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A semi-free weighting matrices approach for neutral-type delayed neural networks

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ABSTRACT

In this paper, a new approach is proposed for stability issues of neutral-type neural networks (DNNs) with constant delay. First, the semi-free weighting matrices are proposed and used instead of the known free weighting matrices to express the relationship between the terms in the Leibniz–Newton formula to simplify the system synthesis and to obtain less computation demand. Second, global exponential stability conditions which are less conservative and restrictive than the known results are derived. At the same time, based on the above approach, fewer variable matrices are introduced in the construction of the Lyapunov functional and augmented Lyapunov functional. Two examples are given to show their effectiveness and advantages over others.

1. Introduction

It is well known that delayed neural networks (DNNs) share the potential benefits of neural networks (NNs) such as parallel distributed processing, high computation rates, fault tolerance, and adaptive capability. In the past decades, DNNs have been extensively used in various engineering and scientific fields for the above benefits, and the trait of time delay has been found in engineering applications, for example, solving linear projection equations and quadratic optimization problems [5], component detection of medical signal [26], automatic control engineering [27], analysis of moving images or speech [6], biological simulation [28] and so on. However, the existence of time delays in these DNN models indicates that time delays are dependent on the past state. In fact, many practical delay systems can be modeled as differential systems of neutral type, whose differential expression concludes not only the derivative term of the current state but also concludes the derivative term of the past state, such as partial element equivalent circuits [7] and transmission lines [8] in electrical engineering, controlled constrained manipulators in mechanical engineering [9], population dynamics system [1] and so on. It is natural and important that systems should contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [10].

However, few researchers studied the stability for a class of DNNs which is described by nonlinear delay differential equations of the neutral type [10,11,29]. Park et al. studied the global asymptotic stability of delayed neural networks of neutral type in [10]. Two sufficient globally exponential stability conditions for such systems were proposed in [11,29]. In [11], an inequality was used to derive the result, which made the criteria more conservative. However, there are too many matrices included in the stability criterion in [10,29]. Motivated by the aforementioned discussion, the exponential stability of DNNs of neutral type which gives a fast convergence rate and estimates the exponential convergence rates for the system is considered in this paper.

In order to derive some delay-dependent stability criteria, a model transformation [2,12,13] and the bounded technique on the cross term are used in [14–16]. In addition, a descriptor model transformation which is also based on the substitution

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of $x(t) - \int_{t-\tau}^{t} \dot{x}(\xi)\mathrm{d}\xi$ for $x(t-\tau)$ is applied to derive less conservative conditions in [17–19]. Gu proposed the discretized Lyapunov functional method in [3,20,21]. Recently He et al. proposed a new approach which is, applying free weighting matrices into the Leibniz–Newton formula [22,23]. Both of them are very efficient. However, many researchers have found that too many free variables introduced in the free weighting matrices method will make system synthesis more complex and consequently lead to a significant increase in the computational demand [24].

He has also noted in [22] that the $-\int_{-\tau}^{-d(t)} \dot{y}^{\mathrm{T}}(t+\xi)Z\dot{y}(t+\xi)\mathrm{d}\xi$ in $\dot{V}(y(t))$, where $d(t)$ satisfies that $0 < d(t) \le \tau$, $0 < |\dot{d}(t)| \le u$ is a time-varying differentiable function, was not considered in most of the previous works of time-varying delayed systems, which leads to considerable conservativeness. The new free weighting matrices are introduced to estimate the upper bound of the derivative of the Lyapunov functional. Inspired by this, we consider this approach to deal with the augmented term related to $d(t)$, which can not only be used for a time-varying delayed system, but also for a constant delay system. In this paper, we consider the time-varying functions $d_i(t), i = 1, 2, \ldots, m-1$, which satisfy $0 \le d_{m-1}(t) \le d_{m-2}(t) \le \cdots \le d_1(t) \le \tau, \tau > 0, \max\left\{|\dot{d}_1(t)|, |\dot{d}_2(t)|, \ldots, |\dot{d}_{m-1}(t)|\right\} \le u, u > 0$ and are independent of the original system (1) (see Section 2). By dividing the delay interval $[-\tau, 0]$, we construct the following Lyapunov–Krasovskii functional for system (1)

$$
\begin{aligned}
V_2(y(t)) = {} & \int_{-\tau}^{-d_1(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi)M_1 y(t+\xi)\mathrm{d}\xi + \int_{-d_1(t)}^{-d_2(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi)M_2 y(t+\xi)\mathrm{d}\xi \\
& + \int_{-d_2(t)}^{-d_3(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi)M_3 y(t+\xi)\mathrm{d}\xi + \cdots + \int_{-d_{m-2}(t)}^{-d_{m-1}(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi)M_{m-1} y(t+\xi)\mathrm{d}\xi \\
& + \int_{-d_{m-1}(t)}^{0} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi)M_m y(t+\xi)\mathrm{d}\xi + \int_{-\tau}^{0} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi)Q y(t+\xi)\mathrm{d}\xi,
\end{aligned}
$$

$M_i, i = 1, 2, \ldots, m$, and Q satisfy the following equations:

$$
\begin{aligned}
& 2[y^{\mathrm{T}}(t), y^{\mathrm{T}}(t-d_{m-1}(t)), y^{\mathrm{T}}(t-d_{m-2}(t)), \ldots, y^{\mathrm{T}}(t-d_1(t)), y^{\mathrm{T}}(t-\tau)] \\
& \quad \times M_i \left[\mathrm{e}^{2\beta(t-d_i(t))} y(t-d_i(t)) - \mathrm{e}^{2\beta(t-d_{i-1}(t))} y(t-d_{i-1}(t)) - \int_{-d_{i-1}(t)}^{-d_i(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi \right] = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& 2[y^{\mathrm{T}}(t), y^{\mathrm{T}}(t-d_{m-1}(t)), y^{\mathrm{T}}(t-d_{m-2}(t)), \ldots, y^{\mathrm{T}}(t-d_1(t)), y^{\mathrm{T}}(t-\tau)] \\
& \quad \times Q \left[y(t) - \mathrm{e}^{-2\beta\tau} y(t-\tau) - \int_{-\tau}^{0} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi \right] = 0.
\end{aligned} \tag{1'}
$$

Since the number m is to be determined, and the $M_i, i = 1, 2, 3, \ldots, m$ and Q are put in the Lyapunov–Krasovskii functional $V_2(y(t))$ which are different from the free matrices in [10,29], we call the $M_i, i = 1, 2, \ldots, m$, and Q semi-free weighting matrices. Numerical examples are given to show their effectiveness and advantages over others.

2. Problem formulation and preliminaries

R^n denotes the n dimensional Euclidean space, and $R^{m\times n}$ is the set of all $m\times n$ real matrices. $\|\cdot\|$ denotes the Euclidean norm in R^n or $R^{m\times n}$. Let the Euclidean norm $\|\phi\|_\tau = \sup_{-\tau\le\theta\le 0} \|x(\theta)\|$, $\|\phi^*\|_\tau = \sup_{-\tau\le\theta\le 0} \|\dot{x}(\theta)\|$ for a given continuous function. A^{T} denotes the transpose of matrix A, $\lambda_{\mathrm{Max}}(M)$ denotes the maximum eigenvalue of M, $\lambda_{\mathrm{Min}}(M)$ denotes the minimum eigenvalue of M.

