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Symmetry properties of chordal rings of degree 3☆

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Abstract

This paper presents the main properties of chordal rings of degree 3. This family of graphs is strongly related to circulant graphs, which are actually often called chordal rings too. The use of triangles in the plane to represent the vertices allows one to associate a plane tessellation to every chordal ring. By using this geometrical approach, we study the recognition and the isomorphism problems for this class of graphs. A polynomial-time algorithm to recognize chordal rings and a polynomial-time algorithm to decide isomorphism between two chordal rings, given by its adjacency list, are presented. Both algorithms are based on the study of the 4- and 6-cycles of the graph. This approach is also applied to the characterization of the automorphisms-group of chordal rings. We believe that these results produce useful tools for further works as, in particular, the study of compact routing schemes, and the study of optical routing protocols in edge-transitive chordal rings.

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1. Introduction

Many definitions of chordal rings have been proposed in the literature. Roughly speaking, a chordal ring is obtained by adding chords to a cycle in order to get compact

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and reliable networks by decreasing their diameter and increasing their connectivity. For instance, chordal rings are defined in [26] as rings with non-crossing chords. However, in order to preserve the symmetry [10] it is preferable to define chordal rings as Cayley graphs on $\mathbb{Z}_n$, that is as circulant graphs, for which 1 and -1 belong to the set of generators. These graphs are obtained from the cycle by adding chords to each vertex in a regular manner.

In this context, different problems have been studied. A large number of papers are devoted to the isomorphism problem for circulant graphs [1,18,25,28], and to the recognition of circulant graphs [22]. Other papers focus on particular families of chordal rings, namely those of degree 4 or 6. Their aim is to study problems such as leader election [3,19], independent spanning trees [14], routing and fault-tolerance [9,17], compact or interval routings [15,23], optical routings [24], etc.

Most of the works about routing in chordal rings of degree 4 or 6 make use of a very helpful geometrical tool, that is the correspondence between a plane tessellation and a chordal ring. The construction of the tessellation associated to a given chordal ring is based on a representation of every vertex of degree d, $d=4$ or 6, by a d-gon of the plane (that is, squares and hexagons), in such a way that the adjacent polygons correspond to the adjacent vertices. This construction is helpful to visualize metric properties of the graph [20,29,30].

This paper deals with chordal rings of degree 3, that is the third and last family of chordal rings that can be represented by a plane tessellation with regular polygons (each vertex is represented by a triangle). Chordal rings of degree 3 are obtained from an even-order cycle by adding a single chord to every vertex. All the new chords have the same length and connect an even vertex to an odd vertex [2,7]. Although chordal rings of degree 3 are not circulant graphs, they are related to circulant graphs in several ways. Indeed, they can be viewed as bipartite incident graphs of circulant digraphs of degree 3. Moreover, one can associate a chordal ring of degree 6 to every chordal ring of degree 3, with a strong relation between their associated tessellations. However, chordal rings of degree 3 are less "symmetric" than chordal rings of degree 4 or 6 because they are Cayley graphs on a non-commutative group.

Chordal rings of degree 4 and 6 are well known than those of degree 3, and there is some confusion coming from the fact that chordal rings of degree 3 are called chordal rings although they are not circulant graphs. Nevertheless, there are previous works in the literature about chordal rings of degree 3. In [12,13,31], some results on connectivity and fault-tolerance are presented. In the other references known by the author, the association of the triangular tessellation with a chordal ring of degree 3 is presented and used to get results about the diameter [20,21,30], to derive broadcast or gossip algorithms [4], and to study the existence of fault-tolerant routing [5].

The aim of this paper is to improve the knowledge of the chordal rings of degree 3, and it was motivated by two related problems. The first fundamental question is to figure out in which way the tessellation determines the graph. The second problem comes from the study of routing problems in optical networks using WDM technology. In [6,11], the forwarding index and the wavelength index are introduced. It is often conjectured that these two parameters are equal (for all-to-all routing). This conjecture is still open. In order to use the plane tessellation to compute bounds of these indices in chordal rings of degree 3, as it was done in [24] for optimal chordal rings of degree 4, we need to determine which type of chordal rings of degree 3 are edge-transitive. Moreover, the tessellation has been also used to derive compact routing schemes in chordal rings of degree 4 [23]. We believe that a better knowledge of the structural properties of chordal rings of degree 3 will help to solve the compact routing problem for them.

More precisely, we have considered the following problems. Given a graph G of order $2n$ and degree 3, is G isomorphic to a chordal ring? Given two chordal rings G_1 and G_2, are G_1 and G_2 isomorphic? Are chordal rings edge-transitive? We propose two algorithms in $O(n)$ time solving the two first problems. These algorithms are based in the study of cyclical structure of the graphs, by using the tessellations of the plane. For the third problem, the same geometrical approach allowed us to completely characterize the automorphisms-group of every chordal ring of degree 3.

Our results: The main results in this paper can be summarized as follows:

- Theorem 9 states that there is a linear time algorithm for solving the recognition problem for chordal rings of degree 3.
- Theorem 11 shows that the isomorphism problem between two chordal rings of degree 3 can be solved in linear time.
- In Theorem 16 we give a characterization of the automorphisms-group of chordal rings of degree 3.

All of this three theorems are based on the fact that, for almost all the chordal rings of degree three, the tessellation completely determines the graph. This is equivalent to say that, except for some particular cases, chordal rings of degree 3 are CI-graphs [16].

The paper is organized as follows. Section 2 gives the formal definition, and the basic properties of chordal rings of degree 3, simply called chordal rings from now. In Section 3, we present the characterization of chordal rings according to the number of existing 4- and 6-cycles. This result allowed us to go deeper in the description of chordal rings. Section 4 proposes a linear algorithm to recognize whether a given graph is a chordal ring. A variant of the recognizing algorithm gives a new algorithm to decide whether two chordal rings are isomorphic or not. Finally, Section 5 presents a characterization of the group of automorphisms of chordal rings.

2. Definition and basic properties

The family of graphs we are interested in is a generalization of the chordal ring graphs introduced in [7], and defined by Arden and Lee [2] as graphs obtained from a $2n$-undirected cycle by adding chords in a regular manner. More precisely:

Definition 1. Let $n \geqslant 3$ be an integer, and let a, b, and c be three distinct odd integers in $[1, 2n-1]$. The *chordal ring* of order $2n$ and chords a, b, and c, is denoted by

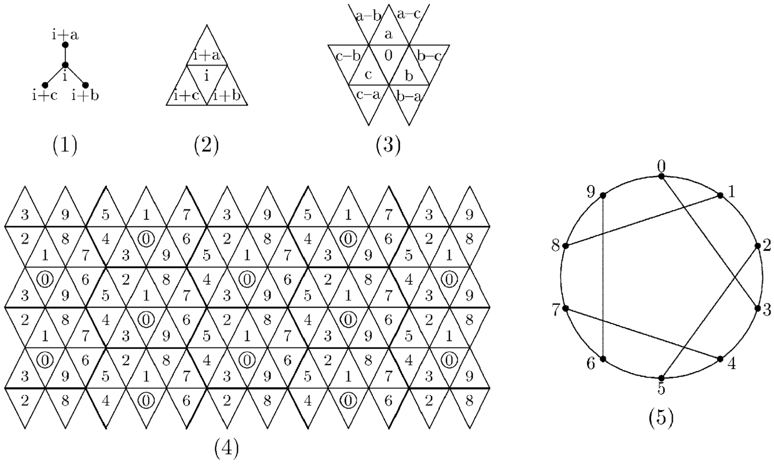

Fig. 1. Representation of a chordal ring by triangulated polygons. The adjacencies (1) are represented by the four triangles (2). As one can check in (3), the three different chords correspond to three directions in the plane. The tessellation in (4) shows the representation of $C_{10}(1,-1,3)$, also drawn in (5).

$C_{2n}(a,b,c)$, and is defined as the graph with vertex set $\mathbb{Z}_{2n}$, and adjacencies given by $i \sim i+a, i \sim i+b$ and $i \sim i+c$ for all even vertex i.

From the definition, chordal rings are 3-regular. They are also bipartite since even vertices are pairwise independent, and so are the odd vertices. Note that every odd vertex i of $C_{2n}(a,b,c)$ is adjacent to $i-a, i-b$, and $i-c$. Therefore, the bijection $\alpha_w : \mathbb{Z}_{2n} \to \mathbb{Z}_{2n}$ defined by $\alpha_w(i) = w + i$ if w is even, and $\alpha_w = w - i$ if w is odd is an automorphism of $C_{2n}(a,b,c)$. It maps the vertex 0 onto the vertex w. Consequently, chordal rings are vertex-transitive. Moreover, the three values of the chords give a proper coloring of the edges of the graph. In what follows, we will say that edges corresponding to the chord a (respectively, b or c) are edges of color **a** (respectively, **b** or **c**).

