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Global Solutions of a High Dimensional System for Korteweg Materials

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The global existence of solutions to a high-dimensional system of Korteweg materials is established when the initial data are small. The complete model was proposed by Dunn and Serrin and the local existence of the solution was obtained in an earlier paper by the authors. © 1996 Academic Press, Inc.

1. INITIAL VALUE PROBLEM

In [3], a high dimensional system for materials of Korteweg type was studied. The system is a simplified isothermal version of Dunn and Serrin's model [2]. In [3], the local classical solution for the initial value problem of the system is obtained. In this paper, we extend the results obtained in [3] and establish the global existence of the solution for the system. To simplify the notation, as in [3], we will write down explicitly the system with two space variables. The discussion of the general high dimension case can be performed in exactly the same manner. The only modification in the general high dimensional case is the regularity requirement on the initial data, which is demanded by the Sobolev imbedding theorem and Banach algebra properties for H^k. The necessary adjustments for the general high dimensional case are pointed out in all the theorems of the paper.

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The system for isothermal motion of the Korteweg type studied in [3] is the following:

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2)_x + (\rho uv)_y = -p_x + \nu\rho\Delta\rho_x + \mu\Delta u \\ \quad +(\mu/3)(u_{xx} + v_{xy}), \\ (\rho v)_t + (\rho uv)_x + (\rho v^2)_y = -p_y + \nu\rho\Delta\rho_y + \mu\Delta v \\ \quad +(\mu/3)(u_{xy} + v_{yy}) \end{cases} \tag{1.1}$$

with the initial data

$$(\rho, u, v)(x, y, 0) = (\rho_0, u_0, v_0)(x, y). \tag{1.2}$$

Here (ρ, u, v) denote the density and the (x, y)-components of velocity.

In the system (1.1), we assume that p is a smooth function of ρ and satisfies

$$p'(\rho) > 0, \qquad p''(\rho) > 0. \tag{1.3}$$

Let $\bar{\rho}_0$ be a positive constant and define function $H(\rho)$ by

$$H'(\rho) = h(\rho) = \int_{\bar{\rho}_0}^{\rho} \frac{p'(s)}{s}\, ds \tag{1.4}$$

and

$$H(\bar{\rho}_0) = 0. \tag{1.5}$$

Then we have $H'(\rho_0) = 0$, $H''(\rho) > 0$. Therefore in the neighborhood of $\bar{\rho}_0$, we have

$$H(\rho) \geq \gamma(\rho - \bar{\rho}_0)^2, \tag{1.6}$$

where γ is a positive number.

Set $\phi \equiv \rho - \bar{\rho}_0$ and $w \equiv (\phi, u, v)$. Let H^k be the usual Sobolev space and $\|\cdot\|_k$ the standard kth order Sobolev norm. We introduce the norm

$$|||w|||_{0,T}^2 \equiv \sup_{0 \leq t \leq T} \left(\|w(t)\|_0^2 + \|\nabla\rho(t)\|_0^2 \right) + \int_0^T \left(\|\nabla w(t)\|_0^2 + \|\nabla^2\phi(t)\|_0^2 \right) dt \tag{1.7}$$

and

$$|||w|||_{k,T}^2 = \sum_{|j| \leq k} |||\partial_{x,y}^j w|||_{0,T}^2. \tag{1.8}$$

In [3], the following local existence of solutions for (1.1), (1.2) was proved:

THEOREM 1.1. *For the Cauchy problem* (1.1), (1.2), *assume that the initial data* (ρ_0, u_0, v_0) *satisfy* $\rho_0 \geq \delta_0 > 0$ *and* $\phi_0 = \rho_0 - \bar{\rho}_0 \in H^{k+1}(R^2)$, $(u_0, v_0) \in H^k(R^2)\,(k \geq 3)$. *Then, there exists a* $T > 0$ *such that in* $0 \leq t \leq T$, *problem* (1.1), (1.2) *has a unique solution* (ρ, u, v) *such that* $\phi = \rho - \bar{\rho}_0 \in L^\infty([0, T]; H^{k+1}(R^2))$, $(u, v) \in L^\infty([0, T]; H^k(R^2))$, *and*

$$|||\, w \,|||^2_{k,T} \geq C_{k,T}\left(\|w_0\|_k^2 + \|\phi_0\|_{k+1}^2\right). \tag{1.9}$$

Here $\phi_0 = \rho_0 - \bar{\rho}_0$.

For the n*-dimensional case* R^n, *one should have* $k \geq 2 + n/2$.

Remark 1.1. The norm $|||\, w \,|||^2_k$ in the energy estimate in [3] actually does not include the integration in t of $\nabla\rho, \nabla^2\rho$ and hence is different from (1.7), (1.8) above. However, from the derivation of (1.9) in Section 3, it is easy to see that the (1.9) form of energy estimate is also true in [3]. In addition, the requirement $k \geq 3$ in Theorem 1.1 is a relaxation of the requirement $k \geq 4$ in [3]. From the above improved estimate and the proof in this paper, it is easy to see that this relaxation is justified.

