On Solutions to Nonlinear Reaction–Diffusion–Convection Equations with Degenerate Diffusion

Yunguang Lu

Departamento de Matemáticas y Estadística, Universidad Nacional de Colombia, Bogotá, Colombia
E-mail: yglu@matematicas.unal.edu.co

and

Willi Jäger

Institut für Angewandte Mathematik, Universität Heidelberg, Germany

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This paper consists of three parts. In Section 2, the Cauchy problem for general reaction-convection equations with a special diffusion term $G(u) = u^m$ in multi-dimensional space is studied and Hölder estimates of weak solutions with explicit Hölder exponents are obtained by applying the maximum principle. In Section 3, for any nondecreasing smooth function $G$, the sharp regularity estimate $G(u) \in C^{1,1}$ up to the boundaries for the radial solution $u$ of the general equation of Newtonian filtration is obtained by applying the maximum principle with the Minty's device. A direct by-product is the sharp regularity estimate of the temperature to the classical two-phase Stefan model. In Section 4, the Hölder continuity of weak solutions of the initial-boundary value problem for general nonlinear reaction-diffusion-convection equations is considered. Under the critical condition on the diffusion function $G$: $\text{meas}[u : G(u) = g(u) = 0] = 0$, we obtain a Hölder continuous solution $u$ and the sharp regularity estimate $G(u) \in C^{1,1}$ up to the boundaries. Our proof is based on the maximum principle.

1. INTRODUCTION

This paper is concerned with the regularity of solutions to the general reaction-diffusion-convection equation

$$u_t = AG(u) + \sum_{i=1}^{N} f_i(u) u_i + h(u),$$

(1.1)
where $G(u)$ is a nondecreasing smooth function, and $N$ denotes the space dimension. The Eq. (1.1) arises in several applications in which $u$ stands for a nonnegative quantity. For example, the equation

$$u_t = A(u^m)$$  \hspace{1cm} (1.2)

models the non-stationary flow of a compressible Newtonian fluid in a porous medium under polytropic conditions. The value of $u \geq 0$ is proportional to the density of the fluid. If the flow is not polytropic, (1.2) is replaced by the more general equation of Newtonian filtration

$$u_t = AG(u).$$  \hspace{1cm} (1.3)

If the medium has heat sources, then (1.3) is replaced by an equation of the form

$$u_t = AG(u) + h(u).$$  \hspace{1cm} (1.4)

A model of the process of heat propagation in a moving non-linear medium is given by the non-linear heat equation with transfer

$$u_t = AG(u) + \sum_{i=1}^{N} f_i(u) x_i.$$  \hspace{1cm} (1.5)

Equation (1.1) is parabolic at points where $g(u) = G'(u) > 0$, but which degenerate when $g(u) = 0$. If $g(u) \equiv 0$ and one of $f_i(u)$ is nonlinear, (1.1) is the standard nonlinear hyperbolic conservation law with a source. The continuous solution of (1.1), in general, exists only locally in time. This means that shock waves appear in the solution in finite time. We refer the readers to the book [27] and the references therein for the details about the theory of shock waves.

It is well-known that degenerate parabolic equations do not possess a classical solution. The solution $u$ in general fails to be smooth at the interface between a parabolic region and a region of parabolic degeneracy. Fortunately, if the diffusion function $G(u)$ has only one degenerate point, such as the porous media equation (1.2), the following estimate is known to be optimal,

$$|(u^m - 1)_x| \leq M,$$  \hspace{1cm} (1.6)

for solutions of the Cauchy problem (1.2) with initial data in the one dimensional space [1]. However the regularity property of solutions for the Cauchy problem in the multidimensional space or for the initial-boundary value problem for arbitrary $N \geq 1$ is different. We can see the differences from the following facts.
Consider the porous media equation (1.2) with bounded, continuous, nonnegative function \( u_0(x_1, x_2, \ldots, x_N), N \geq 2 \) on the line \( t = 0 \). A numerical example constructed by Graveleau shows that if there are holes in \( \text{supp} \ u_0 \), then it is possible for \( \nabla u^{m-1} \) to blow up. The existence and uniqueness of Graveleau’s solution was proved later by Aronson and Graveleau by a construction of radially symmetric solutions (See [2]).

On the other hand, Caffarelli, Vazquez and Wolanski in [8] showed that \( \{u_m\} \) is bounded in \( \mathbb{R}^N \) for a suitable large time \( T \). In some sense, this estimate is best possible as is shown by Graveleau’s solution.

From the above two basic results, it is natural to ask what is the optimal regularity estimate of solutions in the space variables for \( t > 0 \).

In [22], Lu studied a special case \( m = 2, N = 2 \) and obtained a Hölder solution with the regularity estimate \( |u| \leq M \) in \( \mathbb{R}^N \times (0, \infty) \). This is the first Hölder solution with explicit Hölder exponent in the multidimensional space. Of course, this does not imply the optimal regularity of solutions in the multidimensional space. However it partially improved previous results such as the global continuity of weak solutions with a logarithmic modulus of continuity by Caffarelli and Friedman [7] and the local Hölder continuity of weak solutions with implicit Hölder exponents in [4, 7, 12, 26, 28] (See also the references of the review paper [20]).

Consider
\[
\begin{align*}
\begin{cases}
 u_t &= (u^m)_{xx}, \\
 u(x, 0) &= x^m, \\
 u(0, t) &= 0, u(1, t) &= 1.
\end{cases}
\end{align*}
\]

The solution of (1.7) is \( u = x^m \). Then \( |(u^s)_x| = \frac{m}{2} x^{(s/m) - 1} = \infty \) at \( x = 0 \) for any \( s < m \). So the sharp regularity estimate of solutions for (1.2) with initial-boundary values is \( |(u^m)_x| \leq M \), even in the one dimensional space, which is much weaker than the estimate (1.6) for solutions of the Cauchy problem.

If the diffusion function \( G(u) \) has infinite degenerate points, the phenomena of solutions are more complicated.

