A Sheaf of Bicomodules over the Incidence Coalgebra

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To each partition \( \pi \) of a locally finite partially ordered set \( P \) into convex subsets is associated a \( C(P) \)-bicomodule \( S(\pi) \) where \( C(P) \) is the incidence coalgebra of \( P \) over some commutative ring. The complete lattice of all partitions of \( P \) into convex subsets is given a structure of Grothendieck topology in such a way that the association of \( S(\pi) \) to \( \pi \) yields a sheaf of \( C(P) \)-bicomodules. Some applications of parts of this machinery are given including a proof of Rota's Main Theorem.

1. Preliminaries

All partially ordered sets under consideration are locally finite and will be referred to simply as posets. \( K(P) \) will denote the set of all partitions of the poset \( P \) into convex subsets. For \( x \in P \) and \( \pi \in K(P) \), \( \pi(x) \) will denote the member of \( \pi \) which contains \( x \). \( K(P) \) is a complete meet semilattice; for an arbitrary family \( \{\pi_j\} \) of members of \( K(P) \) and for \( x \in P \),

\[
\left( \bigwedge \pi_i \right)(x) = \bigcap \pi_i(x).
\]

\( \{P\} \in K(P) \), and, hcncc, \( K(P) \) is a complete lattice (\( K(P) \) is not, in general, a sublattice of the lattice of all partitions of \( P \)). For any increasing map \( \sigma : P \to Q \) of posets, we let \( \sigma \) also denote the associated partition

\[
\{\sigma^{-1}(q) : q \in \sigma(P)\}
\]

of \( P \) into convex subsets.

\( k \) will denote a fixed commutative ring with unit. R. Heyneman noticed

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that the incidence algebra of a poset $P$ is the linear dual of a coalgebra associated to $P$; namely, let $C(P)$ denote the free $k$-module on the set 
$$\{(x, y) : x, y \in P, x \leq y\}.$$ 
Then the $k$-linear maps 
$$\Delta : C(P) \to C(P) \otimes C(P),$$ 
$$\epsilon : C(P) \to k,$$ 
defined by 
$$\Delta(x, y) = \sum_{x \leq t \leq y} (x, t) \otimes (t, y),$$ 
$$\epsilon(x, y) = \delta_{x, y} \text{ (Kronecker's delta)},$$ 
give $C(P)$ a structure of $k$-coalgebra [3], the incidence coalgebra of $P$ over $k$. That is, the following diagrams commute:

(In the last two diagrams, the maps on the bottom are the canonical isomorphisms.) The incidence algebra of $P$ over $k$ is the $k$-linear dual of $C(P)$, $A(P) = \text{hom}_k (C(P), k)$, with its inherited structure [3] of $k$-algebra. $\zeta_p$ and $\mu_p$, or simply $\zeta$ and $\mu$ when no confusion arises, will denote the zeta and Möbius functions, respectively, of $A(P)$.

2. CONSTRUCTION OF A SHEAF OF $C(P)$-BICOMODULES

Let $\pi \in K(P)$ and let $S(\pi)$ denote the $k$-module $C(P)$ (forgetting the $k$-coalgebra structure of $C(P)$). Then the $k$-linear map 
$$\rho : S(\pi) \to S(\pi) \otimes C(P),$$ 
defined by 
$$\rho(x, y) = \sum_{\substack{x \leq t \leq y \\pi(t) = \pi(y)}} (x, t) \otimes (t, y),$$
gives \( S(\pi) \) a structure of right \( C(P) \)-comodule [6]. That is, the following diagrams commute:

\[
\begin{array}{cccc}
S(\pi) & \xrightarrow{\rho} & S(\pi) \otimes C(P) & \\
\downarrow & & \downarrow \rho \otimes I \\
S(\pi) \otimes C(P) & \xrightarrow{I \otimes \rho} & S(\pi) \otimes C(P) \otimes C(P) & \\
\end{array}
\]

(Again, the bottom map of the last diagram is the canonical isomorphism.) Similarly, the \( k \)-linear map

\[
\lambda : S(\pi) \rightarrow C(P) \otimes S(\pi),
\]

defined by

\[
\lambda(x, y) = \sum_{\pi(x) = \pi(t)} (x, t) \otimes (t, y),
\]

gives \( S(\pi) \) a structure of left \( C(P) \)-comodule [6]. Moreover, \( (I \otimes \rho) \circ \lambda = (\lambda \otimes I) \circ \rho, \)

and so, in analogy with the concept of bimodule for an algebra, let us refer to \( S(\pi) \) as a \( C(P) \)-bicomodule. When more than one member \( \pi \in K(P) \) is given, we let \( \rho(\pi) = \rho \) and \( \lambda(\pi) = \lambda \) to avoid confusion.

