Exponential Dichotomies and Transversal Homoclinic Points

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1. INTRODUCTION

Let $F$ be a diffeomorphism on a two-dimensional manifold. Smale [24] shows that if $F$ has a transversal homoclinic point there is a Cantor-like set near it on which some iterate of $F$ is invariant and isomorphic to the Bernoulli shift on a finite number of symbols. Examples arise as period maps of periodic systems of differential equations. Melnikov [17], Chow, Hale and Mallet-Paret [5] and Holmes [13, 14] have considered periodic perturbations of two-dimensional autonomous systems of the form

$$\dot{z} = g(z) + \mu h(t, z, \mu) ,$$

(1)

where the autonomous system has a saddle point with a homoclinic orbit through it and show, with suitable conditions on $h$, that for $\mu$ sufficiently small the period map for the perturbed system has a transversal homoclinic point.

In Section 2 of this paper, it is shown that $y_0$ is a transversal homoclinic point of the period map corresponding to an $n$-dimensional periodic system of differential equations

$$\dot{x} = f(t, x)$$

(2)

if and only if this system has a periodic solution $x(t)$ such that

$$| y(t) - x(t) | \to 0 \quad \text{as} \quad |t| \to \infty$$

(here $y(t)$ is the solution of (2) with $y(0) = y_0$) and such that the variational equation

$$\dot{x} = f_x(t, y(t)) x$$

has an exponential dichotomy (Coppel [10]) on $(-\infty, \infty)$. 

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In Section 3 a classical theorem (Coppel [9]) on the existence of bounded solutions of quasi-linear systems with linear part having an exponential dichotomy is used to prove the existence of a solution of a nonautonomous system near a piecewise continuous vector function with small jumps which satisfies the differential equation between its points of discontinuity. This result is analogous to the so-called “shadow lemma” for diffeomorphisms with a hyperbolic invariant set (Anosov [11], Bowen [4], Conley [7], Franke and Selgrade [11], Robinson [22]). From it we deduce a theorem for periodic systems, which we apply to show that if the period map of a periodic system has a transversal homoclinic point, then there is for each positive integer $N$ a Cantor-like subset of $\mathbb{R}^n$ on which some iterate of the period map is invariant and isomorphic to the Bernoulli shift on $N$ symbols. This result has usually been proved by the horseshoe construction.

In Section 4 the work of Melnikov and others mentioned earlier is generalized using an approach suggested by the article [5] of Chow, Hale and Mallet-Paret. We consider an $n$-dimensional system of the form (1), where the autonomous system

$$\dot{z} = g(z)$$

is assumed to have a bounded solution $\zeta(t)$ such that the corresponding variational equation has an exponential dichotomy on both half-lines but not on $(-\infty, \infty)$ (for autonomous systems, this is necessarily so). Using an extension of the implicit function theorem in Banach spaces suggested by a theorem of Coppel's [8], we show for $\mu \neq 0$ sufficiently small and under certain conditions on $h$ that the perturbed system (1) has a bounded solution near $\zeta(t)$ such that the corresponding variational equation does have an exponential dichotomy on $(-\infty, \infty)$. The results of Melnikov and Holmes are shown to be special cases of this.

Finally in Section 5 the sinusoidally forced pendulum equation, as studied by Kirchgraber [15], is examined and his results are shown to follow from ours.

This paper was stimulated by the reading of [15] and the author would like to thank Dr. Kirchgraber for sending him a copy of that preprint.

2. Characterization of Transversal Homoclinic Points in Terms of Exponential Dichotomy

Let $A(t)$ be a real $n \times n$ matrix function, piecewise continuous on an interval $\mathcal{S}$. Denote by $X(t)$ a fundamental matrix for the system

$$\dot{x} = A(t)x.$$ (3)
This system is said to have an exponential dichotomy on the interval $\mathcal{I}$ if there is a projection $P$ and constants $K \geq 1$, $a > 0$ such that

$$|X(t)PX^{-1}(s)| \leq Ke^{-\alpha(t-s)} \quad \text{for} \quad s \leq t,$$

$$|X(t)(I-P)X^{-1}(s)| \leq Ke^{-\alpha(s-t)} \quad \text{for} \quad s \geq t. \quad (4)$$

We use the notation $\mathcal{P}(t)$ for the projection matrix function $X(t)PX^{-1}(t)$. Note that for all $t$ and $s$

$$X(t)PX^{-1}(s) = X(t)X^{-1}(s)\mathcal{P}(s) = \mathcal{P}(t)X(t)X^{-1}(s), \quad (5)$$

$$X(t)(I-P)X^{-1}(s) = X(t)X^{-1}(s)(I-\mathcal{P}(s)) = (I-\mathcal{P}(t))X(t)X^{-1}(s) \quad (6)$$

and that $\mathcal{P}(t)$ is a solution of the matrix system

$$\dot{X} = A(t)X - XA(t) \quad (7)$$

so that it is determined by its value at one point. When $\mathcal{I} = [0, \infty)$ the range of $\mathcal{P}(0)$ must be the stable subspace

$$\{\xi \in \mathbb{R}^n : \sup_{t \geq 0} |X(t)X^{-1}(0)\xi| < \infty\}$$

but the kernel may be any complementary subspace; when $\mathcal{I} = (-\infty, 0]$ the kernel of $\mathcal{P}(0)$ must be the unstable subspace

$$\{\xi \in \mathbb{R}^n : \sup_{t < 0} |X(t)X^{-1}(0)\xi| < \infty\}$$

but the range may be any complementary subspace. So when $\mathcal{I} = (-\infty, \infty), \mathcal{P}(t)$ is uniquely determined.

In [10, p. 19] Coppel shows the following result.

**Proposition 2.1.** Let $A(t)$ be an $n \times n$ matrix function, defined and continuous on $(-\infty, \infty)$. Then system (3) has an exponential dichotomy on $(-\infty, \infty)$ if and only if it has an exponential dichotomy on both $[0, \infty)$ and $(-\infty, 0]$, and $\mathbb{R}^n$ is the direct sum of the stable and unstable subspaces.

When $A(t) = A$ is constant, system (3) has an exponential dichotomy on an infinite interval if and only if the eigenvalues of $A$ have nonzero real parts and when $A(t)$ is periodic, system (3) has an exponential dichotomy on an infinite interval if and only if the Floquet multipliers lie off the unit circle. The dimensions of the stable subspaces are respectively the number of eigenvalues with negative real parts and the number of multipliers inside the unit circle.
Now let \( O \subset \mathbb{R}^n \) be open and \( f: \mathbb{R} \times O \to \mathbb{R}^n \) a continuous function with period \( T \) in \( t \) and continuous partial derivative \( f_x(t, x) \). Denote by \( \Phi(t, \xi) \) the unique solution of the differential equation

\[
\dot{x} = f(t, x)
\]

with \( \Phi(0, \xi) = \xi \). The set \( \tilde{O} \) of \( \xi \) for which the maximal interval of existence of \( \Phi(t, \xi) \) contains \([0, T]\) is an open subset (of \( O \)) which we assume to be nonempty. Then the mapping \( F \) of \( \tilde{O} \) into \( \mathbb{R}^n \) defined by

\[
F(\xi) = \Phi(T, \xi)
\]

is a \( C^1 \)-diffeomorphism called the **period map** and it is well known that a point \( \xi_0 \) is a fixed point of \( F \) if and only if \( \Phi(t, \xi_0) \) is defined for all \( t \) and has period \( T \). In this case \( F'(\xi_0) = \Phi_t(T, \xi_0) \) is the monodromy matrix for the variational equation

\[
\dot{x} = f_x(t, \Phi(t, \xi_0))x,
\]

so that the eigenvalues of \( F'(\xi_0) \) are just the Floquet multipliers for (9). It follows that \( \xi_0 \) is a hyperbolic fixed point (that is, the eigenvalues of \( F'(\xi_0) \) lie off the unit circle) if and only if the variational equation (9) has an exponential dichotomy on \((-\infty, \infty)\).