In the next section, we state and prove the main theorem of this paper on the global existence of classical solutions to the problem (1.1), (1.2). The proof of the main theorem is based on the local result Theorem 1.1 obtained in [3] and a method employed in [7] which, in the context of this paper, is formulated as Theorem 2.2. Section 3 is devoted to the proof of Theorem 2.2.

2. MAIN THEOREM

The main theorem of this paper is the following

THEOREM 2.1. *Consider the problem* (1.1), (1.2). *In addition to the assumptions in Theorem* 1.1, *we assume that*

$$\|w_0\|_3^2 + \|\phi_0\|_4^2 \leq \varepsilon \tag{2.1}$$

with $\varepsilon \ll 1$. *Then there exists a unique classical solution* (ρ, u, v) *in* $[0, \infty)$ *such that*

$$\phi \equiv \rho - \bar{\rho}_0 \in L^\infty\big([0, \infty); H^4(R^2)\big), \qquad (u, v) \in L^\infty\big([0, \infty); H^3(R^2)\big). \tag{2.2}$$

Besides, we have the estimate

$$|||\, w \,|||^2_{3,\infty} \leq C\left(\|w_0\|_3^2 + \|\phi_0\|_4^2\right) \tag{2.3}$$

with $w \equiv (\phi, u, v) \equiv (\rho - \bar{\rho}_0, u, v)$.

The proof of Theorem 2.1 is based upon the existence of local solutions in Theorem 1.1 and the following lemma:

THEOREM 2.2. *Let* w *be a sufficiently smooth solution for the problem* (1.1), (1.2). *Then there exists* $\delta \ll 1$ *such that if* w *satisfies*

$$\sup_{0 \le t \le T} \left(\| w(t) \|_3 + \| \phi \|_4 \right) \le \delta, \tag{2.4}$$

then we have the estimate

$$|||\, w \,|||_{3,T}^2 \le C_\delta \left(\| w_0 \|_3^2 + \| \phi_0 \|_4^2 \right). \tag{2.5}$$

Here C_δ *depends only on* δ *and is independent of* T.

Remark 2.1. For general R^n, Sobolev spaces H^3, H^4 in Theorems 2.1 and 2.2 should be replaced by H^k and H^{k+1} with $k \ge 2 + n/2$.

Assuming Theorem 2.2, we can choose ε in Theorem 2.1 so small that

$$C_\delta \varepsilon \le \delta. \tag{2.6}$$

Therefore, for all the initial data (ρ_0, u_0, v_0) satisfying (2.1), the existence of the local solution is guaranteed by Theorem 1.1. Because of (2.6), we will always have our local solution w satisfying (2.4), and therefore (2.5). The standard continuation argument then yields the global existence of the solution.

Remark 2.2. From the proof in the following, it is easy to see that if the initial data are C^∞ smooth, then under the same "smallness" requirement (2.1) we can obtain the global existence of C^∞ smooth solutions.

Remark 2.3. Another interesting observation is that in establishing the existence of a global solution, the assumption on the convexity of $p(\rho)$ in (1.3) is used only in the derivation of the 0-order estimate of Theorem 2.2. The higher order estimates can be derived simply by induction assumption.

3. PROOF OF THEOREM 2.2

Since μ and ν are two fixed constants in our discussion, we will in the following let them be equal to 1 to simplify the notation.

We will use induction on k $(0 \le k \le 4)$ to prove the following estimate under the assumption (2.4):

$$|||\, w \,|||_{k,T}^2 \le C_\delta \left(\| w_0 \|_k^2 + \| \phi_0 \|_{k+1}^2 \right). \tag{2.5$'$}$$

3.1. *Estimate for $k = 0$*

First of all, taking the inner product of the second and third equations in (1.1) with (u, v) in the space $L^2((0,t) \times R^2)$, we have

$$
\begin{aligned}
&(\rho u_t + \rho u u_x + \rho v u_y, u) + (\rho v_t + \rho u v_x + \rho v v_y, v) \\
&\quad + (p_x, u) + (p_y, v) \\
&\quad - \left(\Delta u + \tfrac{1}{3}(u_{xx} + v_{xy}), u\right) - \left(\Delta v + \tfrac{1}{3}(u_{xy} + v_{yy}), v\right) \\
&\quad - (\rho \Delta \rho_x, u) - (\rho \Delta \rho_y, v) \\
&\quad \equiv I_1 + I_2 + I_3 + I_4 = 0. \qquad (3.1)
\end{aligned}
$$

The four terms I_i $(i = 1, 2, 3, 4)$ are estimated separately as follows.