A very nice feature of chordal rings is that we can represent the adjacencies as a figure in the plane, by drawing a triangle to represent a vertex, and by placing adjacent triangles to represent adjacent vertices [20]. This representation of the graph gives a tile which periodically tessellates the plane when the vertices are repeated in the plane according to a bi-dimensional lattice. The set of triangles containing vertex 0 is called the 0-lattice (see Fig. 1).

Given a chordal ring $C_{2n}(a,b,c)$ with its associated tessellation, a question that naturally arises is: *In which way the lattice is related to the parameters of the graph* (*its order* $2n$, *and its chords* a, b, *and* c)? Note that a succession of adjacent triangles in the tessellation can be viewed either as a succession of chords, that is a path in the graph, or as a vector in the plane. Note also that the 0-lattice is generated by two vectors with their two end-points in different triangles, both containing the vertex 0. As a consequence of these two remarks, the lattice can be described by using some

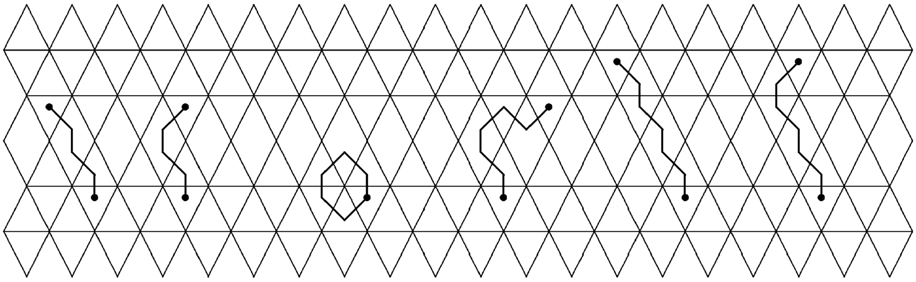

Fig. 2. The shape of 4- and 6-cycles that a chordal ring could have.

special cycles of the graph. (Since a chordal ring $C_{2n}(a,b,c)$ is bipartite, the length of every cycle is even.)

Let us denote by A, B and C the three integers such that $2A=b-c, 2B=c-a$ and $2C=a-b$. Then the relation $2C+2B+2A=0$ is always satisfied, and it can be viewed as the succession $(a,-b,c,-a,b,-c)$, which corresponds to a cycle in every chordal ring. Indeed, the representation of this cycle is an hexagon, and it is not a lattice generator (see Fig. 2). Moreover, a cycle can be expressed as a linear equation modulo $2n$ using $2A, 2B$ and $2C$, or as a linear equation modulo n using A, B and C. For instance, $xA-yB=0$ in $\mathbb{Z}_n$, means that $xA-yB$ is a multiple of n. It is satisfied if and only if the succession $\mathbf{bc}, \mathbf{bc} \ldots \mathbf{bc}, \mathbf{ac} \ldots \mathbf{ac}$, where $\mathbf{bc}$ appears x times and $\mathbf{ac}$ appears y times, is a cycle.

For any integer k, we denote by $\equiv_k$ the equality in $\mathbb{Z}_k$, or the equality modulo k. With this notation, we can associate two equations to the 0-lattice of a chordal ring (or more than two linearly dependent equations). By looking again at the example in Fig. 1, one can check that the lattice equations of $C_{10}(1,-1,3)$ are

$$C-B \equiv_5 0,$$

$$B-2A \equiv_5 0.$$

This is equivalent to say that the 0-lattice of $C_{10}(1,-1,3)$ is generated by the vectors $(0,-1,1)$ and $(-2,1,0)$. As a consequence of the definition of the associated tessellation, if two chordal rings have the same 0-lattice then they are isomorphic. The isomorphism can be defined as follows: the image of a vertex in the first chordal ring is the vertex of the second chordal ring that is located at the same triangle. For more details in this construction, see [5,30].

The following properties will be useful for the purpose of our calculations:

Lemma 2 (Morillo [20]). *$C_{2n}(a,b,c)$ is connected if and only if* $\gcd(a-b,b-c,2n)=\gcd(a-b,b-c,c-a,2n)=2$. *If* $\gcd(a-b,b-c,2n)=2k$, $k>1$, *then $C_{2n}(a,b,c)$ is the union of k connected components, each of them being isomorphic to $C_{2m}(a',b',c')$, where $m=n/k$ and $k(a'-b')=a-b$, $k(b'-c')=b-c$, and $k(c'-a')=c-a$.*

Unless specified, the chordal rings of this paper are always assumed to be connected. Therefore, the connectivity condition is a necessary condition for a set of integers $\{n, a, b, c\}$ to define a chordal ring of order $2n$ and chords a, b, and c.

Lemma 3 (Morillo [20]). *$C_{2n}(a,b,c)$ and $C_{2n}(a',b',c')$ satisfy the same lattice equations if and only if $a'-b' \equiv_n \ell(a-b)$, and $b'-c' \equiv_n \ell(b-c)$, for some $\ell \in \mathbb{Z}_n^*$.*

A first consequence of Lemma 3 is that we can arbitrarily fix one of the chords. In some cases it will be useful to take $a=1$. Another consequence of the same lemma is that, if $\gcd(a-b,2n)=2k$, then there is an odd integer $c' \in \mathbb{Z}_{2n}$ such that $C_{2n}(a,b,c)$ is isomorphic to $C_{2n}(2k-1,-1,c')$. Thus, we can assume $n/k(a-b)=2n$. Note that if $k=1$ this gives that there is an odd integer $c' \in \mathbb{Z}_{2n}$ such that $C_{2n}(a,b,c)$ is isomorphic to $C_{2n}(1,-1,c')$.

Lemma 4. *Given two integer vectors, $v_1=(\lambda_1,\lambda_2,\lambda_3)$ and $v_2=(\gamma_1,\gamma_2,\gamma_3)$, if there exist a chordal ring of order $2n$ with lattice equations $\lambda_1 A+\lambda_2 B+\lambda_3 C \equiv_n 0$ and $\gamma_1 A+\gamma_2 B+\gamma_3 C \equiv_n 0$ then n is the absolute value of* $\det \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$.

This follows from the already known fact that the value of n is the absolute value of the determinant of the generating vectors of the lattice [8]. Notice that, in this case, since in our reference system $(1,1,1)=0$, the two vectors v_1 and v_2 are independent if and only if in a three-dimensional space the vectors v_1, v_2, and $(1,1,1)$ are independent.

Lemma 5. *$C_{2n}(a,b,c)$ is isomorphic to a Cayley graph on the dihedral group, generated by three axial symmetries.*

Proof. For every u, v in $\mathbb{Z}_{2n}$, let $u * v$ denote an addition if u is even, and a subtraction if u is odd. It follows from the definition that $C_{2n}(a,b,c)$ is the Cayley graph on the group $(\mathbb{Z}_{2n}, *)$, with the set of generators $\{a,b,c\}$.

Let us denote by D_n the dihedral group, composed of the n-gon symmetries, and by r the $2\pi/n$-rotation. The proof is concluded by noticing that, for every axial symmetry $s \in D_n$, there is a group automorphism $\phi : \mathbb{Z}_{2n} \to D_n$ defined by $\phi(2i)=r^i$ and $\phi(1)=s$, such that $\phi(2i+1)$ is an axial symmetry. □

From the characterization of Cayley graphs [27], given a chordal ring $G=C_{2n}(a,b,c)$, the set of automorphisms which preserve the edge-coloring is the so-called *group of translations*, $G_L=\{\alpha_w,\ w \in \mathbb{Z}_{2n}\}$, with $\alpha_w(i)=w+(-1)^w i$ for any $i \in \mathbb{Z}_{2n}$. G_L is a subgroup of $Aut(G)$, that is the group of automorphisms of G. For any two vertices u and v of G, we denote by $\alpha_{u,v}$ unique translation of $C_{2n}(a,b,c)$ which maps u onto v. It is easy to see that $\alpha_{u,v}=\alpha_w$, with $w=v-u$ if $u \equiv_2 v$, and $w=v+u$, otherwise.

Since the three colors of the edges correspond to the three directions given by adjacent triangles in the plane, the group of translations has also a graphical representation. For any two vertices u and v, if u and v have the same parity, then $\alpha_{u,v}$ is a translation of the plane, otherwise $\alpha_{u,v}$ is a translation followed by a π-rotation.

3. Characterization of the 4- and 6-cycles in $C_{2n}(a, b, c)$

In the previous section we have seen that, in some sense, the tessellation determines the graph. One of our objectives is to determine whether there exist two isomorphic chordal rings with essentially different associated tessellation. For this purpose next we give a characterization of the 4- and 6-cycles of a chordal ring, in terms of the three values of the chords.

As we noticed before, a cycle can be described by a succession of colors **a**, **b** or **c**. In the geometrical representation, a cycle can be either a path with its origin and its end at the same triangle or a path whose two end-points are located at different triangles representing the same vertex. For instance, in $C_{2n}(a,b,c)$, **abcabc** represents the cycle $(0, a, a-b, a-b+c, c-b, c)$, and **ababab** represents a 6-cycle if and only if $3(a-b) \equiv_{2n} 0$. We can also write $3C \equiv_n 0$.