If \( \xi_0 \) is a hyperbolic fixed point of \( F \), the stable manifold \( W^s(\xi_0) \) is defined to be the set of those \( y \) such that \( F^m(y) \to \xi_0 \) as \( m \to \infty \). It is known that \( W^s(\xi_0) \) is an immersed submanifold of dimension \( k \), where \( k \) is the number of eigenvalues of \( F'(\xi_0) \) inside the unit circle; so \( k \) is also the dimension of the stable subspace for (9). Moreover, if we denote by \( V \) the tangent space to \( W^s(\xi_0) \) at \( y \), then the operator norm of the restriction of \( (F^m)'(\xi_0) \) to \( V \) approaches zero exponentially as \( m \to \infty \).

Since \( f \) has period \( T \) in \( t \) the identity

\[
\Phi(t + mT, \xi) = \Phi(t, \Phi(mT, \xi))
\]

holds when both sides are defined. Let \( y \in W^s(\xi_0) \). Then it follows that \( F^m(y) = \Phi(mT, y) \) and for \( mT \leq t < (m + 1)T \)

\[
\Phi(t, y) = \Phi(t - mT, F^m(y)).
\]

Using the fact that

\[
\sup_{0 \leq t \leq T} |\Phi(t, F^m(y)) - \Phi(t, \xi_0)| \to 0 \quad \text{as} \quad m \to \infty,
\]

we deduce that \( |\Phi(t, y) - \Phi(t, \xi_0)| \to 0 \) as \( t \to \infty \). This also means that for large \( t \) the coefficient matrix in the variational equation

\[
\dot{x} = f_x(t, \Phi(t, y))x
\]
is uniformly close to the coefficient matrix of (9) and so it follows from the roughness theorem for exponential dichotomies (Coppel [10, p. 34]) that (11) has an exponential dichotomy on $[t_0, \infty)$ for large $t_0$ and hence on $[0, \infty)$ ([10, p. 13]); moreover, the stable subspace has dimension $k$, the same as for (9).

Now if $\eta \in V$, the tangent space to $W^s(\xi_0)$ at $y$, we already know that

$$(F^m)'(y)\eta \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.$$  

Then using (10) we have for $mT \leq t < (m + 1)T$

$$\Phi_t(t, y)\eta = \Phi_t(t - mT, F^m(y))(F^m)'(y)\eta.$$  

Since $|\Phi_t(t, F^m(y))|$ is uniformly bounded with respect to $t$ in $[0, T]$ and $m$, it follows that

$$\Phi_t(t, y)\eta \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$  

so that $\eta$ is in the stable subspace for (11) (observe that $\Phi_t(t, y)$ is the fundamental matrix for (11) with $\Phi_t(0, y) = I$). So $V$ is a subspace of the stable subspace for (11). But both spaces have the same dimension $k$ and so they must coincide.

We need one more definition. A point $y$ is said to be a transversal homoclinic point with respect to the hyperbolic fixed point $\xi_0$ if $y \in W^u(\xi_0) \cap W^s(\xi_0)$ (here $W^u(\xi_0)$ is the unstable manifold, those $y$ such that $F^m(y) \rightarrow \xi_0$ as $m \rightarrow -\infty$) and $\mathbb{R}^n$ is the direct sum of the tangent spaces to $W^s(\xi_0)$ and $W^u(\xi_0)$ at $y$.

We have just about proved the main proposition of this section.

**Proposition 2.2.** Let $O \subset \mathbb{R}^n$ be open and let $f: \mathbb{R} \times O \rightarrow \mathbb{R}^n$ be a continuous function with period $T$ in $t$ and with continuous partial derivative $f_\xi(t, x)$. Denote by $\Phi(t, \xi)$ the unique solution of (8) satisfying $\Phi(0, \xi) = \xi$.

(i) $\xi_0 \in O$ is a hyperbolic fixed point of the period map if and only if $\Phi(t, \xi_0)$ is a $T$-periodic solution of (8) and the variational equation (9) has an exponential dichotomy on $(-\infty, \infty)$. $y \in W^s(\xi_0)$ (resp. $W^u(\xi_0)$) if and only if $|\Phi(t, y) - \Phi(t, \xi_0)| \rightarrow 0$ as $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$) and, in this case, the variational equation (11) has an exponential dichotomy on $[0, \infty)$ (resp. $(-\infty, 0]$) with the tangent space to $W^s(\xi_0)$ (resp. $W^u(\xi_0)$) at $y$ as stable (resp. unstable) subspace.

(ii) $y \in O$ is a transversal homoclinic point with respect to a hyperbolic fixed point $\xi_0$ if and only if $|\Phi(t, y) - \Phi(t, \xi_0)| \rightarrow 0$ as $|t| \rightarrow \infty$ and the variational equation (11) has an exponential dichotomy on $(-\infty, \infty)$.

**Proof.** The statements in (i) have already been proved, apart from those for $W^u(\xi_0)$, which are proved analogously to those for $W^s(\xi_0)$. Statement (ii) follows from (i) and Proposition 2.1.
3. A Shadow Lemma for Nonautonomous Systems

To prove the main theorem in this section we need a new result about exponential dichotomies (Lemma 3.2) and to prove it we need the following lemma.

**Lemma 3.1.** For each integer \( k \) let \( A_k(t) \) be a bounded, continuous \( n \times n \) matrix function such that the system

\[
\dot{x} = A_k(t)x
\]

has an exponential dichotomy on an interval \([t_{k-1}, t_k]\) with constants \( K \geq 1, \alpha > 0 \) and projection matrix function \( \mathcal{P}_k(t) \). Then if for all \( k \)

(i) \( t_k - t_{k-1} \geq 2\alpha^{-1} \log K \),
(ii) \( \mathcal{P}_k(t_{k-1}) = \mathcal{P}_{k-1}(t_{k-1}) \),

system (3), where

\[
A(t) = A_k(t) \quad \text{for} \quad t_{k-1} \leq t < t_k,
\]

has an exponential dichotomy on \((-\infty, \infty)\) with constants \( K', \alpha/2 \).

**Proof.** Let \( X(t) \) be the fundamental matrix for (3) with \( X(0) = I \). Define the continuous function \( \mathcal{P}(t) \) on \((-\infty, \infty)\) by taking it as \( \mathcal{P}_k(t) \) on \([t_{k-1}, t_k]\). Then \( \mathcal{P}(t) \) is a solution of the matrix system (7) and so by uniqueness

\[
\mathcal{P}(t) = X(t)PX^{-1}(t) \quad \text{for all} \quad t,
\]

where \( P = \mathcal{P}(0) \).

Suppose \( s \leq t \). There exist integers \( k \leq l \) such that

\[
l_{k-1} \leq s < t_k, \quad t_{l-1} \leq t < t_l.
\]

Then

\[
|X(t)PX^{-1}(s)| \leq |X(t)PX^{-1}(t_{l-1})| |X(t_{l-1})PX^{-1}(t_{l-2})| \ldots |X(t_{k+1})PX^{-1}(t_k)| |X(t_k)PX^{-1}(s)|
\]

\[
= |X(t)X^{-1}(t_{l-1}) \mathcal{P}(t_{l-1})| |X(t_{l-1})X^{-1}(t_{l-2}) \mathcal{P}_{l-1}(t_{l-2})| \ldots |X(t_{k+1})X^{-1}(t_k) \mathcal{P}_k(t_k)| |X(t_k)X^{-1}(s)| \mathcal{P}(s)|
\]

\[
\leq Ke^{\alpha(t_{l-1} - t_{l-2})} Ke^{\alpha(t_{l-2} - t_{l-3})} \ldots Ke^{\alpha(t_{k+1} - t_k)} Ke^{\alpha(t_k - s)} = Ke^{-\alpha(t-s)} = K^2 e^{-\alpha(t-s)/2}
\]

since \( t - s \geq t_{l-1} - t_k \geq (l - k - 1) 2\alpha^{-1} \log K \).
Similarly we show that for $s > t$
\[ |X(t)(I - P) X^{-1}(s)| \leq K^2 e^{-\alpha(s-t)/2}. \]

**Lemma 3.2.** Let the same conditions hold as in Lemma 3.1 except that we require
\[ |A_k(t)| \leq N \quad \text{for all } t \text{ and } k \]
and (i), (ii) are replaced by
\[ (i)' \quad t_k - t_{k-1} \geq 2\alpha^{-1} \log 3K, \quad (ii)' \quad |\varphi_k(t_{k-1}) - \varphi_{k-1}(t_{k-1})| \leq \delta. \]

Then if
\[ \delta < \min\{1/4K, \alpha/2592K^3[2N + \alpha(2 \log 3K)^{-1}]\}, \]
system (3), defined as in Lemma 3.1, has an exponential dichotomy on $(\infty, \infty)$ with constants depending only on $K$ and $\alpha$.