When the measure (let it be \( Z \)) of degenerate points of \( G(u) \) is zero, i.e.,
\[
Z = \text{meas}\{u : G'(u) = g(u) = 0\} = 0,
\]
then in the one dimensional space, the solutions \( u \) of the Cauchy problem (1.1) with a smooth initial function is shown to be Hölder continuous and \( G(u) \in C^{1,1} \). (See [18, 19] for the details.)
Remark 1. A function $h(r, t)$ belongs to $C^{(1)}$ if $|h(r, t)| \leq M$ and

$$\frac{|h(r_2, t_2) - h(r_1, t_1)|}{|r_2 - r_1| + |t_2 - t_1|^{1/2}} \leq M$$

(1.9)

for any points $(r_2, t_2)$ and $(r_1, t_1); h(r) \in C^{(1)}$ means that (1.9) is satisfied for $t_1 = t_2$.

When the measure $Z > 0$, the continuous solution of (1.1), in general, exists only locally in time. However if $f_i(u) \equiv 0$ and $h(u) \equiv 0$ in (1.1), then $G(u)$, which is corresponding to the temperature in the classical two-phase Stefan problem, is continuous for the unique solution $u$ of (1.1) with initial-boundary values [6].

In this paper we mainly focus our study on the regularity of solutions to (1.1) in three respects.

In Section 2, we extend the results of [22] to the more general equation (2.1). Specially we obtain a Hölder solution with explicit Hölder exponent for the Cauchy problem (2.1) and (2.2). In Section 3, for any nondecreasing smooth function $G$, we obtain the sharp regularity estimate $G(u) \in C^{(1)}$ up to the boundaries for the radial solution $u$ of the general equation of Newtonian filtration (3.1) with initial boundary values (3.2) by applying the maximum principle with the Minty’s device [14]. A direct by-product is the sharp regularity estimate of the temperature of the classical two-phase Stefan model. In Section 4, the Hölder continuity of weak solutions of the initial-boundary value problem for general nonlinear reaction–diffusion–convection equations is considered. Under the critical condition on the diffusion function $G$: $\text{meas}\{u : G'(u) = g(u) = 0\} = 0$, a Hölder continuous solution $u$ and the sharp regularity estimate $G(u) \in C^{(1)}$ up to the boundaries are obtained by applying the maximum principle.

2. CAUCHY PROBLEM IN MULTI-DIMENSIONAL SPACE

In this part, we study the Hölder estimates of solutions to the degenerate parabolic equation in the $N$-dimensional space

$$u_t = A(u^m) + \sum_{i=1}^{N} f_i(u)x_i$$

(2.1)

with initial data

$$u(x, 0) = u_0(x_1, x_2, ..., x_N) \geq 0.$$  

(2.2)
The main results in this part are in the following

**Theorem 1.** There exists a weak solution $u$ of the Cauchy problem (2.1), (2.2) which satisfies the regularity estimates in (I), (II).

(I) Let $|f'(u)| \leq M u^{m-2}$ for $0 \leq u \leq L$ with $L$ being the upper bound of $u_0$ and $u_0(x_1, x_2, ..., x_N) \) a suitable smooth function.

(i) If $1 < m < 1 + (1/\sqrt{N-1})$, then $u^k$ is uniformly Lipschitz continuous in the space variables for any $t > 0$, where

$$l_1 \geq m - \frac{1}{2} - \sqrt{2 - 2(m - 1)^2 (N - 1)}/4.$$  

(ii) If $1 < m < 1 + (1/\sqrt{N})$, then $u^k$ is uniformly Lipschitz continuous in the space variables for any $t > 0$, where

$$l_2 = \min \left\{ l_1, m - \frac{1}{2} \sqrt{1 - N(m - 1)^2}/2 \right\}.$$  

(II) $u^\beta$ is Hölder continuous with respect to $t$ with exponent $\frac{1}{2}$, where $\gamma = 2\beta - m + 1$ and $\beta = l_1$ for case (i) and $\beta = l_2$ for case (ii).

**Proof.** To prove Theorem 1, one can add a small positive constant $\varepsilon$ to the initial data and consider the problem in uniformly parabolic region $u^\varepsilon \geq \varepsilon$. After obtaining Hölder estimates on $u^\varepsilon$, one can extract a convergent subsequence $u^k$ such that the limit function $u$ of $u^k$ is a weak solution of the Cauchy problem (2.1), (2.2). For simplicity, we omit this standard process and only give necessary uniform Hölder estimates on $u^\varepsilon$.

Let $v = u^\varepsilon$. Then

\[
(u^m)_{xx} = (u^m - n v)_{xx} = u^m - n v_{xx} + 2(u^{m-n})_x v_x + \varepsilon (u^{m-n})_{xx}
\]

\[
= u^m - n v_{xx} + 2(m - n) u^{m-n - 1} u_x v_x + \varepsilon ((m - n) u^{m-n - 1} u_x)_x
\]

\[
= u^m - n v_{xx} + 2(m - n) u^{m-2} v_x^2 + \frac{m - n}{n} \varepsilon (u^{m-2} v_x)_x
\]

\[
= u^m - n v_{xx} + 2(m - n) u^{m-2} v_x^2 + \frac{m - n}{n} u^{m-2} v_{xx}
\]

\[
+ \frac{(m - n)(m - 2n)}{n} v_x u^{m-n-1} u_x
\]

\[
= \frac{m}{n} u^m - n v_{xx} + \frac{m(m - n)}{n^2} u^{m-2} v_x^2,
\]

(2.3)
and
\[
A(u^n) = \frac{m}{n} u^{m-n} A v + \frac{m(m-n)}{n^2} u^{m-2n} \sum_{i=1}^{N} v_{x_i}^2. \tag{2.4}
\]

Let \( w = \frac{1}{2} \sum_{i=1}^{N} v_{x_i}^2 \). It follows from (2.3), (2.4) that
\[
v_t = mw^{n-1} A v + \sum_{i=1}^{N} f'_i(u) v_{x_i}
= mw^{m-n} A v + \frac{2m(m-n)}{n} w^{m-n} + \sum_{i=1}^{N} f'_i(u) v_{x_i}. \tag{2.5}
\]