- Estimate of I_1: Using the first equation in (1.1) and integrating by parts, we obtain

$$
\begin{aligned}
I_1 &\equiv (\rho u_t + \rho u u_x + \rho v u_y, u) + (\rho v_t + \rho u v_x + \rho v v_y, v) \\
&= \tfrac{1}{2} \int \rho (u^2 + v^2)\, dx\, dy \Big|_0^t. \qquad (3.2)
\end{aligned}
$$

- Estimate of I_2: From (1.4), we have $p'(\rho) = h'(\rho)/\rho$. Therefore we derive

$$
\begin{aligned}
I_2 &\equiv (p_x, u) + (p_y, v) = \left(h_x(\rho), \rho u\right) + \left(h_y(\rho), \rho v\right) \\
&= \int H(\rho)\, dx\, dy \Big|_0^t \geq \gamma \int |\phi|^2\, dx\, dy \Big|_0^t. \qquad (3.3)
\end{aligned}
$$

- Estimate of I_3: Integration by parts gives

$$
\begin{aligned}
I_3 &\equiv -\left(\Delta u + \tfrac{1}{3}(u_{xx} + v_{xy}), u\right) - \left(\Delta v + \tfrac{1}{3}(u_{xy} + v_{yy}), v\right) \\
&\geq \tfrac{1}{2} \int_0^t \left(\|\nabla u\|_0^2 + \|\nabla v\|_0^2\right) ds. \qquad (3.4)
\end{aligned}
$$

- Estimate of I_4: Integrating by parts and using the first equation in (1.1), we obtain

$$
\begin{aligned}
I_4 &\equiv -(\rho \Delta \rho_x, u) - (\rho \Delta \rho_y, v) = +\left(\Delta \rho, (\rho u)_x + (\rho v)_y\right) \\
&= \tfrac{1}{2} \int |\nabla \rho|^2\, dx\, dy \Big|_0^t. \qquad (3.5)
\end{aligned}
$$

Combining (3.2)–(3.5), we obtain

$$\frac{1}{2}\Big(\big\|\sqrt{\rho}u(t)\big\|_0^2+\big\|\sqrt{\rho}v(t)\big\|_0^2\Big)+\int_0^t\big(\|\nabla u\|_0^2+\|\nabla v\|_0^2\big)\,ds+\frac{1}{2}\|\nabla\phi(t)\|_0^2$$
$$+\gamma\|\phi(t)\|_0^2$$
$$\le\frac{1}{2}\Big(\|\sqrt{\rho}u_0\|_0^2+\|\sqrt{\rho}v_0\|_0^2\Big)+\frac{1}{2}\|\nabla\phi_0\|_0^2+\gamma\|\phi_0\|_0^2. \tag{3.6}$$

For $\delta \ll 1$ in (2.4), the estimate (3.6) implies

$$\|w(t)\|_0^2+\|\phi(t)\|_1^2+\int_0^t\big(\|\nabla u(s)\|_0^2+\|\nabla v(s)\|_0^2\big)\,ds$$
$$\le C_\delta\big(\|w_0\|_0^2+\|\phi_0\|_1^2\big). \tag{3.7}$$

Next, we take the inner product of the second and the third equations in (1.1) with (ρ_x,ρ_y) in the space $L^2((0,t)\times R^2)$ to derive the estimate of $\int_0^t\|\phi(s)\|_1^2\,ds$:

$$(\rho u_t+\rho u u_x+\rho v u_y,\rho_x)+(\rho v_t+\rho u v_x+\rho v v_y,\rho_y)$$
$$+(p_x,\rho_x)+(p_y,\rho_y)$$
$$-\Big(\Delta u+\frac{1}{3}(u_{xx}+v_{xy}),\rho_x\Big)-\Big(\Delta v+\frac{1}{3}(u_{xy}+v_{yy}),\rho_y\Big)$$
$$-(\rho\Delta\rho_x,\rho_x)-(\rho\Delta\rho_y,\rho_y)$$
$$\equiv J_1+J_2+J_3+J_4=0. \tag{3.8}$$

The four terms in the (3.8) are estimated as follows.

- Estimate of J_1: Using the first equation in (1.1), we have

$$(\rho u_t,\rho_x)=\Big(u_t,\frac{1}{2}\partial_x\rho^2\Big)$$
$$=(\rho u,\rho_x)_0^t+(\rho u_x,\rho_t)$$
$$=(\rho u,\rho_x)_0^t-\big(\rho u_x,(\rho u)_x+(\rho v)_y\big).$$

Therefore

$$\big|(\rho u_t+\rho u u_x+\rho v u_y,\rho_x)\big|\le C_\delta\big(\|\phi(s)\|_1^2+\|u(s)\|_0^2\big)_0^t$$
$$+C_\delta\int_0^t\big(\|\nabla u(s)\|_0^2+\|\nabla v(s)\|_0^2\big)\,ds$$
$$+C\delta\int_0^t\|\nabla\phi(s)\|_0^2\,ds, \tag{3.9}$$

where the constant C is independent of the small choice of δ. Obviously, we have a similar estimate for $(\rho v_t + \rho u v_x + \rho v v_y, \rho_y)$. Hence we obtain

$$|J_1| \leq C_\delta\Big(\|\phi(t)\|_1^2 + \|\phi_0\|_1^2 + \|w(t)\|_0^2 + \|w_0\|_0^2\Big) + C_\delta \int_0^t \Big(\|\nabla u(s)\|_0^2 + \|\nabla v(s)\|_0^2\Big)\,ds + C\delta\int_0^t \|\nabla\phi(s)\|_0^2\,ds. \quad (3.10)$$

- Estimate of J_2: From the assumption $p'(\rho) > 0$, we have

$$J_2 \geq c_0 \int_0^t \|\nabla\phi(s)\|_0^2\,ds, \quad (3.11)$$

where the constant c_0 is independent of δ for $\delta \ll 1$.