**Proof.** Write $P_k = \varphi_k(t_{k-1})$, $Q_k = \varphi_{k-1}(t_{k-1})$ and put
\[ J_k = P_k Q_k + (I - P_k)(I - Q_k). \quad (13) \]
Then using $P_k^2 = P_k$,
\[ |J_k - I| = |P_k(Q_k - P_k) + (I - P_k)(P_k - Q_k)| \leq (|P_k| + |I - P_k|)|P_k - Q_k| \leq 2K\delta < 1/2. \]

So $J_k$ is invertible and
\[ |J_k^{-1}| = |(I - (I - J_k))^{-1}| \leq \sum_{l=0}^{\infty} |I - J_k|^l \leq (1 - 2K\delta)^{-1}. \]

Moreover, it follows from (13) that $P_k J_k = J_k Q_k$ so that
\[ P_k = J_k Q_k J_k^{-1}. \quad (14) \]

Now write
\[ S_k(t) = I + (t_k - t_{k-1})^{-1}(t - t_{k-1})(J_{k+1} - I) \text{ for } t_{k-1} \leq t \leq t_k. \]
Then
\[ |S_k(t) - I| \leq |J_{k+1} - I| \leq 2K\delta < 1/2. \]
So \( S^{-1}_k(t) \) exists and
\[
|S^{-1}_k(t)| \leq (1 - 2K\delta)^{-1} < 2.
\]

Also
\[
|\dot{S}_k(t)| = (t_k - t_{k-1})^{-1} |J_{k+1} - I| \leq 2(2 \log 3K)^{-1}aK\delta.
\]

For each integer \( k \) let \( X_k(t) \) be the fundamental matrix for system (12) satisfying \( X_k(t_{k-1}) = I \) so that
\[
\mathcal{P}_k(t) = X_k(t) P_k X^{-1}_k(t).
\]

Then \( Y_k(t) = S_k(t) X_k(t) \) is a fundamental matrix for the system
\[
y = B_k(t)y,
\]
where
\[
B_k(t) = \dot{Y}_k(t) Y^{-1}_k(t) = S_k(t) A_k(t) S^{-1}_k(t) + \dot{S}_k(t) S^{-1}_k(t).
\]

So if \( t_{k-1} \leq t \leq t_k \),
\[
|B_k(t) - A_k(t)| \leq |S_k(t) - I||A_k(t)||S^{-1}_k(t)|
+ |A_k(t)||S^{-1}_k(t) - I| + |\dot{S}_k(t)||S^{-1}_k(t)|
\leq 4K\{2N + a(2 \log 3K)^{-1}\} \delta. \tag{16}
\]

Now note that system (15) has an exponential dichotomy on \([t_{k-1}, t_k]\) since if \( t_{k-1} \leq s < t < t_k \)
\[
|Y_k(t) P_k Y^{-1}_k(s)| \leq |S_k(t)||X_k(t) P_k X^{-1}_k(s)||S^{-1}_k(s)|
\leq 3Ke^{-a(t-s)}
\]
and similarly, if \( t_{k-1} \leq t < s \leq t_k \),
\[
|Y_k(t)(I - P_k) Y^{-1}_k(s)| \leq 3Ke^{-a(s-t)}.
\]

The corresponding projection matrix function is
\[
\mathcal{Z}_k(t) = Y_k(t) P_k Y^{-1}_k(t).
\]

Using (14) we see that for all integers \( k \)
\[
\mathcal{Z}_{k-1}(t_{k-1}) = S_{k-1}(t_{k-1}) \mathcal{P}_{k-1}(t_{k-1}) S^{-1}_{k-1}(t_{k-1})
= J_k Q_k J^{-1}_k = P_k = \mathcal{Z}_k(t_{k-1}).
\]
Hence if we define $B(t)$ on $(-\infty, \infty)$ by taking it as $B_k(t)$ on $[t_{k-1}, t_k)$ it follows from Lemma 3.1 that the system

$$j' = B(t)j$$

has an exponential dichotomy on $(-\infty, \infty)$ with constants $9K^2$ and $\alpha/2$.

But it follows from (16) and the assumptions on $\delta$ that

$$\sup_t |A(t) - B(t)| < \alpha/648K^4.$$ 

So the roughness theorem ([10, p. 34]) implies that (3) also has an exponential dichotomy on $(-\infty, \infty)$ with constants depending only on $K$ and $\alpha$. (Note that the roughness theorem still holds with the same proof even when the coefficient matrices are only piecewise continuous).

It is noted here that the method used in this proof was suggested by Millionščikov's "rotation" method [18, 19].

**Theorem 3.3.** Let $f(t, x)$ be a vector function, defined and continuous in $\mathbb{R} \times O$, where $O$ is an open subset of $\mathbb{R}^n$. Suppose also that the partial derivative $f_x(t, x)$ exists, is bounded, continuous in $t$ for each fixed $x$ and in $x$ uniformly with respect to $t$ and $x$.

Suppose for each integer $k$ the system

$$\dot{x} = f(t, x)$$

has a solution $w_k(t)$ defined on an interval $[t_{k-1}, t_k]$ such that

(i) the variational equation

$$\dot{x} = f_x(t, w_k(t))x$$

has an exponential dichotomy on $[t_{k-1}, t_k]$ with projection matrix function $P_k(t)$ and constants $K, \alpha$;

(ii) $|w_{k-1}(t_{k-1}) - w_k(t_{k-1})| < \delta$;

(iii) $|P_{k-1}(t_{k-1}) - P_k(t_{k-1})| < \delta$;

(iv) $t_k - t_{k-1} > \tau$.

Then there exist positive constants $\varepsilon_0, \tau_0$ and a function $\delta_0(\varepsilon)$ such that if $\tau \geq \tau_0$, $0 < \varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0(\varepsilon)$, system (17) has a unique solution $x(t)$ satisfying

$$|x(t) - w_k(t)| \leq \varepsilon \quad \text{for } t_{k-1} \leq t \leq t_k \text{ and all } k.$$ 

**Proof.** Define

$$\omega(\Delta) = \sup\{|f_x(t, x) - f_x(t, y)|: -\infty < t < \infty, |x - y| \leq \Delta\}.$$
By hypothesis \( \omega(\Delta) \to 0 \) as \( \Delta \to 0 \). Now we take

\[
\tau_0 = \max\{1, 2\alpha^{-1} \log 3K\}
\]

and we choose \( \varepsilon_0 > 0 \) so that

\[
4K_1 \alpha_1^{-1} \omega(2\varepsilon_0) \leq 1,
\]

where \( K_1, \alpha_1 \) are constants depending only on \( K, \alpha \) and will be defined later. Then we choose \( \delta_0 = \delta_0(\varepsilon) \) to satisfy

\[
2K\delta_0 < 1, \quad 2592K^4[2N + \omega(2 \log 3K)^{-1}] \delta_0 < \alpha, \quad 4K_1^2 \omega(\delta_0) < \alpha_1, \quad 8K_2 \alpha_2^{-1}(N + 1) \delta_0 \leq \varepsilon, \quad 2\delta_0 \leq \varepsilon,
\]

where

\[
N = \sup |f_\lambda(t, x)|
\]

and \( K_2, \alpha_2 \) are constants like \( K_1, \alpha_1 \).