One can check by exhaustive study that, up to a permutation of the chords, a chordal ring can only have the following cycles of length 4 or 6: at most two different types of 4-cycles, **abab** if the chords satisfy $2(a-b) \equiv_{2n} 0$, that is $2C \equiv_n 0$, and **abac** if the chords satisfy $(a-b)+(a-c) \equiv_{2n} 0$, that is $C - B \equiv_n 0$; and at most four different types of 6-cycles, **ababab**, if the chords satisfy the equation $3C \equiv_n 0$, **ababac**, if the chords satisfy the equation $2C - B \equiv_n 0$, **abacbc**, if the chords satisfy the equation $C - B + A \equiv_n 0$, and **abcabc**, corresponding to the equation $A + B + C \equiv_n 0$ that is always satisfied. The specific cycles **abcabc** will be called "hexagons" in the following. Moreover, the graph has an **abacbc** cycle if and only if it has an **acac** cycle, which is a cycle of the same types that of **abab**. Fig. 2 shows these cycles.

Let us first consider graphs $C_{2n}(a,b,c)$ containing 4-cycles.

3.1. Chordal rings with 4-cycles

The tessellation and the graph in Fig. 3(3) shows the possible existence of 4-cycles of type **abab** in a chordal ring. Fig. 3(4) shows the structure of two 4-cycles of type **abac** sharing an edge. We will show that $C_{2n}(a,b,c)$ contains 4-cycles of two types only if the order is 6 or 8 (see Fig. 3(1) and (2)). All these cases are covered by the following lemma:

Lemma 6. *If a chordal ring $G = C_{2n}(a,b,c)$ has a cycle of length 4, then, up to a permutation of the chords, exactly one of the following possibilities is satisfied:*

(1) $G = C_6(1,-1,3) = K_{3,3}$.
(2) $G = C_8(1,-1,3) = K_{4,4} - F$, *where F is a 1-factor.*

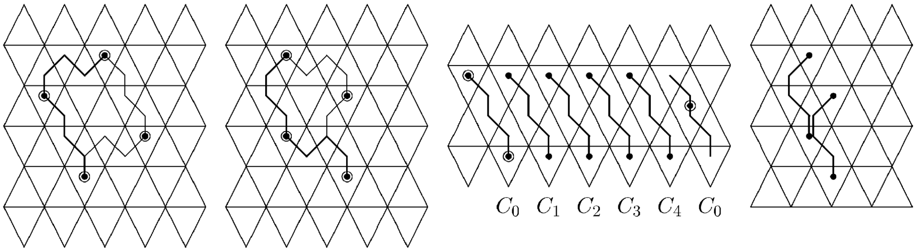

Fig. 3. (1) and (2) The tessellations associated to chordal rings of order 6 and order 8, respectively. (3) A chordal ring of order 20, $C_{20}(11, 1, -1)$, with five 4-cycles **abab**. (4) Two 4-cycles **abac** sharing one edge of color **a**. In all the cases, the triangles containing $\odot$ represent the vertex 0, and a cycle represented in thick lines is the same that the parallel cycle represented in thin lines.

(3) *The lattice of G is generated by the vectors* $(0, 0, 2)$ *and* $(0, k, 1)$. *Every edge of color* **a** *or* **b** *is contained in one* 4-*cycle*, *and the edges of color* **c** *are contained in none* 4-*cycle*.
(4) *The lattice of G is generated by the vectors* $(0, 0, n)$ *and* $(0, -1, 1)$. *Every edge of color* **b** *or* **c** *is contained in one* 4-*cycle*, *and every edge of color* **a** *is contained in two* 4-*cycle*.

Proof. Let us consider chordal rings with two type of 4-cycles. If the graph contains 4-cycles of types **abac** and **bcba**, we can set the lattice equations

$$(a - b) + (a - c) \equiv_{2n} 0,$$

$$(b - c) + (b - a) \equiv_{2n} 0,$$

with the connectivity condition $(b-c, c-a, a-b, 2n) = 2$. The first equation is $a - b \equiv_{2n} c - a$. This yields $(a - b, 2n) = 2$ and we can assume $a = 1$ and $b = -1$, which forces the value for the third chord, $c = 3$. With these values for the chords in the second equation the equivalence $-1 - 3 - 1 - 1 \equiv_{2n} 0$ is obtained, and this implies that the order is 6. $C_6(1, -1, 3) = K_{3,3}$ is the only chordal ring with such number of vertices. The reasoning for the second case is similar. If the graph contains 4-cycles of types **abab** and **cbca**, then the equations $2(a - b) \equiv_{2n} 0$ and $(c - b) + (c - a) \equiv_{2n} 0$ with the connectivity condition $\gcd(b - c, c - a, a - b, 2n) = 2$, imply that its order is 8. $C_8(1, -1, 3) = K_{4,4} - F$, where F is a 1-factor, is the only chordal ring of order 8. The associated tessellations for these two cases are represented in Fig. 3(1) and (2).

Now, let us assume that $G = C_{2n}(a, b, c)$ contains 4-cycles of type **abab** only. Hence, $2(a - b) \equiv_{2n} 0$ holds, and it implies that $2n = 4k$ for some k, because a and b are odd. Moreover, for the second lattice equation, we can assume, w.l.o.g., that A does not appear, the coefficient of B is positive, and the coefficient of C is 0 or 1. The lattice equations are then

$$2C \equiv_{2k} 0,$$

$$\alpha B + \ell C \equiv_{2k} 0.$$

The determinant of this system is 2α. Therefore, from Lemma 4, $2k = 2\alpha$, and thus, $\alpha = k$. Now we have only to observe that, if k is even, the connectivity condition is satisfied only if $\ell = 1$, otherwise, if k is odd and $\ell = 0$ then the equation $kA + C \equiv_{2k} 0$ is also satisfied and by changing the order of the chords, we can obtain $\ell = 1$. These two equations correspond to a lattice generated by $(0, 0, 2)$ and $(0, k, 1)$. The proof of case 3 is concluded by noticing that, in this case, the graph can be decomposed into a disjoint union of 4-cycles of type **abab**, connected by edges of color **c**.

The last case is $G = C_{2n}(a, b, c)$ containing only 4-cycles of type **abac**. The existence of the cycle **abac** is equivalent to $(a - b) + (a - c) \equiv_{2n} 0$. However, the graph is connected if and only if $\gcd(a - b, b - c, c - a, 2n) = 2$. Hence, we can conclude that $\gcd(a - b, 2n) = 2$. This implies that the equations

$$C - B \equiv_n 0,$$

$$nC \equiv_n 0$$

generate the lattice. The corresponding generating vectors are $(0, 0, n)$ and $(0, -1, 1)$. Since every 4-cycle shares two edges of color **a** with two different 4-cycles, we can conclude that every edge of color **a** is contained in two 4-cycle and every edge of color **b** or **c** is contained in one 4-cycle, which completes the proof. □

Lemma 6 shows that $C_{2n}(a, b, c)$ contains a 4-cycle for specific values of a, b, and c. In general, $C_{2n}(a, b, c)$ contains no 4-cycles. This is why we turn our attention to 6-cycles.

3.2. Chordal ring with no 4-cycles

The existence of cycles of type **abacbc** implies the existence of 4-cycles. Thus, we focus on the three different types of 6-cycles we can found when there are no 4-cycles. By using the plane tessellations, Fig. 4(1) represents the cyclic structure for graphs containing hexagons and 6-cycles of type **ababab**, and Fig. 4(2) represents the cyclic structure for graphs containing hexagons and 6-cycles of type **ababac**. The following lemma describes all the possible cases.

