Short cycles of Poncelet’s conics
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ABSTRACT
Main results of this paper are the following:
1. A closed $N$-gon interscribed between two conics exists if and only if a specially constructed polygon with a smaller number of sides ($n$) is closed. To verify the closure of this $n$-gon, we need to find a periodic solution of a dynamical system of order $n$. The proof is based on the connection of Poncelet’s curves and matrices that admit unitary bordering [4,9,10,16]. Application of this criterion makes sense when $n \ll N$, in particular when $n \approx \log_2 N$ (see Table 4 where $n = m_1$). So for example we may say that a polygon with 2049 sides interscribed between two circles is closed if and only if some specially constructed 11-gon is closed.

2. A closed $N$-gon interscribed between two confocal ellipses (the billiard case) exists if and only if an $N$-gon interscribed between two special nested circles is closed.

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1. Introduction and preliminaries

Jean-Victor Poncelet proved his closure theorem as a corollary of the so-called “general theorem” (see [2,3]):

Poncelet’s general theorem. Let $\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{n-1}$ be conics from one pencil. Consider a closed inscribed $n$-gon of $\mathcal{C}$ whose first vertex is $P$ and whose first $n - 1$ sides are tangent to the successive $\mathcal{D}_j$. Then if $P$ moves along $\mathcal{C}$ the $n$th side $L$ of the polygon will envelop a curve which is again a conic belonging to the same pencil.

Corollary: Poncelet’s closure theorem. Let $\mathcal{K} \subset \mathcal{C}$ be a conic. If there exists a closed $N$-gon inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{K}$, then there exists an infinite number of such interscribed $N$-gons and any point on $\mathcal{C}$ may be chosen as a vertex.

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Let \( C \) be the unit circle, and \( K \) – a conic. We say that \( C \) and \( K \) form a Poncelet pair of rank \( N \) if there exists a closed \( N \)-gon inscribed in \( C \) and circumscribed about \( K \) (i.e., closed polygon interscribed between \( C \) and \( K \)).

The general theorem suggests we try to find a closed \( n \)-gon interscribed between \( C \) and some special set of conics which form with \( C \) Poncelet’s pairs of rank \( N > n \). Following this suggestion, we reduce the necessary and sufficient condition of the closure of an \( N \)-sided polygon to the condition of the closure of a polygon with a smaller number of sides, \( n \) (where for the case of circles, \( \log_2 N \leq n \leq N/2 \)). Description of this \( n \)-gon is given in Section 2.1. To verify the closure of the \( n \)-gon, we need to find a periodic solution of a dynamical system of order \( n \). Another interpretation of this \( n \)-gon is based on the group law on an elliptic curve (see Remark 2.1.19 and Tables 6 and 7).

The construction of our \( n \)-gon is based on the behavior of the foci of the chosen set of conics similarly to the vertices of polygons interscribed between \( C \) and the conics (see [11–14] and Remark 1.3). The conics are considered either in the real plane or, equivalently, in complex plane \( \mathbb{C}^1 \). We consider here in detail the case of Poncelet’s circles of an odd rank (Section 2.1). The case of an even rank (Section 2.2) and the case of concentric ellipses (Section 3) are reduced to the case of circles of an odd rank. Section 2.3 presents some way to make calculation numerically stable.

The construction of the \( n \)-gons is also possible for a general location of ellipses or hyperbolas (intersected with \( C \) or separated from \( C \)). However, these cases require equations that are less instructive and illustrative than the cases considered here (see Remark 2.1.3 and Appendix).

Let \( K_1 \) be a conic with foci \( g_1, g_2 \) and minor semi-axis \( b_1 \). The following equation is for the ends of a chord \([z, w]\) of \( C \) when the chord is tangent to \( K_1 \) [11]:

\[
(\bar{g}_1 wz - w - z + g_1)(\bar{g}_2 wz - w - z + g_2) - 4b_1^2 wz = 0. \tag{1}
\]

We have for \( C \) and the conic \( K_1 \) the following proposition:

**Proposition 1.1.** Let \( N + 1 \) numbers \( z_0, z_1, \ldots, z_N \) (not necessarily lying on \( C \)) be such that \( z = z_j \) and \( w = z_{j+1} \) \((j = 0, \ldots, N - 1)\) satisfy (1). \( C \) and \( K_1 \) form Poncelet’s pair of rank \( N \) if and only if \( z_N = z_0 \).

**Proof.** For \( z_0 \in C \) this is just a formulation of Poncelet’s closure theorem. Since \( z_j \) are analytic functions of \( z_0 \), the statement is correct for any complex \( z_0 \). \( \square \)

**The main definition.** Let \( P_1 \) be a closed \( N \)-gon interscribed between \( C \) and \( K_1 \). Without loss of generality, we may assume that \( P_1 \) is convex. Consider its diagonals that leap over \( j - 1 \) vertices \((j = 2, \ldots, [N/2])\). (Here \([x]\) is the greatest integer not larger than \( x \)). These diagonals form one or several closed polygons \( P_j \). There are also infinitely many such polygons and the envelope of the sides of these polygons is a conic \( K_j \). The set of such conics is called a **package of Poncelet’s conics of rank \( N \)**. For odd \( N \), say \( N = 2K + 1 \), there are \( K \) conics in a package. For even \( N \), say \( N = 2K \), there are \( K - 1 \) conics in a package along with a “degenerate” conic – the intersection point of the diagonals that leap over \( K - 1 \) vertices. These conics are nested: \( C \supset K_1 \supset \cdots \supset K_{[N/2]} \). Consideration of the entire package is, in a sense, simpler than consideration of a single conic.

Let \( C \) and \( K_1 \) form a Poncelet pair of rank \( N \). The sequence of \( z_j \) of Proposition 1.1 may be obtained by letting \( z_0 = 0 \) and \( z_1 = \) a focus of \( K_1 \). The following statement is proved in [11], and [12] explains why it is correct:

**Proposition 1.2.** If \( C \) and \( K_1 \) form Poncelet’s pair of rank \( N \) and \( z_0 = 0 \), then the sequence \( z_1, \ldots, z_{N-1} \) is the set of all foci of the smallest package containing \( K_1 \). If this package has rank \( m < N \), then each focus is repeated \( N/m \) times. We will stipulate this special case only when it is essential.

**Remark 1.3.** The numbers \( z_0, \ldots, z_N \) were introduced in Proposition 1.1 noting that in the case \( z_j \in C \), these numbers are vertices of an interscribed polygon which satisfy the closure theorem. If \( z_j \notin C \), they are parameters that define vertices of such polygons in the complex plane \( \mathbb{C}^2 \) (see [13]).
Below we choose a convenient numeration of the foci of a package. The major axes of the conics of a package are parallel [11]. Therefore, without loss of generality, we may assume that the major axes of the conics of a package are horizontal. For the case of nested ellipses, let the numbers of ellipses be assigned according to the way they are nested: \( K_1 \supset K_2 \supset \cdots \supset K_{[N/2]} \). If \( f \) and \( g \) are the foci of the \( j \)th ellipse \( K_j \) and \( \Re f > \Re g \), then we call \( f \) a “right focus,” assigning it the number \( j \) and we call \( g \) a “left focus,” assigning it the number \(-j\). Thus all foci are denoted \( f_1, f_2, f_2, \ldots, f_{(N-1)/2}, f_{-(N-1)/2}, f_N/2 \) for odd \( N \), \( f_1, f_2, f_2, \ldots, f_{(N-1)/2}, f_{-(N-1)/2}, f_N/2 \) for even \( N \), and the sequence \( z_1, \ldots, z_{N-1} \) of Proposition 1.2 is a permutation of these foci. These permutations may be introduced by an \( [(N - 1)/2] \times (N - 1) \) matrix:

**Theorem 1.4** (Invariance of focus numbers). If \( z_1 = f_i \), then the \( l \)th row of the matrix \( K^{(N-1)} \) defines the numbers of the foci \( z_t \) of Proposition 1.2:

\[
K_{lt}^{(N-1)} := \begin{cases} 
  s & \text{if } s \leq N/2; \\
  s - N & \text{if } s \geq (N + 1)/2,
\end{cases}
\]

where \( s \equiv tl \pmod{N} \), \( 1 \leq s \leq N \). See [11] for the proof. The case \( tl \equiv N \pmod{N} \) (i.e., \( K_{lt}^{(N-1)} = 0 \)) means that the \( l \)th conic belongs to the package of smaller rank than \( N \).

**Corollary 1.5.** \( K_{12}^{(N-1)} \equiv 2l \pmod{N} \).

The convenience of the integer matrix \( K^{(N-1)} \) becomes apparent when we need to determine some features of the set of a package’s foci (see Sections 2 and 3 below). It allows us to work with the focus numbers (integers) instead of the focus values (real or complex numbers).