- Estimate of J_3: From integrating by parts, we have

$$|J_3| \leq \frac{C}{\delta_0}\int_0^t \Big(\|\nabla u(s)\|_0^2 + \|\nabla v(s)\|_0^2\Big)\,ds + \delta_0 \int_0^t \|\nabla^2\phi(s)\|_0^2\,ds. \quad (3.12)$$

- Estimate of J_4: Again from integrating by parts, we have

$$J_4 = (\rho\nabla\rho_x, \nabla\rho_x) + (\rho\nabla\rho_y, \nabla\rho_y) + (\nabla\rho_x, [\nabla, \rho]\nabla\rho_x) + \big(\nabla\rho_y, [\nabla, \rho]\nabla\rho_y\big) \geq c_0\int_0^t \|\nabla^2\phi(s)\|_0^2\,ds - \overline{C}\delta\int_0^t \|\nabla\phi(s)\|_0^2\,ds, \quad (3.13)$$

where $\overline{C}$ is independent of δ.

Now we combine (3.8), (3.10)–(3.13) and take δ_0 and δ in (3.12), (3.13) so small that $\delta_0, \delta < \frac{1}{2}c_0$. Using the estimate (3.7), we obtain (2.5′) for $k = 0$:

$$|||\, w\, |||_{0,T}^2 \leq C_\delta\Big(\|w_0\|_0^2 + \|\phi_0\|_1^2\Big). \quad (3.14)$$

3.2. *Estimate of Higher Order $k \geq 1$*

First of all, we state a special case of the Nirenberg inequality and its corollaries which will be used in our estimate. For the most general form of the inequality, the reader is referred to [8].

LEMMA 3.1. *Let* $u \in L^\infty(R^n) \cap H^m(R^n)$. *Then for all* $0 \le j \le m$, *the following inequality holds*:

$$|D^j u|_{L^p} \le C\|u\|_m^a |u|_{L^\infty}^{1-a}, \tag{3.15}$$

where

$$a = \frac{j}{m} = \frac{2}{p}. \tag{3.16}$$

It is worth to pointing out that with the special choice of a in (3.16), all the parameters in (3.15) are independent of the space dimension n except the constant C.

Let q be the conjugate number of p: $1/p + 1/q = 1$. By (3.15), using the standard Hölder inequality, we have for $1/\bar{p} = j/m$ and $1/\bar{q} = (m - j)/m$:

$$\begin{aligned} \left|(D^j u)(D^{m-j} v)\right|_{L^2} &\le C|D^j u|_{L^{2\bar{p}}}|D^{m-j} v|_{L^{2\bar{q}}} \\ &\le C\|u\|_m^{1/p}|u|_{L^\infty}^{1/q}\|v\|_m^{1/q}|v|_{L^\infty}^{1/p} \\ &= C(\|u\|_m|v|_{L^\infty})^{1/p}(|u|_{L^\infty}\|v\|_m)^{1/q}. \end{aligned}$$

Then, from the inequality

$$A^{1/p}B^{1/q} \le \frac{A}{p} + \frac{B}{q}, \qquad A, B > 0,$$

we obtain the following corollaries:

LEMMA 3.2. *Let* u, v *belong to* $L^\infty \cap H^m$ *and* F *be a smooth function. Then we have*

$$\|uv\|_m \le C(\|u\|_m|v|_{L^\infty} + \|v\|_m|u|_{L^\infty}), \tag{3.17}$$

$$\|F(u)\|_m \le C(1 + \|u\|_m), \tag{3.18}$$

where the constant C *in* (3.18) *depends on* $|u|_{L^\infty}$.

Now we prove (2.5′) by induction on k. Assume that (2.5′) is true for all $j \le k - 1$. The proof consists of two steps.