Lemma 7. *If a chordal ring $G = C_{2n}(a, b, c)$ has no cycles of length* 4, *then, up to a permutation of the chords, exactly one of the following possibilities is satisfied*:

(1) *The order of G is $2n = 14, 16$ or 18, and $G = C_{2n}(1, -1, 5)$.*
(2) *The lattice of G is generated by the vectors $(0, 0, 3)$ and $(0, k, \pm 1)$, where the sign depends on the value of k. Every edge of color* **a** *or* **b** *is contained in two* 6-*cycles, and the edges of color* **c** *are contained in one* 6-*cycle, the hexagons.*
(3) *The lattice of G is generated by the vectors $(0, 0, n)$ and $(0, -1, 2)$. Every edge of color* **a** *is contained in five* 6-*cycles, every edge of color* **b** *is contained in four* 6-*cycles, and every edge of color* **c** *is contained in three* 6-*cycles.*
(4) *The* 6-*cycles in G are hexagons only.*

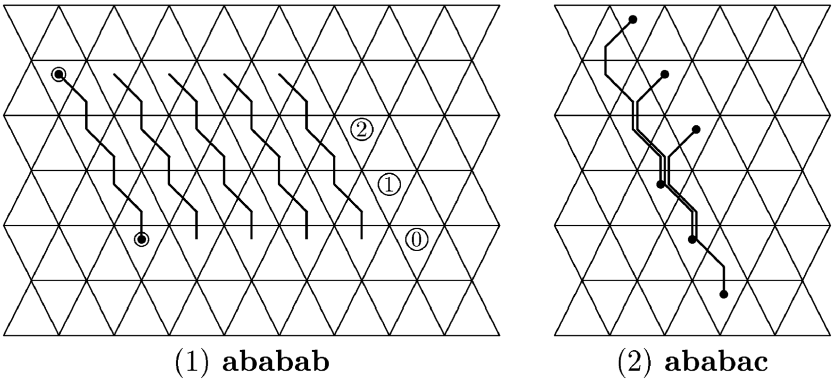

Fig. 4. (1) There are three possible tessellations associated to a chordal ring of order 30 with 6-cycles **ababab**, which correspond to the three possible values of ℓ in Lemma 7: if $\ell = 0$, the triangles that represent the vertex 0 are the triangles containing $\odot$ and the triangle marked with ■; if $\ell = 1$, the triangles that represent the vertex 0 are the triangles containing $\odot$ and the triangle marked with ■; if $\ell = -1$, the triangles that represent the vertex 0 are the triangles containing $\odot$ and the triangle marked with ■. Three 6-cycles **ababac** sharing one edge of color **a**. The edges of color **b** are contained in exactly two cycles **ababac**, and the edges of color **c** are contained in exactly one cycle **ababac**. As in Fig. 3(3), two triangles in the same path containing • represent the same vertex.

Proof. Let us assume that 6-cycles in $C_{2n}(a,b,c)$ are not only hexagons, but also cover two other types of cycles. There are three possible cases: (1) **ababab** and **bcbcbc**, (2) **ababab** and some 6-cycles of type **ababac**, (3) different 6-cycles of type **ababac**. The existence of 6-cycles **ababab** and **bcbcbc** implies $a-b=b-c$ and $\gcd(a-b,b-c,2n)=2$. Thus $a-b=b-c=2$ and the order must be 6. However, $C_6(a,b,c)$ contains 4-cycles, that is case (1) is impossible. In the same way as in Lemma 6 one can show, by examination of cases (2) and (3), that G is one of the three graphs $C_{14}(1,-1,5)$, $C_{16}(1,-1,5)$, and $C_{18}(5,-1,1)$. This completes the proof of case 1.

Case 2 corresponds to chordal rings containing hexagons and 6-cycles of type **ababab** only. The lattice equations are obtained as in case 3 of Lemma 6. Because of the connectivity condition, not all the values of ℓ are possible. One can see that all the possible values can be reduced to $\ell = 1$ or -1. Since the graph can be decomposed into the disjoint union of 6-cycles of type **ababab**, connected by edges of color **c**, we have that every edge of color **a** or **b** is contained in two 6-cycles, and the edges of color **c** are contained in one 6-cycle, the hexagons. This completes the proof of case 2.

Case 3 corresponds to chordal rings containing hexagons and 6-cycles **ababac** only. The proof of this Case 3 is similar to the proof of case 4 of Lemma 6. By examination of the intersection of 6-cycles of type **ababac**, we can conclude that every edge of color **a** is contained in five 6-cycles, every edge of color **b** is contained in four 6-cycles, and every edge of color **c** is contained in three 6-cycles.

Finally, we have to include the general case, that is, chordal rings with only hexagons. The only possible conclusion here is that, every edge is contained in exactly two 6-cycles. □

As a consequence of Lemmas 6 and 7 we have the following:

Corollary 8. *Let G be a chordal ring. Then, up to a permutation of the chords, there are three integers x, k and ℓ such that $k\ell = n$, and $2k$ is the length of the simple cycle containing only colors* **a** *and* **b**, *and the lattice equations are of the form*

$$kC \equiv_n 0,$$

$$xC - \ell B \equiv_n 0.$$

In the next section, we use the knowledge about the structure of $C_{2n}(a,b,c)$ provided by the presented results to recognize chordal rings and to determine whether two given chordal rings are isomorphic. Notice that Lemmas 6 and 7 show that at least in the cases where 6-cycles different from the hexagons or 4-cycles exist, the graph is completely determined by the equations corresponding to these existing cycles. Moreover, this depends only on the number of 6-cycles, or 4-cycles if any, passing through every edge of the graph.

4. Recognizing chordal rings in linear time

The recognition problem consists of deciding whether a graph belongs or not to a given family. In this section we concentrate on this problem for chordal ring graphs.

For a given graph G of order $2n$ and degree 3, we want to know whether there exist three different odd integers a, b, and c such that G is isomorphic to $C_{2n}(a,b,c)$ and, in the affirmative, which are these integers. The algorithm solving this problem is based on the results in Section 3. It performs in three phases. First, from the number of 4- or 6-cycles (if there are not 4-cycles) passing through every edge, we classify the edges by colors **a**, **b** and **c**. Then, according to the different possible cases, the values for the chords are set. The last step of the algorithm is the labeling of the vertices and the verification of the adjacencies, that is, if there is an edge of color **s** between the vertices labeled $2i$ and j, then we have to verify that $j - 2i = s$, where $s \in \{a,b,c\}$. The algorithm can be formally described as follows.

4.1. Algorithm RCR

Let G be a graph of order $N = 2n$ and degree 3, given by its adjacency list. The following algorithm returns a, b and c if G is isomorphic to $C_{2n}(a,b,c)$.

1. If $N = 6, 8$ or 10 then check whether G is isomorphic to $C_N(1,-1,3)$.

2. If $N = 14, 16$ or 18, and G contains no 4-cycles then check whether G is isomorphic to $C_N(1, -1, 5)$.

3. If $N \geqslant 12$ and G contains a 4-cycle.

3.a For $i \in \{0, 1, 2\}$, let E_i be the set of edges that belong exactly to i cycles of length 4.

3.b Case 1. $E_0 = \emptyset$ and $|E_1| = 2|E_2| = N = 2n$.

Then assign color **a** to E_2. To assign colors **b** and **c** to the remaining edges, we can use the fact that all 4-cycles must be of type **abac** in the following way. Starting in a 4-cycle, choose arbitrarily which of the two possible edges has color **b**. This choice forces the color of all the remaining edges in E_1. The values for the chords can be $a = 1$, $b = -1$ and $c = 3$.

Case 2. $N = 4k$ for some k, $E_2 = \emptyset$, and $|E_1| = 2|E_0| = N$.

Then assign color **c** to E_0, and colors **a** and **b** respectively to the pairs of opposite edges in the 4-cycles **abab**. Due to the structure of the graph (see Fig. 3(3) and Fig. 5) this choice is arbitrary for every cycle except when k is even. In this case, the choice in the last 4-cycle is forced by the choices in the preceding one and in the first one. The equation $k(b - c) + (a - b) \equiv_N 0$ holds if there is a cycle $\mathbf{bc}, \ldots, \mathbf{bc}, \mathbf{ab}$, with k times **bc**. Then, w.l.o.g., the chords can be $a = 2k + 1$, $b = 1$ and $c = -1$. If $k(a - c) + (b - a) \equiv_N 0$ then the chords can be $a = 1$, $b = 2k + 1$ and $c = -1$.

3.c Label the vertices starting by 0 at any vertex and using the assigned values for the chords. Then, check for every edge that its color correspond to its assigned value.

4. If $N \geqslant 20$ and G contains no 4-cycles.

4.a For $i \in \{2, 3, 4, 5\}$, let E_i be the set of edges that belong exactly to i cycles of length 6.

4.b Case 1. $E_2 = E$ and $E_i = \emptyset$ for $i \neq 2$.

Then, starting in a 6-cycle, assign colors to the six edges following the pattern **abcabc**. The starting point can be arbitrarily choosen, as well as the direction in which the cycle is traversed. This choice forces the color of all the remaining edges of the graph. Once every edge has been assigned one of the three colors, the equation $k(a - b) \equiv_N 0$ holds if the alternating path **abab**..., with k times **ab** is a cycle. Thus, we can find the smallest integer k satisfying $k(a-b) \equiv_N 0$. Then, w.l.o.g., first two chords can be $a = N/k - 1$ and $b = -1$ (see Lemma 3). By traversing an appropriate path, we can find now the smallest positive integer x such that $x(a - b) - n/k(c - a) \equiv_N 0$. Since $a - b = N/k$, the value of the third chord can be $c = a + 2x - 2k$.

Case 2. $E_2 = \emptyset$ and $|E_3| = |E_4| = |E_5| = n$.

Then assign **a** to E_5, **b** to E_4 and **c** to E_3. The chords are set to be $a = 1$, $b = -1$ and $c = 5$.

Case 3. $N = 6k$ for some k, $E_4 = E_5 = \emptyset$ and $|E_3| = 2|E_2| = N = 2n$.