**Example 1.6.** Let \( N = 17 \). Then

\[
K^{(16)} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \\
2 & 4 & 6 & 8 & -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 & -8 & -6 & -4 & -2 \\
3 & 6 & -8 & -5 & -2 & 1 & 4 & 7 & -7 & -4 & -1 & 2 & 5 & 8 & -6 & -3 \\
4 & 8 & -5 & -1 & 3 & 7 & -6 & -2 & 2 & 6 & -7 & -3 & 1 & 5 & -8 & -4 \\
5 & -7 & -2 & 3 & 8 & -4 & 1 & 6 & -6 & -1 & 4 & -8 & -3 & 2 & 7 & -5 \\
6 & -5 & 1 & 7 & -4 & 2 & 8 & -3 & 3 & -8 & -2 & 4 & -7 & -1 & 5 & -6 \\
7 & -3 & 4 & -6 & 1 & 8 & -2 & 5 & -5 & 2 & -8 & -1 & 6 & -4 & 3 & -7 \\
8 & -1 & 7 & -2 & 6 & -3 & 5 & -4 & 4 & -5 & 3 & -6 & 2 & -7 & 1 & -8
\end{bmatrix}.
\]

Here for example \( K_{7,4}^{(16)} = -6 \); this means that if \( z_1 \) is the right focus of the 7th conic, then \( z_4 \) is the left focus of the 6th conic.

**Example 1.7.** Let \( N = 15 \). Then

\[
K^{(14)} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \\
2 & 4 & 6 & -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 & -6 & -4 & -2 \\
3 & 6 & -6 & -3 & 0 & 3 & 6 & -6 & -3 & 0 & 3 & 6 & -6 & -3 \\
4 & -7 & -3 & 1 & 5 & -6 & -2 & 2 & 6 & -5 & -1 & 3 & 7 & -4 \\
5 & -5 & 0 & 5 & -5 & 0 & 5 & -5 & 0 & 5 & -5 & 0 & 5 & -5 \\
6 & -3 & 3 & -6 & 0 & 6 & -3 & 3 & -6 & 0 & 6 & -3 & 3 & -6 \\
7 & -1 & 6 & -2 & 5 & -3 & 4 & -4 & 3 & -5 & 2 & -6 & 1 & -7
\end{bmatrix}.
\]
Example 1.8. Let $N = 20$. Then

$$
K^{(19)} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & 1 \\
3 & 6 & 9 & -8 & -5 & -2 & 1 & 4 & 7 & 10 & -7 & -4 & -1 & 2 & 5 & 8 & -9 & -6 & -3 \\
4 & 8 & -8 & -4 & 0 & 4 & 8 & -8 & -4 & 0 & 4 & 8 & -8 & -4 & 0 & 4 & 8 & -8 & -4 \\
5 & 10 & -5 & 0 & 5 & 10 & -5 & 0 & 5 & 10 & -5 & 0 & 5 & 10 & -5 & 0 & 5 & 10 & -5 \\
7 & -6 & 1 & 8 & -5 & 2 & 9 & -4 & 3 & 10 & -3 & 4 & 9 & -2 & 5 & -8 & -1 & 6 & -7 \\
8 & -4 & 4 & -8 & 0 & 8 & -4 & 4 & -8 & 0 & 8 & -4 & 4 & -8 & 0 & 8 & -4 & 4 & -8 \\
9 & -2 & 7 & -4 & 5 & -6 & 3 & -8 & 1 & 10 & -1 & 8 & -3 & 6 & -5 & 4 & -7 & 2 & -9
\end{bmatrix}.
$$

Rewrite (1) in the following form:

$$
w^2 - w \left[ \frac{g_1 - z}{1 - g_1 z} + \frac{g_2 - z}{1 - g_2 z} + \frac{4b_1^2 z}{(1 - g_1 z)(1 - g_2 z)} \right] + \frac{g_1 - z}{1 - g_1 z} \frac{g_2 - z}{1 - g_2 z} = 0. \quad (3)
$$

Without loss of generality (see [11, Section 3.1]) we may assume that the origin is not among the foci of the package and that no two ellipses share a focus. Then the sequence $z_j (j = 0, \ldots, N - 1)$ of Proposition 1.2 may be determined by the following recursive calculations (see [11,12]):

Step 1. $z = z_0 = 0, \ z_1 := w = g_1$;

Step 2. $z = z_1, \ z_2 := w = \frac{4b_1^2 g_1 + (g_2 - g_1)(1 - \left| g_1 \right|^2)}{(1 - \left| g_1 \right|^2)(1 - g_1 g_2)}$; \quad (4)

Step $t(\geq 3)$. $z = z_{t-1}, \ z_t := w = \frac{(z_{t-1} - g_1)(z_{t-1} - g_2)}{z_{t-2}(1 - g_1 z_{t-1})(1 - g_2 z_{t-1})}$. \quad (5)

In accordance with the closure theorem and Proposition 1.2, $z_{N-1} = g_2$ and $z_0 = 0$.

Remark 1.9. Eq. (4) is a kind-of duplication formula. What is doubled the focus’ number, not the focus’ value. See Corollary 1.5.

Now let us consider a package of circles.

2. Package of circles

Without loss of generality (see [11]), we may consider the centers of all the circles in a package real. Denote $c_j = c_{-j}$ and $r_j$ the center and radius of the $j$th circle, respectively. For the chord $[z, w]$ tangent to the $j$th circle, (1) yields

$$
(c_j w z - w - z + c_j)^2 - 4r_j^2 w z = 0. \quad (6)
$$

Notice that all the circles of a package belong to one pencil [1], and an invariant of the pencil containing the unit circle and the $j$th circle is

$$
I := \frac{1 + c_j^2 - r_j^2}{2c_j}. \quad (7)
$$

$I$ and $c_j$ have the same sign. Therefore, without loss of generality, we may assume that the centers of all the circles in a package are positive.

Notations: A circle with center $c$ of an invariant $I$ is denoted by $K(c, I)$ (so its radius $r = \sqrt{c^2 + 1 - 2Ic}$). A package of circles of rank $N$ and the invariant $I$ is denoted $\Pi(I, N)$.

Since a circle has just one center, rather than left and right foci, we may replace the negative integers of matrix $K^{(N-1)}$ by their absolute values. Consequently, we have instead of Examples 1.6–1.8,
Example 2.1. Let $N = 17$. Then
\[
K^{(16)} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
2 & 4 & 6 & 8 & 7 & 5 & 3 & 1 & 1 & 3 & 5 & 7 & 8 & 6 & 4 & 2 \\
3 & 6 & 8 & 5 & 2 & 1 & 4 & 7 & 7 & 4 & 1 & 2 & 5 & 8 & 6 & 3 \\
4 & 8 & 5 & 1 & 3 & 7 & 6 & 2 & 2 & 6 & 7 & 3 & 1 & 5 & 8 & 4 \\
5 & 7 & 2 & 3 & 8 & 4 & 1 & 6 & 6 & 1 & 4 & 8 & 3 & 2 & 7 & 5 \\
6 & 5 & 1 & 7 & 4 & 2 & 8 & 3 & 3 & 8 & 2 & 4 & 7 & 1 & 5 & 6 \\
7 & 3 & 4 & 6 & 1 & 8 & 2 & 5 & 5 & 2 & 8 & 1 & 6 & 4 & 3 & 7 \\
8 & 1 & 7 & 2 & 6 & 3 & 5 & 4 & 4 & 5 & 3 & 6 & 2 & 7 & 1 & 8
\end{bmatrix}.
\]

Example 2.2. Let $N = 15$. Then
\[
K^{(14)} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
2 & 4 & 6 & 7 & 5 & 3 & 1 & 3 & 5 & 7 & 6 & 4 & 2 \\
3 & 6 & 6 & 3 & 0 & 3 & 6 & 6 & 3 & 0 & 3 & 6 & 6 & 3 \\
4 & 7 & 3 & 1 & 5 & 6 & 2 & 2 & 6 & 5 & 1 & 3 & 7 & 4 \\
5 & 5 & 0 & 5 & 5 & 0 & 5 & 5 & 0 & 5 & 5 & 0 & 5 & 5 \\
6 & 3 & 3 & 6 & 0 & 6 & 3 & 3 & 6 & 0 & 6 & 3 & 3 & 6 \\
7 & 1 & 6 & 2 & 5 & 3 & 4 & 4 & 3 & 5 & 2 & 6 & 1 & 7
\end{bmatrix}.
\]