Step One. To simplify the notation, we will denote by ∇^k the operator vector with components consisting of all the differential operators D^α with multi-index $|\alpha| = k$. Applying the operator $\rho\nabla^k\rho^{-1}$ to the second and third equations in (1.1) and then taking the inner product with $(\nabla^k u, \nabla^k v)$ in the space $L^2((0,t)\times R^2)$, we have

$$
\begin{aligned}
&\left(\rho\nabla^k(u_t+uu_x+vu_y),\nabla^k u\right)+\left(\rho\nabla^k(v_t+uv_x+vv_y),\nabla^k v\right)\\
&\qquad+\left(\rho\nabla^k\rho^{-1}p_x,\nabla^k u\right)+\left(\rho\nabla^k\rho^{-1}p_y,\nabla^k v\right)\\
&\qquad-\left(\rho\nabla^k\rho^{-1}\left(\Delta u+\tfrac{1}{3}(u_{xx}+v_{xy})\right),\nabla^k u\right)\\
&\qquad-\left(\rho\nabla^k\rho^{-1}\left(\Delta v+\tfrac{1}{3}(u_{xy}+v_{yy})\right),\nabla^k v\right)\\
&\qquad-\left(\rho\nabla^k\Delta\rho_x,\nabla^k u\right)-\left(\rho\nabla^k\Delta\rho_y,\nabla^k v\right)\\
&\quad\equiv I_{k1}+I_{k2}+I_{k3}+I_{k4}=0.
\end{aligned}
\tag{3.19}
$$

The four terms $I_{ki}(i=1,2,3,4)$ are estimated separately as follows.

- Estimate of I_{k1}: From

$$
\left(\rho\nabla^k u_t,\nabla^k u\right)=\left(\rho,\tfrac{1}{2}\partial_t(\nabla^k u)^2\right),
$$

$$
\left(\rho\nabla^k uu_x,\nabla^k u\right)=\left(\rho[\nabla^k,u]u_x,\nabla^k u\right)-\tfrac{1}{2}\left((\rho u)_x,(\nabla^k u)^2\right),
$$

$$
\left(\rho\nabla^k vu_y,\nabla^k u\right)=\left(\rho[\nabla^k,v]u_y,\nabla^k u\right)-\tfrac{1}{2}\left((\rho v)_y,(\nabla^k u)^2\right),
$$

and using the first equation in (1.1), we have

$$
\begin{aligned}
&\left(\rho\nabla^k(u_t+uu_x+vu_y),\nabla^k u\right)\\
&\qquad=\tfrac{1}{2}\int\rho(\nabla^k u)^2\,dx\,dy\Big|_0^t+\left(\rho[\nabla^k,u]u_x+\rho[\nabla^k,v]u_y,\nabla^k u\right).
\end{aligned}
\tag{3.20}
$$

A similar identity also holds for $(\rho\nabla^k(u_t+uu_x+vu_y),\nabla^k u)$:

$$
\begin{aligned}
&\left(\rho\nabla^k(v_t+uv_x+vv_y),\nabla^k v\right)\\
&\qquad=\tfrac{1}{2}\int\rho(\nabla^k v)^2\,dx\,dy\Big|_0^t+\left(\rho[\nabla^k,u]v_x+\rho[\nabla^k,v]v_y,\nabla^k v\right).
\end{aligned}
\tag{3.21}
$$

Combining (3.20), (3.21), we have

$$
\begin{aligned}
I_{k1}\ge{}& C_\delta^{-1}\left(\|\nabla^k u(t)\|_0^2+\|\nabla^k v(t)\|_0^2-\|w_0\|_k^2\right)\\
&-\Big|\left(\rho[\nabla^k,u]u_x+\rho[\nabla^k,v]u_y,\nabla^k u\right)\\
&\qquad+\left(\rho[\nabla^k,u]v_x+\rho[\nabla^k,v]v_y,\nabla^k v\right)\Big|.
\end{aligned}
\tag{3.22}
$$

Because $[\nabla^k, u]u_x$ is the summation of the terms

$$\partial^{k-s} u \partial^s u_x = \partial^{k-1-s}(\partial u)\partial^s u_x \qquad (0 \le s \le k-1),$$

we have from (3.17) and (2.4):

$$\big\|[\nabla^k, u]u_x\big\|_0 \le C\|\nabla u\|_{k-1}|\nabla u|_{L^\infty} \le C\|\nabla u\|_{k-1}\|\nabla u\|_2 \le C_\delta\|\nabla u\|_{k-1}.$$

Similar estimates hold for $[\nabla^k, v]u_y$, $[\nabla^k, u]v_x$, and $[\nabla^k, v]v_y$. Hence we have

$$I_{k1} \ge C_\delta^{-1}\Big(\big\|\nabla^k u(t)\big\|_0^2 + \big\|\nabla^k v(t)\big\|_0^2 - \|w_0\|_k^2\Big) - C_\delta\int_0^t \big\|\nabla w(s)\big\|_{k-1}^2\, ds. \tag{3.23}$$

- Estimate of I_{k2}: From (1.4), we have

$$I_{k2} = \big(\nabla^k h_x(\rho), \rho\nabla^k u\big) + \big(\nabla^k h_y(\rho), \rho\nabla^k v\big). \tag{3.24}$$