Let \( h(v) = mv^{(m-1)/n} \). Then
\[
(h(v) A v)_{x_i} v_{x_i} = (h(v)(A v) v_{x_i})_{x_i} - h(v)(A v) v_{x_i}
= \sum_{j \neq i} (h(v) v_{x_j} v_{x_i})_{x_j} + \left( h(v) \left( \frac{v_{x_j}^2}{2} \right)_{x_j} \right) - h(v)(A v) v_{x_i}
= \sum_{j \neq i} \left[ v_{x_j} h'(v) v_{x_j}^2 + h(v) v_{x_j} v_{x_i} v_{x_j} \right] + h(v) \left( \frac{v_{x_j}^2}{2} \right)_{x_j}
+ h(v) v_{x_i} h'(v) v_{x_i} - h(v)(A v) v_{x_i}
= \sum_{j=1}^{N} \left[ h'(v) v_{x_j}^2 v_{x_i} + h(v) \left( \frac{v_{x_j}^2}{2} \right)_{x_j} \right] - h(v) v_{x_i}^2. \tag{2.6}
\]

It follows from (2.5) that
\[
(v_{x_i})_{x_i} v_{x_i} = (h(v)(A v)(v_{x_i}))_{x_i} - h(v)(A v) v_{x_i}
= \sum_{j=1}^{N} \left[ h'(v) v_{x_j}^2 v_{x_i} + h(v) \left( \frac{v_{x_j}^2}{2} \right)_{x_j} \right] - h(v) v_{x_i}^2.
\]

\[ \begin{align*}
&= (h(v) \Delta v)_{x_i} v_{x_i} + \frac{2(m-n)}{n} e^{(m-n-1)h v_{x_i} w_{x_i}} \\
&+ \frac{2m(m-n)(m-n-1)}{n^2} e^{(m-2n-1)h v^2 w} \\
&+ \sum_{j=1}^{N} f'_j(u) \left( \frac{v_{x_j}^2}{2} \right) + \sum_{j=1}^{N} \frac{f''_j(u)}{m^2} v_{x_j}^2 v_{x_j}. 
\end{align*} \tag{2.7} \]

Combining (2.3)–(2.7) and using (2.1) we obtain the following equation in the unknown \( w \)

\[ \begin{align*}
&= 2h'(v)(\Delta v) w + h(v) \Delta w - \sum_{i,j=1}^{N} h(v) v_{x_i x_j} \\
&+ \frac{2m(m-n)}{n} e^{(m-n-1)h} \sum_{i=1}^{N} v_{x_i} w_{x_i} + \frac{4m(m-n)(m-n-1)}{n^2} e^{(m-2n-1)h y_i^2} \\
&+ \sum_{j=1}^{N} f'_j(u) w_{x_j} + \sum_{j=1}^{N} \frac{2f''_j(u)}{m^2} v_{x_j} w. 
\end{align*} \tag{2.8} \]

Let \( P = \Delta v. \) Then

\[ \begin{align*}
&= \sum_{i,j=1}^{N} h(v) v_{x_i x_j} \geq h(v) \sum_{i=1}^{N} v_{x_i}^2 \geq \frac{h(v)}{N} \left( \sum_{i=1}^{N} v_{x_i} \right)^2 = \frac{h(v)}{N} P^2. \tag{2.9} \]

For any constant \( s, \) it follows from (2.5) that

\[ \begin{align*}
(v^s)_i &= sm^{(m-1)/n} + s - 1 \Delta v + \frac{2ms(m-n)}{n} e^{(m-n-1)/n-1} (v^s) \\
&+ \sum_{i=1}^{N} f'_j(u)(v^s)_{x_i}. \tag{2.10} \]

Let \( v^s w = z. \) Then it follows from (2.8), (2.10) that

\[ \begin{align*}
z_i &= \left( s + \frac{2(m-1)}{n} \right) m^{(m-1)/n-1} z \Delta v \\
&+ \left[ \frac{4m(m-n)(m-n-1)}{n^2} + \frac{2ms(m-n)}{n} \right] e^{(m-2n-1)/n-1} z^2 \\
&- \sum_{j=1}^{N} m^{(m-1)/n} + s v_{x_j x_j} + \sum_{j=1}^{N} f'_j(u) z_{x_j} + \sum_{j=1}^{N} \frac{2f''_j(u)}{m^2} u^{1-n} v_{x_j} z \\
&+ m^{(m-1)/n-1} A(z^{-1}) + \frac{2m(m-n)}{n} e^{(m-n-1)/n-1} \sum_{i=1}^{N} v_{x_i} (z^{-1} z)_{x_i}, \tag{2.11} \]
Since

\[ A(e^{-z}) = \sum_{j=1}^{N} (e^{-z})_{x_j} = \sum_{j=1}^{N} \left[ (v^{-s})_{x_j} z + 2(v^{-s})_{x_j} z + v^{-s} x_j \right] \]

\[ = \sum_{j=1}^{N} \left[ (-s) v^{-s-1} x_j + s(s+1) v^{-s-2} x_j^2 z + 2(v^{-s})_{x_j} z + v^{-s} x_j \right] \]

\[ = (-s v^{-s-1} A + 2(s+1) v^{-s-2} w) z \]

\[ + \sum_{j=1}^{N} \left[ -2 s v^{-s-1} x_j z + v^{-s} x_j \right] \]

(2.12)

and

\[ \sum_{i=1}^{N} v_{i}(v^{-s})_{x_i} = \sum_{i=1}^{N} \left[ (-s) v^{-s-1} x_i^2 z + v^{-s} x_i \right] \]

\[ = -2 s v^{-s-1} w z + \sum_{i=1}^{N} v^{-s} x_i z_i \]