Example 2.3. Let $N = 20$. Then
\[
K^{(19)} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 9 & 8 & 7 \\
2 & 4 & 6 & 8 & 10 & 8 & 6 & 4 & 2 & 0 & 2 & 4 & 6 \\
3 & 6 & 9 & 8 & 5 & 2 & 1 & 4 & 7 & 10 & 7 & 4 & 1 \\
4 & 8 & 8 & 4 & 0 & 4 & 8 & 8 & 4 & 0 & 4 & 8 & 8 \\
5 & 10 & 5 & 0 & 5 & 10 & 5 & 0 & 5 & 10 & 5 & 0 & 5 \\
6 & 8 & 2 & 4 & 10 & 4 & 2 & 8 & 6 & 0 & 6 & 8 & 2 \\
7 & 6 & 1 & 8 & 5 & 2 & 9 & 4 & 3 & 10 & 3 & 4 & 9 \\
8 & 4 & 4 & 8 & 0 & 8 & 4 & 4 & 8 & 0 & 8 & 4 & 4 \\
9 & 2 & 7 & 4 & 5 & 6 & 3 & 8 & 1 & 10 & 1 & 8 & 3 \\
6 & 5 & 4 & 3 & 2 & 1 \\
8 & 10 & 8 & 6 & 4 & 2 \\
2 & 5 & 8 & 9 & 6 & 3 \\
4 & 0 & 4 & 8 & 8 & 4 \\
10 & 5 & 0 & 5 & 10 & 5 \\
4 & 10 & 4 & 2 & 8 & 6 \\
2 & 5 & 8 & 1 & 6 & 7 \\
8 & 0 & 8 & 4 & 4 & 8 \\
6 & 5 & 4 & 7 & 2 & 9
\end{bmatrix}.
\]

Below we need the following statement:

**Proposition 2.4.** For any $I_0 > 1$ and integer $N \geq 3$, there exists a package $\Pi(I_0, N)$ nested in $C$.

**Proof.** Consider packages of rank $N$. Since
\[
\lim_{c \to +0} I(c) = \infty, \quad \lim_{c \to 1-0} I(c) = 1 + 0
\]
and $I(c)$ is a continuous function for $c \in (0, 1)$, there exists $c_0 \in (0, 1)$ such that $I(c_0) = I_0$, i.e., that the circle centered in $c_0$ of invariant $I_0$ belongs to the package of rank $N$. □

Consider a package of circles $\Pi(I, N)$. Let $\xi_{\min}(I, N)$ denote the center of the circle closest to the origin. The circle $K(\xi, I)$, where $\xi = \xi_{\min}(I, N)$, is the largest circle in the package, i.e., circle #1 for our
numeration of the package’s circles (i.e., according to the way they are nested). Therefore the N-gon interscribed between C and \( K(\xi, I) \) is convex. The Jacobi proof of Poncelet’s closure theorem may be interpreted as based on the fact that each side of such a polygon supports an arc \( [e^{i\varphi_1}, e^{i\varphi_2}] \) of C of “measure” \( 1/N \) (see [5,6]):
\[
\int_{\varphi_1}^{\varphi_2} \frac{C\varphi}{\sqrt{1-\cos \varphi}} = \frac{1}{N},
\]
where \( C^{-1} = \int_0^{2\pi} d\varphi / \sqrt{1-\cos \varphi}. \) The point \((x_0 = I - \sqrt{I^2 - 1}, 0)\) is the degenerate Poncelet circle for the given I and \( N = 2 \) (the smallest \( N \)). Any chord of C passing through \((x_0, 0)\) breaks up the circle C in two parts of measure 1/2.

**Proposition 2.5.** If a circle \( K(c, I) \) is nested in C and \( c > 0 \), then \( c \leq x_0 \).

**Proof.** Indeed, (7) yields \( r^2 = 1 + c^2 - 2lc \) and \( c \leq x_0 \) since \( r^2 \) is not negative. □

The following two simple statements establish the base for the main results of this paper (Theorems 2.1.13 and 2.1.17):

**Proposition 2.6.** If \( N_1 < N_2 \) then \( \xi_{\min}(I, N_1) > \xi_{\min}(I, N_2) \).

**Proof.** It follows from (7) that the derivative \((c + r)' = (c + r - I)/r. \) Since \( c + r < 1 < I \), this derivative is negative. Hence \( c + r < 1 < I \) falls when \( c \) grows, i.e., if \( \Delta \xi > 0 \) (so \( \Delta r < 0 \)) then the circle \( K_1 = K(\xi + \Delta \xi, I) \) is inside \( K_2 = K(\xi, I) \). Consequently by the “measure equation” (8), the closed convex polygon interscribed between C and \( K_2 \) should have more sides than the closed convex polygon interscribed between C and \( K_1 \). □

**Proposition 2.7.** Let \( M_1 = 2^m - 1, M_2 = 2^m + 1 \) \((m = 2, 3, \ldots)\) and packages \( \Pi(I, M_1) \) and \( \Pi(I, M_2) \) of circles be nested in C. Then these circles are alternating in the following sense:

\[
K\left(w^{M_2}_1, I\right) \supset K\left(w^{M_1}_1, I\right) \supset K\left(w^{M_2}_2, I\right) \supset K\left(w^{M_1}_2, I\right) \\
\supset \cdots \supset K\left(w^{M_1}_{(M_1-1)/2}, I\right) \supset K\left(w^{M_2}_{(M_2-1)/2}, I\right),
\]

where \( w^j_{k} \) is the center of the \( j \)th circle of the package \( \Pi(I, N) \).

**Proof.** Recall the way of numeration of circles \( K_j \) in a package that envelope polygons \( P_j \) (see The Main definition in Section 1). It follows from (8) that each side of \( P_j \) supports an arc of C of measure \( j/N \). On the other hand, for any \( 1 \leq j < 2^{m-1} \), we have
\[
\frac{j}{2^m + 1} < \frac{j}{2^{m-1}} < \frac{j + 1}{2^m + 1}.
\]
Consequently \( w^{M_2}_1 < w^{M_1}_1 < w^{M_2}_2 < w^{M_1}_2 < \cdots < w^{M_1}_{(M_1-1)/2} < w^{M_2}_{(M_2-1)/2} \) and the order of nested circles is as claimed in the proposition. □

2.1. Odd number of polygon sides

Let us first deal with the numbers of centers as they are enumerated.

**Lemma 2.1.1.** Consider the sequence \( L_k \equiv 2^kj \pmod N \) for \( j = 1 \leq j \leq (N - 1)/2 \), where \( k = 0, 1, 2, \ldots \) and \( |L_k| \leq (N - 1)/2 \), and let \( l_k = |L_k| \). All j’s \( 1 \leq j \leq (N - 1)/2 \) may be broken into a finite number of disjoint sequences \( j = l_0, l_1, \ldots, l_m = l_0 \), where \( 1 \leq l_k \leq (N - 1)/2 \) and \( l_{k+1} = \text{either} 2l_k \text{ or } -2l_k \pmod N \).
Lemma 2.1.8. other cycles
number of disjoint sequences
dance with Euler’s generalization of Fermat’s Little Theorem, 2
Example 2.1.4. Notation:
\[ I \sum \]
Example 2.1.5. Proof.
\[ \mu(\) \]
Lemma 2.1.9. Remark 2.1.3. Corollary 2.1.2. The partition into a disjoint union of cycles may be done also by the partition of the permutation \((j = 1, 2, \ldots, (N − 1)/2)\)
\[ k = \begin{cases} j/2, & \text{if } j \text{ is even} \\ (N − j)/2, & \text{if } j \text{ is odd,} \end{cases} \]
or the partition of the inverse permutation \((k = 1, 2, \ldots, (N − 1)/2)\)
\[ j = \begin{cases} 2k, & \text{if } 2k \leq (N − 1)/2 \\ (N − 2k)/2, & \text{if } 2k > (N − 1)/2. \end{cases} \]

Remark 2.1.3. The number \(m\) of Lemma 2.1.1 has the same sense as \(n\) in the Introduction. In the general case of conics, we may have bigger \(n\) if we look for cycles of a package’s foci. However, we may find cycles for the centers of conics. The length of such cycles is still \(m\), but the corresponding equations are less illuminating than for circles. See Appendix.

For a given \(N\), denote by \(I_m^N\) the number of cycles of length \(m\) among \(1 \leq j \leq (N − 1)/2\). Then
\[ \sum_{i=1}^{I_m} m_i \times I_m^N = (N − 1)/2. \]

Notation:
\[ \mu(N) \] is the set of cycle lengths \(m_i\) corresponding to \(N\).
\[ m_1(N) := \max\{m \in \mu(N)\}. \] Sometimes we omit \(N\) when its value is clear.

Example 2.1.4. \(N = 11\); there is one cycle of length \(m = 5\) : \((1, 2, 4, −8 \equiv 3, −6 \equiv 5, −10 \equiv 1)\) (mod 11); \(\mu(11) = \{5\}\).

Example 2.1.5. \(N = 31\); there are 3 cycles of length \(m = 5\) each: \((1, 2, 4, −16 \equiv 15, −30 \equiv 1), (3, 6, 12, −24 \equiv 7, 14, −28 \equiv 3), (5, 10, −20 \equiv 11, −22 \equiv 9, 18, 36 \equiv 5)\) (mod 31); \(\mu(31) = \{5\}\).