We rewrite the terms in (3.24) as follows:

$$\begin{aligned}\big(\nabla^k h_x(\rho), \rho\nabla^k u\big) &= -\big(\nabla^k h(\rho), \partial_x \rho\nabla^k u\big)\\ &= -\big(h'(\rho)\nabla^k\rho, \nabla^k(\rho u)_x\big)\\ &\quad -\big(\big[\nabla^{k-1}, h'(\rho)\big]\nabla\rho, \nabla^k(\rho u)_x\big)\\ &\quad -\big(\nabla^k h(\rho), \partial_x\big[\rho, \nabla^k\big]u\big).\end{aligned} \tag{3.25}$$

Similarly, we have

$$\begin{aligned}\big(\nabla^k h_y(\rho), \rho\nabla^k v\big) &= -\big(h'(\rho)\nabla^k\rho, \nabla^k(\rho v)_y\big)\\ &\quad -\big(\big[\nabla^{k-1}, h'(\rho)\big]\nabla\rho, \nabla^k(\rho v)_y\big)\\ &\quad -\big(\nabla^k h(\rho), \partial_y\big[\rho, \nabla^k\big]v\big).\end{aligned} \tag{3.26}$$

Combining (3.25), (3.26) and using the first equations in (1.1), we obtain

$$I_{k2} = \big(h'(\rho)\nabla^k\rho, \partial_t\nabla^k\rho\big) + R_{k2}, \tag{3.27}$$

where

$$\begin{aligned}R_{k2} = &-\big(\big[\nabla^{k-1}, h'(\rho)\big]\nabla\rho, \nabla^k((\rho u)_x + (\rho v)_y)\big)\\ &-\big(\nabla^k h(\rho), \partial_x\big[\rho, \nabla^k\big]u + \partial_y\big[\rho, \nabla^k\big]v\big).\end{aligned} \tag{3.27}$$

By Lemma 3.2 and $h'(\rho) > 0$, it is easy to derive

$$\left(h'(\rho)\nabla^k\rho, \partial_t\nabla^k\rho\right) \geq C_\delta^{-1}\left(\|\nabla^k\rho(t)\|_0 - \|\phi_0\|_k\right) - C\int_0^t\|\nabla\phi(s)\|_{k-1}^2\,ds, \tag{3.28}$$

$$|R_{k2}| \leq C\int_0^t\left(\|\nabla w(s)\|_{k-1}^2 + \delta_0\left(\|\nabla u(s)\|_k^2 + \|\nabla v(s)\|_k^2\right)\right)ds. \tag{3.29}$$

Combining (3.28) and (3.29), we obtain the estimate for I_{k2}:

$$I_{k2} \geq C_\delta^{-1}\left(\|\rho(t)\|_k - \|w_0\|_k\right) - C_\delta\int_0^t\|\nabla w(s)\|_{k-1}^2\,ds - \delta_0\int_0^t\left(\|\nabla u(s)\|_k^2 + \|\nabla v(s)\|_k^2\right)ds \tag{3.30}$$

where δ_0 is a small number to be chosen later.

- Estimate of I_{k3}: Since

$$\begin{aligned}\left(\rho\nabla^k\rho^{-1}\left(\Delta u + \tfrac{1}{3}(u_{xx} + v_{xy})\right), \nabla^k u\right) &= \left(\Delta\nabla^k u + \tfrac{1}{3}\left(\nabla^k u_{xx} + \tfrac{1}{3}(u_{xx} + v_{xy})\nabla^k v_{xy}\right), \nabla^k u\right)\\ &\quad + \left(\rho\left[\nabla^k, \rho^{-1}\right]\left(\Delta u + \tfrac{1}{3}(u_{xx} + v_{xy})\right), \nabla^k u\right),\end{aligned}$$

and similarly for $(\rho\nabla^k\rho^{-1}(\Delta u + \frac{1}{3}(u_{xx} + v_{xy})), \nabla^k u)$ we have

$$\begin{aligned}I_{k3} &\geq \int_0^t\left(\|\nabla^{k+1}u(s)\|_0^2 + \|\nabla^{k+1}v(s)\|_0^2\right)ds\\ &\quad - C\int_0^t(\|\nabla u(s)\|_k + \|\nabla v(s)\|_k + \|\rho(s)\|_k)\|\nabla w(s)\|_{k-1}\,ds\\ &\geq \delta_0\int_0^t\left(\|\nabla u(s)\|_k^2 + \|\nabla v(s)\|_k^2\right)ds - C_\delta\int_0^t\|\nabla w(s)\|_{k-1}^2\,ds.\end{aligned} \tag{3.31}$$

- Estimate of I_{k4}: From

$$\left(\rho\nabla^k\Delta\rho_x, \nabla^k u\right) = \left(\nabla^{k+1}\rho, \nabla^{k+1}(\rho u)_x\right) + \left(\nabla^{k+1}\rho, \partial_x\nabla\left[\nabla^k, \rho^{-1}\right](\rho u)\right)$$

and a similar identity for $(\rho\nabla^k\Delta\rho_x, \nabla^k u)$, by using the first equation in (1.1) we have

$$I_{k4} = \left(\nabla^{k+1}\rho, \nabla^{k+1}\rho_t\right) - \left(\nabla^{k+1}\rho, \partial_x\nabla\left[\nabla^k, \rho^{-1}\right]\rho u + \partial_y\nabla\left[\nabla^k, \rho^{-1}\right]\rho v\right). \tag{3.32}$$