Now we assume \( \tau \geq \tau_0, 0 < \varepsilon \leq \varepsilon_0, \delta \leq \delta_0(\varepsilon) \) and define a function \( w(t) \) on \( (-\infty, \infty) \) by taking \( w(t) = w_k(t) \) when \( t_{k-1} \leq t < t_k \). Then \( w(t) \) is continuous except at the points \( t_k \), where it has the jumps

\[
\beta_k = w_{k+1}(t_k) - w_k(t_k).
\]

It follows from our assumptions and Lemma 3.2 that the system

\[
\dot{x} = f_x(t, w(t))x
\]

has an exponential dichotomy on \( (-\infty, \infty) \) with constants \( K_1, \alpha_1 \) depending only on \( K, \alpha \).

We approximate \( w(t) \) by a continuous function \( z(t) \) defined on \( (-\infty, \infty) \) by

\[
z(t) = \begin{cases} 
    w(t) + (t_k - t_{k-1})^{-1}(t - s_k) \beta_{k-1} & \text{if } t_{k-1} \leq t \leq s_k \\
    w(t) + (t_k - t_{k-1})^{-1}(t - s_k)\beta_k & \text{if } s_k \leq t \leq t_k,
\end{cases}
\]

when \( s_k = (t_{k-1} + t_k)/2 \). \( z(t) \) is continuous with a piecewise continuous derivative such that

\[
|z(t) - w(t)| \leq \delta, \quad |z(t) - \dot{w}(t)| \leq \tau^{-1} \delta \leq \delta,
\]

the latter inequality holding except for \( t = t_k \). Now for all \( t \)

\[
|f_x(t, z(t)) - f_x(t, w(t))| \leq \omega(\delta).
\]
Under our assumptions it follows from the roughness theorem for exponential dichotomies ([10, p. 34]) that the system

\[ \dot{x} = f_z(t, z(t))x \]

has an exponential dichotomy on \((-\infty, \infty)\) with constants \(K_2, \alpha_2\) depending only on \(K_1, \alpha_1\).

We introduce the new variable \(v\) through \(x = z(t) + v\) so that (17) becomes

\[ \dot{v} = f_z(t, z(t))v + g(t, v), \]  

where

\[ g(t, v) = \{f(t, z(t) - \hat{z}(t)) + [f(t, z(t) + v) - f(t, z(t)) - f_z(t, z(t))v]\}. \]

Except at the points \(t = t_k\),

\[ |g(t, 0)| \leq |f(t, z(t)) - f(t, w(t))| + |\dot{w}(t) - \dot{z}(t)| \leq (N + 1)\delta \]

and for all \(t\) and \(v\)

\[ |g_v(t, v)| = |f_z(t, z(t) + v) - f_z(t, z(t))| \leq \omega(|v|). \]

With the assumption we have made the conditions of Theorem 4 in Coppel [9, p. 137] are satisfied with \(\rho = 2\varepsilon\), \(K = 2K_2\alpha_2^{-1}\) and \(\gamma = \alpha_2/4K_2\). So (18) has a unique solution \(v(t)\) satisfying \(|v(t)| \leq 2\varepsilon\) for all \(t\). Moreover for all \(t\)

\[ |v(t)| \leq 4K_2\alpha_2^{-1} \text{ ess sup } |g(t, 0)|. \]

Then \(x(t) = z(t) + v(t)\) is a solution of (17) such that for all \(t\)

\[ |x(t) - w(t)| \leq |v(t)| + |z(t) - w(t)| \leq 4K_2\alpha_2^{-1}(N + 1)\delta + \delta \leq \varepsilon. \]

If \(\bar{x}(t)\) is another solution of (17) with the same property, \(\bar{v}(t) = \bar{x}(t) - z(t)\) is a solution of (18) satisfying

\[ |\bar{v}(t)| \leq |\bar{x}(t) - w(t)| + |w(t) - z(t)| \leq \varepsilon + \delta \leq \varepsilon + \varepsilon/2 < 2\varepsilon. \]

Thus \(\bar{v}(t) = v(t)\) and hence \(\bar{x}(t) = x(t)\). So we get uniqueness and the proof of the theorem is complete.

**Remark.** Note that Theorem 4 in [9, p. 137] can still be applied even when, as above, the vector field is only piecewise continuous in \(t\). The
function $x(t) = z(t) + v(t)$ is continuous and satisfies the differential equation (17) except perhaps at the $t_k$. But since $f(t, x)$ is continuous, it must also satisfy the equation at the $t_k$ too.

Our next theorem is a corollary of the last one but in order to prove it we need another lemma, which does not seem to have been noticed previously.

**Lemma 3.4.** Let $A(t)$ and $B(t)$ be $n \times n$ matrix functions, bounded and continuous on $[t_0, \infty)$, such that system (3) has an exponential dichotomy on $[t_0, \infty)$ with projection matrix function $P(t)$ and such that $B(t) \to 0$ as $t \to \infty$. Then the perturbed system

$$\dot{x} = [A(t) + B(t)]x \quad (19)$$

also has an exponential dichotomy on $[t_0, \infty)$ and if $P(t)$ is a corresponding projection matrix function

$$|P(t) - P(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

**Proof:** Let $K, \alpha$ be the constant involved in the exponential dichotomy for (3). There exists $\tau_0$ such that

$$|B(t)| < \alpha/4K^2$$

for $t \geq \tau_0$. For any $\tau \geq \tau_0$ it follows from the roughness theorem [10, p. 34] that (19) has an exponential dichotomy on $[\tau, \infty)$ with constants $K_1, \alpha_1$, depending only on $K, \alpha$ and the associated projection matrix function $P_\tau(t)$ can be so chosen that for $t \geq \tau$

$$|P_\tau(t) - P(t)| \leq 4\alpha^{-1}K^3 \delta(\tau),$$

where

$$\delta(\tau) = \sup_{t \geq \tau} |B(t)|.$$

It follows from [10, p. 13] that (19) has an exponential dichotomy on $[t_0, \infty)$ since it has one on $[\tau_0, \infty)$. Let $P(t)$ be any allowable projection matrix function with associated constants $L, \beta$. $P(t)$ and $P_\tau(t)$ must have the same range for all $t$ so that

$$P(t) P_\tau(t) = P_\tau(t) P(t),$$

Then if $Y_\tau(t)$ is the fundamental matrix for (19) with $Y_\tau(\tau) = I$,

$$|P_\tau(t) - P(t)| = |P_\tau(t)(I - P(t))|$$

$$= |Y_\tau(t) P_\tau(t)(I - P(t)) Y_\tau^{-1}(t)|$$

$$\leq |Y_\tau(t)| |P_\tau(t) - P(t)| |(I - P(t)) Y_\tau^{-1}(t)|$$

$$\leq K_1 e^{-\alpha_1(\tau - \tau)} Le^{-\beta(t - \tau)}.$$
for $t \geq \tau \geq \tau_0$, where we have used (5) and (6). So if $\tau \geq \tau_0$ and $t \geq 2\tau$,
\[
|\mathcal{L}(t) - \mathcal{P}(t)| \leq |\mathcal{L}(t) - \mathcal{L}_x(t)| + |\mathcal{L}_x(t) - \mathcal{P}(t)| \\
\leq K_1 L e^{-(\alpha_1+\beta)\tau} + 4\alpha^{-1}K^3 \delta(2\tau) \\
\rightarrow 0 \quad \text{as} \quad \tau \rightarrow \infty.
\]

It follows that $|\mathcal{L}(t) - \mathcal{P}(t)| \rightarrow 0$ as $t \rightarrow \infty$.