Then assign color **c** to E_2, and colors **a** and **b** to the remaining edges according to an initial arbitrary choice, as in the previous cases. If there is a cycle $\mathbf{bc}, \ldots, \mathbf{bc}$, with k times **bc**, then exchange colors **a** and **b**. Thus, we can assume

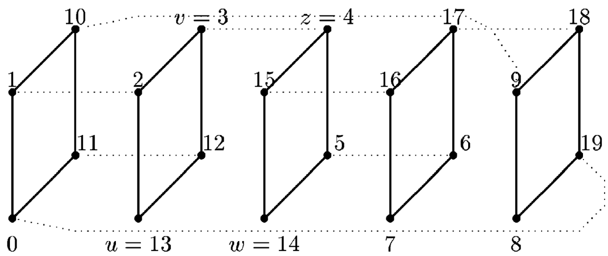

Fig. 5. Chordal ring $C_{20}(11, 1, -1)$: the automorphism β_1 exchanges the vertices u and v and the vertices w and z. All the remaining vertices are invariant. In this graph, there are hexagons sharing two edges: for instance, the successions of vertices 2, 3, 4, 15, 14, 13, 2 and 12, 3, 4, 5, 14, 13, 12 constitute two hexagons that share the two edges of color c, $\{3, 4\}$ and $\{13, 14\}$.

that the equation $k(b - c) + \ell(a - b) \equiv_N 0$, for $\ell = 1$ or $\ell = 2$, holds, that is, there is a cycle **bc**, $\ldots$, **bcab**, or **bc**, $\ldots$, **bcabab**, with k times **bc**. Therefore the chords are set to be $a = 2k + (-1)^\ell$, $b = -1$ and $c = 1$.

4.c Label the vertices starting by 0 at any vertex and using the assigned values for the chords. Then, check for every edge that its color correspond to its assigned value.

Theorem 9. *Let G be a graph of order $N = 2n$ and degree 3 given by its adjacency list. The algorithm RCR returns a, b and c if $G = C_{2n}(a, b, c)$, and stops otherwise, in time* $O(n)$.

Proof. First, we derive from Lemmas 6, 7 and Corollary 8, that the algorithm *RCR* returns a, b and c if $G = C_{2n}(a, b, c)$, and stops otherwise.

1 follows from case 1 of Lemma 6 and from the fact that there is only one chordal ring of order 10. **2** follows from case 1 of Lemma 7. In these two cases, the checking can be done in constant time. In **3.a**, the sets E_i for $i \in \{0, 1, 2\}$ are defined in order to find which types of 4-cycles are contained in the graph. In **3.b** we set the values for the chords. **Case 1** of **3.b** follows from case 4 in Lemma 6, and **Case 2** of **3.b** follows from case 3 in Lemma 6. If neither **Case 1** nor **Case 2** are satisfied, then the graph is not isomorphic to a chordal ring. In **3.c** we label the vertices and check the adjacencies. In **4.a**, as in **3.a**, the sets E_i for $i \in \{2, 3, 4, 5\}$ are defined in order to find which types of 6-cycles are contained in the graph. In **4.b**, as in **3.b**, we set the values for the chords. **Case 1**, **Case 2**, and **Case 3** of **4.b** follow from Corollary 8, case 2 and case 3 of Lemma 7, respectively. If none of the three cases **Case 1**, **Case 2** or **Case 3**, is satisfied, then the graph is not isomorphic to a chordal ring. In **4.c**, as in **3.c**, we label the vertices and check the adjacencies.

Finally, we have to compute the complexity of this algorithm. For a graph G satisfying the conditions of **1** or **2**, the time of the algorithm is constant. If G satisfies the condition of **3** or **4**, the time of the algorithm is the following. The computation of the sets E_i, either in **3.a** or **4.a**, takes $O(n)$-time, because to compute the number of 4- or 6-cycles passing through a given edge can be done in constant time. To assign values to the chords takes $O(n)$-time in **Cases 1** and **2** of **3.b**, because we have to explore $O(n)$ 4-cycles. It takes also $O(n)$-time in **Cases 1** and **3** of **4.b**, because we have to explore a cycle of length at most $2n$. And it takes $O(1)$-time in the remaining case, **Case 3** of **4.b**. The labeling of the vertices and the checking of the adjacencies, either in **3.c** or **4.c**, take $O(n)$-time. Therefore, the global time for the algorithm *RCR* is $O(n)$. □

4.2. Isomorphism between chordal rings

We concentrate now on the following problem. Let G_1 and G_2 be two chordal rings, given by its adjacency list. Are G_1 and G_2 isomorphic?

A consequence of Lemmas 6 and 7 is the following:

Remark 10. The chordal rings $C_{2n}(a,b,c)$ and $C_{2n}(a',b',c')$ are isomorphic if and only if there exists $\ell \in \mathbb{Z}_n^*$ and a permutation of a', b' and c', $\{a'',b'',c''\} = \{a',b',c'\}$, such that $a'' - b'' \equiv_{2n} \ell(a-b)$, and $b'' - c'' \equiv_{2n} \ell(b-c)$.

This property says that there is a bijection between chordal rings and lattices. That is, two chordal rings are isomorphic if and only if the associated tessellation is the same, up to a permutation of the chords. This follows from the fact that the number of 4- or 6-cycles passing through every edge allows one to determine the partition of edges into colors. Thus, if two chordal rings are isomorphic, the correspondence between the two partitions is, in fact, a permutation of the chords. In the plane, this corresponds to a transformation of the lattice that preserves distances. The study of such transformations is used in the following section to find the automorphisms-group of a chordal ring.

Thanks to Lemma 10, we can use the recognizing algorithm *RCR* to determine whether two chordal rings are isomorphic or not, as shown in the following theorem.

Theorem 11. *Let G_1 and G_2 be two chordal rings of the same order $2n$, given by its adjacency list. There exists a $O(n)$-time algorithm with input G_1 and G_2, and output "yes" if G_1 and G_2 are isomorphic chordal rings, and "not" otherwise.*

Proof. If the algorithm *RCR*, with input graph G_1 returns the values, a, b, and c, then G_1 is isomorphic to $C_{2n}(a,b,c)$. If the algorithm *RCR*, with input graph G_2 returns the values, a', b', and c', then G_2 is isomorphic to $C_{2n}(a',b',c')$. There are six possible permutations of a', b' and c', and for each permutation a'', b'' and c'', we have to check if there exists $\ell \in \mathbb{Z}_n^*$ such that $a'' - b'' \equiv_n \ell(a-b)$, and $b'' - c'' \equiv_n \ell(b-c)$. Since $|\mathbb{Z}_{2n}^*| = \Phi(n) < n$, this can be done in $O(n)$-time. □

5. Group of automorphisms of chordal rings

The study of cycles in the plane tessellation is not only fruitful to recognize chordal rings, but also to describe the automorphisms-group of a chordal ring. In Section 3 we have shown that, by looking at the cycles of length 4 and cycles of length 6, we can set the lattice equations. Here, we will show that, apart few cases, an automorphism preserves the partition of the edges into colors, up to a permutation of these colors, and that these automorphisms are actually the isometric transformations of the plane which preserve the lattice structure.

Recall that, if $G=C_{2n}(a,b,c)$, the translation-group $G_L=\{\alpha_w, w\in\mathbb{Z}_{2n}\}$ is a subgroup of $Aut(G)$ that can be represented in the plane: α_w is a translation, followed if w is odd by a π-rotation. The identity will be denoted Id. Let $\mathscr{A}_0\subset Aut(G)$ be the subgroup of automorphisms of G which let the vertex 0 unchanged. Since $Aut(G)=G_L\circ\mathscr{A}_0$, one can restrict the study of $Aut(G)$ to the study of $\mathscr{A}_0$. Let us denote the *origin* an arbitrary fixed point of the 0-lattice. By looking at an automorphism as a bijection between triangles, the elements of $\mathscr{A}_0$ can be represented in the plane up to a translation, as those bijections which let the origin unchanged.

Lemmas 12 and 13 show that, for almost all chordal rings, an automorphism is an isometry of the plane.

Lemma 12. *Let $G=C_{2n}(a,b,c)$. An automorphism $\alpha\in Aut(G)$ is an isometry if and only if two different edges of same color are mapped by α onto two different edges of same color.*

Proof. Let us assume that α is an isometry. Then α preserves parallel lines. Moreover, the colors are directions in the tessellation. Thus, two edges of the same color are mapped by α onto two edges of the same color.

Conversely, we can assume w.l.o.g. that α. One can check that, if all edges of the same color are mapped by α onto edges of same color, the images of vertices a,b, and c determine α. Thus, we can distinguish three cases. Firstly, if α fixes a,b and c then $\alpha=Id$. Secondly, if α fixes only one $x\in\{a,b,c\}$ then α is the axial symmetry with respect to the line passing through the origin and the triangle x adjacent to the origin. Thirdly, if the only fixed point of α is the origin, then α is a $\pm 2\pi/3$-rotation. All of these transformations are isometries. □

Lemma 13. *Let $G=C_{2n}(a,b,c)$ be a chordal ring of order $2n\geqslant 20$. $\mathscr{A}_0$ contains only isometries if and only if the graph contains no 4-cycles of type* **abab**.