In this paper, we study the following class of delayed neural networks described by a nonlinear neutral delay differential equation:

$$
\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + C\dot{x}(t-\tau) + J, \tag{1}
$$

with the initial condition

$$
x(\theta) = \phi(\theta), \quad \forall\theta \in [-\tau, 0],
$$

where n is the number of neurons in the indicated neural network, $x(t) = [x_1(t), x_2(t), x_3(t), \ldots, x_n(t)]^{\mathrm{T}}$ is the state vector of network at time t, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), f_3(x_3(t)), \ldots, f_n(x_n(t))]^{\mathrm{T}} \in R^n$ is the activation functions and $J = [J_1, J_2, J_3, \ldots, J_n]^{\mathrm{T}}$ is the external constant inputs. $D = \mathrm{diag}(d_i) > 0, A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}, C = (c_{ij})_{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons. Throughout this paper, we always assume that the activation functions are bounded and satisfy Lipschitz's condition, i.e.

(H) There exist constants $L_i > 0$ such that $\|f_i(x) - f_i(y)\| \le L_i \|x - y\|$, for any

$$
x, y \in R, i = 1, 2, 3, \ldots, n.
$$

It is easy to see that the condition (H) implies that the activation functions are continuous but not always monotonic. It is not difficult to prove that system (1) has at least an equilibrium point by using the well-known Brouwer's fixed-point theorem.

Suppose that $x^* = (x_1^*, x_2^*, \dots, x_n^*)^{\mathrm{T}} \in R^n$ is an equilibrium point of system (1), and let $y_i(t) = x_i(t) - x^*, i = 1, 2, 3, \dots, n$, then system (1) can be rewritten as

$$\begin{cases} \dot{y}(t) = -Dy(t) + Ag(y(t)) + Bg(y(t-\tau)) + C\dot{y}(t-\tau), \\ y(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{2}$$

where $y(t)$ is the state vector of the transformed equation (2) and

$$g_i(y_i(t)) = f_i(y_i(t) + x_i^*) - f_i(x_i^*), \quad i = 1, 2, 3, \dots, n. \tag{3}$$

Obviously, the origin of system (2) is one of the equilibrium points (if many) of system (1). It is obvious that neural network activation functions can be exactly "linearized" and shifted into an interval linear system under the condition (H). Recently, the following "linearized" method was used in [13,30].

We define the time-varying bounded functions $s_i(t)$, for $i = 1, 2, 3, \dots, n$,

$$s_i(t) = \begin{cases} \dfrac{g_i\,(y_i(t))}{y_i(t)}, & y_i(t) \neq 0, \\ 0, & y_i(t) = 0. \end{cases} \tag{4}$$

Obviously, $s_i(t)$ is piecewise continuous on R.

From (4) and the condition (H), we have $0 \le s_i \le L_i$. Furthermore, system (2) can be rewritten as

$$\begin{cases} \dot{y}(t) = -Dy(t) + As(t)y(t) + Bs(t-\tau)y(t-\tau) + C\dot{y}(t-\tau), \\ y(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0], \end{cases} \tag{5}$$

where $s(t) = \mathrm{diag}(s_i(t)) \le \mathrm{diag}(L_i) = L$.

Definition 1. System (1) is said to be exponentially stable if there exist $\mu > 1$, $k > 0$, such that for every solution $x(t)$ of system (1), we have

$$\|x(t)\| \le \mu \mathrm{e}^{-kt} \max\{\|\phi\|_\tau, \|\phi^*\|_\tau\}.$$

Lemma 1 (*[4]*). *The following LMI*

$$\begin{bmatrix} Q(x) & S(x) \\ S^{\mathrm{T}}(x) & R(x) \end{bmatrix} > 0,$$

where $Q(x) = Q^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x)$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0, \qquad Q(x) - S(x)R^{-1}(x)S^{\mathrm{T}}(x) > 0.$$

Lemma 2 (*[25]*). *Given a matrix D, let there be a positive-definite matrix S and there exist a positive scalar $\eta \in (0, 1)$ such that*

$$D^{\mathrm{T}}SD - \eta^2 S < 0. \tag{6}$$

Then, the matrix D satisfies the following bound

$$\|D^i\| \le \chi \mathrm{e}^{-\lambda i} \quad \textit{with } \chi = \sqrt{\frac{\lambda_{\mathrm{Max}}(S)}{\lambda_{\mathrm{Min}}(S)}}$$

and $\lambda = -\ln(\eta)$.

Lemma 3 (*[25]*). *Consider the system*

$$x(t) + Dx(t-\tau) = f(t), \quad t \ge 0 \tag{7}$$

where the matrix $D \in R^{n\times n}$ is Schur stable, and

$$\|f(t)\| \le \theta \mathrm{e}^{-\beta t}, \quad t \ge 0, \theta > 0, \beta > 0.$$

Then, for any initial function $\varphi \in C([-\tau, 0], R^n)$, the solution $x(t, \varphi)$ of system (7) satisfies the inequality

$$\|x(t,\varphi)\| \le \left[\chi \|\varphi\|_h + \theta + \frac{\chi}{\tau\varepsilon r\mathrm{e}}\theta\right] \mathrm{e}^{-(1-\varepsilon)rt}, \quad t \ge 0, \tag{8}$$

where $r \triangleq \min\{\lambda/\tau, \beta\}$, $\varepsilon \in (0, 1)$, χ and λ are determined by using Lemma 2.

3. Main results

Theorem 1. *Suppose that the condition* (H) *is satisfied. Then, the delay neural network of neutral-type system* (1) *is globally exponentially stable if there exists a time-variant function $d_i(t)$, $i = 1, 2, \ldots, m-1$, which satisfies $0 \le d_{m-1}(t) \le d_{m-2}(t) \le \cdots \le d_1(t) \le \tau$, $\tau > 0$, $\max\left\{|\dot{d}_1(t)|, |\dot{d}_2(t)|, \ldots, |\dot{d}_{m-1}(t)|\right\} \le u$, $u \ge 0$, there exist positive constant $\beta > 0$ and positive scalar $\eta \in (0, 1)$, and positive-definite matrix S and symmetrical positive-definite matrices P, Q, Z, M_i, $i = 1, 2, 3, \ldots, m$, where m is to be determined, $0 < M_1 \le M_2 \le \cdots \le M_{m-1} \le M_m$, such that*

$$\Omega_1 = \begin{bmatrix} \Omega_2 & X_1 & X_2 & X_3 & X_4 & X_5 & \cdots & X_{m+2} \\ X_1^{\mathrm{T}} & -2\tau Z & 0 & 0 & 0 & 0 & \cdots & 0 \\ X_2^{\mathrm{T}} & 0 & -Z & 0 & 0 & 0 & \cdots & 0 \\ X_3^{\mathrm{T}} & 0 & 0 & -Z & 0 & 0 & \cdots & 0 \\ X_4^{\mathrm{T}} & 0 & 0 & 0 & -Z & 0 & \cdots & 0 \\ X_5^{\mathrm{T}} & 0 & 0 & 0 & 0 & -Z & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{m+2}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & \cdots & -Z \end{bmatrix} < 0,$$

$$C^{\mathrm{T}}SC - \eta^2 S < 0, \tag{9}$$

where

$$X_1 = \begin{bmatrix} 2\tau Z \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} \sqrt{(1-\mathrm{e}^{-2\beta\tau})}Q \\ 0 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} \sqrt{(1-\mathrm{e}^{-2\beta\tau})}M_1 \\ 0 \\ 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} \sqrt{(1-\mathrm{e}^{-2\beta\tau})}M_2 \\ 0 \\ 0 \end{bmatrix},$$