Example 2.1.6. \(N = 33\); there are 3 cycles of length \(m_1 = 5\) each and \(m_2 = 1\) cycle of length \(1\) (a fixed point): \((1, 2, 4, 8, 16, −32 \equiv 1), (3, 6, 12, −24 \equiv 9, −18 \equiv 15, −30 \equiv 3), (5, 10, −20 \equiv 13, −26 \equiv 7, 14, −28 \equiv 5), (11, −22 \equiv 11)\) (mod 33); \(\mu(33) = \{5, 1\}\).

Example 2.1.7. \(N = 45\); there are 5 cycles of lengths \(m_1 = 12, m_2 = 4, m_3 = 3, m_4 = 2, m_5 = 1; \mu(45) = \{12, 4, 3, 2, 1\}.

Lemma 2.1.8. \(m_1(N)\) is a divisor of Euler’s function \(\phi(N)\).

Proof. Consider the sequence \(L_k \equiv 2^k\) (mod \(N\)) where \(k = 0, 1, \ldots\) and \(|L_k| \leq (N − 1)/2\). In accordance with Euler’s generalization of Fermat’s Little Theorem, \(2^{\phi(N)} \equiv 1\) (mod \(N\)). Let \(K\) be the smallest positive integer such that \(L_K \equiv 1\) (mod \(N\)). Then \(K\) divides \(\phi(N)\). The maximum length of cycles of Lemma 2.1.1, i.e., \(m_1(N)\) is either equal to \(K\) or equal to \(K/2\).

Lemma 2.1.9. The maximum value of the cycle length for \(M_1 = 2^{m_1} − 1\) and \(M_2 = 2^{m_1} + 1\) is \(m_1\). All other cycles (if they exist) have lengths \(m_i\) \((i = 2, \ldots)\) that are divisors of \(m_1\) or \(1\).
Proof. Consider the sequence $1, 2, 4, \ldots, 2^{m_1}$. Its last number is $\equiv 1 \pmod{M_1}$ and $\equiv -1 \pmod{M_2}$. So we have a cycle of length $m_1$. Similarly, for $1 < j \leq 2^{m_1 - 1} - 1$, the sequence $j, 2j, 4j, \ldots, 2^{m_1}j$ of $(\text{mod } M_1)$ or $(\text{mod } M_2)$ also may be a cycle either of length $m_1$ or $m$ which divides $m_1$ (by Lagrange's theorem).

Now let us consider the center of a circle of a package, say the $j$th center. For the recursive calculations of the introduction (Steps 1 – $N$), we have $z_1 = c_j$ and $z_2 = 4r_j^2c_j/(1 - c_j^2)^2$. Since $z_2$ is the center of either the 2nd circle or of the $(N - 2j)$th circle of the package (see (2)), we have $z_2 = c_{l_1}$, where $l_1$ is defined in Lemma 2.1.1. Substituting, in accordance with (7), $r_j^2 = 1 + c_j^2 - 2lc_j$, we have

$$c_{l_1} = \frac{4c_j(1 - 2lc_j + c_j^2)}{(1 - c_j^2)^2}. \tag{9}$$

Similarly, we can derive

$$c_{l_2} = \frac{4c_{l_1}(1 - 2lc_{l_1} + c_{l_1}^2)}{(1 - c_{l_1}^2)^2},$$

where $l_2 \equiv 2l_1 \equiv -2l_1 \pmod{N}$, and so on. For $m$ defined in Lemma 2.1.1, we have $l_m = j$ and $c_{l_m} = c_j$. Based on (9), we may describe the centers $c_j = c_{l_0}, c_{l_1}, c_{l_2}, \ldots, c_{l_m} = c_{l_0}$ as follows. Consider the function

$$f(x) = \frac{4x(x^2 - 2lx + 1)}{(1 - x^2)^2} \tag{10}$$

in the interval $0 < x < x_0 = 1 - \sqrt{l^2 - 1}$. This function is unimodal (one-humped) with zeros at the ends of the interval, $f(0) = f(x_0) = 0$. It attains the maximum for

$$x = \bar{x}_0 = \frac{1}{x_0} \left(1 - \sqrt{1 - x_0^2}\right). \tag{11}$$

This maximum $f(\bar{x}_0) = x_0$, i.e., the graph of $f(x)$ is inscribed in the square $x_0 \times x_0$ (see Fig. 1). It follows from (9) and (10) that

$$c_{l_1} = f(c_{l_0}), c_{l_2} = f(c_{l_1}), \ldots, c_{l_m} = f(c_{l_{m-1}}), c_j = c_{l_0} = f(c_{l_m}),$$

or $c_j$ satisfies the equation

$$f^{(m)}(c) = c, \tag{12}$$

where $f^{(1)}(x) = f(x), f^{(2)}(x) = f(f(x))$, and so on. Since the graph of $f(x)$ is inscribed in the square $x_0 \times x_0$, for any $x \in (0, x_0)$ and any positive integer $m$ we have $f^{(m)}(x) \in (0, x_0)$. Figs. 1–3 show the solution of (12) for $l = 71/64$ and several $m$ and $N$. The sense of (12) is that this equation provides us with a necessary condition to form a Poncelet pair:

**Theorem 2.1.10.** Let the circle $\mathcal{K} = \mathcal{K}(c, l)$ form with $c$ a Poncelet pair of rank $N$. Then there exists a positive integer $m \in \mu(N)$ such that $c$ satisfies (12).

Proof. The circle $\mathcal{K}$ belongs to some package of rank $N$. This circle has a number, say $j$, in the package. For $N$, find the cycle containing $j$. $m$ is the length of this cycle.

It turns out that the condition of Theorem 2.1.10 is also sufficient to form a Poncelet pair. Let us first prove the following lemma:

**Lemma 2.1.11.** The function $f^{(m)}(x)$ has $2^{m-1}$ maxima and $2^{m-1} + 1$ minima for $0 \leq x \leq x_0$. Each maximum of $f^{(m)}(x)$ is equal to $x_0 = 1 - \sqrt{l^2 - 1}$, each minimum of $f^{(m)}(x)$ is equal to zero.
Fig. 1. The root of $f(x) = x$ for $N = 3$ and $I = 71/64$.

Fig. 2. The roots of $f^{[4]}(x) = x$, $f^{[2]}(x) = x$ and $f(x) = x$ for $N = 15$ and $I = 71/64$; $w^{(15)}_1 = 0.03432$, $w^{(15)}_2 = 0.12728$, $w^{(15)}_4 = 0.38599$, $w^{(15)}_7 = 0.62373$; $w^{(15)}_3 = w^{(5)}_1 = 0.25403$, $w^{(15)}_6 = w^{(5)}_2 = 0.58162$, $w^{(3)}_1 = 0.5$.

**Proof.** Let us prove the lemma by induction. For $m = 1$, the assertion is mentioned above (see [10, 11]). Assume $0 = t_0 < t_1 < t_2 < \cdots < t_{2^m-1} < t_{2^m} = x_0$ are such that $f^{(m)}(t_{2j}) = 0$ ($j = 0, 1, \ldots, 2^{m-1}$) and $f^{(m)}(t_{2j+1}) = x_0$ ($j = 0, 1, \ldots, 2^{m-1}$). Since $f(0) = f(x_0) = 0$ and $f(\tilde{x}_0) = x_0$, we have $f^{(m+1)}(t_{2j}) = f^{(m+1)}(t_{2j+1}) = 0$, and for some $\tilde{t} \in (t_{2j}, t_{2j+1})$, $f^{(m+1)}(\tilde{t}) = x_0$. Therefore, $f^{(m+1)}(x)$ has twice maxima as $f^{(m)}(x)$, i.e., $2^m$. Consequently, there are $2^m + 1$ minima of $f^{(m+1)}(x)$ for $0 \leq x \leq x_0$. □

**Remark 2.1.12.** Such a big number of extrema of $f^{(m)}(x)$ makes calculation of $f^{(m)}(x)$ unstable for large $m$. The main source of calculation instability is the accumulation of round-off errors in the course of calculation of $f^{(m)}$. Separation of the smallest positive root of (12) (see Theorem 2.1.17) makes determination of this root significantly more stable. Then the rest of centers can be determined by
Fig. 3. The roots of $f^{(4)}(x) = x$ for $N = 17$ and $I = 71/64$; $w_1^{(17)} = 0.02687, w_2^{(17)} = 0.10130, w_4^{(17)} = 0.32491, w_8^{(17)} = 0.62491; w_3^{(17)} = 0.20728, w_6^{(17)} = 0.52780, w_5^{(17)} = 0.43613, w_7^{(17)} = 0.59204.$

(4,5). Very effective way to make the calculation of $f^{(m)}(x)$ stable is to reduce the number of circles in the cycle (see Section 2.3).