Proof. Let $G=C_{2n}(a,b,c)$ be a chordal ring of order $2n\geqslant 20$, and let $\alpha\in\mathscr{A}_0$. Let us prove that, if G contains no 4-cycles of type **abab**, then α maps edges of the same color onto edges of the same color.

If G contains the other type of 4-cycles, namely **abac**, the image of one of these 4-cycles is also a 4-cycle, so α(**abac**) = **abac** or α(**abac**) = **acab**. Therefore, the color a is preserved and the colors b and c are either preserved or exchanged. However, if α preserves b and c in one of the cycles, then α preserves b and c in any adjacent 4-cycles (see Fig. 3(4)). So, edges of same color map onto edges of same color.

The reasoning is analogous for G without 4-cycles. Roughly speaking, the existence of 6-cycles of type **ababac** forces the preservation of the colors. The existence of 6-cycles **ababab** implies that α preserves the color c but not necessarily colors a and

b that can be exchanged. Finally, if the 6-cycles in G are hexagons only, then there is no restriction on the permutation of colors. In any case, since α maps hexagons onto hexagons, edges of the same color are mapped onto edges of the same color. Indeed, two adjacent hexagons share only one edge.

Conversely, as we have seen in Lemma 6, the chordal rings of order $2n \geqslant 20$ containing 4-cycles **abab** are the chordal rings $C_{4k}(2k+1,1,-1)$, for all $k \geqslant 5$. Such graphs contain adjacent hexagons sharing the two opposite edges of color c. Therefore, there is an automorphism of $C_{4k}(2k+1,1,-1)$ which is not an isometry (see Fig. 5). Indeed, let us denote by C_i, $0 \leqslant i \leqslant k-1$, the k pairwise disjoint cycles **abab** in $C_{4k}(2k+1,1,-1)$. For any i, $0 \leqslant i \leqslant k-1$, $C_i = \{2i, 2i+2k+1, 2i+2k, 2i+1\}$. Let us define the bijection $\beta_i, 0 \leqslant i \leqslant k-2$, of $\mathbb{Z}_{4k}$, as follows:

$$\beta_i(2i+2k+1) = 2i+1,$$

$$\beta_i(2i+1) = 2i+2k+1,$$

$$\beta_i(2(i+1)) = 2(i+1)+2k,$$

$$\beta_i(2(i+1)+2k) = 2(i+1)$$

and all the remaining points are unchanged. One can check that, for every i, β_i is an automorphism of G. However, β_i is not an isometry because some edges of color a are mapped onto edges of color a, whereas some others are mapped onto edges of color b.

For $x, y \in \{a,b,c\}$ let us denote by π_{xy} the axial symmetry that maps x onto y (and y onto x). The $2\pi/3$-rotation and $4\pi/3$-rotation are denoted by r and r^2, respectively. Then, the set $\{Id, \pi_{ab}, \pi_{bc}, \pi_{ca}, r, r^2\}$ is the set of the isometries preserving the triangles of the tessellation of $C_{2n}(a,b,c)$ and which let the origin unchanged. An element of this set is an automorphism of G if it preserves the 0-lattice. Indeed, it must preserve the lattice equations and zero must stay unchanged. In other words, one of these transformations is an automorphism of G if it induces a bijection of the vertices.

In the two following lemmas we use the argument that if an automorphism is an isometry of the plane then it preserves the lattice equations. This yields symmetry conditions on the equations, which allow us to characterize the graph. Since the lattice equations are given in terms of $2A, 2B$, and $2C$, the symmetry conditions we are looking for follow from the images of vertices $2A, 2B$ and $2C$. □

Lemma 14 characterizes the chordal ring $C_{2n}(a,b,c)$ when its group of automorphisms contains an axial symmetry. The existence of a rotation in this group is studied in Proposition 15.

Lemma 14. *Let $G = C_{2n}(a,b,c)$. There is an axial symmetry in $Aut(G)$ if and only if there exist two positive integers ℓ and m satisfying one of the two following conditions*:

(1) $n = 2\ell m$, $\gcd(\ell,m) = 1$, *and* $G = C_{4\ell m}(4m+1, 1, 2m-2\ell+1)$, *and*

(2) $n=\ell m$, ℓ *and* m *have the same parity*, $\gcd(m,(\ell-m)/2)=1$, *and* $G=C_{2\ell m}(2m+1,1,m-\ell+1)$.

Proof. Let us assume that $\pi_{ab}\in\mathscr{A}_0$. It follows from $\pi_{ab}(a)=b$, $\pi_{ab}(b)=a$, and $\pi_{ab}(c)=c$ that $\pi_{ab}(2A)=-2B$, $\pi_{ab}(2B)=-2A$, and $\pi_{ab}(2C)=-2C$. Therefore, $xA+yB+zC\equiv_n 0$ if and only if $-yA-xB-zC\equiv_n 0$. By adding these two equations, we get $(x-y)A+(y-x)B\equiv_n 0$. Let m and ℓ be the two smallest positive integers such that $mA-mB\equiv_n 0$ and $\ell C\equiv_n 0$. Now we have to distinguish two cases. In the first case, the lattice generated by these equations is the 0-lattice of the chordal ring. In the second case, the points satisfying the given equations belong to the 0-lattice, but constitute only a proper subset of it.

If the lattice equations are

$$\ell C=0,$$

$$mA-mB=0,$$

then G can be decomposed into $2m$ pairwise disjoint 2ℓ-cycles containing edges of colors a and b only. Thus $2n=4m\ell$ and the connectivity condition is satisfied if and only if $\gcd(\ell,m)=1$. By choosing $b=1$, a solution of the lattice equations is $a=4m+1$ and $c=2m-2\ell+1$. Thus, the graph is $C_{4\ell m}(4m+1,1,2m-2\ell+1)$. Fig. 6 shows the tessellation of $C_{48}(13,1,-1)$, with $m=3$ and $\ell=4$.

If the equations $\ell C=0$ and $mA-mB=0$ partially generate the lattice, then in order to preserve the symmetry, we get $mA-[(\ell-m)/2]C=0$, and ℓ and m must have the same parity. The lattice equations are then

$$mA-mB=0,$$

$$mA-\frac{\ell-m}{2}C=0$$

and the value of the order is $2n=2\ell m$. The connectivity condition is satisfied if and only if $\gcd(m,(\ell-m)/2)=1$. By choosing $b=1$, a solution for the lattice equations is $a=2m+1$ and $c=m-\ell+1$. Thus, the graph is $C_{2\ell m}(2m+1,1,m-\ell+1)$. The associated tessellation to $C_{30}(7,1,-1)$ is shown in Fig. 7.

Conversely, the lattice equations of the chordal ring $C_{4\ell m}(4m+1,1,2m-2\ell+1)$ are

$$\ell C=0,$$

$$mA-mB=0$$

and the lattice equations of the chordal ring $C_{2\ell m}(2m+1,1,m-\ell+1)$ are

$$mA-mB=0,$$

$$mA-\frac{\ell-m}{2}C=0.$$

In both cases, the symmetry π_{ab} preserves the lattice. $\square$

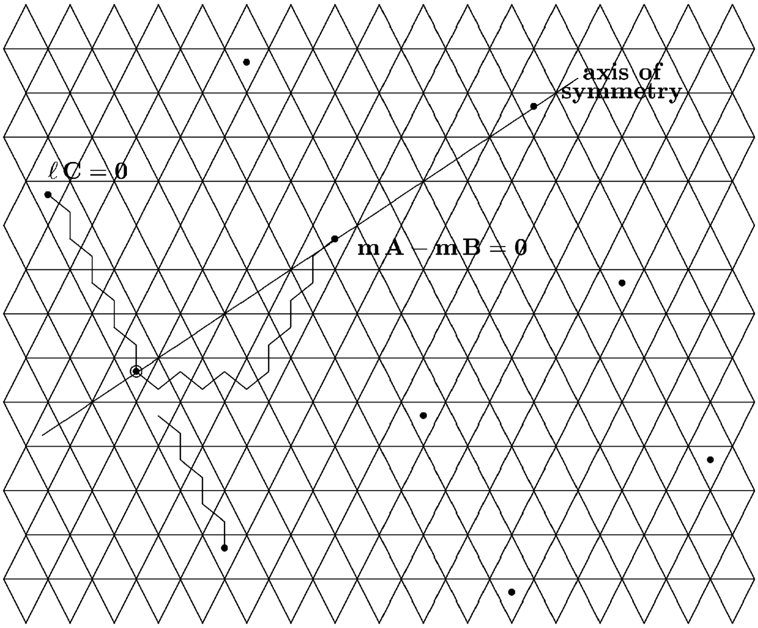

Fig. 6. The tessellation associated to the chordal ring $C_{48}(13,1,-1)$, that corresponds to $m=3$ and $\ell=4$ in case 1 of Lemma 14. The lattice equations are $\ell C \equiv_{24} 0$, and $mA - mB \equiv_{24} 0$. The triangles containing $\bullet$ are the triangles that represent the vertex 0, and the triangle which contains $\odot$ is the origin.