$$X_5 = \begin{bmatrix} \sqrt{(1-\mathrm{e}^{-2\beta\tau})}M_3 \\ 0 \\ 0 \end{bmatrix}, \quad X_{m+2} = \begin{bmatrix} \sqrt{(1-\mathrm{e}^{-2\beta\tau})}M_m \\ 0 \\ 0 \end{bmatrix},$$

and

$$\Omega_2 = \left[\begin{matrix} 2\beta P + 2P(-D+AL) + 3M_m - Q & (M_{m-1} - M_m) \\ M_{m-1}^{\mathrm{T}} - M_m^{\mathrm{T}} & (1-u)(M_{m-1} - M_m) \\ M_{m-2}^{\mathrm{T}} - M_{m-1}^{\mathrm{T}} & 0 \\ \vdots & \vdots \\ M_1^{\mathrm{T}} - M_2^{\mathrm{T}} & 0 \\ -2\beta PC + LB^{\mathrm{T}}P + (Q^{\mathrm{T}} - M_1^{\mathrm{T}})\mathrm{e}^{-2\beta\tau} + (D^{\mathrm{T}} - LA^{\mathrm{T}})PC & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} M_{m-2} - M_{m-1} & \cdots & M_1 - M_2 & \begin{matrix} C^{\mathrm{T}}P(D-AL) + PBL - 2\beta C^{\mathrm{T}}P \\ +(Q - M_1)\mathrm{e}^{-2\beta\tau} \end{matrix} \\ 0 & \cdots & 0 & 0 \\ (1-u)(M_{m-2} - M_{m-1}) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & (1-u)(M_1 - M_2) & 0 \\ 0 & \cdots & 0 & \begin{matrix} 2(\beta C^{\mathrm{T}}PC - C^{\mathrm{T}}PBL) \\ -M_1\mathrm{e}^{-2\beta\tau} - Q\mathrm{e}^{-2\beta\tau} \end{matrix} \end{matrix}\right].$$

Moreover,

$$\|y(t)\| \le \left[\chi\|\varphi\|_h + \theta + \frac{\chi}{\tau\varepsilon re}\theta\right]\mathrm{e}^{-(1-\varepsilon)rt}, \quad t \ge 0, \tag{10}$$

where

$$\theta \triangleq \frac{1}{\sqrt{\lambda_{\mathrm{Min}}(P)}}\left[2\lambda_{\mathrm{Max}}(P) + 2\lambda_{\mathrm{Max}}(P)C^{\mathrm{T}}C + \lambda_{\mathrm{Max}}(M_{\mathrm{Max}})\frac{1-\mathrm{e}^{-2\beta\tau}}{2\beta} + \lambda_{\mathrm{Max}}(Q)\frac{1-\mathrm{e}^{-2\beta\tau}}{2\beta}\right.$$
$$\left. + 2\lambda_{\mathrm{Max}}(Z)\frac{2\beta\tau - 1 + \mathrm{e}^{-2\beta\tau}}{4\beta^2}\right]^{\frac{1}{2}} \max\{\|\phi\|_\tau, \|\phi^*\|_\tau\},$$

$$\chi = \sqrt{\frac{\lambda_{\mathrm{Max}}(S)}{\lambda_{\mathrm{Min}}(S)}} \quad \textit{and} \quad \omega = -\ln(\eta)$$

$$r \triangleq \min\{\omega/\tau, \beta\}, \quad \varepsilon \in (0, 1),$$
$$\lambda_{\mathrm{Max}}(M_{\mathrm{Max}}) = \max\{\lambda_{\mathrm{Max}}(M_1), \lambda_{\mathrm{Max}}(M_2), \lambda_{\mathrm{Max}}(M_3), \ldots, \lambda_{\mathrm{Max}}(M_m)\}.$$

Proof. Let the Lyapunov functional candidate be

$$V(y(t)) = V_1(y(t)) + V_2(y(t)) + V_3(y(t)),$$

where

$$\begin{aligned}
V_1(y(t)) &= \mathrm{e}^{2\beta t}[y(t) - Cy(t-\tau)]^{\mathrm{T}} P[y(t) - Cy(t-\tau)],\\
V_2(y(t)) &= \int_{-\tau}^{-d_1(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi) M_1 y(t+\xi)\mathrm{d}\xi + \int_{-d_1(t)}^{-d_2(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi) M_2 y(t+\xi)\mathrm{d}\xi\\
&\quad + \int_{-d_2(t)}^{-d_3(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi) M_3 y(t+\xi)\mathrm{d}\xi + \cdots + \int_{-d_{m-2}(t)}^{-d_{m-1}(t)} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi) M_{m-1} y(t+\xi)\mathrm{d}\xi\\
&\quad + \int_{-d_{m-1}(t)}^{0} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi) M_m y(t+\xi)\mathrm{d}\xi + \int_{-\tau}^{0} \mathrm{e}^{2\beta(t+\xi)} y^{\mathrm{T}}(t+\xi) Q y(t+\xi)\mathrm{d}\xi,\\
V_3(y(t)) &= \int_{-\tau}^{-d_1(t)} \int_{t+\theta}^{t} \mathrm{e}^{2\beta\xi} \dot{y}^{\mathrm{T}}(\xi) Z \dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta + \int_{-d_1(t)}^{-d_2(t)} \int_{t+\theta}^{t} \mathrm{e}^{2\beta\xi} \dot{y}^{\mathrm{T}}(\xi) Z \dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta\\
&\quad \times \int_{-d_2(t)}^{-d_3(t)} \int_{t+\theta}^{t} \mathrm{e}^{2\beta\xi} \dot{y}^{\mathrm{T}}(\xi) Z \dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta + \cdots + \int_{-d_{m-2}(t)}^{-d_{m-1}(t)} \int_{t+\theta}^{t} \mathrm{e}^{2\beta\xi} \dot{y}^{\mathrm{T}}(\xi) Z \dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta\\
&\quad + \int_{-d_{m-1}(t)}^{0} \int_{t+\theta}^{t} \mathrm{e}^{2\beta\xi} \dot{y}^{\mathrm{T}}(\xi) Z \dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta + \int_{-\tau}^{0} \int_{t+\theta}^{t} \mathrm{e}^{2\beta\xi} \dot{y}^{\mathrm{T}}(\xi) Z \dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta.
\end{aligned} \tag{11}$$