**Theorem 3.5.** Let $f(t, x)$ be as in the first paragraph of Theorem 3.3 and assume, in addition, that $f$ has period $T > 0$ in $t$. Suppose system (17) has a doubly infinite sequence \(\{u_k(t)\}_{k \in \mathbb{Z}}\) of $T$-periodic solutions and another sequence \(\{v_k(t)\}\) of bounded solutions such that

(i) for each $k$ the variational equation
\[
\dot{x} = f_x(t, v_k(t))x
\]
has an exponential dichotomy on $(-\infty, \infty)$ with constants $K, \alpha$ independent of $k$;

(ii) $|v_k(t) - u_{k-1}(t)| \rightarrow 0$ as $t \rightarrow -\infty$ and $|v_k(t) - u_k(t)| \rightarrow 0$ as $t \rightarrow \infty$ uniformly with respect to $k$.

Then there are a number $\epsilon_0 > 0$ and a function $M_0(\epsilon)$ such that when $0 < \epsilon \leq \epsilon_0$ and $m$ is a positive integer satisfying $m > M_0(\epsilon)$, system (17) has a unique solution $x(t)$ defined on $(-\infty, \infty)$ satisfying
\[
|x(t + (2k - 1)mT) - v_k(t)| \leq \epsilon
\]
for $-mT \leq t \leq mT$ and all $k$.

**Proof.** Using (ii) and the continuity properties of $f_x$,
\[
|f_x(t, v_k(t)) - f_x(t, u_k(t))| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty
\]
uniformly with respect to $k$. It follows from the roughness theorem [10, p. 34] that the variational equation
\[
\dot{x} = f_x(t, u_k(t))x
\]
has, for $t_0$ sufficiently large, an exponential dichotomy on $[t_0, \infty)$ with constants depending only on $K, \alpha$. Since the coefficient matrix here has period $T$, Propositions 3 and 4 in [10, pp. 70–72] imply that (21) has an exponential dichotomy on $(-\infty, \infty)$ with the same constants and the corresponding projection matrix function $\mathcal{P}_k(t)$ has period $T$. Then it follows
from Lemma 3.4 and its proof that if $\mathcal{P}_k(t)$ is the projection matrix function corresponding to the exponential dichotomy for (20),

$$|\mathcal{P}_k(t) - \mathcal{P}_k(t)| \to 0 \quad \text{as} \quad t \to \infty,$$

uniformly with respect to $k$. Similarly, we find that

$$|\mathcal{P}_k(t) - \mathcal{P}_{k-1}(t)| \to 0 \quad \text{as} \quad t \to -\infty,$$

uniformly with respect to $k$.

Let $\varepsilon_0$, $\tau_0$ and $\delta_0(\varepsilon)$ be the constants and function from Theorem 3.3 and choose the positive integer $M_0(\varepsilon)$ so that when $m \geq M_0(\varepsilon)$,

$$2M_0(\varepsilon)T \geq \tau_0,$$

$$|v_{k-1}(mT) - u_{k-1}(mT)| + |v_k(-mT) - u_{k-1}(-mT)| \leq \delta_0(\varepsilon),$$

$$|\mathcal{P}_{k-1}(mT) - \mathcal{P}_{k-1}(mT)| + |\mathcal{P}_k(-mT) - \mathcal{P}_{k-1}(-mT)| \leq \delta_0(\varepsilon)$$

for all $k$. Now for each integer $k$ define $t_k = 2kmT$ and

$$w_k(t) - v_k(t - (2k - 1)mT),$$

where $0 < \varepsilon \leq \varepsilon_0$ and $m \geq M_0(\varepsilon)$. Then condition (i) of Theorem 3.3 is satisfied with projection matrix function $\mathcal{P}_k(t) = \mathcal{P}_k(t - (2k - 1)mT)$. Also

$$|w_{k-1}(t_{k-1}) - w_k(t_{k-1})| = |v_{k-1}(mT) - v_k(-mT)|$$

$$\leq |v_{k-1}(mT) - u_{k-1}(mT)| + |v_k(-mT) - u_{k-1}(-mT)|$$

$$\leq \delta_0(\varepsilon),$$

$$|\mathcal{P}_{k-1}(t_{k-1}) - \mathcal{P}_k(t_{k-1})| = |\mathcal{P}_{k-1}(mT) - \mathcal{P}_k(-mT)|$$

$$\leq |\mathcal{P}_{k-1}(mT) - \mathcal{P}_{k-1}(mT)| + |\mathcal{P}_k(-mT) - \mathcal{P}_{k-1}(-mT)|$$

$$\leq \delta_0(\varepsilon)$$

and

$$t_k - t_{k-1} = 2mT \geq \tau_0.$$

So all the hypotheses of Theorem 3.3 are satisfied and the present theorem follows at once from it.

We now define the Bernoulli shift (cf. Smale [24], Moser [20, p. 62], Arnol'd and Avez [3]). Let $N$ be a positive integer and $\mathcal{S}_N$ the set of all doubly infinite sequences

$$a = (..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...),$$
where \( a_k \in \{0, 1, \ldots, N - 1\} \). With the product topology \( S_N \) becomes a totally disconnected compact Hausdorff space (Cantor set). The homeomorphism \( \beta \) of \( S_N \) onto itself defined by

\[
(\beta(a))_k = a_{k+1}
\]

is the Bernoulli shift (note usually \( (\beta(a))_k = a_{k-1} \) but it seems more convenient to take it as \( a_{k+1} \) here).

**Corollary 3.6.** Let \( f(t, x) \) be as in the first paragraph of Theorem 3.3 and assume, in addition, that \( f \) has period \( T \) in \( t \). Suppose system (17) has a \( T \)-periodic solution \( u(t) \) and another solution \( v(t) \) such that

(i) the variational equation

\[
\dot{x} = f_x(t, v(t))x
\]

has an exponential dichotomy on \( (-\infty, \infty) \):

(ii) \( |v(t) - u(t)| \to 0 \) as \( |t| \to \infty \).

Then there exists a constant \( \varepsilon_0 > 0 \) and for each positive integer \( N \) a function \( M_N(\varepsilon) \) such that given \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \) and a positive integer \( m \geq M_N(\varepsilon) \), system (17) has for each \( a \) in the Cantor set \( S_N \) a unique solution \( x_a(t) \) satisfying

\[
|x_a(t + (2k - 1)mT) - v(t + a_k T)| \leq \varepsilon
\]

for \( -mT \leq t \leq mT \) and all \( k \).

The correspondence \( \phi(a) = x_a(0) \) is a homeomorphism onto a compact subset of \( \mathbb{R}^n \) on which the \( 2m \)th iterate \( F^{2m} \) of the period map \( F \) is invariant and satisfies

\[
F^{2m} \circ \phi = \phi \circ \beta,
\]

where \( \beta \) is the Bernoulli shift on \( S_N \).

**Proof.** If for each integer \( k \) we define

\[
v_k(t) = v(t + a_k T), \quad u_k(t) = u(t)
\]

then the existence of \( \varepsilon_0, M_N(\varepsilon) \) and the unique solution \( x_a(t) \) satisfying (17) follow from Theorem 3.3. Let us now adjust \( \varepsilon_0 \) so that

\[
\varepsilon_0 < \left( \frac{1}{3} \right) \min \{|v(iT) - v(0)| : i = (N - 1), \ldots, (N - 1)\}
\]

and adjust \( M_N(\varepsilon) \) so that

\[
M_N(\varepsilon) \geq N - 1.
\]
For the continuity of the mapping \( \phi \) at the point \( a \) we have to show given \( \delta > 0 \) that there exists \( K \) such that

\[
|\phi(a') - \phi(a)| < \delta
\]

when \( a'_k = a_k \) for \( |k| < K \). Suppose this is not true. Then there exists \( \delta > 0 \) and a sequence \( a^{(i)} \) in \( \mathcal{S}_N \) such that for \( l = 1, 2, \ldots \)

\[
a^{(i)}_k = a_k \quad \text{for} \quad |k| < l
\]

but

\[
|\phi(a^{(i)}) - \phi(a)| \geq \delta.
\]