Suppose that \( \pi \leq \theta \) in \( K(P) \) (i.e., \( \pi(x) \subseteq \theta(x) \) for all \( x \in P \)). Then the \( k \)-linear map

\[
S(\pi, \theta) : S(\theta) \rightarrow S(\pi),
\]

defined by

\[
S(\pi, \theta)(x, y) = \begin{cases} 
(x, y), & \text{if } \pi(x) = \theta(x) \text{ and } \pi(y) = \theta(y), \\
0, & \text{otherwise},
\end{cases}
\]

is easily seen to be a map of right \( C(P) \)-comodules. That is, the following diagram commutes:

\[
\begin{array}{cccc}
S(\theta) & \xrightarrow{S(\pi, \theta)} & S(\pi) & \\
\downarrow \rho(\theta) & & \downarrow \rho(\pi) \\
S(\theta) \otimes C(P) & \xrightarrow{S(\pi, \theta) \otimes I} & S(\pi) \otimes C(P) & \\
\end{array}
\]
It is equally easy to prove that $S(\pi, \theta)$ is a map of left $C(P)$-comodules and, hence, a map of $C(P)$-bicomodules. Notice that $S(\pi, \pi) = I$ and that, for $\eta \leq \pi \leq \theta$ in $K(P)$,

$$S(\eta, \theta) = S(\eta, \pi) \circ S(\pi, \theta).$$

The complete lattice $K(P)$ may be viewed as a category whose morphisms consist of all $(\pi, \theta)$ where $\pi \leq \theta$ in $K(P)$. Thus, where convenient, we may write $\pi \to \theta$ rather than $\pi \leq \theta$ or $(\pi, \theta)$. The diagram

$$\begin{array}{ccc}
\pi & \rightarrow & \theta \\
\downarrow & & \downarrow \\
\pi & \rightarrow & \eta
\end{array}$$

is easily seen to be a fibered product diagram, and, more generally, arbitrary fibered products exist in $K(P)$. From the preceding paragraph, we see that $S$ is a contravariant functor from $K(P)$ to the category of $C(P)$-bicomodules.

Now we define covering in $K(P)$ in such a way that the functor $S$ becomes a sheaf of $C(P)$-bicomodules on the Grothendieck topology $K(P)$ [1]. We say that a family $\{\pi_i \leq \pi\}$ is a covering in $K(P)$ if and only if it is a finite family of morphisms in $K(P)$ and for all $x \leq y$ in $P$ there exists $\pi_i \leq \pi$ in the family such that $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$.

If $\{\pi_i \leq \pi\}$ is a covering in $K(P)$, then $\pi = \lor \pi_i$ since for any $x \in P$ there exists $\pi_i \leq \pi$ in the covering such that $\pi_i(x) = \pi(x)$. Let $\{\pi_i \leq \pi\}$ be a covering and suppose that $\theta \leq \pi$ in $K(P)$. For $x \leq y$ in $P$, $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$ for some $\pi_i \leq \pi$ in the covering. Hence,

$$(\theta \land \pi_i)(x) = \theta(x) \land \pi(x) = \theta(x),$$

and similarly $(\theta \land \pi_i)(y) = \theta(y)$, proving that $\{\theta \land \pi_i \leq \theta\}$ is also a covering. It follows that $K(P)$ is a Grothendieck topology [1] and that $S$ is a presheaf on $K(P)$.

We wish to prove that $S$ is a sheaf of $C(P)$-bicomodules, and so we need:

**Proposition 1.** Finite products exists in the category of $C$-bicomodules where $C$ is any $k$-coalgebra.

**Proof.** Let $\{M_i\}$ be any finite family of $C$-bicomodules with structure maps

$$\rho_i : M_i \to M_i \otimes C, \lambda_i : M_i \to C \otimes M_i.$$ 

It is easily verified that the $k$-linear maps

$$\begin{array}{c}
\Pi M_i \xrightarrow{\Pi \rho_i} \Pi(M_i \otimes C) \to (\Pi M_i) \otimes C, \\
\Pi M_i \xrightarrow{\Pi \lambda_i} \Pi(C \otimes M_i) \to C \otimes \Pi M_i,
\end{array}$$

exist.