The time derivative of the Lyapunov–Krasovskii functional along the trajectories of system (6) is derived as follows:

$$\begin{aligned}
\dot{V}_1(y(t)) &= 2\beta \mathrm{e}^{2\beta t}[y(t) - Cy(t-\tau)]^{\mathrm{T}} P[y(t) - Cy(t-\tau)]\\
&\quad + 2\mathrm{e}^{2\beta t}[y(t) - Cy(t-\tau)]^{\mathrm{T}} P[-Dy(t) + As(t)y(t) + Bs(t-\tau)y(t-\tau)]\\
&< 2\mathrm{e}^{2\beta t}[y^{\mathrm{T}}(t)(\beta P - PD + PAL)y(t) + y^{\mathrm{T}}(t-\tau)(-2\beta PC + PBL + C^{\mathrm{T}}PD - C^{\mathrm{T}}PAL)y(t)\\
&\quad + y^{\mathrm{T}}(t-\tau)(\beta C^{\mathrm{T}}PC - C^{\mathrm{T}}PBL)y(t-\tau)],
\end{aligned} \tag{12}$$

$$\begin{aligned}
\dot{V}_2(y(t)) &< (1-u)\mathrm{e}^{2\beta(t-d_1(t))} y^{\mathrm{T}}(t-d_1(t))(M_1 - M_2)y(t-d_1(t))\\
&\quad + (1-u)\mathrm{e}^{2\beta(t-d_2(t))} y^{\mathrm{T}}(t-d_2(t))(M_2 - M_3)y(t-d_2(t))\\
&\quad + \cdots + (1-u)\mathrm{e}^{2\beta(t-d_{m-1}(t))} y^{\mathrm{T}}(t-d_{m-1}(t))(M_{m-1} - M_m)y(t-d_{m-1}(t))\\
&\quad + \mathrm{e}^{2\beta(t)} y^{\mathrm{T}}(t) M_m y(t) - \mathrm{e}^{2\beta(t-\tau)} y^{\mathrm{T}}(t-\tau) M_1 y(t-\tau)\\
&\quad + \mathrm{e}^{2\beta(t)} y^{\mathrm{T}}(t) Q y(t) - \mathrm{e}^{2\beta(t-\tau)} y^{\mathrm{T}}(t-\tau) Q y(t-\tau)\\
&< \mathrm{e}^{2\beta t}[(1-u)y^{\mathrm{T}}(t-d_1(t))(M_1 - M_2)y(t-d_1(t)) + (1-u)y^{\mathrm{T}}(t-d_2(t))(M_2 - M_3)y(t-d_2(t))\\
&\quad + \cdots + (1-u)y^{\mathrm{T}}(t-d_{m-1}(t))(M_{m-1} - M_m)y(t-d_{m-1}(t))\\
&\quad + y^{\mathrm{T}}(t) M_m y(t) - \mathrm{e}^{-2\beta\tau} y^{\mathrm{T}}(t-\tau) M_1 y(t-\tau) + y^{\mathrm{T}}(t) Q y(t) - \mathrm{e}^{-2\beta\tau} y^{\mathrm{T}}(t-\tau) Q y(t-\tau)],
\end{aligned} \tag{13}$$

$$\begin{aligned}
\dot{V}_3(y(t)) &\le 2\tau \mathrm{e}^{2\beta t} \dot{y}(t)^{\mathrm{T}} Z \dot{y}(t) - \int_{-\tau}^{-d_1(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi) Z \dot{y}(t+\xi)\mathrm{d}\xi\\
&\quad - \int_{-d_1(t)}^{-d_2(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi) Z \dot{y}(t+\xi)\mathrm{d}\xi - \int_{-d_2(t)}^{-d_3(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi) Z \dot{y}(t+\xi)\mathrm{d}\xi\\
&\quad - \cdots - \int_{-d_{m-2}(t)}^{-d_{m-1}(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi) Z \dot{y}(t+\xi)\mathrm{d}\xi\\
&\quad - \int_{-d_{m-1}(t)}^{0} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi) Z \dot{y}(t+\xi)\mathrm{d}\xi - \int_{-\tau}^{0} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi) Z \dot{y}(t+\xi)\mathrm{d}\xi.
\end{aligned} \tag{14}$$

Adding (12)–(14), we obtain

$$\begin{aligned}
\dot{V}(y(t)) &= \dot{V}_1(y(t)) + \dot{V}_2(y(t)) + \dot{V}_3(y(t)) + 2\eta^{\mathrm{T}}(t) M_1 \Bigg[\mathrm{e}^{2\beta(t-d_1(t))} y(t-d_1(t)) - \mathrm{e}^{2\beta(t-\tau)} y(t-\tau)\\
&\quad - \int_{-\tau}^{-d_1(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi\Bigg] + 2\eta^{\mathrm{T}}(t) M_2 \Bigg[\mathrm{e}^{2\beta(t-d_2(t))} y(t-d_2(t))\\
&\quad - \mathrm{e}^{2\beta(t-d_1(t))} y(t-d_1(t)) - \int_{-d_1(t)}^{-d_2(t)} \mathrm{e}^{2\beta(t+\xi)} \dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+2\eta^{\mathrm{T}}(t)M_3\left[\mathrm{e}^{2\beta(t-d_3(t))}y(t-d_3(t))-\mathrm{e}^{2\beta(t-d_2(t))}y(t-d_2(t))-\int_{-d_2(t)}^{-d_3(t)}\mathrm{e}^{2\beta(t+\xi)}\dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi\right]\\
&+\cdots+2\eta^{\mathrm{T}}(t)M_{m-1}\left[\mathrm{e}^{2\beta(t-d_{m-1}(t))}y(t-d_2(t))\right.\\
&\left.-\mathrm{e}^{2\beta(t-d_{m-2}(t))}y(t-d_1(t))-\int_{-d_{m-2}(t)}^{-d_{m-1}(t)}\mathrm{e}^{2\beta(t+\xi)}\dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi\right]\\
&+2\eta^{\mathrm{T}}(t)M_m\left[\mathrm{e}^{2\beta t}y(t)-\mathrm{e}^{2\beta(t-d_{m-1}(t))}y(t-d_{m-1}(t))\right.\\
&\left.-\int_{-d_{m-1}(t)}^{0}\mathrm{e}^{2\beta(t+\xi)}\dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi\right]-2\eta^{\mathrm{T}}(t)Q\left[\mathrm{e}^{2\beta t}y(t)-\mathrm{e}^{2\beta(t-\tau)}y(t-\tau)-\int_{-\tau}^{0}\mathrm{e}^{2\beta(t+\xi)}\dot{y}^{\mathrm{T}}(t+\xi)\mathrm{d}\xi\right]\\
<\;&2\tau\mathrm{e}^{2\beta t}\dot{y}(t)^{\mathrm{T}}Z\dot{y}(t)+\mathrm{e}^{2\beta t}\eta^{\mathrm{T}}(t)[(\mathrm{e}^{-2\beta d_1(t)}-\mathrm{e}^{-2\beta\tau})M_1Z^{-1}M_1^{\mathrm{T}}+(\mathrm{e}^{-2\beta d_2(t)}-\mathrm{e}^{-2\beta d_1(t)})M_2Z^{-1}M_2^{\mathrm{T}}\\
&+(\mathrm{e}^{-2\beta d_3(t)}-\mathrm{e}^{-2\beta d_2(t)})M_3Z^{-1}M_3^{\mathrm{T}}+\cdots+(\mathrm{e}^{-2\beta d_{m-1}(t)}-\mathrm{e}^{-2\beta d_{m-2}(t)})M_{m-1}Z^{-1}M_{m-1}^{\mathrm{T}}\\
&+(\mathrm{e}^{0}-\mathrm{e}^{-2\beta d_{m-1}(t)})M_mZ^{-1}M_m^{\mathrm{T}}+(\mathrm{e}^{0}-\mathrm{e}^{-2\beta\tau})QZ^{-1}Q^{\mathrm{T}}+\Omega_2]\eta(t)\\
&-\int_{-\tau}^{-d_1(t)}\mathrm{e}^{2\beta(t+\xi)}[\eta^{\mathrm{T}}(t)M_1+\dot{y}^{\mathrm{T}}(t+\xi)Z]Z^{-1}[\eta^{\mathrm{T}}(t)M_1+\dot{y}^{\mathrm{T}}(t+\xi)Z]^{\mathrm{T}}\mathrm{d}\xi\\
&-\int_{-d_1(t)}^{-d_2(t)}\mathrm{e}^{2\beta(t+\xi)}[\eta^{\mathrm{T}}(t)M_2+\dot{y}^{\mathrm{T}}(t+\xi)Z]Z^{-1}[\eta^{\mathrm{T}}(t)M_2+\dot{y}^{\mathrm{T}}(t+\xi)Z]^{\mathrm{T}}\mathrm{d}\xi\\
&-\cdots-\int_{-d_{m-2}(t)}^{-d_{m-1}(t)}\mathrm{e}^{2\beta(t+\xi)}[\eta^{\mathrm{T}}(t)M_{m-1}+\dot{y}^{\mathrm{T}}(t+\xi)Z]Z^{-1}[\eta^{\mathrm{T}}(t)M_{m-1}+\dot{y}^{\mathrm{T}}(t+\xi)Z]^{\mathrm{T}}\mathrm{d}\xi\\
&-\int_{-d_{m-1}(t)}^{0}\mathrm{e}^{2\beta(t+\xi)}[\eta^{\mathrm{T}}(t)M_m+\dot{y}^{\mathrm{T}}(t+\xi)Z]Z^{-1}[\eta^{\mathrm{T}}(t)M_m+\dot{y}^{\mathrm{T}}(t+\xi)Z]^{\mathrm{T}}\mathrm{d}\xi\\
&-\int_{-\tau}^{0}\mathrm{e}^{2\beta(t+\xi)}[\eta^{\mathrm{T}}(t)Q-\dot{y}^{\mathrm{T}}(t+\xi)Z]Z^{-1}[\eta^{\mathrm{T}}(t)Q-\dot{y}^{\mathrm{T}}(t+\xi)Z]^{\mathrm{T}}\mathrm{d}\xi.
\end{aligned}
\tag{15}
$$