Now for all \( l \)

\[
|\phi(a^{(i)})| \leq \sup_{-\infty < t < \infty} |v(t)| + \varepsilon < \infty.
\]

So taking a subsequence if necessary, we can assume that \( \phi(a^{(i)}) \to \xi \) as \( l \to \infty \) with \( |\phi(a) - \xi| \geq \delta \). It follows from standard theorems on continuous dependence on initial values that \( x_{a^{(i)}}(t) \to x(t) \) as \( l \to \infty \) uniformly on all compact \( t \)-intervals, where \( x(t) \) is the solution of (17) satisfying \( x(0) = \xi \). But for \( |k| < l \) and \(-mT \leq t \leq mT\)

\[
|x_{a^{(i)}}(t + (2k - 1)mT) - v(t + a_k T)| = |x_{a^{(i)}}(t + (2k - 1)mT) - v(t + a^{(i)}_k T)|
\]

\[
\leq \varepsilon.
\]

Letting \( l \to \infty \) we find that for all \( k \) and \(-mT \leq t \leq mT \),

\[
|x(t + (2k - 1)mT) - v(t + a_k T)| \leq \varepsilon.
\]

By the uniqueness of \( x_a(t) \), it follows that \( x(t) = x_a(t) \), contradicting \( x(0) = \xi \neq \phi(a) = x_a(0) \). This contradiction shows that \( \phi \) is continuous, as asserted.

To show that \( \phi \) is one to one, let \( a \neq a' \). Then \( a_k \neq a'_k \) for some \( k \). Since \( |a_k| \leq N - 1 \leq M_N(\varepsilon) \leq m \), we may take \( t = a_k T \) in (22) and its analogue for \( a' \) to get

\[
|x_a((2k - 1)mT - a_k T) - x_a((2k - 1)mT - a_k T)|
\]

\[
\geq |v(0) - v((a'_k - a_k)T)| - |x_a((2k - 1)mT - a_k T) - v(0)|
\]

\[
- |x_a((2k - 1)mT - a_k T) - v((a'_k - a_k)T)|
\]

\[
> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon,
\]

where we have used (24). So the solutions \( x_a(t) \) and \( x_{a'}(t) \) are distinct and hence \( \phi(a) \neq \phi(a') \).
Thus $\phi$ is continuous and one to one. Since $\mathcal{S}$ is compact Hausdorff, it follows that $\phi$ is a homeomorphism onto its image.

The identity (23) follows from the observation that the solution $x(t) = x_0(t + 2mT)$ satisfies

$$|x(t + (2k - 1)mT) - v(t + a_{k+1}T)| = |x_0(t + (2k + 1 - 1)mT) - v(t + a_{k+1}T)| \leq \varepsilon$$

for $-mT \leq t \leq mT$ and all $t$. By uniqueness $x(t) = x_{0(a)}(t)$ and so

$$\phi(\beta(a)) = x(0) = x_0(2mT) = F^{2m}(\phi(a)).$$

Remarks. (a) We can interpret this corollary as follows: The solution $v(t)$ has $N$ arcs corresponding to the time segments $[-mT, mT]$, $[-(m - 1)T, (m + 1)T]$, ..., $[-(m - N + 1)T, (m + N - 1)T]$. What we have shown is the existence of solutions which "shadow" one of these arcs in each time segment $[(2k - 2)mT, 2kmT]$ but switch randomly from one arc to another.

(b) Let $F: M \to M$ be a diffeomorphism on a compact manifold $M$ with a hyperbolic fixed point $x_0$ and a transversal homoclinic point $y$ with respect to $x_0$. Then the orbit of $y$ together with $x_0$ form a hyperbolic set $\mathcal{S}$ and so the "shadow" lemma holds. That is, given $\varepsilon > 0$ there is $\delta > 0$ such that if $\{y_k\}$ is a $\delta$-pseudo orbit in $\mathcal{S}$ ($d(F(y_k), y_{k+1}) < \delta$ for all $k$ in $\mathcal{F}$, where $d(\cdot, \cdot)$ is a metric on $\mathcal{M}$) there is a unique orbit $\{F^k(x)\}$ which $\varepsilon$-shadows $\{y_k\}$ ($d(F^k(x), y_k) < \varepsilon$ for all $k$).

Now let $N$ be a positive integer and let $a$ be in the Cantor set $\mathcal{S}$. For $m$ sufficiently large the sequence

$$\ldots, y_{a_1+m}, y_{a_1+m+1}, \ldots, y_{a_0+m}, y_{a_0-m+1}, \ldots, y_{a_0},$$

$$\ldots, y_{a_0+m}, y_{a_1-m+1}, \ldots, y_{a_1}, \ldots, y_{a_1+m}, \ldots,$$

where $y_k = F^ky$, forms a $\delta$-pseudo orbit in $\mathcal{S}$. Define $\phi(a)$ as the point close to $y_{a_0}$ in the unique $\varepsilon$-shadowing orbit. Then if $\varepsilon$ is sufficiently small and $m$ sufficiently large, $\phi$ will give a homeomorphism of $\mathcal{S}$ onto a compact subset of $M$ on which $F^{2m}$ is invariant and such that $F^{2m} \circ \phi = \phi \circ \beta$. This is usually proved by the horseshoe construction. The proof just sketched is probably known but there seems to be no explicit reference to it in the literature.

Suppose the system

\[ \dot{z} = g(t, z) \]  

has a solution \( \zeta(t) \) bounded on \((-\infty, \infty)\) such that the variational equation

\[ \dot{z} = g_z(t, \zeta(t))z \]

has an exponential dichotomy on \((-\infty, \infty)\). Then it is a consequence of Theorem 4 in [9, p. 137] that a perturbed system

\[ \dot{z} = g(t, z) + \mu h(t, z, \mu) \]

has, for \( \mu \) sufficiently small, a unique bounded solution near \( \zeta(t) \). However, if (25) is autonomous so that \( g(t, z) = g(z) \), this theorem cannot be applied because the variational equation now has the nontrivial bounded solution \( \zeta(t) \) and hence cannot have an exponential dichotomy on \((-\infty, \infty)\). So we have a critical case and must impose additional conditions on the perturbation term to ensure the existence of a bounded solution near \( \zeta(t) \).

First we use the Lyapunov–Schmidt method (cf. Chow and Hale [6, p. 33]) to prove an extension of the implicit function theorem suggested by a special case in Coppel [8]. Note that if \( L \) is a linear operator, we denote by \( \mathcal{N}(L) \) its kernel and by \( \mathcal{R}(L) \) its range.

**Theorem 4.1.** Let \( \mathcal{E}, \mathcal{F} \) be Banach spaces and \( f: \mathcal{E} \times \mathbb{R} \to \mathcal{F} \) a \( C^2 \)-mapping defined on a neighborhood of \((0, 0)\) such that \( f(0, 0) = 0 \) and \( L = f_z(0, 0) \) is Fredholm with index zero. Then if

(i) \( f(x, 0) = 0 \) for \( x \in \mathcal{M} \), a \( C^2 \) submanifold of \( \mathcal{E} \) containing 0 with \( \mathcal{N}(L) \) as tangent space at 0,

(ii) \( Lx = -f_{\mu}(0, 0) \) has a solution \( p \),

(iii) \( y \in \mathcal{N}(L) \) and \( \{f_{xx}(0, 0)p + f_{\mu x}(0, 0)y\} y \in \mathcal{R}(L) \Rightarrow y = 0 \), there is a neighborhood of 0 in \( \mathcal{E} \) such that for \( \mu \) sufficiently small the equation

\[ f(x, \mu) = 0 \]  

has a solution \( x(\mu) \) in this neighborhood which is unique when \( \mu \neq 0 \). Moreover \( x(0) = 0 \), \( x(\mu) \) is \( C^1 \) and when \( \mu \neq 0 \) \( f_x(x(\mu), \mu) \) is invertible.