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give $\Pi M_i$ the structure of $C$-bicomodule where

$$\Pi (M_i \otimes C) \to (\Pi M_i) \otimes C$$

is the well-known $k$-linear isomorphism. (We have used the fact that for a finite family $\{M_i\}$ of $k$-modules, $\Pi M_i \simeq \Pi M_i$.)

**Proposition 2.** $S$ is a sheaf of $C(P)$-bicomodules on $K(P)$.

**Proof.** Let $\{\pi_i \leq \pi\}$ be a covering in $K(P)$. We must prove that the following diagram is exact:

$$S(\pi) \stackrel{\phi}{\to} \prod S(\pi_i) \stackrel{\phi_1}{\to} \prod S(\pi_i \wedge \pi_j),$$

where $\phi$ is the map induced by the $S(\pi_i, \pi)$ (i.e., $\phi = \Pi S(\pi_i, \pi)$) and $\phi_1$ is the map induced by applying $S$ to the projections on the first factor ($S(\pi_i \wedge \pi_j, \pi_i)$) and $\phi_2$ is induced by the projections on the second factor.

Suppose $\phi(x, y) = 0$. For each $(x, y)$ there exists $\pi_i \leq \pi$ in the covering such that $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$. So, from the definition of $S(\pi_i, \pi)$, it follows that $a_{x,y} = 0$. Hence $\phi$ is injective.

Suppose $\phi(x, y) = 0$. For each $(x, y)$ there exists $\pi_i \leq \pi$ in the covering such that $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$. So, from the definition of $S(\pi_i, \pi)$, it follows that $a_{x,y} = 0$. Hence $\phi$ is injective.

Suppose that $(\alpha_i) \in \Pi S(\pi_i)$ and that $\phi_1(\alpha_i) = \phi_2(\alpha_i)$. We must show that $(\alpha_i) = \phi(\alpha)$ for some $\alpha \in S(\pi)$. Let $\alpha_i = \sum a(i)_{x,y} (x, y)$. For $a(i)_{x,y} \neq 0$, choose $\pi_i \leq \pi$ in the covering such that $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$. Then $(x, y)$ occurs in the $S(\pi_i \wedge \pi_j)$-component of $\phi_1(\alpha_i)$ with coefficient $a(i)_{x,y} \neq 0$ since $\pi_i(x) = (\pi_i \wedge \pi_j)(x)$ and $\pi_i(y) = (\pi_i \wedge \pi_j)(y)$. But $\phi_1(\alpha_i) = \phi_2(\alpha_i)$; hence, $\pi_i(x) = (\pi_i \wedge \pi_j)(x)$ and $\pi_i(y) = (\pi_i \wedge \pi_j)(y)$, from which it follows that $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$. This proves that, if $(x, y)$ occurs in $\alpha_i$ with non-zero coefficient, then $\pi_i(x) = \pi(x)$ and $\pi_i(y) = \pi(y)$. But, for any $(x, y)$ and any $n$ and $j$ such that $(x, y)$ occurs with non-zero coefficient in the $S(\pi_n \wedge \pi_j)$-component of $\phi_1(\alpha_i) = \phi_2(\alpha_i)$, we must have $a(n)_{x,y} = a(j)_{x,y}$. It follows that $(\alpha_i) = \phi(\alpha)$ for some $\alpha \in S(\pi)$ and, hence, that $S$ is a sheaf of $C(P)$-bicomodules.

### 3. Applications

Since $S(\pi)$ is a $C(P)$-bicomodule for each $\pi \in K(P)$, there is induced on $S(\pi)$ an $A(P)$-bimodule structure [6] where the right and left actions of $F \in A(P)$ on a basis element $(x, y) \in S(\pi)$ are given respectively by

$$(x, y) \cdot F = \sum_{x \leq t \leq y \atop \pi(t) = \pi(x)} F(x, t)(t, y),$$

$$F \cdot (x, y) = \sum_{x \leq t \leq y \atop \pi(t) = \pi(y)} F(t, y)(x, t).$$
Hence, $S(\pi)^* = \text{hom}_k(S(\pi), k)$ is, in a natural way, an $A(P)$-bimodule for each $\pi \in K(P)$. Let this $A(P)$-bimodule structure be determined by the $k$-algebra maps