**Theorem 2.1.13.** If $\xi$ a root of (12), then there exists an integer $3 \leq N \leq 2^m + 1$ such that the circle $K = K(\xi, I)$ forms with $C$ a Poncelet pair of rank $N$. Moreover, if $\frac{d}{d\xi}f^{(m)}(\xi) > 0$, then $N$ is a divisor of $M_1 = 2^m - 1$. If $\frac{d}{d\xi}f^{(m)}(\xi) < 0$, then $N$ is a divisor of $M_2 = 2^m + 1$.

**Proof.** Let us count the number of roots of (12). Consider the intersection points of the line $y = x$ and the graph $y = f^{(m)}(x)$ for $0 < x < x_0$ (see Figs. 4–6). The line has two common points with each wave of the graph, except the first wave, where there is only one common point of the line and the graph (since we do not count the origin). In accordance with Lemma 2.1.11, there are $2^{m-1}$ waves of the graph. Hence, there are $2^m - 1$ common points, i.e., (12) has $2^m - 1$ roots.

Fig. 4. The roots of $f^{(2)}(x) = x$ for $N = 3, 5$, and $I = 71/64 w_j^{(N)}$ is the center of the $j$th circle of the package of rank $N$. 
Fig. 5. The roots of $f^{(3)}(x) = x$ for $N = 7, 9$, and $I = 71/64$ $w_j^{(N)}$ is the center of the $j$th circle of the package of rank $N$, $w_1^{(9)} = 0.09912, w_1^{(7)} = 0.14400, w_2^{(9)} = 0.29875, w_2^{(7)} = 0.42119, w_3^{(9)} = w_3^{(9)} = 0.5, w_3^{(7)} = 0.60473, w_4^{(9)} = 0.61431.$

On the other hand, consider packages of invariant $I$ of circles of rank $M_1$ and $M_2$. Such packages exist by Proposition 2.4. The first package contains $2^{m-1} - 1$ circles, whereas the second contains $2^{m-1}$ circles. In total there are $2^m - 1$ circles, i.e., the number of these circles and the number of the roots of (12) coincide. Lemma 2.1.9 and Theorem 2.1.17 yield that all centers of these circles satisfy (12). Thus, for any root $c$ of (12) a circle centered at $c$ of the invariant $I$ belongs either to the package of rank $M_1$ or to the package of rank $M_2$. We may say more due to Propositions 2.6 and 2.7. The smallest positive root of (12) is $\xi_{\min}(I, M_2)$, which lies on the intersection of the line $y = x$ and the first descending branch of $y = f^{(m)}(x)$. Then all other intersection points $(\xi, \xi)$ on the descending branches of $y = f^{(m)}(x)$ represent other centers $\xi$ of the package $\Pi(I, M_2)$ whilst the intersection points of the line and the ascending branches of the graph represent the centers of the package $\Pi(I, M_1)$. Therefore the last claim of the theorem is valid. □

Corollary 2.1.14. If the circle $K(c, I)$ belongs to package $\Pi(I, N)$ where $N \neq M_1, N \neq M_2$ and the center $c$ satisfies (12), then this circle belongs also to a package either of rank $M_1$ or $M_2$.

Corollary 2.1.15. $m_1(N) = \min\{m : N \text{ divides either } 2^m - 1 \text{ or } 2^m + 1\}$.

It is convenient for calculation to have a way to evaluate $\xi_{\min}(I, M_2)$. For this evaluation, we need the inverse function $f^{(-1)}(x)$ of the function $f(x)$.

Lemma 2.1.16. The inverse function of $f(x)$ is: for the ascending branch, $0 < C \leq x_0, 0 < f^{(-1)}(C) \leq \tilde{x}_0$. 
Fig. 6. The roots of $f^{(4)}(x) = x$ for $N = 15, 17$, and $l = 71/64 w_j^{(N)}$ is the center of the $j$th circle of the package of rank $N$, $w_j^{(17)} = 0.02687, w_1^{(15)} = 0.03432, w_2^{(17)} = 0.10130, w_3^{(15)} = 0.12728, w_3^{(17)} = 0.20728, w_4^{(15)} = w_3^{(15)} = 0.25403, w_3^{(17)} = 0.32491, w_4^{(15)} = 0.38599, w_5^{(17)} = 0.43613, w_4^{(15)} = w_5^{(15)} = 0.5, w_5^{(17)} = 0.52780, w_6^{(15)} = w_5^{(15)} = 0.58162, w_7^{(17)} = 0.59204, w_5^{(15)} = 0.62373, w_6^{(17)} = 0.62491$. 

$$f_{1}^{(-1)}(C) := \left[ \sqrt{\frac{1 + R + C}{2C}} - \sqrt{\frac{1 + R - C}{2C}} \right]^2$$

where $R = \sqrt{1 + C^2 - 2IC}$, and for the descending branch, $0 < C < x_0, x_0 > f_{2}^{(-1)}(C) > \bar{x}_0$.

$$f_{2}^{(-1)}(C) := \left[ \sqrt{\frac{1 - R + C}{2C}} - \sqrt{\frac{1 - R - C}{2C}} \right]^2$$

Proof. May be established by direct verification.

**Theorem 2.1.17.** Let $x$ be such that $x < f(x) < f^{(2)}(x) < \cdots < f^{(m-1)}(x) < x_0$ and $f^{(m)}(x) = x$. (This means that $f(x), \ldots, f^{(m-1)}(x)$ are on the ascending branch of the graph of $f(x)$ and $f^{(m)}(x)$ is on the descending branch.) Then $x$ is the smallest positive root of (12). In accordance with Lemma 2.1.16, $x > f_{1}^{(-m-1)}(x_0)$.

Proof. Indeed, if $0 < z < x$, then $f^{(m-1)}(z) < f^{(m-1)}(x)$ on the ascending branch and $f^{(m)}(z) > f^{(m)}(x)$ on the descending branch. Hence $f^{(m)}(z) \neq z$.

**Corollary 2.1.18.** Let $2^{m-1} + 2 < N < 2^m$. Then $\bar{\xi}_{\min}(I, 2^m + 1) < \bar{\xi}_{\min}(I, N) < \bar{\xi}_{\min}(I, 2^{m-1} + 1)$. 


Proof. Follows from Proposition 2.6.

Figs. 4–6 show the graphs \( y = f^{(m)}(x) \) and the roots of (12) for \( m = 2, 3, 4 \) and corresponding \( N \)'s. There \( w_j^{(N)} \) is the center of the \( j \)th circle of a package of rank \( N \). Some results of calculations are presented in Tables 1–5. Table 1 presents the intervals for the smallest positive root of (12) determined in accordance with Theorem 2.1.17, as well as the values of these roots. Table 2 presents the lengths of the cycles, \( 1 \leq m \leq 11 \), and the corresponding odd numbers of polygon sides (\( N \)'s). Table 3 presents the centers \( z_j \) of a cycle of length 11 for \( N = 2049 \) and \( I = 71/64 \). Table 4 presents various odd \( N \)'s, Euler's function \( \phi(N) \) and the corresponding lengths \( m_j \) of cycles and the quantity \( f_j^N \) of these cycles. Note that \( N \) divides \( 2^{2m_1} - 1 \) in accordance with Euler's generalization of Fermat's Little Theorem since \( m_1 \) either = \( \phi(N)/2 \) or is a divisor of this number (see Lemma 2.1.8 and Corollary 2.1.15 above). Table 5 presents the maximum length \( m_1 \) of cycles and centers \( \xi_{\text{min}}(I, N) \) for \( 33 \leq N \leq 65 \).

Remark 2.1.19. By substituting

\[
 w = \frac{2r_j \sqrt{c_j} y + (1 - c_j z)(c_j - z) + 2r_j^2 z}{(1 - c_j z)^2},
\]

we have from (6) \( y^2 = z(z^2 - 2Iz + 1) \), where \( I = (1 + c_j^2 - r_j^2)/(2c_j) \). Here \( y = y(z) \) is an elliptic curve with the addition rule and the zero point \( \emptyset = (\ldots, \infty) \). Using group law language, we may say that to determine the roots \( z_j \) of (12) means to determine points \( p_j = (x_j, y_j) \) of finite order on this curve such that \( p_{j+1} = p_j \oplus p_j \), \((z_j, y_j) = p_j \oplus (z_j, y_j) \) and either \( p_m \oplus p_1 = \emptyset \) or \( p_m \ominus p_1 = \emptyset \). Tables 6 and 7 present the results of calculations of \( p_j, z_j, y_j \) for \( N = 33 \) and \( m = 5 \). Using this language, we may reformulate Theorems 2.1.10 and 2.1.13.