(2.13)

it follows from (2.11) that

\[ z_s = \frac{2m(m-1)}{n} e^{(m-n-1)/n} z A v - \sum_{i,j=1}^{N} m e^{(m-1)/n} + e^{2 x_j} \]

\[ + 2 ms(s+1) - \frac{2ms(m-n)}{n} + \frac{4m(m-n)(m-n-1)}{n} \]

\[ e^{(m-2n-1)/n} - z_s^2 \]

\[ + \sum_{i=1}^{N} \left[ f'_i(u) + 2m \frac{(m-n)}{n} e^{(m-n-1)/n} v_{x_i} \right] z_i \]

\[ + m n_{(m-1)/n} A z + \sum_{i=1}^{N} \frac{2}{n} f'_i(u) u^{1-n} v_{x_i} z \]

\[ \leq \frac{2m(m-1)}{n} e^{(m-n-1)/n} z p - \frac{m}{N} e^{((m-1)/n) + s p^2} \]

\[ + 2 ms(s+1) - \frac{2ms(m-n)}{n} + \frac{4m(m-n)(m-n-1)}{n} \]

\[ e^{(m-2n-1)/n} - z_s^2 \]

\[ + \sum_{i=1}^{N} \left[ f'_i(u) + 2m \frac{(m-n)}{n} e^{(m-n-1)/n} v_{x_i} \right] z_i \]

\[ + m n_{(m-1)/n} A z + \sum_{i=1}^{N} \frac{2}{n} f'_i(u) u^{1-n} v_{x_i} z. \]

(2.14)
If we assume that
\[ \frac{2}{N} \left( s(m-n) - s(s+1) - \frac{2(m-n)(m-n-1)}{n^2} \right) > \frac{(m-1)^2}{n^2}, \] (2.15)
or equivalently
\[ 2n^2 s^2 + 2n(2n-m) s + 4(m-n)(m-n-1) + (m-1)^2 N < 0, \] (2.16)
since the left side of (2.16) is a parabola in the s-variable, we see that
\[ (2n-m)^2 - 8(m-n)(m-n-1) - 2(m-1)^2 N > 0. \] (2.17)
Setting \( l = m-n \) in (2.17), we find after some algebra
\[ 4l^2 - 4(2-m) l + 2(m-1)^2 N - m^2 < 0. \] (2.18)
Then
\[ (2-m)^2 - 2(m-1)^2 N + m^2 > 0. \] (2.19)
This implies
\[ (m-1)^2 (N-1) < 1. \] (2.20)
If we only consider the case \( m > 1 \), then
\[ 1 < m < 1 + \frac{1}{\sqrt{N-1}}. \] (2.21)
From the inequality (2.18), we get
\[ \frac{3m - 2 - \sqrt{2 - 2(m-1)^2 (N-1)}}{2} < n < \frac{3m - 2 + \sqrt{2 - 2(m-1)^2 (N-1)}}{2}. \] (2.22)
For simplicity, we choose \( n = (3m-2)/2 \). Then it follows from (2.16) that
\[ (ns)^2 + 2(m-1)(ns) - m \left( 1 - \frac{m}{2} \right) + \frac{N(m-1)^2}{2} < 0. \] (2.23)
This implies
\[
-2(m-1) - \sqrt{2 - 2(N-1)(m-1)^2} \over 2 < ns < -2(m-1) + \sqrt{2 - 2(N-1)(m-1)^2} \over 2.
\] (2.24)

If \(|f''(u)| \leq M u^{-2}, n = (3m-2)/2\) and \(ns\) satisfies (2.24), we have from (2.14) that
\[
z_s \leq m u^{(m-1)/n} \Delta z + \sum_{i=1}^{N} \left[ f'_i(u) + 2m \left( {m-n \over n} - s \right) v^{(m-n-1)/n}z_{x_i} \right] z_{x_i} \\
- c v^{(m-2n-1)/n} - s^2 z + \sum_{i=1}^{N} \frac{2}{n} f''_i(u) u^{1-n}z_{x_i} z
\] (2.25)
for a suitable positive constant \(c\) depending only on \(m\) and \(s\).

Since
\[s > -2(m-1) - \sqrt{2 - 2(N-1)(m-1)^2} \over 3m-2 > -2,
\]
and
\[
\left| \sum_{i=1}^{N} \frac{2}{n} f''_i(u) u^{1-n}z_{x_i} z \right| \leq d v^{(m-n-1)/n} z_{2} \over 2 z_{3/2} \over 2 (2.26)
\]
for positive constants \(d\) and \(d_1\), we get \(z_s \leq M_1\), for a suitable large positive constant \(M_1\), by applying the maximum principle to (2.25). Thus \(|\nabla u^{(\nu,s+1)}| \leq M_1\). This and the boundedness of \(u\) yield the boundedness \(\nabla u^{\nu}\) for any
\[l_1 > \left( {s \over 2} + 1 \right) n > m - 1 \over 2 - \sqrt{2 - 2(N-1)(m-1)^2} \over 4.
\]
So \(u^{\nu}\) is uniformly Lipschitz continuous for any
\[l_1 \geq m - 1/2 - \sqrt{2 - 2(N-1)(m-1)^2} \over 4,
\]
which is the conclusion (i) of (I) in Theorem 1.
If \(1 < m < 1 + 1/\sqrt{N}\), we consider (2.8) directly. Since
\[
2\partial_t (w) = - \sum_{i,j=1}^N h_t (v_{x_i x_j}) + \frac{4m(m-n)(m-n-1)}{n^2} v^{(m-2n-1)w^2} - 2h_t (v_{x_j}) = - 2h_t (w) \frac{h(v)}{N} p^2 - \frac{4m(m-n)(n-(m-1))}{n^2} v^{(m-2n-1)w^2}
\]
\[
= - m e^{(m-2n-1)n} \left[ \frac{1}{N} e^{2p^2} - \frac{2(m-1)}{n} v w + \frac{4(m-n)(n-(m-1))}{n^2} w^2 \right],
\]
(2.27)
we choose \(n \in (m-1, m)\) and assume that
\[
\frac{1}{N} \frac{4(m-n)(n-(m-1))}{n^2} > (m-1)^2.
\]
(2.28)
Then
\[
\left( m - n - \frac{1}{2} \right) - 1 + \frac{(m-1)^2}{4} N > 0.
\]
(2.29)
The inequality (2.29) implies that
\[
m - 1 + \sqrt{1 - (m-1)^2} \frac{N}{2} < n < m - 1 + \sqrt{1 - (m-1)^2} \frac{N}{2}.
\]
(2.30)
If \(|f_t(u)| \leq Mu^{m-2}\) and \(n\) satisfies (2.30), it follows from (2.8), (2.27) and (2.28) that
\[
w_t \leq h_t (v_{x_j}) + \frac{2m(m-n)}{n} v^{(m-n-1)n} \sum_{i=1}^N v_{x_i} w_{x_i} + \sum_{j=1}^N f_j(u) w_{x_j} - cv^{(m-2n-1)n} w^2 + \sum_{j=1}^N \frac{2f_j(u)}{mu^{m-2}} v_{x_j} w,
\]
(2.31)
for a suitable positive constant \(c\) depending only on \(m\) and \(n\).
Since
\[
n > m - 1 + \sqrt{1 - (m-1)^2} \frac{N}{2} > 0,
\]
then
\[
\sum_{j=1}^N \frac{2f_j(u)}{mu^{m-2}} v_{x_j} w \leq cv^{(m-2n-1)n} w^{3/2}
\]
(2.32)
for a suitable positive constant $d$ depending on $m$ and $n$. Applying the maximum principle to (2.31) we get $w \leq M_1$ for a suitable, large positive constant $M_1$. So $\nabla u^*$ is bounded in $\mathbb{R}^N \times (0, \infty)$ for any