$$R(\pi) : A(P)^0 \to \text{end}_k(S(\pi)^*),$$

$$L(\pi) : A(P) \to \text{end}_k(S(\pi)^*),$$

where $A(P)^0$ is the algebra opposite to $A(P)$. That is, for $F \in A(P), f \in S(n^*), \text{ and } x \leq y$ in $P$,

$$R(\pi)(F)(x, y) = \sum_{x \leq t, t \leq y \atop \pi(t) = \pi(y)} f(x, t) F(t, y),$$

$$L(\pi)(F)(x, y) = \sum_{x \leq t, y \leq t \atop \pi(x) = \pi(t)} F(x, t) f(t, y).$$

For $\pi \leq \theta$ in $K(P)$, the map

$$S(\pi, \theta) : S(\theta) \to S(\pi)$$

of $C(P)$-bicomodules induces a map

$$S(\pi, \theta)^* : S(\pi)^* \to S(\theta)^*$$

of $A(P)$-bimodules where $S(\pi, \theta)^*(f) = f \circ S(\pi, \theta)$ for any $f \in S(\pi)^*$.

The $A(P)$-bimodules described above yield generalized Möbius inversion formulas; namely, for $\pi \in K(P)$ and $x \leq y$ in $P$, if

$$f(x, y) = \sum_{x \leq t, t \leq y \atop \pi(t) = \pi(y)} f'(x, t),$$

$$g(x, y) = \sum_{x \leq t, t \leq y \atop \pi(x) = \pi(t)} g'(t, y),$$

$$h(x, y) = \sum_{x \leq t, t \leq s \leq y \atop \pi(x) = \pi(s) = \pi(t)} h'(t, s),$$

then

$$f'(x, y) = \sum_{x \leq t, t \leq y \atop \pi(t) = \pi(y)} f(x, t) \mu(t, y),$$

$$g'(x, y) = \sum_{x \leq t, t \leq y \atop \pi(x) = \pi(t)} \mu(x, t) g(t, y),$$

$$h'(x, y) = \sum_{x \leq t, t \leq s \leq y \atop \pi(x) = \pi(s) = \pi(t)} \mu(x, t) h(t, s) \mu(s, y).$$
Let $\sigma : P \to Q$ be an increasing map. We say that $\sigma$ is final on the interval $[x, y]$ of $P$ provided

for any $u \in [\sigma(x), \sigma(y)]$, there exists $\bar{u} \in [x, y]$ such that $\sigma(\bar{u}) = u$ and $\sigma^{-1}[\sigma(x), u] \cap [x, y] = [x, \bar{u}]$.

$\sigma$ is said to be initial on $[x, y]$ if $\sigma$ is final on $[y, x]$ as a map of the corresponding dual posets ($\sigma : P^* \to Q^*$). That is,

for any $u \in [\sigma(x), \sigma(y)]$, there exists $\bar{x} \in [x, y]$ such that $\sigma(\bar{x}) = u$ and $\sigma^{-1}[u, \sigma(y)] \cap [x, y] = [\bar{x}, y]$.

**Proposition 3.** Suppose that an increasing map $\sigma : P \to Q$ is final on an interval $[x, y]$ of $P$. Then for $\sigma(x) \leq u \leq w \leq \sigma(y)$,

$$R_\sigma(\zeta_P) \mu_P(\bar{u}, \bar{w}) = \mu_Q(u, w).$$

**Proof.** For any $\sigma(x) \leq u \leq w \leq \sigma(y)$ in $Q$ with $\sigma$ final on $[x, y]$,

$$\sum_{u \leq z \leq w} R_\sigma(\zeta_P) \mu_P(\bar{u}, \bar{z}) = \sum_{u \leq z \leq w} \sum_{\tilde{t} \leq \bar{t} \leq \bar{w}} \mu_P(\bar{u}, \bar{t}) = \sum_{\tilde{u} \leq \tilde{t} \leq \bar{w}} \mu_P(\tilde{u}, \bar{t}) = \delta_{\tilde{u}, \bar{w}}.$$

The assertion follows since a left inverse for $\zeta_Q$ restricted to the interval $[u, w]$ is also a right inverse.

Now we state, without proof, the analogous result for initial maps but, henceforth, do so only when such a result will be used.