**Proof.** Note that since \( f(x, 0) \) is constant on \( \mathcal{M} \),

\[ \{f_{xx}(0, 0)y\} z = 0 \]
for all $y, z$ in $\mathcal{H}(L)$. It follows that if (iii) holds for one solution $p$ of $Lp = -f_p(0, 0)$ then it holds for all such $p$.

Now let $P$ and $Q$ be continuous projections on $E$ and $F$, respectively, such that $\mathcal{H}(P) = \mathcal{H}(L)$ and $\mathcal{H}(Q) = \mathcal{H}(L)$, respectively. We can write (26) as

$$Lx = h(x, \mu),$$

where

$$h(x, \mu) = f'_x(0, 0)x - f(x, \mu).$$

If we put

$$x = Px + (I - P)x = \xi + \eta$$

then according to the Lyapunov–Schmidt method Eq. (27) is equivalent to the two equations

$$\eta = KQh(\xi + \eta, \mu),$$

$$\eta = (I - Q)h(\xi + \eta, \mu) = 0,$$

where $K: \mathcal{H}(L) \to \mathcal{H}(P)$ is the inverse of $L: \mathcal{H}(P) \to \mathcal{H}(L)$.

Now since

$$F(\xi, \eta, \mu) = \eta - KQh(\xi + \eta, \mu)$$

is a $C^2$ function with

$$F(0, 0, 0) = 0, \quad F_\eta(0, 0, 0) = I,$$

the usual implicit function theorem implies that there is a neighborhood of zero in $\mathcal{H}(P)$ such that when $\xi$ and $\mu$ are sufficiently small (28) has a unique solution $\eta = \eta(\xi, \mu)$ in this neighborhood. Moreover, $\eta(0, 0) = 0$ and $\eta(\xi, \mu)$ is a $C^2$ function. When we substitute $\eta(\xi, \mu)$ back into (29), we obtain the equation

$$G(\xi, \mu) = (I - Q)h(\xi + \eta(\xi, \mu), \mu) = 0.$$  

The manifold $\mathcal{H}$ in hypothesis (i) can be represented locally as

$$\mathcal{H} = \{\xi + \phi(\xi): \xi \in \mathcal{H}(L), \|\xi\| < \delta\},$$

where $\phi$ is a $C^2$ function into $\mathcal{H}(P)$ satisfying $\phi(0) = 0$, $\phi_\xi(0) = 0$. Since $x = \xi + \phi(\xi)$ satisfies (27) for $\mu = 0$, (28) and (29) must also be satisfied for $\mu = 0$ with $\eta = \phi(\xi)$. If $\delta$ is sufficiently small it follows from the uniqueness of $\eta(\xi, \mu)$ that for $\|\xi\| < \delta$,

$$\eta(\xi, 0) = \phi(\xi) \quad \text{and} \quad G(\xi, 0) = 0.$$
Then the function $H(\xi, \mu)$ defined by

$$H(\xi, \mu) = \begin{cases} \mu^{-1}G(\xi, \mu) & \text{for } \mu \neq 0 \\ G_\mu(\xi, 0) & \text{for } \mu = 0 \end{cases}$$

is $C^1$.

Solving (30) for $\mu \neq 0$ is equivalent to solving

$$H(\xi, \mu) = 0. \quad (31)$$

To apply the implicit function theorem to this equation, we show that $H(0, 0) = G_\mu(0, 0) = 0$ and that $H_{\mu}(0, 0) = \frac{\partial G_\mu}{\partial \nu}(0, 0)$ is invertible. Now

$$G_\mu(\xi, 0) = (I - Q)(h_\mu(\xi + \phi(\xi), 0) \eta_\mu(\xi, 0) + h_\mu(\xi + \phi(\xi), 0)). \quad (32)$$

So

$$G_\mu(0, 0) = -(I - Q)f_\mu(0, 0) = 0,$$

since, by (ii), $f_\mu(0, 0)$ is in $\mathcal{A}(L) = \mathcal{A}(Q)$. Differentiating (32) with respect to $\xi$ in the direction $y$ and putting $\xi = 0$, we obtain for all $y \in \mathcal{N}(L)$ that

$$G_{\mu}(0, 0)y = -(I - Q)(f_{xx}(0, 0)\eta_\mu(0, 0) + f_{ux}(0, 0)y)$$

$$= -(I - Q)(f_{xx}(0, 0)\eta_\mu(0, 0) + f_{ux}(0, 0))y,$$

using the symmetry of second order derivatives. To evaluate $\eta_\mu(0, 0)$, differentiate (28) with respect to $\mu$ and put $\xi = 0$, $\mu = 0$ to get

$$\eta_\mu(0, 0) = -KQf_\mu(0, 0) = -Kf_\mu(0, 0).$$

So $\eta_\mu(0, 0)$ is a solution $p$ of $Lp = -f_\mu(0, 0)$. It follows from hypothesis (iii) that $G_{\mu}(0, 0)y = 0$ only if $y = 0$. Since the domain $\mathcal{N}(L)$ and range $\mathcal{N}(Q)$ of $G_{\mu}(0, 0)$ are finite-dimensional subspaces of the same dimension, this means that $G_{\mu}(0, 0)$ is invertible.

Hence we may apply the usual implicit function theorem to deduce that there is a neighborhood of 0 in $\mathcal{A}(L)$ such that for $\mu$ sufficiently small (31) has a unique solution $\xi(\mu)$ in this neighborhood. Moreover, $\xi(0) = 0$ and $\xi'(\mu)$ is $C^1$.

Then $x(\mu) = \xi(\mu) + \eta(\xi(\mu), \mu)$ is a solution of (27) with the required properties except that we still need to verify that $f_s(\mu, \mu)$ is invertible for $\mu \neq 0$ sufficiently small. Since $f_s(\mu, \mu) \to L$ as $\mu \to 0$, $f_s(\mu, \mu)$ is Fredholm of index zero for $\mu$ sufficiently small and hence invertible if it has the trivial kernel. So we have to show that the equation

$$f_s(\mu, \mu)x = 0 \quad (33)$$

has the unique solution $x = 0$ for $\mu \neq 0$ sufficiently small.
Before proceeding with the proof of this, let us make two observations. First, by putting $x = x(\mu)$ in (26), differentiating with respect to $\mu$ and setting $\mu = 0$, we see that the derivative $x_{\mu}(0)$ is a solution of $Lx = -f_{\mu}(0, 0)$. Second, note that if we do not assume the existence of $f_{\mu\mu}$, the above proof still goes through but we can only conclude that $x(\mu)$ is continuous.

We want to apply what we proved above to Eq. (33). Now $g(x, \mu) = f_{x}(x(\mu), \mu)x$ is $C^2$ near $(0, 0)$ except that $g_{\mu\mu}$ may not exist. Also it satisfies hypothesis (i) with $\mathscr{M} = \mathscr{N}(L)$; (ii) holds since $g_{\mu}(0, 0) = 0$ and if $Lp = 0$ and $y \in \mathscr{N}(L)$,

\[
\{g_{xx}(0, 0) p + g_{\mu x}(0, 0)\} y = \{f_{xx}(0, 0) x_{\mu}(0) + f_{\mu x}(0, 0)\} y
\]

so that (iii) is also satisfied. So we conclude that there is a neighborhood of 0 in $\mathscr{E}$ such that for $\mu \neq 0$ sufficiently small (33) has a unique solution in this neighborhood. Since (33) is linear in $x$, this implies that (33) has the unique solution $x = 0$ for $\mu \neq 0$ sufficiently small. This completes the proof of the theorem.

We now introduce the linear operator $L$ we will be concerned with and show that it is Fredholm. In the following lemma $C^k(\mathbb{R}, \mathbb{R}^n)$ denotes the Banach space of bounded continuous $\mathbb{R}^n$-valued functions whose derivatives up to order $k$ exist and are bounded and continuous on $(-\infty, \infty)$.