Since $[\nabla^k, \rho^{-1}]$ is an operator or order $k-1$, the total derivatives in $\partial_x \nabla [\nabla^k, \rho^{-1}] \rho u$ and $\partial_y \nabla [\nabla^k, \rho^{-1}] \rho v$ are no higher than $k+1$. In particular, at least one derivative is on ρ. Consequently, we have by Lemma 3.2

$$\left| \left(\nabla^{k+1} \rho, \partial_x \nabla \left[\nabla^k, \rho^{-1} \right] \rho u + \partial_y \nabla \left[\nabla^k, \rho^{-1} \right] \rho v \right) \right| \le C \|\nabla \rho\|_k^2 + \|\nabla w\|_{k-1}^2.$$

Therefore, we obtain

$$I_{k4} \ge \tfrac{1}{2} \left\| \nabla^{k+1} \rho(t) \right\|_0^2 - C \|\phi_0\|_{k+1}^2 - C_\delta \int_0^t \left(\| \nabla w(s) \|_{k-1}^2 + \| \nabla \phi(s) \|_k^2 \right) ds. \tag{3.33}$$

Combining (3.23), (3.30), (3.31), (3.33), and the induction assumption, we obtain by choosing $\delta_0 \ll 1$

$$\| w(t) \|_k^2 + \| \rho(t) \|_{k+1}^2 + \int_0^t \left(\| \nabla u(s) \|_k^2 + \| \nabla v(s) \|_k^2 \right) ds \le C_\delta \left(\|w_0\|_k^2 + \|\phi_0\|_{k+1}^2 \right). \tag{3.34}$$

This concludes the first step of the proof.

Step Two. Applying the operator $\rho \nabla^k \rho^{-1}$ to the second and third equations in (1.1) and then taking the inner product with $(\nabla^k \rho_x, \nabla^k \rho_y)$ in the space $L^2((0,t) \times R^2)$, we have

$$\begin{aligned} &\left(\rho \nabla^k (u_t + u u_x + v u_y), \nabla^k \rho_x \right) + \left(\rho \nabla^k (v_t + u v_x + v v_y), \nabla^k \rho_y \right) \\ &\quad + \left(\rho \nabla^k \rho^{-1} p_x, \nabla^k \rho_x \right) + \left(\rho \nabla^k \rho^{-1} p_y, \nabla^k \rho_y \right) \\ &\quad - \left(\rho \nabla^k \rho^{-1} \left(\Delta u + \tfrac{1}{3} (u_{xx} + v_{xy}) \right), \nabla^k \rho_x \right) \\ &\quad - \left(\rho \nabla^k \rho^{-1} \left(\Delta v + \tfrac{1}{3} (u_{xy} + v_{yy}) \right), \nabla^k \rho_y \right) \\ &\quad - \left(\rho \nabla^k \Delta \rho_x, \nabla^k \rho_x \right) - \left(\rho \nabla^k \Delta \rho_y, \nabla^k \rho_y \right) \\ &\equiv J_{k1} + J_{k2} + J_{k3} + J_{k4} = 0. \end{aligned} \tag{3.35}$$

The four terms in (3.35) are estimated as follows.

- Estimate of J_{k1}: First of all we note that integration by parts gives:

$$\left(\rho \nabla^k u_t, \nabla^k \rho_x \right) = \left(\nabla^k u, \rho \nabla^k \rho_x \right)_0^t - \left(\rho_t \nabla^k u, \nabla^k \rho_x \right) + \left(\partial_x \rho \nabla^k u, \nabla^k \rho_t \right).$$

Using the first equation in (1.1) and Lemma 3.2, we have

$$\begin{aligned} &\left| \left(\rho \nabla^k (u_t + u u_x + v u_y), \nabla^k \rho_x \right) \right| \\ &\quad \le C_\delta \left(\| \nabla w(t) \|_{k-1}^2 + \| \nabla \phi(t) \|_k^2 + \|w_0\|_k^2 + \|\phi_0\|_{k+1}^2 \right) \\ &\qquad + C_\delta \int_0^t \left(\| \nabla w(s) \|_k^2 + \| \nabla \phi(s) \|_k^2 \right) ds. \end{aligned}$$

The same estimate also holds for $(\rho\nabla^k(v_t + uv_x + vv_y), \nabla^k\rho_y)$. Therefore, we have

$$|J_{k1}| \le C_\delta\Big(\|\nabla w(t)\|_{k-1}^2 + \|\nabla\phi(t)\|_k^2 + \|w_0\|_k^2 + \|\phi_0\|_{k+1}^2\Big) + C_\delta\int_0^t\Big(\|\nabla w(s)\|_k^2 + \|\nabla\phi(s)\|_k^2\Big)\,ds. \tag{3.36}$$