**Proposition 3'.** Suppose that an increasing map is initial on an interval $[x, y]$ of $P$. Then for $\sigma(x) \leq u \leq w \leq \sigma(y)$,

$$L_\sigma(\zeta_P) \mu_P(\bar{u}, \bar{w}) = \mu_Q(u, w).$$

**Corollary 1.** If $\sigma : P \to Q$ is an increasing map which is final on $[x, y]$ and if $\sigma(x) \neq \sigma(y)$, then

$$\sum_{\sigma(x) \leq z \leq \sigma(y)} \mu_P(\bar{z}, \bar{t}) = 0.$$
Proof. Since \( \sigma(x) \neq \sigma(y) \),

\[
\sum_{\sigma(z) \leq \sigma(y)} \mu(o, \sigma(y)) = 0.
\]

Apply Proposition 3 to each term of this sum.

**Corollary 2.** Let \( \sigma : M \to N \) be a lattice homomorphism (i.e., meet and join preserving) of locally finite lattices. Suppose that \( \sigma \) is convex on the interval \([x, y]\); that is, \( \sigma([x, y]) = [\sigma(x), \sigma(y)] \). Then \( \sigma \) is both final and initial on \([x, y]\), and, hence, for any \( \sigma(x) \leq u \leq w \leq \sigma(y) \),

\[
R_o(\xi_M) \mu_M(u, w) = \mu_N(u, w) = L_o(\xi_M) \mu_M(u, w).
\]

Proof. For \( u \in [\sigma(x), \sigma(y)] \), let

\[
\begin{align*}
\bar{u} &= \bigvee \{ t \in [x, y] : \sigma(t) = u \}, \\
y &= \bigwedge \{ t \in [x, y] : \sigma(t) = u \}.
\end{align*}
\]

**Corollary 3 (Rota [2, Th. 1]).** Let \( J : P \to P \) be a closure operator on a poset \( P \), and let \( Q \) be the poset of \( J \)-closed points of \( P \). Then for any \( J(x) \leq J(y) \),

\[
R_J(\xi_P) \mu_P(J(x), J(y)) = \mu_Q(J(x), J(y)).
\]

Proof. The increasing map \( J : P \to Q \) is final on the interval \([J(x), J(y)]\) of \( P \). Apply Proposition 3.

**Corollary 3'.** Let \( J : P \to P \) be a coclosure operator on a poset \( P \), and let \( Q \) be the poset of \( J \)-coclosed points of \( P \). Then, for any \( J(x) \leq J(y) \),

\[
L_J(\xi_P) \mu_P(J(x), J(y)) = \mu_Q(J(x), J(y)).
\]

**Corollary 4 (Rota [4, Th. 1, p. 347]).** Let \( Q \) and \( P \) be posets having a 0 and a 1, respectively, and let

\[
\tau : Q \to P, \sigma : P \to Q
\]

be increasing maps satisfying

\begin{align*}
(1) & \quad \sigma(\tau(q)) \geq q, \quad \text{for all } q \in Q, \\
(2) & \quad \tau(\sigma(p)) \leq p, \quad \text{for all } p \in P.
\end{align*}

Then both \( P \) and \( Q \) have both a 0 and a 1, and if, moreover,

\[
(3) \quad \tau(q) = 1 \text{ if and only if } q = 1,
\]
then
\[ \mu_\sigma(\sigma(0), 1) = \sum_{\sigma(p) = \sigma(0)} \mu_p(p, 1). \]

**Proof.** Conditions (1) and (2) imply that the pair \( \sigma, \tau \) is a Galois connection between \( Q \) and \( P^* \), the dual of \( P \). It follows that \( J' = \sigma \circ \tau \) is a closure operator on \( Q \), that \( J = \tau \circ \sigma \) is a coclosure operator on \( P \), and that \( J(P) \) and \( J'(Q) \) are isomorphic posets. Moreover,

\[ J(P) = \{ \tau(q) : q \in Q \}, \]
\[ J'(Q) = \{ \sigma(p) : p \in P \}. \]

Notice that \( \tau(q) \leq p \) if and only if \( q \leq \sigma(p) \). So, since \( 0 \in Q \) and \( 1 \in P \), it follows that \( \tau(0) = 0 \in P \) and \( \sigma(1) = 1 \in Q \). Hence, both \( P \) and \( Q \) are finite since in this note, poset is synonymous with locally finite partially ordered set. Also,

\[ J(0) = J(\tau(0)) = \tau(0) = 0. \]