### Table 1

<table>
<thead>
<tr>
<th>Length of cycles ((m))</th>
<th>Intervals for the smallest positive root of (12) based on Theorem 2.1.17</th>
<th>The smallest positive root of (12) ((m))</th>
<th>Length of cycles ((m))</th>
<th>Intervals for the smallest positive root of (12) based on Theorem 2.1.17</th>
<th>The smallest positive root of (12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.3539, 0.6291)</td>
<td>0.50000000</td>
<td>7</td>
<td>(0.00003819, 0.0004834)</td>
<td>0.47598057E−3</td>
</tr>
<tr>
<td>2</td>
<td>(0.1130, 0.3539)</td>
<td>0.25403106</td>
<td>8</td>
<td>(0.00009555, 0.0001209)</td>
<td>0.11995457E−3</td>
</tr>
<tr>
<td>3</td>
<td>(0.07478, 0.1132)</td>
<td>0.091117233</td>
<td>9</td>
<td>(0.00002388, 0.00003023)</td>
<td>0.30107672E−4</td>
</tr>
<tr>
<td>4</td>
<td>(0.02401, 0.03026)</td>
<td>0.026869798</td>
<td>10</td>
<td>(0.000005969, 0.000007557)</td>
<td>0.75417377E−5</td>
</tr>
<tr>
<td>5</td>
<td>(0.006085, 0.007694)</td>
<td>0.0072370236</td>
<td>11</td>
<td>(0.000001492, 0.000001889)</td>
<td>0.18872831E−5</td>
</tr>
<tr>
<td>6</td>
<td>(0.001526, 0.001932)</td>
<td>0.0018728056</td>
<td>12</td>
<td>(0.000003730, 0.000004723)</td>
<td>0.47205162E−6</td>
</tr>
</tbody>
</table>

### Table 2

Length of cycles versus \( N \).

<table>
<thead>
<tr>
<th>Length of cycles ((m))</th>
<th>Numbers of polygon sides ((N)'s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3, 9, 15, ...</td>
</tr>
<tr>
<td>2</td>
<td>5, 15, 25, ...</td>
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<tr>
<td>3</td>
<td>7, 9, 21, 27, ...</td>
</tr>
<tr>
<td>4</td>
<td>15, 17, 45, 51, ...</td>
</tr>
<tr>
<td>5</td>
<td>11, 31, 33, 55, 77, 93, 99, 121, 143, 155, 165, 187, 209, 217, ...</td>
</tr>
<tr>
<td>6</td>
<td>13, 21, 39, 63, 65, ...</td>
</tr>
<tr>
<td>7</td>
<td>43, 127, 129, ...</td>
</tr>
<tr>
<td>8</td>
<td>51, 85, 153, 255, 257, ...</td>
</tr>
<tr>
<td>9</td>
<td>19, 27, 57, 73, 81, 95, 133, 135, 171, 189, 209, 219, 247, 285, 311, 513, ...</td>
</tr>
<tr>
<td>10</td>
<td>25, 41, 75, 93, 123, 125, 341, ..., 1023, 1025, ...</td>
</tr>
<tr>
<td>11</td>
<td>23, 69, 115, ..., 2047, 2049, ...</td>
</tr>
</tbody>
</table>
Theorem 2.1.20. \( \text{The center } L_t \text{ for Table 3} \)

**Remark 2.1.21.** By the addition law, we may determine \( p_0 = (\xi, y_1) \oplus (0, 0) \).

**Theorem 2.1.20.** \( \mathcal{K} \) and \( \mathcal{C} \) form a Poncelet pair of rank \( N \) if and only if \( p_m = p_1 \) or \( p_m = -p_1 \), where \( p_{j+1} = p_j \oplus p_j = 2^{j+1}p_0, j = 0, 1, \ldots, m - 1 \) and \( m \in \mu(N) \).

**Remark 2.1.21.** Eqs. (4) and (5), as well as (10), are rational expressions. It means that if one conic in a package has rational parameters, then all conics of this package also have rational parameters. On the other hand, some of these parameters can be expressed through other parameters, using quadratic roots. This provides us with conditions which are necessary for Poncelet's polygons to close. These conditions are rather constrictive and may result in limitations of the rank of a package of rational conics. These limitations are known from a theorem of Malyshev [8]. However, Malyshev's theorem is based on a very beautiful, but complex theorem of Mazur. If Malyshev's theorem may be proved independently, applying approach presented here (and in [11–13]), then perhaps Mazur's theorem could be easier to prove through Malyshev's theorem.

**2.2. Even number of polygon sides**

Consider the sequence \( L_k \equiv 2^k j \pmod{2^{m+1}}, k = 1, 2, \ldots \). If \( j \) is odd then \( L_{m+1} \equiv 0 \pmod{2^{m+1}} \) and \( L_k \) is not \( 0 \pmod{2^{m+1}} \) for all \( k \leq m \). If \( j \) is even then \( L_t \equiv 0 \pmod{2^{m+1}} \) for some \( t \leq m \). For the center \( c_j \) of the \( j \)th circle of the package \( \Pi(I, 2^{m+1}) \) we have obviously \( f^{(1)}(c_j) = 0 \) if and only if \( L_t \equiv 0 \pmod{2^{m+1}} \). Therefore we have the following lemma:

**Lemma 2.2.1.** The package \( \Pi(I, 2^{m+1}) \) contains exactly \( 2^{m-1} \) circles such that their rank = \( 2^{m+1} \) and their centers \( c_j \) \((j = 1, 3, \ldots, 2^m - 1)\) satisfy the equation for \( x \)

\[
 f^{(m)}(x) = x_0, \quad (15)
\]

where \( x_0 = 1 - \sqrt{1^2 - 1} \). The rest of the circles of \( \Pi(I, 2^{m+1}) \) belongs to the packages of smaller ranks.

**Lemma 2.2.2.** The roots \( x_k^{(m)} \) of the equation \( f^{(m)}(x) = x_0 \) \((k = 1, 2, \ldots, 2^{m-1})\) can be determined by the following recurrent procedure:

Let the distinct \( x_j^{(m-1)} \) be such that \( f^{(m-1)}(x_j^{(m-1)}) = x_0 \) \((m \geq 2, j = 1, 2, \ldots, 2^{m-2})\). Then...
\[ x_j^{(m)} = f^{(1)}_1(x_j^{(m-1)}), x_{2m-1+j}^{(m)} = f^{(1)}_2(x_j^{(m-1)}). \]

Here, \( x_1^{(1)} = \xi_0 = \left(1 - \sqrt{1 - x_0^2}\right) /x_0, x_1^{(2)} = f^{(1)}_1(x_1^{(1)}), x_2^{(2)} = f^{(1)}_2(x_1^{(1)}), \) and so on.

**Proof.** is obvious since \( f^{(m)} f^{(-1)}_1 = f^{(m)} f^{(-1)}_2 = f^{(m-1)}. \) See Fig. 7. \( \square \)

**Theorem 2.2.3.** The circle \( K_0 = \mathcal{K}(c_0, I) \) and the unit circle \( c \) form a Poncelet pair of rank \( 2^{m+1} \), where \( m \geq 1 \), if and only if \( f^{(m)}(c_0) = x_0. \)

**Proof.** Necessity. For \( m = 1 \), it is given that \( K_0 \in \Pi(I, 4) \), i.e., the quadrangle with vertices \((\pm 1, 0)\) and \((x_0, \pm \sqrt{1 - x_0^2})\) is inscribed between \( K_0 \) and \( c \). Then it is easy to verify that \( f(c_0) = x_0. \) Let \( m > 1 \). Consider the \( j \)-th circle \( K_0 = \mathcal{K}(c_0, I) \) of the package \( \Pi(I, 2^{m+1}) \). It follows from (2) and (9) that the circle \( K(c_1 = f(c_0), I) \) of \#1 \( \equiv 2j \) or \(-2j \) (mod \( N \)) belongs to the package \( \Pi(I, 2^m) \). Continuing this procedure, we have finally the point \((x_0, 0)\) where \( x_0 = f^{(m)}(c_0) \).

Sufficiency. Let \( f^{(m)}(c_0) = x_0. \) For \( m = 1 \) it gives \( f(c_0) = x_0 \) and therefore the point \((f(c_0), 0)\) is the degenerate Poncelet circle of \( \Pi(I, 4) \) and \( K_0 \) is inscribed in the quadrangle with the vertices \((\pm 1, 0)\) and \((x_0, \pm \sqrt{1 - x_0^2})\), i.e., \( K_0 \in \Pi(I, 4) \). Let \( m > 1 \) and \( f^{(m)}(c_0) = x_0. \) We shall prove that \( K_0 \) and \( c \) form a Poncelet pair of rank \( 2^{m+1} \). For that, like in the proof of Theorem 2.1.13, let us count the number of solutions of an equation, here Eq. (15). The number of its solutions is \( 2^{m-1} \) in \((0, x_0)\) by Lemma 2.1.11. On the other hand, there are \( 2^{m-1} \) circles in the package \( \Pi(I, 2^{m+1}) \) such that their centers satisfy (15) by Lemma 2.2.1. Hence, again, as in the proof of Theorem 2.1.13, each solution of (15) is the center of a circle of the package \( \Pi(I, 2^{m+1}) \), i.e., \( K_0 \) and \( c \) form a Poncelet pair of rank \( 2^{m+1} \). \( \square \)

Now the case of a polygon with an even number of sides may be reduced to the case of with an odd number.