- Estimate of J_{k2}: Obviously we have

$$|J_{k2}| \le C_\delta\int_0^t\|\nabla\phi(s)\|_{k-1}^2\,ds. \tag{3.37}$$

- Estimate of J_{k3}: Integrating by parts and using Lemma 3.2, we readily obtain

$$|J_{k3}| \le \frac{C}{\delta_0}\int_0^t\|\nabla w(s)\|_k^2\,ds + \delta_0\int_0^t\|\nabla^2\phi(s)\|_k^2\,ds. \tag{3.38}$$

- Estimate of J_{k4}: Integrating by parts and using Lemma 3.2, we have

$$\begin{aligned} J_{k4} &= \big(\rho\nabla^{k+1}\rho_x, \nabla^{k+1}\rho_x\big) + \big(\rho\nabla^{k+1}\rho_y, \nabla^{k+1}\rho_y\big) \\ &\quad + \big(\nabla^{k+1}\rho_x, [\nabla, \rho]\nabla^k\rho_x\big) + \big(\nabla^{k+1}\rho_y, [\nabla, \rho]\nabla^k\rho_y\big) \\ &\ge c_0\int_0^t\|\nabla^2\phi(s)\|_k^2\,ds - C_0\delta\int_0^t\|\nabla\phi(s)\|_k^2\,ds, \end{aligned} \tag{3.39}$$

where c_0 and C_0 are independent of δ for $\delta \ll 1$.

Combining (3.36)–(3.39) and taking $\delta, \delta_0 \ll 1$, we have

$$\begin{aligned} \int_0^t\|\nabla^2\phi(s)\|_k^2\,ds &\le C_\delta\int_0^t\Big(\|\nabla w(s)\|_k^2 + \|\nabla\phi(s)\|_k^2\Big)\,ds \\ &\le C_\delta\Big(\|\nabla w(t)\|_{k-1}^2 + \|\nabla\phi(t)\|_k^2 + \|w_0\|_k^2 + \|\phi_0\|_{k+1}^2\Big). \end{aligned} \tag{3.40}$$

Now combining (3.34), (3.40), and the induction assumption, we obtain (2.5). This finishes our proof of Theorem 2.2.

Remark. In our proof it is easy to see that we only need to use the norms

$$|\nabla u|_{L^\infty}, \qquad |\nabla v|_{L^\infty}, \qquad |\nabla^2 \phi|_{L^\infty}.$$

Consequently, by Sobolev imbedding theorem, we need to have the index $k \geq 3$ in Theorem 2.2.

REFERENCES

1. G. Andrews and J. M. Ball, Asymptotic behavior and change of phase in one-dimensional nonlinear viscoelasticity, *J. Differential Equations* **44** (1982), 306–341.
2. J. E. Dunn and J. Serrin, On the thermodynamics of interstitial working, *Arch. Rational Mech. Anal.* **88** (1985), 95–133.
3. H. Hattori and D. Li, Solutions for two dimensional system for materials of Korteweg type, *SIAM J. Math. Anal.* **25** (1994), 85–98.
4. H. Hattori and K. Mischaikow, A dynamical systems approach to a phase transition problem, *J. Differential Equations* **94** (1991), 340–378.
5. D. J. Korteweg, Sur la forme que prennent les équations des movement des fluides si l'on tient comple des forces capillaires par des variations de densité, *Arch. Nederl. Sci. Exactes. Nat. Ser. II* **6** (1901), 1–24.
6. J. L. Lions and E. Magenes, "Nonhomogeneous Boundary Value Problems and Applications I," Springer-Verlag, New York, 1972.
7. A. Matsumura and T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* **20** (1980), 67–104.
8. L. Nirenberg, On elliptic partial differential equations, *Ann. Scuol. Norm. Sup. Pisa* **13** (1959), 116–162.
9. J. Serrin, The form of interfacial surfaces in Korteweg's theory of phase equilibria, *Quart. J. Appl. Math.* **41** (1983), 357–364.
10. J. Serrin, Phase transition and interfacial layers for van der Waals fluids, *in* "Proceedings of SAFA IV Conference, Recent Methods in Nonlinear Analysis and Applications, Naples, 1980" (A. Camfora, S. Rionero, C. Sbordone, and C. Trombetti, Eds.).
11. M. Slemrod, Admissibility criteria for propagating phase boundaries in a van der Waals fluid, *Arch. Rational Mech. Anal.* **81** (1983), 301–315.
12. M. Slemrod, Dynamic phase transitions in a van der Waals fluid, *J. Differential Equations* **52** (1984), 1–23.
13. J. Sprekels and S. Zheng, Global solutions to the equations of a Ginsburg-Landau theory for structural phase transitions in shape memory alloys, *Physica D* **39** (1989), 59–76.