Now we prove that \( \tau : Q \to J(P) \) is final on \([0, 1] = Q\). But

\[ [\tau(0), \tau(1)] = [0, 1] = J(P). \]

Let \( p \in J(P) \). Let \( \bar{p} = \sigma(p) \in Q \). Then \( \tau(\bar{p}) = J(p) = p \). Moreover, if for \( q \in Q \), \( \tau(q) \leq p \), then \( q \leq \sigma(\tau(q)) \leq \sigma(p) = \bar{p} \). Hence, \( \tau \) is final on \([0, 1] = Q\) and, from Proposition 3,

\[ R_\sigma(\zeta_\sigma) \mu_\sigma(\sigma(0), 1) = \mu_{J(P)}(0, 1). \]

But, from condition (3), we have

\[ R_\sigma(\zeta_\sigma) \mu_\sigma(\sigma(0), 1) = \mu_\sigma(\sigma(0), 1). \]

Applying Corollary 3' to \( J : P \to J(P) \) and the points

\[ 0 = J(0) \leq J(1) = 1, \]

we have

\[ L_J(\zeta_p) \mu_p(0, 1) = \mu_{J(P)}(0, 1); \]

combining equalities, we have

\[ \mu_\sigma(\sigma(0), 1) = L_J(\zeta_p) \mu_p(0, 1) = \sum_{0 \leq p \leq 1, J(p) = J(0)} \mu_p(p, 1). \]

But \( J(0) = 0 \), and \( J(p) = 0 \) if and only if \( \sigma(p) = \sigma(0) \) from which the corollary follows.
Finally, we prove a result which involves the topology of $K(P)$. Recall that, if $\sigma : P \to Q$ is an increasing map, we abuse notation by letting $\sigma$ also denote the corresponding member of $K(P)$. So if $\tau : P \to Q'$ is another increasing map and if $x \in P$, then $\sigma(x) = \tau(x)$ refers, not to equality of function values which is meaningless unless $Q = Q'$, but to equality of the $\sigma$-class of $x$ and the $\tau$-class of $x$.

**Corollary 5**. Let $\sigma : P \to Q$ be an increasing map. Let $\{\sigma_i : P \to Q_i\}$ be a finite family of increasing maps such that $\sigma_i \leq \sigma$ in $K(P)$ for each $i$ and the family $\{\sigma_i \leq \sigma\}$ is a covering in $K(P)$. Suppose that, if $x \leq y$ in $P$ and if $\sigma(x) = \sigma(y)$, then $\sigma_i$ is final on $[x, y]$. Then, for any $x \leq y$ in $P$, there exists $\sigma_i$ in the family such that

$$R_{\sigma_i}(\zeta_P) \mu_P(\sigma_i(x), y) = \mu_{Q_i}(\sigma_i(x), \sigma_i(y)).$$

**Proof**. Let $x \leq y$. Since $\{\sigma_i \leq \sigma\}$ is a covering, there exists a member $\sigma_i \leq \sigma$ such that $\sigma_i(x) = \sigma(x)$ and $\sigma_i(y) = \sigma(y)$. By assumption, $\sigma_i$ is final on $[x, y]$. From Proposition 1,

$$R_{\sigma_i}(\zeta_P) \mu_P(\sigma_i(x), y) = \mu_{Q_i}(\sigma_i(x), \sigma_i(y)).$$

Applying the map $S(\sigma_i, \sigma)^* \text{ of } A(P)$-bimodules to this equality gives the desired result.

Given an increasing map $\sigma : P \to Q$, Smith defines [5] an operator $B$ on $A(P)$. It is easily seen that $B = R_\sigma(\zeta)$. In fact, many of the results of [5] are analogous to results on the $A(P)$-bimodules $S(\pi)^*$, $\pi \in K(P)$. We give one such example here. Since, for any $\pi, \theta \in K(P)$, $S(\pi) = S(\theta)$ as $k$-modules, any $k$-linear map of $S(\pi)$ is also a $k$-linear map of $S(\theta)$.