$$n > m - \frac{1 + \sqrt{1 - (m - 1)^2 N}}{2}.$$

At the same time we see that $u^*$ is uniformly Lipschitz continuous for any $n \geq m - \frac{1 + \sqrt{1 - (m - 1)^2 N}}{2}$.

Combining this with the conclusion of (i), we get the proof of (ii).

Finally replace $n, v$ in (2.5) by $\gamma, \phi$, we have

$$
\phi_t = m u^{m-1} A \phi + \frac{m(m-\gamma)}{\gamma} u^{m-\gamma-1} \sum_{j=1}^{N} \phi_j^2 + \sum_{j=1}^{N} f_j(u) \phi_j
$$

and

$$
= m u^{m-1} A \phi + \frac{\gamma(m-\beta)}{\beta^2} u^{\beta + m-2\beta-1} \sum_{j=1}^{N} ((u^\beta)_x)_j^2 + \sum_{j=1}^{N} f_j(u) \phi_j,
$$

where $\beta = l_1$ or $l_2$. Set $\gamma = 2\beta - m + 1$. Since

$$
(m - \gamma) \sum_{j=1}^{N} ((u^\beta)_x)_j^2
$$

is bounded, a result in [15] yields that $\phi$ is Hölder continuous with respect to $t$ with exponent $\frac{1}{2}$. Hence the conclusions of (II) in Theorem 1 are proved.

**Remark 2.** If $f_i(u) \equiv 0$ for $i = 1, 2, 3, \ldots, N$, then the conclusions of (I), (II) in Theorem 1 hold for $1 < m \leq 1 + (1/\sqrt{N} - 1)$ or $1 < m \leq 1 + (1/\sqrt{N})$ respectively.

### 3. RADIAL SOLUTION FOR CAUCHY–DIRICHLET PROBLEM

In this section we study the radial solution for the following Cauchy–Dirichlet problem

$$
\begin{align*}
\phi_t &= A \phi(u(x, t)), \quad (x, t) \in Q = \Omega \times (0, T), \\
\phi(x, 0) &= u_0(x), \quad x \in \Omega, \\
\phi(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T),
\end{align*}
$$

and
where $G(u)$ is a nondecreasing smooth function, $\Omega$ is the region $\{x : 1 < |x| < 2\}$ in $\mathbb{R}^N$ ($N \geq 1$) and $u_0(x) = u_0(r)$, $r = |x|$ is a given smooth function.

For a radial solution, the problem (3.1), (3.2) is equivalent to the following problem

$$u_t = G(u)_r + \frac{N-1}{r} G(u)_r, \quad t > 0, \quad 1 < r < 2$$

(3.3)

with initial-boundary values

$$
\begin{align*}
&u(r, 0) = u_0(r), &1 \leq r \leq 2 \\
&u(i, t) = 0, &i = 1, 2; \quad t \geq 0.
\end{align*}
$$

(3.4)

**Theorem 2.** If $G(u)$ is a nondecreasing smooth function, $|u_0(r)| \leq M$ and $G(u_0(r)) \in C^{(1)}$, then the Cauchy–Dirichlet problem (3.3), (3.4) has a weak solution $u(r, t)$, which satisfies $G(u(r, t)) \in C^{(1)}(\bar{S})$, where $\bar{S} = [1, 2] \times [0, T]$.

To prove Theorem 2, we add a small perturbation $\varepsilon > 0$ to (3.3) and consider solutions $u^\varepsilon$ for the following problem

$$
\begin{align*}
&u_t^\varepsilon = G_1(u^\varepsilon)_r + \frac{N-1}{r} G_1(u^\varepsilon)_r, &t > 0, \quad 1 < r < 2 \\
&u(r, 0) = G_1^{-1}(w_0^\varepsilon), &1 \leq r \leq 2 \\
&u(i, t) = 0, &i = 1, 2; \quad t \geq 0,
\end{align*}
$$

(3.5)

where $G_1(u^\varepsilon) = \int_{S^\varepsilon} g_1(s) \, ds$, $g_1(s) = g(s) + \varepsilon$, $w_0^\varepsilon = G(u_0) * J^\varepsilon$ and $J^\varepsilon$ is a mollifier.

Since the equation in (3.5) is strictly parabolic for any fixed $\varepsilon > 0$ and $u^\varepsilon(r, 0)$ is smooth, we have the following theorem 3 from [21] directly.

**Theorem 3.2.** If the conditions of Theorem 2 are satisfied, then for any fixed $\varepsilon > 0$, the Cauchy–Dirichlet problem (3.5) has a unique classical smooth solution $u^\varepsilon$ in $\bar{S}$, which satisfies

$$|u^\varepsilon| \leq M.$$  

(3.6)

**Remark 3.** For the porous media equation $G(u) = u^m$, $m > 1$, we consider nonnegative solutions with nonnegative initial data. We may add a positive perturbation $\varepsilon$ to (3.4) such that $u^\varepsilon \geq \varepsilon$ by applying the maximum principle to (3.3), (3.4). So the function $G(u^\varepsilon)$ is smooth for all $m > 1$.