**Theorem 2.2.4.** The circle \( \mathcal{K}(c_0, I) \) and the unit circle \( c \) form a Poncelet pair of rank \( N = 2^k N_0 \), where \( N_0 \) is an odd number, if and only if the circle \( \mathcal{K}(c = f^{(k)}(c_0), I) \) forms with \( c \) a Poncelet pair of rank \( N_0 \).

**Proof.** Applying the same procedure as in Theorem 2.2.3, we should stop after \( k \) steps. Then we have the claim of the theorem. \( \square \)

### 2.3. How to shorten a cycle

Table 4 shows that for some \( N \) the “short” cycles are relatively long. They may be significantly shortened by a combination of (4) and (5) (for \( t = 3 \)). For such cases, (5) suggests considering the function

\[ g(x) := x \left[ \frac{f(x)/x - 1}{1 - xf(x)} \right]^2 \]

in the interval \( 0 < x < x_1 = f(x_1) \). This function is also unimodal (like \( f(x) \)) with zeros on the ends of the interval. Its maximum is also equal to \( x_0 = I - \sqrt{I^2 - 1} \). Note that \( (fg)(x) = (gf)(x) \) since doubling and tripling are commutative. Indeed, in notations adopted in Section 2, let \( x \) be the center of the \( j \)-th circle of a package of rank \( N \geq 6 \). (For any fixed \( I > 1 \), there is infinite number of such \( x \)’s in \((0, x_0)\).) \( (fg)(x) \), as well as \( (gf)(x) \), is the center of the 6\( j \)-th circle. Since \( x_1 < x_0 \), not any collection of \( f \) and \( g \) is appropriate for \( 0 < x < x_0 \). Further analysis of \( g(x) \) and combinations \( f^{(k)} g^{(l)} \) is out of scope of this paper. We show its application only with examples how to calculate \( \xi_{\min}(l = 71/64, N) \). The proper root of the equation \( g^{(k)} f^{(l)}(x) = x \) is determined applying Corollary 2.1.18. The verification of the rank \( N \) as well as the successive centers of the corresponding package are determined by (4, 5).
<table>
<thead>
<tr>
<th>Number of polygon sides $N$</th>
<th>Euler's function $\phi(N)$</th>
<th>Max length of cycles $m_{i1}$</th>
<th>$\sum_{m_j \in \mu(N)} m_j \times j_j^N = (N - 1)/2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>1</td>
<td>$1 \times 1 = 1$</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>2</td>
<td>$2 \times 1 = 2$</td>
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<tr>
<td>7</td>
<td>6</td>
<td>3</td>
<td>$3 \times 1 = 3$</td>
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<td>6</td>
<td>3</td>
<td>$3 \times 1 + 1 \times 1 = 4$</td>
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<td>5</td>
<td>$5 \times 1 = 5$</td>
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<td>13</td>
<td>12</td>
<td>6</td>
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<td>4</td>
<td>$4 \times 1 + 2 \times 1 + 1 \times 1 = 7$</td>
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<td>16</td>
<td>4</td>
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<td>19</td>
<td>18</td>
<td>9</td>
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<td>$6 \times 1 + 3 \times 1 + 1 \times 1 = 10$</td>
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<td>$11 \times 1 = 11$</td>
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<td>10</td>
<td>$10 \times 1 + 2 \times 1 = 12$</td>
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<td>18</td>
<td>9</td>
<td>$9 \times 1 + 3 \times 1 + 1 \times 1 = 13$</td>
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<td>$10 \times 2 = 20$</td>
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<td>6</td>
<td>$6 \times 5 + 2 \times 1 = 32$</td>
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<td>66</td>
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<td>70</td>
<td>35</td>
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<td>72</td>
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<td>$9 \times 4 = 36$</td>
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<td>88</td>
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Table 4 (continued)

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<th>Number of polygon sides $N$</th>
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<th>$\sum_{m_i \in \mathcal{U}(N)} m_i \times J_1^m = (N - 1)/2$</th>
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<td>12</td>
<td>$12 \times 3 + 6 \times 1 + 4 \times 1 + 3 \times 1 + 2 \times 1 + 1 \times 1 = 52$</td>
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<tr>
<td>107</td>
<td>106</td>
<td>53</td>
<td>53 \times 1 = 53</td>
</tr>
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<td>108</td>
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<td>18 \times 3 = 54</td>
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<td>$36 \times 1 + 18 \times 1 + 1 \times 1 = 55$</td>
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<td>50</td>
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<td>7</td>
<td>$7 \times 9 + 1 \times 1 = 64$</td>
</tr>
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<td>65 \times 1 = 65</td>
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<tr>
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<td>138</td>
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</tr>
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<td>46</td>
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<td>148</td>
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<td>157</td>
<td>156</td>
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<td>$26 \times 3 = 78$</td>
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<td>$33 \times 2 + 11 \times 1 + 3 \times 1 = 80$</td>
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<td>162</td>
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<td>$20 \times 3 + 5 \times 3 + 4 \times 1 + 2 \times 1 + 1 \times 1 = 82$</td>
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<td>167</td>
<td>166</td>
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<td>$83 \times 1 = 83$</td>
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<td>$78 \times 1 + 6 \times 1 = 84$</td>
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<td>$9 \times 9 + 3 \times 1 + 1 \times 1 = 85$</td>
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<tr>
<td>173</td>
<td>172</td>
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<td>179</td>
<td>178</td>
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<td>$89 \times 1 = 89$</td>
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<td>60</td>
<td>$60 \times 1 + 30 \times 1 + 1 \times 1 = 91$</td>
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<td>$40 \times 2 + 5 \times 1 + 4 \times 2 = 93$</td>
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<td>$189 = 3 \times 3 \times 7$</td>
<td>108</td>
<td>18</td>
<td>$18 \times 3 + 9 \times 1 + 6 \times 4 + 3 \times 2 + 1 \times 1 = 94$</td>
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<td>191</td>
<td>190</td>
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<td>$95 \times 1 = 95$</td>
</tr>
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<td>193</td>
<td>192</td>
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<td>$48 \times 2 = 96$</td>
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<td>$195 = 5 \times 39$</td>
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<td>$12 \times 5 + 6 \times 5 + 4 \times 1 + 2 \times 1 + 1 \times 1 = 97$</td>
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<tr>
<td>197</td>
<td>196</td>
<td>98</td>
<td>$98 \times 1 = 98$</td>
</tr>
</tbody>
</table>
For \( N = 53 \) (\( m_1 = 26 \)), we consider the cycle \((1, 2, 6, 18, 54 \equiv 1) \pmod{53}\) that corresponds to the equation \(g^{3}(f(x)) = x\). Its solution \(\xi_{\text{min}} = 0.002814908650\).

For \( N = 59 \) (\( m_1 = 29 \)), we have the cycle \((1, 2, 4, 8, 16, 32 \equiv -27, -54 \equiv 5, 10, 20, 60 \equiv 1) \pmod{59}\) that corresponds to the equation \(gf^{8}(x) = x\). Its solution \(\xi_{\text{min}} = 0.002272410322\).
solution ξ

\[ N = \text{Table 6} \]

Ellipse concentric with 3. Package of concentric ellipses

\[ w_j = \text{equations derived for circles (see [2]) to the case of concentric ellipses. It is also appropriate to recollect} \]

This reduction is considered in the present section. Such a reduction allows one to apply Steiner–Fuss equations derived for circles (see [2]) to the case of concentric ellipses. It is also appropriate to recollect future efforts may explore:

The case of two confocal ellipses (the billiard case – see [3]) may be reduced to the case of a pair of nested circles. The latter case in turn, may be reduced to the case of Poncelet's pairs of rank 1.

1. Whether the roots of equation \( g^{(k)} f^{(l)}(x) = x \) provide us with centers of circles that form with \( C \) Poncelet's pairs of rank \( N \) (similarly to the roots of the equation \( f^{(m)}(x) = x \)),
2. How to determine this \( N \) when such pairs exist, and
3. How to choose \( k \) and \( l \) to get the shortest sequence to calculate \( \xi_{\text{min}}(I, N) \).

### 3. Package of concentric ellipses

The case of two confocal ellipses (the billiard case – see [3]) may be reduced to the case of an ellipse concentric with \( C \). The latter case in turn, may be reduced to the case of a pair of nested circles. This reduction is considered in the present section. Such a reduction allows one to apply Steiner–Fuss equations derived for circles (see [2]) to the case of concentric ellipses.
Table 7

\[ N = 33, I = 71/64 \text{ (see Remark 2.1.19 and Theorem 2.1.20).} \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 2^2 \times 3 )</th>
<th>( j \equiv 2^2 \times 3 \mod 33 )</th>
<th>( z_j )</th>
<th>( w_j = z_{j+1} )</th>
<th>( y_j )</th>
<th>( p_k = 2^k p_0 = (X_k, Y_k) )</th>
</tr>
</thead>
<tbody>
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<td>(16.01889, -59.64230)</td>
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<td>(16.01889, 59.64230)</td>
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</table>

![Fig. 7](image_url)

**Fig. 7.** The roots of \( f^{(3)}(x) = x_0 \) (see Lemma 2.2.2, \( I = 71/64, x_0 = 0.629049 \)).

that Schoenberg [15] proved that two ellipses of general location may be projectively mapped onto a circle and a concentric ellipse.