Since $Z > 0$ and $0 \le d_{m-1}(t) \le d_{m-2}(t) \le \cdots \le d_1(t) \le \tau$, the last $m+1$ parts are less than 0. We can omit them here for the LMI's simplicity, although it will bring more conservatism. By Lemma 1, we immediately obtain

$$
\begin{aligned}
\dot{V}(y(t)) \le\;& 2\tau\mathrm{e}^{2\beta t}\dot{y}(t)^{\mathrm{T}}Z\dot{y}(t)+\mathrm{e}^{2\beta t}\eta^{\mathrm{T}}(t)[(\mathrm{e}^{0}-\mathrm{e}^{-2\beta\tau})M_1Z^{-1}M_1^{\mathrm{T}}\\
&+(\mathrm{e}^{0}-\mathrm{e}^{-2\beta\tau})M_2Z^{-1}M_2^{\mathrm{T}}\ldots+(\mathrm{e}^{0}-\mathrm{e}^{-2\beta\tau})M_{m-1}Z^{-1}M_{m-1}^{\mathrm{T}}\\
&+(\mathrm{e}^{0}-\mathrm{e}^{-2\beta\tau})M_mZ^{-1}M_m^{\mathrm{T}}+(\mathrm{e}^{0}-\mathrm{e}^{-2\beta\tau})QZ^{-1}Q^{\mathrm{T}}+\Omega_2]\eta(t)\\
=\;& \mathrm{e}^{2\beta t}\eta_1^{\mathrm{T}}(t)\Omega_1\eta_1(t),
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
&\eta^{\mathrm{T}}(t)=[y^{\mathrm{T}}(t),y^{\mathrm{T}}(t-d_{m-1}(t)),y^{\mathrm{T}}(t-d_{m-2}(t)),\ldots,y^{\mathrm{T}}(t-d_1(t)),y^{\mathrm{T}}(t-\tau)]^{\mathrm{T}},\\
&\eta_1^{\mathrm{T}}(t)=[y^{\mathrm{T}}(t),y^{\mathrm{T}}(t-d_{m-1}(t)),y^{\mathrm{T}}(t-d_{m-2}(t)),\ldots,y^{\mathrm{T}}(t-d_1(t)),y^{\mathrm{T}}(t-\tau),\dot{y}^{\mathrm{T}}(t),1,1^2,\ldots,1^{m+1}]^{\mathrm{T}}.
\end{aligned}
$$

If Ω_1 holds, then $\dot{V}(y(t)) \le 0$, $V(y(t)) \le V(y(0))$.

However

$$
\begin{aligned}
V(y(0)) =\;& [y(0)-Cy(-\tau)]^{\mathrm{T}}P[y(0)-Cy(-\tau)]+\int_{-\tau}^{-d_1(0)}\mathrm{e}^{2\beta\xi}y^{\mathrm{T}}(\xi)M_1y(\xi)\mathrm{d}\xi\\
&+\int_{-d_1(0)}^{-d_2(0)}\mathrm{e}^{2\beta\xi}y^{\mathrm{T}}(\xi)M_2y(\xi)\mathrm{d}\xi+\int_{-d_2(0)}^{-d_3(0)}\mathrm{e}^{2\beta\xi}y^{\mathrm{T}}(\xi)M_3y(\xi)\mathrm{d}\xi\\
&+\cdots+\int_{-d_{m-1}(0)}^{0}\mathrm{e}^{2\beta\xi}y^{\mathrm{T}}(\xi)M_my(\xi)\mathrm{d}\xi+\int_{-\tau}^{0}\mathrm{e}^{2\beta\xi}y^{\mathrm{T}}(\xi)Qy(\xi)\mathrm{d}\xi\\
&+\int_{-\tau}^{-d_1(0)}\int_{\theta}^{0}\mathrm{e}^{2\beta\xi}\dot{y}^{\mathrm{T}}(\xi)Z\dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta+\int_{-d_1(0)}^{-d_2(0)}\int_{\theta}^{0}\mathrm{e}^{2\beta\xi}\dot{y}^{\mathrm{T}}(\xi)Z\dot{y}(\xi)\mathrm{d}\xi\mathrm{d}\theta
\end{aligned}
$$

$$+\int_{-d_2(0)}^{-d_3(0)}\int_{\theta}^{0} e^{2\beta\xi}\dot{y}^{\mathrm{T}}(\xi)Z\dot{y}(\xi)\mathrm{d}\xi\,\mathrm{d}\theta+\cdots+\int_{-d_{m-2}(0)}^{-d_{m-1}(0)}\int_{\theta}^{0} e^{2\beta\xi}\dot{y}^{\mathrm{T}}(\xi)Z\dot{y}(\xi)\mathrm{d}\xi\,\mathrm{d}\theta$$
$$+\int_{-d_{m-1}(0)}^{0}\int_{\theta}^{0} e^{2\beta\xi}\dot{y}^{\mathrm{T}}(\xi)Z\dot{y}(\xi)\mathrm{d}\xi\,\mathrm{d}\theta+\int_{-\tau}^{0}\int_{\theta}^{0} e^{2\beta\xi}\dot{y}^{\mathrm{T}}(\xi)Z\dot{y}(\xi)\mathrm{d}\xi\,\mathrm{d}\theta.$$