Lemma 15. *Let $G = C_{2n}(a,b,c)$. The rotations r and r^2 are automorphisms of G if and only if there exists a positive integer ℓ satisfying one of the two following conditions*:

(1) $n = 3\ell^2 + 3\ell + 1$ *and* $G = C_{2n}(-1, 6\ell+3, 1)$, *and*
(2) ℓ *is odd*, $n = 3\ell^2 + 3\ell + 4$ *and* $G = C_{2n}(1-2(\ell+2), 2\ell+1, 1)$.

Proof. Let us assume that $r \in \mathscr{A}_0$. It follows from $r(a)=c, r(b)=a$ and $r(c)=b$ that $r(2A)=2C, r(2B)=2A$, and $r(2C)=2B$. Therefore, $xA+yB+zC \equiv_n 0$ if and only if $yA+zB+xC \equiv_n 0$. This implies that the lattice equations are

$$kA - \ell B \equiv_n 0,$$

$$kB - \ell C \equiv_n 0,$$

$$kC - \ell A \equiv_n 0$$

for some positive integers k and ℓ, $k \leqslant \ell$. From these equations, we can compute the order, as a function of k and ℓ. Indeed, we only have to count the number of triangles contained in the region bounded by the cycles $kA - \ell B$, $kC - \ell A$, and $kB - \ell C$

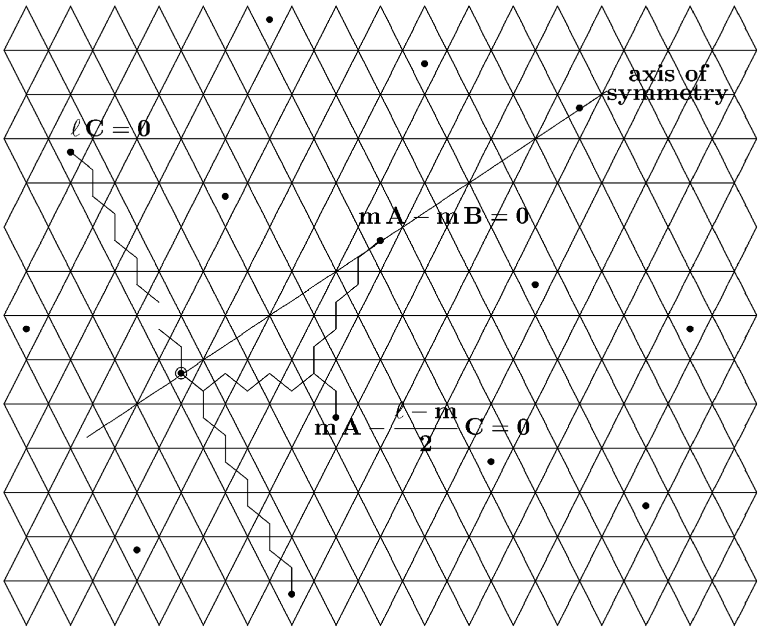

Fig. 7. The tessellation associated to the chordal ring $C_{30}(7,1,-1)$, that corresponds to $m=3$ and $\ell=5$ in case 2 of Lemma 14. The lattice equations are $mA-mB \equiv_{15} 0$, and $mA-[(\ell-m)/2]C \equiv_{15} 0$. The triangles containing • are the triangles that represent the vertex 0, and the triangle which contains ⊙ is the origin.

(see Fig. 8). This counting gives

$$n = \sum_{i=k+1}^{k+\ell+1} i + \sum_{i=\ell+2}^{k+\ell} i - 2\ell - 1 = \frac{1}{2}(k+\ell+1)(k+\ell) + k(\ell+1) - 2k - 1.$$

Let m such that $k=\ell+m$. We get $(\ell+m)A - \ell B \equiv_n 0$, $(\ell+m)B - \ell C \equiv_n 0$, $(\ell+m)C - \ell A \equiv_n 0$, and $n = 3\ell^2 + 3\ell m + (m^2+m)/2 + m - 1$. Moreover, thanks to the relation $A+B+C=0$, we can rewrite the system as

$$(\ell+m)A - \ell B \equiv_n 0,$$
$$\ell A + (2\ell+m)B \equiv_n 0.$$

Finally note that, if $\lambda \in \mathbb{Z}_n^*$, then if A, B, and C is a solution of the system, then $\lambda A, \lambda B$ and λC is also a solution of the system, since the corresponding chordal rings have the same associated 0-lattice. Thus, the system can be simplified to only one equation, stating that the determinant must be either 0 or a multiple of n. The determinant of the system is $3\ell^2 + 3\ell m + m^2$. Since $n = 3\ell^2 + 3\ell m + (m^2+m)/2 + m - 1$, the determinant is 0 in $\mathbb{Z}_n$ if it is exactly n. Therefore $m^2 - 3m + 2 = 0$, and the only possible values for m are $m=1$ or 2.

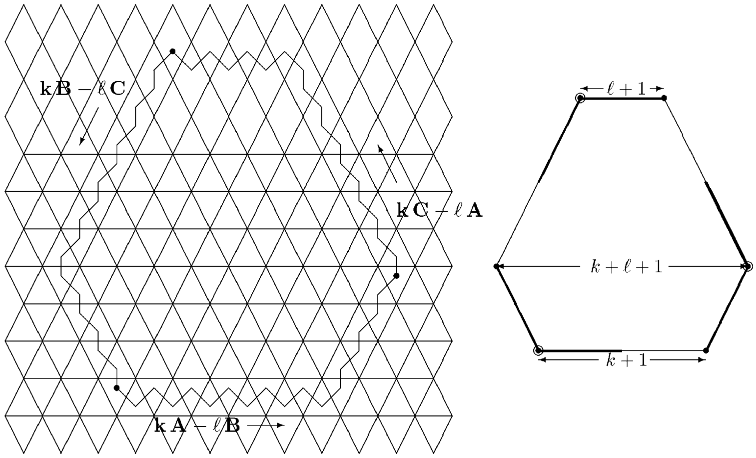

Fig. 8. The region of the plane bounded by the cycles $kA - \ell B$, $kB - \ell C$, and $kC - \ell A$, if $k = 6$ and $\ell = 3$. The triangles containing • are the triangles which represent the vertex 0.

If $m = 1$, then $n = 3\ell^2 + 3\ell + 1$, and the equations are

$$(\ell + 1)A - \ell B \equiv_n 0,$$

$$\ell A + (2\ell + 1)B \equiv_n 0.$$

Therefore, a solution is $a = -1, b = 6\ell + 3$, and $c = 1$. The graph is then $C_{2n}(-1, 6\ell + 3, 1)$.

If $m = 2$, then $n = 3\ell^2 + 6\ell + 4$, and the equations are

$$(\ell + 2)A - \ell B \equiv_n 0,$$

$$\ell A + (2\ell + 2)B \equiv_n 0.$$

The connectivity condition can be satisfied only if ℓ is odd. Thus, a solution in this case is $a = 1 - 2(\ell + 2), b = 2\ell + 1$, and $c = 1$. The graph is then $C_{2n}(1 - 2(\ell + 2), 2\ell + 1, 1)$, for ℓ odd.

Conversely, as in Lemma 14, it is easy to check that r preserves the lattice equations of the graphs $C_{2n}(-1, 6\ell + 3, 1)$ and $C_{2n}(1 - 2(\ell + 2), 2\ell + 1, 1)$.

As a consequence of the previous lemmas, we are able to characterize the group of automorphisms of a chordal ring of order $2n \geqslant 20$, as settled in the following theorem. For almost every chordal ring, the automorphisms are exactly the translations, that is $\mathscr{A}_0 = \{Id\}$. If the group contains not only translations, there are four different possibilities. The three first cases correspond to chordal rings for which all the automorphisms let one of the colors unchanged. The graphs are described assuming that the stable color is c, as in the previous lemmas. The automorphisms β_i's are those defined in the proof of Lemma 13. □

Theorem 16. *Let G be a chordal ring of order $2n$. For any integer $k \geqslant 5$, and for any integer $i, 0 \leqslant i \leqslant k - 2$, let us define β_i, bijections of $\mathbb{Z}_{4k}$, by $\beta_i(2i + 2k + 1) = 2i + 1$,*

$\beta_i(2i+1)=2i+2k+1$, $\beta_i(2(i+1))=2(i+1)+2k$, and $\beta_i(2(i+1)+2k)=2(i+1)$, and all the remaining points are unchanged.