About solutions $u^\varepsilon$ of the Cauchy–Dirichlet problem (3.5), we have the estimates, given in Lemmas 4, 5, which are crucial in the proof of Theorem 2.

For simplicity, we omit the superscript $\varepsilon$ in proofs throughout this section and the next section.
Lemma 4. If the conditions of Theorem 2 are satisfied, then $G_1(u)$, is bounded independent of $\varepsilon$ on the boundaries $r = 1$ and $r = 2$.

Proof. Let $v = G_1(u)$. Then it follows from the equation in (3.5) that

$$v_t = g_1(u) \left( v_{rr} + \frac{N-1}{r} v_r \right). \tag{3.7}$$

Applying the transformation

$$z = v + Me^{-kr},$$

we get

$$\begin{cases} z_t &= g_1(u) \left( z_{rr} + \frac{N-1}{r} z_r - Mk^2e^{-kr} + FN - 1rMke^{-kr} \right) \\ &< g_1(u) \left( z_{rr} + \frac{N-1}{r} z_r \right) \tag{3.8} \end{cases}$$

for a suitable large constant $k$. Thus $z$ attains its greatest value on the boundaries $t = 0, r = 1$ and $r = 2$. Since $v_r(r, 0)$ is bounded, then

$$z_r(r, 0) = v_r(r, 0) - Mke^{-kr} < 0, \quad 1 \leq r \leq 2 \tag{3.9}$$

for a large $M$, and

$$z(1, t) = G_1(0) + Me^{-k} > G_1(0) + Me^{-2k} = z(2, t), \quad t \geq 0. \tag{3.10}$$

Hence it follows from (3.9) and (3.10) that $z$ attains its greatest value on the boundary $r = 1$. But, on $r = 1$, $z$ is a constant. Thus $z(1, t) \leq 0$. This means that

$$v_r = z_r + Mke^{-kr} \leq Mke^{-kr}, \quad \text{on} \quad r = 1. \tag{3.11}$$

Similarly $z_1 = v - M_1 e^{-kr}$ attains its smallest value on $r = 1$, so

$$v_r = z_1 - M_1 e^{-kr} \geq -M_1 e^{-kr}, \quad \text{on} \quad r = 1; \tag{3.12}$$

$z_2 = v + M_2 e^{kr}$ attains its greatest value on $r = 2$, so

$$v_r = z_2 - M_2 e^{kr} \geq -M_2 e^{kr}, \quad \text{on} \quad r = 2; \tag{3.13}$$

$z_3 = v - M_3 e^{kr}$ attains its smallest value on $r = 2$, so

$$v_r = z_3 + M_3 e^{kr} \leq M_3 e^{kr}, \quad \text{on} \quad r = 2. \tag{3.14}$$

Lemma 4 is proved.
Lemma 5. If the conditions of Theorem 2 are satisfied, then $G_1(u)$, is bounded independent of $\epsilon$ in $S$.

Proof. It follows from the equation in (3.5) that, for any constant $q$,

$$
(ru)_t = (rG_1(u))_t + (N - 2)G_1(u) + q,
$$

$$
= \left( rG_1(u) \left( u_t + \frac{(N - 2)G_1(u) + q}{rG_1(u)} \right) \right)_t.
$$

Let

$$
w = u_t + \frac{(N - 2)G_1(u) + q}{rG_1(u)}
$$

and

$$
h(u) = \frac{(N - 2)G_1(u) + q}{g_1(u)}.
$$

Then

$$
(ru)_t = (rG_1(u)w)_t,
$$

or

$$
u_t = \frac{1}{r}(rG_1(u)w)_r.
$$

Differentiating (3.17) with respect to $r$ yields that

$$
(ru)_r + u_t = (rG_1(u)w)_r.
$$

It follows from (3.16), (3.19) that

$$
rw_t + (1 - h'(u)) u_t = (rG_1(u)w)_r.
$$

Substituting $u_t$ in (3.18) into (3.20) yields

$$
w_t + \frac{1 - h'(u)}{r^2} (rG_1(u)w)_r = \frac{1}{r} (rG_1(u)w)_r.
$$

Since $u$ is bounded in $\bar{S}$ and $G_1(u)$, is bounded on the boundaries $t = 0$, $r = 1$ and $r = 2$, one can choose $q$ to be a large positive (or small negative) constant such that $w > 0$ (or $w < 0$) on the boundaries $t = 0$, $r = 1$ and $r = 2$. Therefore applying the maximum principle to (3.21) yields $w > 0$ (or $w < 0$) in $S$. So Lemma 5 is proved.

From Lemmas 4 and 5 and a result in [15], we obtain the following
Theorem 6. If the conditions of Theorem 2 are satisfied, then $G_1(u')$ is bounded, independent of $\varepsilon$, in $C^{11/4}(\bar{S})$.

Using the conclusions in Theorem 6, we can end the proof of Theorem 2.

Proof of Theorem 2. Since $u'$ is bounded independent of $\varepsilon$ and $G_1(u') \in C^{11/4}(\bar{S})$, then there exists a subsequence $\varepsilon_n$ such that $u_n'$ converges weakly to a bounded function $u$ and $G_1(u_n')$ converges to a suitable function $v \in C^{11/4}(\bar{S})$ almost everywhere. Then $G(u_n')$ converges to $v$. Since $G$ is a nondecreasing smooth function, the Minty’s device ([14, p. 261]) yields $G(u) = v$. So $u$ is a weak solution of (3.3) and (3.4). Moreover $G(u) \in C^{11/4}(\bar{S})$. Theorem 2 is proved.

A direct by-product of Theorem 2 is the sharp regularity estimate of the temperature of the classical two-phase Stefan model.