Since

$$[y(0)-Cy(-\tau)]^{\mathrm{T}}P[y(0)-Cy(-\tau)]\le 2y^{\mathrm{T}}(0)Py(0)+2y^{\mathrm{T}}(-\tau)C^{\mathrm{T}}PCy(-\tau)$$

we can obtain that

$$\begin{aligned}V(y(0))\le&\left[2\lambda_{\mathrm{Max}}(P)+2\lambda_{\mathrm{Max}}(P)C^{\mathrm{T}}C+\lambda_{\mathrm{Max}}(M_{\mathrm{Max}})\int_{-\tau}^{0}e^{2\beta\xi}\mathrm{d}\xi+\lambda_{\mathrm{Max}}(Q)\int_{-\tau}^{0}e^{2\beta\xi}\mathrm{d}\xi\right]\|\phi\|_{\tau}^{2}\\&+2\left[\lambda_{\mathrm{Max}}(Z)\int_{-\tau}^{0}\int_{\theta}^{0}e^{2\beta\xi}\mathrm{d}\xi\,\mathrm{d}\theta\right]\|\phi^{*}\|_{\tau}^{2}\\=&\left[2\lambda_{\mathrm{Max}}(P)+2\lambda_{\mathrm{Max}}(P)C^{\mathrm{T}}C+\lambda_{\mathrm{Max}}(M_{\mathrm{Max}})\frac{1-e^{-2\beta\tau}}{2\beta}+\lambda_{\mathrm{Max}}(Q)\frac{1-e^{-2\beta\tau}}{2\beta}\right]\|\phi\|_{\tau}^{2}\\&+2\lambda_{\mathrm{Max}}(Z)\frac{2\beta\tau-1+e^{-2\beta\tau}}{4\beta^{2}}\|\phi^{*}\|_{\tau}^{2}\end{aligned}\tag{17}$$

and

$$e^{2\beta t}\lambda_{\mathrm{Min}}(P)\|y(t)-Cy(t-\tau)\|^{2}\le V(y(t)),\tag{18}$$

Then

$$\begin{aligned}\|y(t)-Cy(t-\tau)\|\le&\frac{1}{\sqrt{\lambda_{\mathrm{Min}}(P)}}\left[2\lambda_{\mathrm{Max}}(P)+2\lambda_{\mathrm{Max}}(P)C^{\mathrm{T}}C+\lambda_{\mathrm{Max}}(M_{\mathrm{Max}})\frac{1-e^{-2\beta\tau}}{2\beta}+\lambda_{\mathrm{Max}}(Q)\frac{1-e^{-2\beta\tau}}{2\beta}\right.\\&\left.+2\lambda_{\mathrm{Max}}(Z)\frac{2\beta\tau-1+e^{-2\beta\tau}}{4\beta^{2}}\right]^{\frac{1}{2}}\max\{\|\phi\|_{\tau},\|\phi^{*}\|_{\tau}\}e^{-\beta t}.\end{aligned}\tag{19}$$

Using Lemma 3, therefore, we immediately obtain

$$\|y(t)\|\le[\chi\|\phi\|_{\tau}+\theta+\frac{\chi}{\tau\varepsilon re}\theta]e^{-(1-\varepsilon)rt},\quad t\ge 0.\tag{20}$$

By Lemma 2, we can get

$$\begin{aligned}\theta=&\frac{1}{\sqrt{\lambda_{\mathrm{Min}}(P)}}\left[2\lambda_{\mathrm{Max}}(P)+2\lambda_{\mathrm{Max}}(P)C^{\mathrm{T}}C+\lambda_{\mathrm{Max}}(M_{\mathrm{Max}})\frac{1-e^{-2\beta\tau}}{2\beta}+\lambda_{\mathrm{Max}}(Q)\frac{1-e^{-2\beta\tau}}{2\beta}\right.\\&\left.+2\lambda_{\mathrm{Max}}(Z)\frac{2\beta\tau-1+e^{-2\beta\tau}}{4\beta^{2}}\right]^{\frac{1}{2}}\max\{\|\phi\|_{\tau},\|\phi^{*}\|_{\tau}\},\end{aligned}$$

and $r\triangleq\min\{\omega/h,\beta\}$, $\varepsilon\in(0,1)$, $\omega=-\ln(\eta)$, $\chi=\sqrt{\frac{\lambda_{\mathrm{Max}}(S)}{\lambda_{\mathrm{Min}}(S)}}$ which satisfies $C^{\mathrm{T}}SC-\eta^{2}S<0$, $\eta\in(0,1)$.

Moreover, inequality (20) implies the global exponential stability of system (1). The proof is thus completed. □

Remark 1. According to the results of Vladimir et al. [25], for $\varepsilon\in(0,1)$, there is the freedom to choose r smaller at the cost of slower decay rate, when ε is close to 1, or if ε is close to 0, a larger r and a faster decay rate is obtained.

Remark 2. In order to reduce conservatism, neither the model transformation approach nor any bounded technique on the cross term is involved. Actually, we can use N_i, $i=1,2,3,\ldots,m$ instead of M_i, $i=1,2,3,\ldots,m$ in (1′) which adds to Eq. (15), and the N_i, $i=1,2,3,\ldots,m$ are the traditional free matrices. The purpose of using the semi-fee weighting matrices is to reduce the number of the variational matrices and to simplify the system synthesis and consequently avoid significant increase in the computational demand which the free matrices will bring.

Remark 3. In Theorem 1, if we set $d_i(t)=\frac{\tau\times(m-i)}{m}$, $i=1,2,\ldots,m-1$, $u=0$, $h=\tau-d_1(t)=d_1(t)-d_2(t)=\cdots=d_{m-2}(t)-d_{m-1}(t)=d_{m-1}(t)=\frac{\tau}{m}$, the approach is similar to a simple generalized discretization scheme, then from Theorem 1 we have the following Theorem 2.

Theorem 2. *Suppose that the condition* (H) *is satisfied. Then, the delay neural network of neutral-type system* (1) *is globally exponentially stable if there exist positive-definite matrix S, positive scalar $\eta \in (0, 1)$, and symmetrical positive-definite matrices P, Q, Z, M_i, $i = 1, 2, 3, \ldots, m$, m to be determined, such that*

$$\Omega_3 = \begin{bmatrix} \Omega_4 & X_1^* & X_2^* & X_3^* & X_4^* & \cdots & X_{m+1}^* & X_{m+2}^* \\ (X_1^*)^{\mathrm{T}} & -2\tau Z & 0 & 0 & 0 & \cdots & 0 & 0 \\ (X_2^*)^{\mathrm{T}} & 0 & \mathrm{e}^{-2\beta\tau} - 1 & 0 & 0 & \cdots & 0 & 0 \\ (X_3^*)^{\mathrm{T}} & 0 & 0 & (\mathrm{e}^{-2\beta\tau} - \mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}})Z & 0 & \cdots & 0 & 0 \\ (X_4^*)^{\mathrm{T}} & 0 & 0 & 0 & (\mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}} - \mathrm{e}^{-2\beta\frac{\tau\times(m-2)}{m}})Z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (X_{m+1}^*)^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & (\mathrm{e}^{-2\beta\frac{2\tau}{m}} - \mathrm{e}^{-2\beta\frac{\tau}{m}})Z & 0 \\ (X_{m+2}^*)^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 & (\mathrm{e}^{-2\beta\frac{\tau}{m}} - 1)Z \end{bmatrix} < 0$$

