int_0^\infty (F, u(t))\, dt/t$. Hence, from (5.2) and (11.3),

$$|(F,f)| \leqslant \lambda \int_0^\infty k_{21}^*(t; F)\, j_{12}(t; u)\, dt.$$

Applying Hölder's inequality and using (11.11), we see that

$$|(F,f)| \leqslant \lambda^2 \rho'(k_{21}^*(t; F))\, \|f\|_{\rho;j},$$

so, letting $\lambda \to 1$, we deduce that the dual norm of F (i.e., the norm of F in $(X_1, X_2)_{\rho;j}^*$) does not exceed $\rho'(k_{21}^*(t; F))$ which is the norm of F in the space $(X_2{}^*, X_1{}^*)_{\rho';k}$. This holds for all $F \in X_1{}^* \cap X_2{}^*$ and all $f \in X_1 \cap X_2$ and hence, using the hypothesis (11.9), for all $F \in (X_2{}^*, X_1{}^*)_{\rho';k}$. The proof is complete.

In connection with the density condition (11.9), the obvious conjecture that $X_1 \cap X_2$ is dense in $(X_1, X_2)_{\rho;k}$ if ρ is absolutely continuous is false. Indeed, if ρ is the L^1-norm $\rho(f) = \int_0^\infty |f(t)|\, dt$, $X_1 = \mathscr{C}[0, 1]$, the continuous functions on $[0, 1]$, $X_2 = L^1[0, 1]$, then $X_1 \cap X_2 = \mathscr{C}[0, 1]$ which is not dense in $(X_1, X_2)_{\rho;k} = \tilde{\mathscr{C}} = L^\infty$ (cf. Theor. 8.10(a)). A more natural conjecture might be that $(X_1 \cap X_2)^\sim$ is dense in $(X_1, X_2)_{\rho;k}$ whenever ρ is absolutely continuous, but we have been unable to decide this. However, if the indices of ρ lie strictly between 0 and 1, then $(X_1, X_2)_{\rho;k} = (X_1, X_2)_{\rho;j}$ by the equivalence theorem so $X_1 \cap X_2$ is dense in $(X_1, X_2)_{\rho;k}$ if ρ is absolutely continuous (Theorem 4.3).

THEOREM 11.3. *Let (X_1, X_2) be a conjugate couple, and let ρ be a reflexive* r.i. *norm whose indices satisfy $0 < \beta \leqslant \alpha < 1$. Then the dual of $(X_1, X_2)_{\rho;j}$ is isometrically isomorphic to $(X_2{}^*, X_1{}^*)_{\rho';k}$.*

Proof. Recall that ρ is reflexive if and only if both ρ and ρ' are absolutely continuous. In view of the remarks above, $X_1 \cap X_2$ is dense in $(X_1, X_2)_{\rho;j}$ and $X_1{}^* \cap X_2{}^*$ is dense in $(X_2{}^*, X_1{}^*)_{\rho';k}$ so the theorem follows from Lemmas 11.1 and 11.2.

At the expense of losing the isometric character of the duality, we can dispense with the restriction that ρ' be absolutely continuous.

THEOREM 11.4 (The Duality Theorem). *Let (X_1, X_2) be a conjugate couple, and suppose that ρ is an absolutely continuous* r.i. *norm whose indices satisfy $0 < \beta \leqslant \alpha < 1$. Then*

$$(X_1, X_2)_{\rho;j}^* = (X_2{}^*, X_1{}^*)_{\rho';k} = (X_2{}^*, X_1{}^*)_{\rho';j} = (X_1, X_2)_{\rho,k}^*,$$

up to equivalence of norms.

Proof. By the equivalence theorem (Theorem 9.3) and Lemma 11.1, we have

$$(X_1, X_2)^*_{\rho;j} = (X_1, X_2)^*_{\rho;k} \subseteq (X_2^*, X_1^*)_{\rho';k} = (X_2^*, X_1^*)_{\rho';j},$$

with equivalent norms. Hence, to complete the proof we need only establish the inclusion $(X_2^*, X_1^*)_{\rho';j} \subset (X_1, X_2)^*_{\rho;k}$.

Let $F \in (X_2^*, X_1^*)_{\rho';j}$, and let $F = \int_0^\infty U(t)\, dt/t$ be a representation of F, where $U(t) \in X_1^* \cap X_2^*$ and the integral converges in $X_1^* + X_2^*$. Then by (11.1), for any $f \in X_1 \cap X_2$,

$$|(F,f)| \leqslant \int_0^\infty |(U(t),f)|\, dt/t \leqslant \int_0^\infty J^*_{12}(t^{-1}; U(t))\, K_{12}(t;f)\, dt/t.$$

Using (4.2) and (11.3), we have

$$|(F,f)| \leqslant \int_0^\infty j^*_{21}(t; U)\, k_{12}(t;f)\, dt \leqslant \rho'(j_{21}(t; U))\, \rho(k_{12}(t;f)),$$

so taking the appropriate infimum over U, we find that

$$\| F \|_{(X_1,X_2)^*_{\rho;k}} \leqslant \| F \|_{(X_2^*,X_1^*)_{\rho';j}},$$

as required. This completes the proof.

COROLLARY 11.5. *Let (X_1, X_2) be a conjugate couple of reflexive Banach spaces, and let ρ be a reflexive* r.i. *norm with indices $0 < \beta \leqslant \alpha < 1$. Then $(X_1, X_2)_{\rho;j}$ and $(X_1, X_2)_{\rho;k}$ are reflexive.*

Proof. By Theorem 11.4, the dual of $(X_1, X_2)_{\rho;j}$ is equivalent to $(X_2^*, X_1^*)_{\rho';k}$. But since X_1 and X_2 are reflexive, the couple (X_2^*, X_1^*) is again conjugate, so by Theorem 11.4, the dual of $(X_2^*, X_1^*)_{\rho';k}$ is equivalent to $(X_1^{**}, X_2^{**})_{\rho;j}$. Thus, under the canonical isomorphism $(X_1, X_2)_{\rho;j}$ is equivalent to its second dual and hence is reflexive. A similar argument applies to show the reflexivity of $(X_1, X_2)_{\rho;k}$.

There are entirely analogous statements to the above for the $(\rho; J)$ and $(\rho; K)$ methods which we omit. However, it is worth pointing out the following corollaries for the $(\theta, q; J)$ and $(\theta, q; K)$ methods which were first established by Lions and Peetre [22] for the so-called "spaces of means."

COROLLARY 11.6. *Let (X_1, X_2) be a conjugate couple, and let $0 < \theta < 1$, $1 < q < \infty$. Then the dual of $(X_1, X_2)_{\theta,q;J}$ is isometrically isomorphic to $(X_2^*, X_1^*)_{1-\theta,q';K}$, where $1/q + 1/q' = 1$.*

COROLLARY 11.7. *Let* (X_1, X_2) *be a conjugate couple, and let* $0 < \theta < 1$, $1 \leqslant q < \infty$. *Then*

$$(X_1, X_2)^*_{\theta,q;J} = (X_2^*, X_1^*)_{1-\theta,q';K} = (X_2^*, X_1^*)_{1-\theta,q';J} = (X_1, X_2)^*_{\theta,q;K},$$

with equivalent norms.

COROLLARY 11.8. *If* (X_1, X_2) *is a conjugate couple of reflexive Banach spaces and* $0 < \theta < 1$, $1 < q < \infty$, *then* $(X_1, X_2)_{\theta,q;J}$ *and* $(X_1, X_2)_{\theta,q;K}$ *are reflexive.*

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