The reduction presented here may be proved applying the Cayley criterion. Therefore this reduction may be known. However, the author did not find in the literature the formulas for any Poncelet’s ellipses of rank \( N > 3 \) (for \( N = 3 \) and general location of an ellipse, the formula is given by Goldberg and Zwas [5]). Here, the connection between any package of concentric ellipses and some special package of circles is established applying (4, 5) and (2).

**Theorem 3.1.** Let a package of odd rank \( N \) of circles \( K_j \) have positive centers \( c_j \) and radii \( r_j \) \( (j = 1, \ldots, (N - 1)/2) \). Then the ellipses \( E^\delta \) with foci \( \pm \sqrt{c_j} \) and the squares of the minor semi-axes \( b^2_j = (1 - c_j + (-1)^j r_j)/2 \) also form a package of rank \( N \). These ellipses are nested as follows:

\[
E_1 = E^2 \supset E_2 = E^4 \supset \cdots \supset E_{(N-1)/4} = E^{2[(N-1)/4]} \supset E_{(N+3)/4} = E^2(\frac{N-2}{4}+1) \supset \cdots \supset E_{(N-3)/2} = E^3 \supset E_{(N-1)/2} = E^1.
\]

Here, the subscript at \( E \) signifies, as usual, the number of the ellipse in the package of ellipses. The superscript at \( E \) signifies the number of the circle in the package of circles associated with the ellipses by this theorem.

**Proof.** Consider a circle \( K \) centered at positive \( c \) with radius \( r \) and an ellipse \( E(\delta) \) with foci \( \pm \sqrt{c} \) and minor semi-axis \( b = \sqrt{(1 - c + \delta r)/2} \). Applying (4) and (5) to \( K \), we have \( z_1 = c, z_2 = 4cr^2/(1 - c^2), z_t = (z_{t-1} - c)^2/[z_{t-2}(1 - cz_{t-1})^2] \) \((t \geq 3)\). Applying (4) and (5) to \( E(\delta) \) we have \( Z_1 = \sqrt{c}, Z_2 = 2\sqrt{c}\delta r/(1 - c^2), Z_t = (z_{t-1}^2 - c)/[z_{t-2}(1 - cz_{t-1}^2)] \) \((t \geq 3)\). Hence for \( \delta^2 = 1, Z_t^2 = z_t \). In particular,
if \( \mathcal{K} \) and \( \mathcal{C} \) form a Poncelet pair of rank \( N \), then either \( \mathcal{E}(1) \) or \( \mathcal{E}(-1) \) also forms with \( \mathcal{C} \) Poncelet’s pair of rank \( N \).

Let us assume that \( \mathcal{K} \) has number \( j \) in the package of circles \( (j = 1, 2, \ldots, (N - 1)/2) \) and \( \mathcal{E}(\delta) \) has number \( k \) in the package of ellipses, where \( k = (N - j)/2 \) if \( j \) is odd, and \( k = j/2 \) if \( j \) is even. Then for odd \( j, k \geq [(N - 1)/4] \) and for even \( j, k < [N - 1]/4 \). It follows from (2) that \( Z_1 = \sqrt{c_2/j} \) is the right focus of \( \mathcal{E}_k \). \( Z_2 \) is either the left focus of \( \mathcal{E}_{2k} \) for an odd \( j \), or the right focus of \( \mathcal{E}_{2k} \) for an even \( j \). Thus \( \delta = (-1)^j \).

**Remark 3.3.** The confocal ellipses \( x^2/(b^2 + f^2) + y^2/b^2 = 1 \) and \( x^2/(b_1^2 + f^2) + y^2/b_1^2 = 1, b_1 < b \), form a Poncelet pair of odd rank \( N \) if and only if the unit circle \( \mathcal{C} \) forms a Poncelet pair of rank \( N \) with the circle centered at \( c = (b_2 - b_1^2)/(b_2 + f^2) \) of radius \( r = |b|^2 - 2b_2b_1^2 - b_1^2f^2/(b_2 + f^2) \).

**Corollary 3.2.** The invariant for the pencil containing the concentric ellipses \( x^2/a^2 + y^2/b^2 = 1 \) is

\[
I_e := \frac{a^2(1 - b^2) + b^2(1 - a^2)}{a^2 - b^2}.
\]

This invariant is equal to the invariant \( I \) of (4) if \( a \) and \( b \) satisfy the conditions of Theorem 3.1. The short cycles of Section 2 may be determined by the equation

\[
F_{2i}^2 = \frac{4F_1^2\left(1 - 2le_2^2 + F_1^2\right)}{(1 - F_1^2)^2} = f(F_1^2),
\]

where \( f(x) \) is defined in (10) with \( I_e \) instead of \( I \).

4. Appendix: Generalization for any real conic and its location

\[
\text{Conic} \left(\frac{(X - \xi)^2}{A} + \frac{(Y - \eta)^2}{B} = 1 \right) \quad (A > B, \, e^2 = A - B, \, \xi \eta \neq 0).
\]

Solving jointly the equation of this conic with the equation of \( \mathcal{C} \left( X^2 + Y^2 = 1 \right) \), we have the following invariants for the pencil containing these conics:

\[
I_1 = \frac{BE_1}{e^2}, \quad I_2 = \frac{A\eta}{e^2}, \quad J = \frac{AN_1^2 + BE_2^2 + A - AB}{e^2},
\]

\[
I_0 = \frac{J - 1}{2I_1} = \frac{AN_1^2 + BE_2^2 + B - AB}{2BE_1}.
\]

Let the given conic of \#k in a package of rank \( N \) have the center \( \xi_1 + i\eta_1 \). Denote \( \xi_2 + i\eta_2 \) the center of the conic of \#2k \: (mod \: N) in this package. Below we show how to determine this center when either \( \xi_1 \) or \( \eta_1 \) is given.

It follows from (4) for \( g_1 = \xi_1 + i\eta_1 + e_1 \) and \( g_2 = \xi_1 + i\eta_1 - e_1 \)

\[
\xi_2 = \frac{I_1\eta_2}{I_2 - \eta_2}, \quad \eta_2 = \frac{4\eta_1A_1}{(1 - \xi_1^2 - \eta_1^2 + e_1^2)^2 + 4\eta_1^2e_1^2},
\]

where

\[
A_1 = \xi_1^2 - 2I_0\xi_1 + 1 + \frac{I_2^2\xi_1^2}{I_1 + \xi_1 I_1}, \quad \eta_1 = \frac{I_2\xi_1}{I_1 + \xi_1}, \quad e_1 = \frac{\xi_1A_1}{I_1 + \xi_1}.
\]

4.1. \( \eta = 0 \)

\[
\text{Conic} : \left(\frac{(X - \xi)^2}{A_1} + \frac{Y^2}{B_1} = 1 \right).
\]
For this case, $l_0 = l_{01} = (J - 1)/(2l_1) = (\xi^2 + 1 - A)/(2\xi)$.

$$\xi_2 = \frac{4\xi_1 B_1}{(1 - \xi^2 - e_1^2)^2 - 4\xi_1^2 e_1^2},$$

where

$$A_1 = \frac{\xi_2^2}{\xi_1} - 2l_{01} \xi_1 + 1, \quad B_1 = A_1 \frac{l_1}{l_1 + \xi_1}, \quad e_1^2 = A_1 - B_1 = \frac{B_1 \xi_1}{l_1}.$$

4.2. $\xi = 0$

Conic : $\frac{X^2}{A_1} + \frac{(Y - \eta_1)^2}{B_1} = 1$.

For this case, we have the invariant $l_{02} = f/(2l_2) = (\eta^2 + 1 - B)/(2\eta)$.

$$\eta_2 = \frac{4\eta_1 A_1}{(1 + \eta_1)^2 + e_1^2} \frac{(1 - \eta_1)^2 + e_1^2}{1}.$$

where

$$B_1 = \eta_1^2 - 2l_{02} \eta_1 + 1, \quad e_1^2 = \frac{B_1 \eta_1}{l_2 - \eta_1}, \quad A_1 = B_1 + e_1^2 = B_1 \frac{l_2}{l_2 - \eta_1}.$$

The case $A = B$ is considered in Section 2. The case $\xi = \eta = 0$ is considered in Section 3. Future efforts may explore constraints on the parameters of conics of this Appendix.

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References