$$C^{\mathrm{T}}SC - \eta^2 S < 0,$$

where

$$X_1^* = \begin{bmatrix} 2\tau Z \\ 0 \\ 0 \end{bmatrix}, \qquad X_2^* = \begin{bmatrix} (\mathrm{e}^0 - \mathrm{e}^{-2\beta\tau})Q \\ 0 \\ 0 \end{bmatrix}, \qquad X_3^* = \begin{bmatrix} (\mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}} - \mathrm{e}^{-2\beta\tau})M_1 \\ 0 \\ 0 \end{bmatrix},$$

$$X_4^* = \begin{bmatrix} (\mathrm{e}^{-2\beta\frac{\tau\times(m-2)}{m}} - \mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}})M_2 \\ 0 \\ 0 \end{bmatrix},$$

$$X_{m+1}^* = \begin{bmatrix} (\mathrm{e}^{-2\beta\frac{\tau}{m}} - \mathrm{e}^{-2\beta\frac{2\tau}{m}})M_{m-1} \\ 0 \\ 0 \end{bmatrix}, \qquad X_{m+2}^* = \begin{bmatrix} (1 - \mathrm{e}^{-2\beta\frac{\tau}{m}})M_m \\ 0 \\ 0 \end{bmatrix},$$

and

$$\Omega_4 = \begin{bmatrix} 2\beta P + 2P(-D + AL) + 3M_m - Q & \mathrm{e}^{-2\beta\frac{\tau}{m}}(M_{m-1} - M_m) & \mathrm{e}^{-2\beta\frac{2\tau}{m}}(M_{m-2} - M_{m-1}) & \mathrm{e}^{-2\beta\frac{3\tau}{m}}(M_{m-3} - M_{m-2}) & \cdots & \mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}}(M_1 - M_2) & \begin{matrix} -2\beta PC + C^{\mathrm{T}}P(D - AL) \\ +PBL + (Q - M_1)\mathrm{e}^{-2\beta\tau} \end{matrix} \\ \mathrm{e}^{-2\beta\frac{\tau}{m}}(M_{m-1}^{\mathrm{T}} - M_m^{\mathrm{T}}) & \mathrm{e}^{-2\beta\frac{\tau}{m}}(M_{m-1} - M_m) & 0 & 0 & \cdots & 0 & 0 \\ \mathrm{e}^{-2\beta\frac{2\tau}{m}}(M_{m-2}^{\mathrm{T}} - M_{m-1}^{\mathrm{T}}) & 0 & \mathrm{e}^{-2\beta\frac{2\tau}{m}}(M_{m-2} - M_{m-1}) & 0 & \cdots & 0 & 0 \\ \mathrm{e}^{-2\beta\frac{3\tau}{m}}(M_{m-3}^{\mathrm{T}} - M_{m-2}^{\mathrm{T}}) & 0 & 0 & \mathrm{e}^{-2\beta\frac{3\tau}{m}}(M_{m-3} - M_{m-2}) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}}(M_1^{\mathrm{T}} - M_2^{\mathrm{T}}) & 0 & 0 & 0 & \cdots & \mathrm{e}^{-2\beta\frac{\tau\times(m-1)}{m}}(M_1 - M_2) & 0 \\ \begin{matrix} -2\beta PC + (Q^{\mathrm{T}} - M_1^{\mathrm{T}})\mathrm{e}^{-2\beta\tau} \\ +(D^{\mathrm{T}} - LA^{\mathrm{T}})PC + LB^{\mathrm{T}}P \end{matrix} & 0 & 0 & 0 & \cdots & 0 & \begin{matrix} 2(\beta C^{\mathrm{T}}PC - C^{\mathrm{T}}PBL) \\ -M_1\mathrm{e}^{-2\beta\tau} - Q\mathrm{e}^{-2\beta\tau} \end{matrix} \end{bmatrix}.$$

Moreover,

$$\|y(t)\| \le \left[\chi\|\phi\|_h + \theta + \frac{\chi}{\tau\varepsilon re}\theta\right] \mathrm{e}^{-(1-\varepsilon)rt}, \quad t \ge 0,$$

where

$$\begin{aligned}
\theta &\triangleq \frac{1}{\sqrt{\lambda_{\mathrm{Min}}(P)}}\Bigg[2\lambda_{\mathrm{Max}}(P) + 2\lambda_{\mathrm{Max}}(P)C^{\mathrm{T}}C + \lambda_{\mathrm{Max}}(M_{\mathrm{Max}})\frac{1-\mathrm{e}^{-2\beta\tau}}{2\beta} + \lambda_{\mathrm{Max}}(Q)\frac{1-\mathrm{e}^{-2\beta\tau}}{2\beta} \\
&\quad + 2\lambda_{\mathrm{Max}}(Z)\frac{2\beta\tau - 1 + \mathrm{e}^{-2\beta\tau}}{4\beta^2}\Bigg]^{\frac{1}{2}} \max\{\|\phi|_\tau, \|\phi^*|_\tau\} \\
\chi &= \sqrt{\frac{\lambda_{\mathrm{Max}}(S)}{\lambda_{\mathrm{Min}}(S)}}, \qquad \omega = -\ln(\eta) \\
r &\triangleq \min\{\omega/\tau, \beta\}, \quad \varepsilon \in (0, 1) \\
\lambda_{\mathrm{Max}}&(M_{\mathrm{Max}}) = \max\{\lambda_{\mathrm{Max}}(M_1), \lambda_{\mathrm{Max}}(M_2), \lambda_{\mathrm{Max}}(M_3), \ldots, \lambda_{\mathrm{Max}}(M_m)\}.
\end{aligned}$$

This proof is similar to that of Theorem 1, and so is omitted here.

Remark 4. Gu's method, which divides the square region $[-\tau, 0] \times [-\tau, 0]$ into $2N^2$ triangular regions by a real number with a piecewise linear kernel, is real discretization. But, our method divides the delay interval $[-\tau, 0]$ by invariant function $d_i(t), i = 1, 2, 3, \ldots, m-1$, in Theorem 1 and $\frac{\tau\times(m-i)}{m}, i = 1, 2, 3, \ldots, m-1$ in Theorem 2. Our semi-free weighting matrices should satisfy the condition $0 < M_1 \le M_2 \le \cdots \le M_{m-1} \le M_m$ in Theorem 1 for $V_2(y(t)) \ge 0$. More importantly, Gu's method leads to more computational effort to obtain less conservative results. When N is larger, the maximum estimate delay for stability is larger and the less conservative results are obtained in [3]. However, while the m is larger, the more corresponding terms

$$-\int_{-\frac{\tau\times(m+1-i)}{m}}^{-\frac{\tau\times(m-i)}{m}} \mathrm{e}^{2\beta(t+\xi)}[\eta^{\mathrm{T}}(t)M_i + \dot{y}^{\mathrm{T}}(t+\xi)Z]Z^{-1}[\eta^{\mathrm{T}}(t)M_i + \dot{y}^{\mathrm{T}}(t+\xi)Z]^{\mathrm{T}}\mathrm{d}\xi,$$

in Eq. (15) are omitted which may bring more conservatism. We have to depend on experiments to find the right m so that we can get the better results.

4. Illustrative examples

In this section, we illustrate the correctness of our results and also compare them with those in Refs. [10,11,29].