**Proposition 4**. For any $\pi \leq \theta$ in $K(P)$,

$$(I \otimes \rho(\pi)) \circ \rho(\theta) = (\rho(\theta) \otimes I) \circ \rho(\pi).$$

**Proof**. For any $x \leq y$ in $P$,

$$\{ (u, t) : x \leq u \leq t \leq y, \theta(u) = \theta(y), \pi(t) = \pi(y) \}$$

$$= \{ (u, t) : x \leq u \leq t \leq y, \theta(u) = \theta(t), \pi(t) = \pi(y) \}$$

since $\pi(t) = \pi(y)$ implies $\theta(t) = \theta(y)$.

Translating Proposition 4 to the $A(P)$-bimodules $S(\pi)^*$ and $S(\theta)^*$ yields:

**Corollary 6**. For $\pi \leq \theta$ in $K(P)$ and for $f, g, h \in A(P)$,

$$R_\pi(f)(R_\theta(g) h) = R_\theta(R_\pi(f) g) h.$$
Taking $\theta = \pi$ in Corollary 6 gives:

**Corollary 7.** The association

$$(f, g) \mapsto R_\pi(g)f$$

gives $S(\pi)^*$ the structure of associative $k$-algebra with right identity $\delta =$ Kronecker's delta.

Only in the special case $\pi = \{P\}$ is $\delta$ also a left identity. In that case, we get the incidence algebra $A(P)$, and we write $f * g$ rather than $R_\pi(g)f$.

**Corollary 8.** For any $\pi \in K(P)$ and any $f, g, h \in A(P)$,

$$R_\pi(f)(h * g) = h * (R_\pi(f)g).$$

Compare Corollary 8 to [5, Th. 1, Cor. 2].

4. **Remarks**

Let $\sigma : P \to Q$ be an increasing map. Then there is induced a map $\sigma^{-1} : K(Q) \to K(P)$ where, for $\pi \in K(Q)$, $\sigma^{-1}(\pi)$ is the convex partition of $P$ consisting of all the distinct non-empty subsets of $P$ of the form $\sigma^{-1}(\pi(q))$, $q \in Q$. If $\pi \leq \theta$ in $K(Q)$, then $\sigma^{-1}(\pi) \leq \sigma^{-1}(\theta)$. It follows that $\sigma^{-1} : K(Q) \to K(P)$ is a functor from $K(Q)$ to $K(P)$. $\sigma^{-1}(\{Q\}) = \{P\}$, and it is easy to show that $\sigma^{-1}$ preserves coverings and fibered products. Hence, $\sigma^{-1}$ is a continuous map [1] from the Grothendieck topology $K(P)$ to the Grothendieck topology $K(Q)$. ($\sigma$ should be thought of as a continuous map from $P$ to $Q$ and $\sigma^{-1}$ as a map of the open sets of $Q$, $K(Q)$, to the open sets of $P$, $K(P)$.) $\sigma^{-1}$ induces, as usual, a map $\sigma_\ast$ from sheaves on $K(P)$ to sheaves on $K(Q)$. So if $S$ is the sheaf on $K(P)$ described in Section 2 and, if $\pi \in K(Q)$, then

$$\sigma_\ast(S)(\pi) = S(\sigma^{-1}(\pi)).$$

$\sigma_\ast(S)$ is a sheaf of $C(P)$-bicomodules on $K(Q)$. This raises the question: can the sheaf $\sigma_\ast(S)$ of $C(P)$-bicomodules on $K(Q)$ be related to the given sheaf of $C(Q)$-bicomodules on $K(Q)$?

Fix a poset $P$. Let $P$ denote the category whose objects are all increasing maps $\sigma : P \to Q$ and whose morphisms consist of all commuting triangles of increasing maps:

$$
\begin{array}{ccc}
Q & \xrightarrow{\tau} & Q' \\
\downarrow{\sigma} & & \downarrow{\sigma'} \\
P & \xleftarrow{\sigma'} & P
\end{array}
$$


An object $o : P \rightarrow Q$ may be associated with the object $o$ of $K(P)$ and the above morphism of $P$ may be associated with the morphism $\sigma \leq \sigma'$ of $K(P)$. That is, there is a natural functor $F$ from $P$ to $K(P)$. For $\sigma$ an object of $P$, define $B(\sigma) = S(F(\sigma))$, and for a morphism $\tau$ of $\sigma$ to $\sigma'$ (as above), define $B(\tau) = S(F(\sigma), F(\sigma'))$. Then $B$ is a contravariant functor from $P$ to the category of $C(P)$-bicomodules. Dualizing, there results a functor from $P$ to the category of $A(P)$-bimodules.

REFERENCES