(1) *If there is an odd integer $k \geqslant 5$ such that G is isomorphic to $C_{4k}(2k+1,1,-1)$, then $\mathscr{A}_0$ is the group generated by the set $\{\beta_i \mid i=0,\dots,k-2\}$.*
(2) *If there is an even integer $k \geqslant 5$ such that G is isomorphic to $C_{4k}(2k+1,1,-1)$, then $\mathscr{A}_0$ is the group generated by the set $\{\pi_{ab}\} \cup \{\beta_i \mid i=0,\dots,k-2\}$.*
(3) *If there are two integers m and ℓ, with $l-m$ odd and $\gcd(\ell,m)=1$, such that G is isomorphic to $C_{4\ell m}(4m+1,1,2m-2\ell+1)$, or if there are two integers m and $\ell \neq 2$, with $l-m$ even and $\gcd(m,(\ell-m)/2)=1$, such that G is isomorphic to $C_{2\ell m}(2m+1,1,m-\ell+1)$, then $\mathscr{A}_0=\{Id,\pi_{ab}\}$.*
(4) *If there is an integer ℓ such that $n=3\ell^2+3\ell+1$ and G is isomorphic to $C_{2n}(-1,6\ell+3,1)$, or if there is an odd integer ℓ such that $n=3\ell^2+3\ell+4$ and G is isomorphic to $C_{2n}(1-2(\ell+2),2\ell+1,1)$, then $\mathscr{A}_0=\{Id,r,r^2\}$.*
(5) *Otherwise $\mathscr{A}_0=\{Id\}$, and the automorphism-group of G is the set of translations G_L.*

Note that in case 2, from Lemma 13, $\mathscr{A}_0$ contains β_i. However, one can check that $C_{4k}(2k+1,1,-1)=C_{4\ell m}(2m+1,1,m-\ell+1)$ with $\ell=2$ and $m=k$ even. Therefore, the lattice equations are invariant by π_{ab} as proved in Lemma 14. The last case of the theorem corresponds to edge-transitive graphs. The group of automorphisms contains only isometries of the tessellation, and $\mathscr{A}_0$ is isomorphic to the cyclic group of order three. The graph $C_{2n}(-1,6\ell+3,1)$, with $n=3\ell^2+3\ell+1$ is the chordal ring of odd diameter $2\ell+1$ and maximum order [5].

References

[1] A. Ádám, Research problem 2–10, J. Combin. Theory 2 (1967) 393.
[2] B.W. Arden, H. Lee, Analysis of chordal ring network, IEEE Trans. Comput. C-30 (4) (1981) 291–295.
[3] H. Attiya, J. van Leeuwen, N. Santoro, S. Zaks, Efficient elections in chordal ring networks, Algorithmica 4 (1989) 437–446.
[4] L. Barrière, J. Cohen, M. Mitjana, Gossiping in chordal rings under the line model, in: J. Hromkovic, W. Unger (Eds.), Proceedings of the MFCS'98 Workshop on Communication, 1998, pp. 37–47.
[5] L. Barrière, J. Fàbrega, E. Simó, Zaragozá, Fault-tolerant routings in chordal ring networks, Networks 36 (3) (2000) 180–190.
[6] B. Beauquier, J.-C. Bermond, L. Gargano, P. Hell, S. Pérennes, U. Vaccaro, Graph problems arising from wavelength-routing in all-optical networks, in: Proceedings of the Second IEEE Workshop on Optics and Computer Science, Part of IPPS'97, Geneva, Switzerland, 1997.
[7] H.S.M. Coxeter, Self dual configurations and regular graphs, Bull. Amer. Math. Soc. 56 (1950) 413–455.
[8] P. Erdös, P.M. Gruber, J. Hammer, Lattice Points, Longman Scientific & Technical, Wiley, Harlow, New York, 1989.
[9] M. Escudero, J. Fàbrega, P. Morillo, Fault-tolerant routings in double-loop networks, Ars Combin. 25A (1988) 187–198.
[10] M.-C. Heydemann, Cayley graphs and interconnection networks, pages 167–224, in: G. Hahn, G. Sabidussi (Eds.), Graph Symmetry. Algebraic Methods and Applications, Kluwer Academic Publishers, Dordrecht, 1997, pp. 167–224.
[11] M.-C. Heydemann, J.C. Meyer, D. Sotteau, On forwarding indices of networks, Discrete Appl. Math. 23 (1989) 103–123.

[12] X.D. Hu, F.K. Hwang, Reliabilities of chordal rings, Networks 22 (1992) 487–501.
[13] F.K. Hwang, P.E. Wright, Survival reliability of some double-loop networks and chordal rings, IEEE Trans. Comput. 44 (12) (1995) 1468–1471.
[14] Y. Iwasaki, Y. Kajiwara, K. Obokata, Y. Igarashi, Independent spanning trees of chordal rings, Lect. Notes Comput. Sci. 1276 (1997) 431–440.
[15] D. Krizanc, F. Luccio, Boolean routing on chordal rings, in: Proceedings of the Colloquium on Structural Information and Communication Complexity, Olympia, Greece, 1995.
[16] C.H. Li, On Isomorphisms of finite Cayey graphs – a survey, Discrete Mathematics 256 (2002) 301–334.
[17] A.L. Liestman, J. Opatrny, M. Zaragozá, Network properties of double and triple fixed step graph, Int. J. Found. Comput. Sci. 9 (1) (1998) 57–76.
[18] B. Mans, F. Pappalardi, I. Shparlinski, On the Ádám conjecture on circulant graphs, in: W.-L. Hsu, M.-Y. Kao (Eds.), Proceedings of the Fourth Annual International Computing and Combinatorics Conference, Cocoon'98, Lecture Notes in Computer Science, Vol. 1449, Springer, Berlin, 1998, pp. 251–260.
[19] B. Mans, N. Santoro, Optimal fault-tolerant leader election in chordal rings, in: Proceedings of the 24th Annual International Symposium on Fault-Tolerant Computing, Los Alamitos, CA, USA, IEEE Computer Society Press, Silver Spring, MD, June 1994, pp. 392–403.
[20] P. Morillo, Grafos y digrafos asociados con teselaciones como modelos para redes de interconexión, Ph.D. Thesis, Universitat Politècnica de Catalunya, Barcelona, Spain, 1987.
[21] P. Morillo, F. Comellas, M.A. Fiol, The optimization of chordal ring networks, in: Q. Yasheng, W. Ziuying (Eds.), Proceedings of the International Conference on Communication Technology ICCF87, World Scientific, Singapore, 1987, pp. 295–299.
[22] M.E. Muzychuk, G. Tinhofer, Recognizing circulant graphs of prime order in polynomial time, Electron. J. Combin. 5 (1) (1998) R25.
[23] L. Narayanan, J. Opatrny, Compact routing on chordal rings of degree four, in: D. Krizanc, D. Peleg (Eds.), Sirocco'97, The Fourth International Colloquium on Structural Information and Communication Complexity, Proceedings in Informatics, Vol. 1, Carleton Scientific, Ascona, Switzerland, 1997, pp. 125–137.
[24] L. Narayanan, J. Opatrny, D. Sotteau, All-to-all optical routing in optimal chordal rings of degree four, in: Proceedings of the 10th ACM-SIAM Symposium on Discrete Algorithms (SODA'99), Baltimore, Maryland, 1999, pp. 695–703.
[25] A. Nayak, V. Accia, P. Gissi, A note on isomorphic chordal rings, Inform. Process. Lett. 55 (6) (1995) 339–341.
[26] A.L. Rosenberg, Theoretical research on networks: models and methodology, in: Sirocco'97, D. Krizanc, D. Peleg (Eds.), The Fourth International Colloquium on Structural Information and Communication Complexity, Proceedings in Informatics, Vol. 1, Carleton Scientific, Ascona, Switzerland, 1997, pp. 283–293.
[27] G. Sabidussi, On a class of fixed-point-free graphs, Proc. Amer. Math. Soc. 9 (1958) 800–804.
[28] S. Toida, A note on Ádám's conjecture, J. Combin. Theory B23 (1977) 247–250.
[29] C.K. Wong, D. Coppersmith, A combinatorial problem related to multimode memory organizations, J. Assoc. Comput. Mach. 21 (1974) 392–402.
[30] J.L.A. Yebra, M.A. Fiol, P. Morillo, I. Alegre, The diameter of undirected graphs associated to plane tessellations, Ars Combin. 20B (1985) 159–171.
[31] G.W. Zimmerman, A.-H. Esfahanian, Chordal rings as fault-tolerant loops, Discrete Appl. Math. 37/38 (1992) 563–573.

For Further reading

L. Babai, Automorphism groups, isomorphism, reconstruction, in: R. Graham, M. Grotschel, L. Lovász (Eds.), Handbook of Combinatorics, Vol. 2, Elsevier and MIT Press, Amsterdam and Cambridge, MA, 1995.