We consider the singular nonlinear partial differential equation

\begin{equation}
\begin{aligned}
&\beta(u(x, t)), \quad (x, t) \in \Omega = \Omega \times (0, T), \\
u(x, t) = 0, \quad (x, t) \in \partial \Omega = \partial \Omega \times (0, T), \\
u(x, 0) = u_0(x), \quad x \in \Omega,
\end{aligned}
\end{equation}

(3.22)

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N (N \geq 1)$, $u_0$ is a given smooth function, and $\beta$ is the multivalued mapping

\[ \beta(x) = \begin{cases} 
ax - 1, & x \leq 0 \quad (a > 0) \\
(-1, 1), & x = 0, \\
bx + 1, & x \geq 0 \quad (b > 0).
\end{cases} \]

(3.23)

Equations (3.22), (3.23) are a formulation of the classical two-phase Stefan problem, describing the flow of heat within a substance (say water) which changes phase (melts or freezes) at the temperature zero. The constants $a$ and $b$ denote the respective thermal conductivities in the ice and water regions, and the jump in $\beta$ at zero corresponds to the latent heat of fusion. The temperature is

\[ T = \begin{cases} 
u, & (u > 0), \\
0, & (u = 0), \\
u, & (u < 0).
\end{cases} \]

(3.24)

The continuity of a unique weak solution of (3.22), (3.23) for all $N \geq 1$ was obtained in [6]. Here if we assume the region $\Omega$ is $\{x : 1 < |x| < 2\}$ and...
u_0(x) = u_0(r), \ r = |x|, then the radial solution \ u(r, t) \ of (3.22), (3.23) is bounded in \ C^{1,1}(\hat{Q}) \ from \ Theorem 2.

4. INITIAL-BOUNDARY VALUE PROBLEM IN ONE-DIMENSIONAL SPACE

We consider the nonlinear reaction–diffusion–convection equation

\[ u_t + f(u)_x = G(u)_{xx} + h(u) \]  

(4.1)

with initial-boundary values

\[
\begin{align*}
\langle u(x, 0) & = u_0(x), \quad 0 \leq x \leq 1 \\
n(i, t) & = 0, \quad i = 0, 1; \quad t \geq 0.
\end{align*}
\]  

(4.2)

When \ h(u) = c_1 u^n, \ f(u) = c_2 u^n \ and \ G(u) = u^m \ (n \geq m > 1, \ h \geq 1), \ the \ equation (4.1) models the process of heat propagation in a moving non-linear medium. When \ f(u) = 0, \ G(u) = u^m \ and \ h(u) = (1 - u)(u - a)u \ for \ 0 < a < 1, \ it \ arises \ in \ the \ Gurten–MacCamy theory of crowd avoiding populations with a nonlinear growth law [2]. The large time behavior of solutions is studied in [3]. When \ h(u) = 0, \ f(u) = \frac{2}{3} \left( 1 + 3u^2 \right)^{3/2} - \left( \frac{1}{3} + u^2/2 \right) \ and \ g(u) = \frac{2}{3} \left( 2 - (1 + 3u^2)^{1/2} / \left(1 + 3u^2\right)^{3/2} \right), \ (4.1) \ is \ related \ to \ the \ Broadwell \ model (See [5, 9, 18]). The last case is interesting since \ g(u) \ has two zero points at \ u = 1 \ and \ u = -1. A unique \ \mathcal{C}^{1,1} \ solution \ for \ the \ Cauchy \ problem \ of \ (4.1) \ is \ proved \ in [18].

When \ G(u) \ has one or finite degenerate points, the regularity of solutions for the problem (4.1) and (4.2) is well studied in many previous papers. Some of which are referenced in the present paper. When the measure of degenerate points of \ G(u) \ is zero, i.e. \ Z = 0, \ the regularity of solutions for the Cauchy problem of (4.1) with smooth initial data is studied in [18]. In this section, under the condition \ Z = 0, \ we prove the sharp estimate \ \mathcal{G}(u) \in \mathcal{C}^{1,1}(\bar{S}) \ for solutions \ u \ of \ (4.1) \ and \ (4.2) \ up \ to \ the \ boundaries \ x = 0 \ and \ x = 1, \ where \ \bar{S} = [0, 1] \times [0, T] \ for \ any \ given \ time \ T > 0.

We consider the following perturbation problem

\[
\begin{align*}
\langle u'_t + f(u')_x & = G_1(u')_{xx} + h(u'), \quad t > 0, \quad 0 < x < 1 \\
n(u'(x, 0) & = G_1^{-1}(w_0'), \quad 0 \leq x \leq 1 \\
n(u'(i, t) & = 0, \quad i = 0, 1; \quad t \geq 0.
\end{align*}
\]  

(4.3)

(4.4)

where \ G_1(u') = \int_0^x g_1(s) \ ds, \ g_1(s) = g(s) + \varepsilon, \ w_0' = G(u_0) * J^\varepsilon \ and \ J^\varepsilon \ is \ a \ mollifier.
Lemma 7. If $G(u)$ is a nondecreasing smooth function, $h(u) \in C^1$ satisfies $h(0) = 0$ and $h'(u) \leq 0$ or has at least two different zero points $u = 0$, $u = u_0$, if $0 < u_0(x) \leq u_0$, then the solution $u'$ of (4.3) and (4.4) satisfies

$$0 \leq u' \leq u_0$$

(4.5)

The proof of Lemma 7 can be obtained by applying the comparison principle to (4.3).

Lemma 8. If $u_0(x)$ and $G(u)$ satisfy the conditions of Lemma 7, moreover if $G(u_0(x)) \in C^1$ and $|f(u)| \leq Cg(u)$ for a positive constant $C$, then $G_1(u')_x$ is bounded independent of $\varepsilon$ on the boundaries $x = 0$ and $x = 1$.

Proof. Let $v = G_1(u)$. It follows from (4.3) that

$$v_t + f'(u) v_x = g_1(u) v_{xx} + g_1(u) h(u).$$

(4.6)

Applying the transformation

$$z = v + Me^{-kx},$$

we get

$$z_t + f'(u)(z_x + Mke^{-kx}) = g_1(u)(z_{xx} - Mk^2e^{-kx} + h(u)).$$

(4.7)

Since $g_1(u) > 0$ and $|f'(u)| \leq Cg(u)$, we have

$$z_t + f'(u) z_x < g_1(u) z_{xx}$$

(4.8)

for a suitable large constant $k$.