Example 1. Consider the following delayed neural network

$$\begin{aligned}
&\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + C\dot{x}(t-\tau) + J \\
&x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0],
\end{aligned} \tag{21}$$

where the activation functions with the Lipschitz coefficient matrix $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$$D = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad A = \begin{bmatrix} \alpha & 0.1 \\ 0.1 & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.16 \\ 0.05 & 0.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad J = \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \quad \tau = 1.$$

The problem is to determine the maximum allowable bound of α for guaranteeing the stability of system (21). This example was considered in [10], where the authors illustrated that the maximum bound of α is 1.198. However, by applying Theorem 1 to system (21), one can see that our criterion is feasible for any α. The obtained results are significantly better than that given in [10].

When $\tau = 1$, $m = 2$, $u = 0.5$, $\alpha = 10\,000$, the corresponding feasible solution of Theorem 1 is

$$P = 10^{-4} \times \begin{bmatrix} 0.1720 & -0.0134 \\ -0.0134 & 0.1714 \end{bmatrix}, \quad Q = 10^{8} \times \begin{bmatrix} 3.3222 & -0.5097 \\ -0.5097 & 3.3201 \end{bmatrix},$$
$$M_1 = 10^{7} \times \begin{bmatrix} 1.2852 & -0.1159 \\ -0.1159 & 3.3201 \end{bmatrix}, \quad M_2 = 10^{7} \times \begin{bmatrix} 2.6590 & -0.1791 \\ -0.1791 & 2.6583 \end{bmatrix},$$
$$Z = 10^{8} \times \begin{bmatrix} 5.8382 & -0.4582 \\ -0.4582 & 5.8365 \end{bmatrix}, \quad S = \begin{bmatrix} 56.2494 & 0 \\ 0 & 56.2494 \end{bmatrix}$$

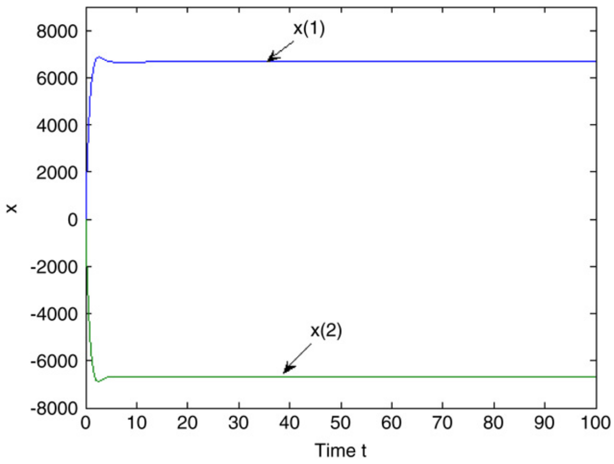

Fig. 1. The solution trajectory of system (21).

Table 1
Maximum estimate delay for different u

$u = 0$	$u = 0.1$	$u = 0.2$	$u = 0.3$	$u = 0.4$	$u = 0.5$
$\tau = 91$	$\tau = 89$	$\tau = 90$	$\tau = 90$	$\tau = 89$	$\tau = 90$
$u = 0.6$	$u = 0.7$	$u = 0.8$	$u = 0.9$	$u \geq 1$	
$\tau = 89$	$\tau = 89$	$\tau = 89$	$\tau = 91$	Infeasible	

with $\eta = 0.8660$ and the exponentially decay rate $r = 0.1438$.

By choosing $x(\theta) = [3,\ -2]^{\mathrm{T}}$, system (21) with $\alpha = 10^4$, $\tau = 1$ is globally asymptotically stable, as shown in Fig. 1.

For $\alpha = 100$, $m = 2$, the upper bounds of the time delay from Theorem 1 are listed in Table 1. The variable u in Table 1 satisfies $\max\left\{|\dot{d}_1(t)|, |\dot{d}_2(t)|, \ldots, |\dot{d}_{m-1}(t)|\right\} \leq u$.

Example 2. Consider the following delayed neural network

$$\begin{aligned} &\dot{x}(t) = -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) + C\dot{x}(t-\tau) + J \\ &x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0] \end{aligned} \tag{22}$$

where

$$D = \begin{bmatrix} 2.7644 & 0 & 0 \\ 0 & 1.0185 & 0 \\ 0 & 0 & 10.2716 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.2651 & -3.1608 & -2.0491 \\ 3.1859 & -0.1573 & -2.4687 \\ 2.0368 & -1.3633 & 0.5776 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.7727 & -0.8370 & 3.8019 \\ 0.1004 & 0.6677 & -2.4431 \\ -0.6622 & 1.3109 & -1.8407 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.2076 & 0.0631 & 0.3915 \\ -0.0780 & 0.3106 & 0.1009 \\ -0.2763 & 0.1416 & 0.3729 \end{bmatrix},$$

$$J = \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$$

with the Lipschitz coefficient matrix

$$L = \begin{bmatrix} 0.1019 & 0 & 0 \\ 0 & 0.3419 & 0 \\ 0 & 0 & 0.0633 \end{bmatrix}.$$

This example was considered in [11] where the authors illustrated that the maximum bound of delay is $\tau = 1.3044$. Compared with this the upper bound of delay is $\tau = 6.5$ in [29]. Our result is listed in Table 2.

From Table 2, we can see that the $m = 3$ in Theorem 2 is the better choice, and we use three semi-free weighting matrices and other similar matrices P, Z, S. Compared with the free matrices method in [10,29], our method uses fewer matrix variables and gives the better results. By choosing $x(\theta) = [5, -2, -1]^{\mathrm{T}}$, system (22) with $\tau = 82$ is exponentially stable, as shown in Fig. 2.

Table 2
Maximum estimate delay for different m

τ	$m = 2$	$m = 3$	$m = 4$	$m = 5$
Theorem 1 with $u = 0.5$	$\tau = 65$	$\tau = 65$	$\tau = 69$	$\tau = 70$
Theorem 2	$\tau = 77$	$\tau = 82$	$\tau = 81$	$\tau = 81$

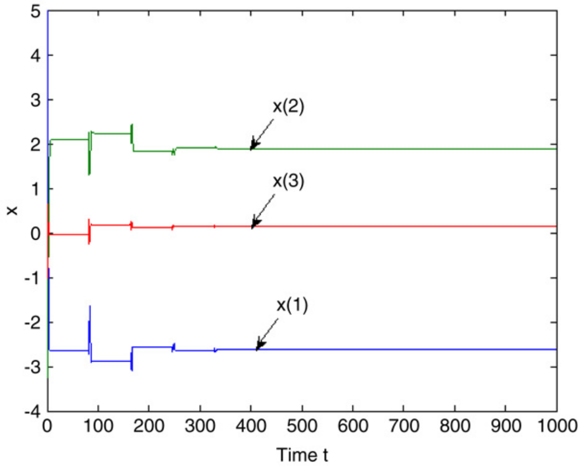

Fig. 2. The solution trajectory of system (22).

5. Conclusions

In this paper, we have investigated the global exponential stability of the DNNs of neutral type by combining several techniques such as the semi-free weighting matrices approach, the Lyapunov–Krasovskii functional, "Linearization" and the linear matrix inequality. The advantages of this method include the use of fewer variable matrices in the construction of the Lyapunov functional. At the same time, the semi-free weighting matrices approach is applied to the Lyapunov functional. Comparing the recent results and illustrated examples shows that the proposed results are less conservative and more convenient.

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