Now the proof of Lemma 8 can be ended in a same fashion to that of Lemma 4.

To estimate $G_1(u')_x$ in $S = (0, 1) \times (0, T)$, we consider the cases $h(u) = 0$, $f(u) \neq 0$; $f(u) = 0$, $h(u) \neq 0$ and $f(u) \neq 0$, $h(u) \neq 0$ respectively.

Lemma 9. If $h(u) = 0$ and the conditions of Lemma 8 are satisfied, then $G_1(u')_x$ is bounded independent of $\varepsilon$ in $S$.

The proof of Lemma 9 is similar to that of Lemma 5.

Lemma 10. If $f(u) = 0$ and $|\frac{h(u)}{g(u)} f'(u)|$ is bounded for $0 \leq u \leq u_0$, then $|G_1(u')_x| \leq Me^\alpha T$ in $S$ for a suitable constant $\alpha$ independent of $\varepsilon$, where $M$ is a bound of $G_1(u)_x$ on $t = 0$, $x = 0$ and $x = 1$.

Proof. Since $f(u) = 0$, it follows from (4.3) that

$$G_1(u)_t = g_1(u) G_1(u)_{xx} + g_1(u) h(u).$$

(4.9)
Let $G_1(u) = v$. Then
\[ v_t = g_1(u) v_{xx} + \frac{g_1'(u)}{g_1(u)} v_x + \left( h(u) + \frac{h(u) g_1'(u)}{g_1(u)} \right) v. \] (4.10)

Since $g_1 = g(u) + \varepsilon$, $g(u) \geq 0$, $|\frac{h(u) x(u)}{g(u)}| \leq M$, then
\[ \left| h'(u) + \frac{h(u) g_1'(u)}{g_1(u)} \right| \leq \alpha \]
for a suitable constant $\alpha$. From the estimates of $v$ on the boundaries $t = 0$, $x = 0$ and $x = 1$, we obtain the proof of Lemma 10 by applying the maximum principle to (4.9).

**Lemma 11.** If the conditions of Lemma 8 are satisfied and
\[ f(u) = F_1(u) + F_2(u) = \bar{F}_1(u) + \bar{F}_2(u), \] (4.11)
where
\[ F_1''(u) \geq 0, \quad f_2(u) = \frac{\bar{F}_2(u)}{g(u) + \varepsilon} \geq 0, \quad f_2'(u) h(u) - h'(u) f_2(u) \geq 0 \] (4.12)
and
\[ \bar{F}_1''(u) \leq 0, \quad \bar{f}_2(u) = \frac{\bar{F}_2(u)}{g(u) + \varepsilon} \leq 0, \quad \bar{f}_2'(u) h(u) - h'(u) \bar{f}_2(u) \leq 0, \] (4.13)
then $G_1(u')$ is bounded independent of $\varepsilon$ in $S$.

The proof of Lemma 11 is similar to that of Lemma 2 given in [18]. We refer the readers to [18] for the details.

Now, we conclude this section by the following

**Theorem 12.** If the conditions of Lemma 7 and 8 are satisfied, moreover if $Z = 0$ and the conditions in one of Lemmas 9–11 are satisfied, then the initial-boundary value problem (4.1) and (4.2) has a continuous solution $u(x, t)$ satisfying the sharp estimate $G(u) \in C^{1,1}(\bar{S})$ up to the boundaries $x = 0$ and $x = 1$.

**Proof.** Since $G_1(u') \in C^{1,1}(\bar{S})$, then there exists a subsequence $\varepsilon_k$ such that $G_1(u^{\varepsilon_k})$ converges to a suitable function $v(x, t) \in C^{1,1}(\bar{S})$ almost everywhere. Moreover since the point measure set $Z = 0$, then for any fixed $(x, t)$, $v(x, t)$ uniquely defines a value $u(x, t)$ by $\int_0^1 g(s) ds = v(x, t)$. Thus
$u^\varepsilon$ converges to $u(x, t)$ almost everywhere on $\tilde{S}$ and $G(u) \in C^{1}(\tilde{S})$. Letting $\varepsilon$ in (4.3) go to zero, we can prove that the limit function $u$ is a weak solution of (4.1) and (4.2). Using the condition $Z = 0$ again and the continuity of $G(u)$ yield the continuity of $u$. So Theorem 12 is proved.

**Remark 4.** It is easy to check that the conditions in Lemma 9 are satisfied for the equation related to the Broadwell model, where $h(u) = 0$, $g(u) = u(2 - (1 + 3u^2)^{1/2})/(1 + 3u^2)^{3/2}$ and $f' (u) = u(2 - (1 + 3u^2)^{1/2})/(1 + 3u^2)^{3/2}$; the conditions in Lemma 10 satisfied for the biological population equation, where $f(u) = u^m$ and $h(u) = (1 - u)(1 - u)u$ for $0 < a < 1$ and the conditions in Lemma 11 satisfied for the equation of infiltration, where $h(u) = c_1u^2$, $f(u) = c_2u^n$ and $G(u) = u^m$ ($c_1 \leq 0$, $h \geq 1$, $n \geq 1$ and $m > 1$).

**Remark 5.** If $h(u) = 0$ and the conditions of Lemma 8 are satisfied, then we can get a weak solution $u$ of the problem (4.1), (4.2) and the sharp estimate $G(u) \in C^{1}(\tilde{S})$ without the condition $Z = 0$.

In fact, under the condition $|f'(u)| \leq Cg(u)$, we can prove $f(u^\varepsilon) \in C^{1}(\tilde{S})$ since $G(u^\varepsilon) \in C^{1}(\tilde{S})$. Set $F(u^\varepsilon) = f(u^\varepsilon) - CG(u^\varepsilon)$. Then $F(u^\varepsilon) \in C^{1}(\tilde{S})$ and $F(u)$ is a nonincreasing function. Using the Minty's device, we have $F(u) = F(u) - CG(u)$, where $u$ denotes the weak limit of $u^\varepsilon$. Then $u$ is a weak solution of the problem (4.1), (4.2) and $G(u) \in C^{1}(\tilde{S})$ by letting $\varepsilon$ in (4.3) go to zero.

**REFERENCES**