**Lemma 4.2.** Let $A(t)$ be an $n \times n$ matrix function bounded and continuous on $(-\infty, \infty)$ such that system (3) has an exponential dichotomy on both half-lines. Then the linear operator

\[
L : \mathscr{E} = C^1(\mathbb{R}, \mathbb{R}^n) \to \mathscr{F} = C^0(\mathbb{R}, \mathbb{R}^n)
\]

defined by

\[
(Lx)(t) = \dot{x}(t) - A(t) x(t)
\]

is Fredholm and $f \in \mathscr{A}(L)$ if and only if

\[
\int_{-\infty}^{\infty} \psi^*(t) f(t) dt = 0 \quad (* = \text{transpose})
\]  

(34)

for all bounded solutions $\psi(t)$ of the adjoint system

\[
\dot{x} = -A^*(t)x.
\]  

(35)

The index of $L$ is $\dim V + \dim W - n$, where $V$ and $W$ are the stable and unstable subspaces for (3).

**Proof.** Let (3) have an exponential dichotomy on $[0, \infty)$ with projection $P$ and on $(-\infty, 0]$ with projection $Q$ and suppose in both cases that the
associated fundamental matrix $X(t)$ satisfies $X(0) = I$. Then the adjoint system (35) has fundamental matrix $X^{-1}(t)$ and by taking transposes in inequalities like (4) we see that (35) has an exponential dichotomy on $[0, \infty)$ with projection $I - P^*$ and on $(-\infty, 0]$ with projection $I - Q^*$.

Now the subspace of initial values (at $t = 0$) of bounded solutions of (3) is $V \cap W$, where $V = \mathcal{R}(P)$ and $W = \mathcal{N}(Q)$, and for (35) it is $V^\perp \cap W^\perp$, where $V^\perp$ (the orthogonal complement of $V$) is $\mathcal{R}(I - P^*)$ and $W^\perp$ is $\mathcal{R}(Q^*)$.

It follows from the previous paragraph that

$$\dim \mathcal{N}(L) = \dim V \cap W.$$

Now let $f \in \mathcal{R}(L)$ so that there exists $x$ in $\mathcal{E}$ satisfying

$$f(t) = \dot{x}(t) - A(t)x(t).$$

Then if $\psi(t)$ is a bounded solution of (35),

$$\int_{-\infty}^{\infty} \psi^*(t) f(t) \, dt = \int_{-\infty}^{\infty} [\psi^*(t) \dot{x}(t) - \psi^*(t) A(t)x(t)] \, dt$$

$$= \int_{-\infty}^{\infty} [\psi^*(t) \dot{x}(t) + \psi^*(t)x(t)] \, dt$$

$$= [\psi^*(t)x(t)]_{-\infty}^{\infty} = 0,$$

since $|\psi(t)| \to 0$ exponentially as $|t| \to \infty$. So if $f \in \mathcal{R}(L)$, (34) holds for all bounded solutions $\psi(t)$ of the adjoint system (35).

Conversely, suppose $f \in \mathcal{F}$ and that (34) holds for all bounded solutions $\psi(t)$ of (35). Note that if $\eta$ is a vector satisfying

$$\eta^*[P - (I - Q)] = 0,$$  \hfill (36)

$\psi(t)$ defined as $X^{-1}(t)(I - P^*)\eta$ for $t \geq 0$ and as $X^{-1}(t)Q^*\eta$ for $t \leq 0$ is a bounded solution of (35). It follows that

$$\eta^* \left[ \int_{-\infty}^{0} QX^{-1}(t)f(t) \, dt + \int_{0}^{\infty} (I - P)X^{-1}(t)f(t) \, dt \right] = 0$$

for all vectors $\eta$ satisfying (36). This means that the linear algebraic equation

$$[P - (I - Q)]\xi = \int_{-\infty}^{0} QX^{-1}(t)f(t) \, dt + \int_{0}^{\infty} (I - P)X^{-1}(t)f(t) \, dt$$
has a solution $\xi$ and then the function $x(t)$, defined for $t \geq 0$ as

$$X(t)P\xi + \int_0^t X(t)PX^{-1}(s)f(s)\,ds - \int_t^{\infty} X(t)(I - P)X^{-1}(s)f(s)\,ds$$

and for $t \leq 0$ as

$$X(t)(I - Q)\xi + \int_{-\infty}^t X(t)QX^{-1}(s)f(s)\,ds - \int_t^0 X(t)(I - Q)X^{-1}(s)f(s)\,ds,$$

is in $\mathcal{F}$ and is a bounded solution of the inhomogeneous linear system

$$\dot{x} = A(t)x + f(t).$$

Thus $Lx = f$ and so $f \in \mathcal{A}(L)$, as required.

To show that $\mathcal{A}(L)$ is closed and has finite codimension, note that each bounded solution $\psi(t)$ of the adjoint system (35) defines a bounded linear functional on $\mathcal{F}$ through

$$f \mapsto \int_{-\infty}^{\infty} \psi^*(t)f(t)\,dt$$

and this correspondence gives an isomorphism between $V^\perp \cap W^\perp$ and a finite-dimensional subspace of the dual space $\mathcal{F}^*$. We have just shown that $\mathcal{A}(L)$ is the subspace of $\mathcal{F}$ annihilated by this finite-dimensional subspace of $\mathcal{F}^*$ and so $\mathcal{A}(L)$ is closed and

$$\text{codim } \mathcal{A}(L) = \dim(V^\perp \cap W^\perp).$$

Hence $L$ is Fredholm, as asserted.

Now the index of $L$ is

$$\dim \mathcal{N}(L) - \text{codim } \mathcal{A}(L),$$

which is

$$\dim V \cap W - \dim(V^\perp \cap W^\perp)$$

$$= \dim V \cap W - \dim(V + W)^\perp$$

$$= \dim V \cap W - \{n - \dim(V + W)\}$$

$$= \dim V \cap W - \{n - [\dim V + \dim W - \dim V \cap W]\}$$

$$= \dim V + \dim W - n,$$

as asserted. So the proof of the lemma is complete. (Note that a related result has been proved by Sacker [23, p. 389].)

We are now ready to give our application of Theorem 4.1.
Corollary 4.3. Let $g(z)$ be a twice continuously differentiable vector function defined in an open subset $U$ of $\mathbb{R}^n$ such that the system

$$\dot{z} = g(z) \quad (37)$$

has a bounded solution $\zeta(t)$, the closure of whose orbit is contained in $U$. Suppose the variational equation

$$\dot{z} = g_z(\zeta(t))z \quad (38)$$

has an exponential dichotomy on both half-lines, that the dimensions of the stable and unstable subspaces have sum $n$ and that $\dot{\zeta}(t)$ is the unique (up to a scalar multiple) bounded solution of $(38)$.

Now let $h(t, z, \mu)$ be a bounded continuous vector function defined for $-\infty < t < \infty$, $|z - \zeta(t)| < \Delta_0$, $|\mu| < \sigma_0 (\mu \in \mathbb{R})$ such that the partial derivatives $h_z, h_\mu, h_{zz}, h_{zz}, h_{z\mu}, h_{\mu\mu}$ exist, are bounded, continuous in $t$ for each fixed $z, \mu$ and continuous in $z, \mu$ uniformly with respect to $t, z, \mu$.

Then if

$$\int_{-\infty}^{\infty} \psi^*(t) h(t, \zeta(t), 0) \, dt = 0, \quad \int_{-\infty}^{\infty} \psi^*(t) h(t, \zeta(t), 0) \, dt \neq 0, \quad (39)$$

where $\psi(t)$ is the unique (up to a scalar multiple) bounded solution of the system adjoint to $(38)$, there exist positive constants $\Delta, \sigma$ such that for $0 < |\mu| < \sigma$ the perturbed system

$$\dot{z} = g(z) + \mu h(t, z, \mu) \quad (40)$$

has a unique solution $z(t, \mu)$ satisfying

$$|z(t, \mu) - \zeta(t)| \leq \Delta$$

for all $t$. Moreover as $\mu \to 0$

$$\sup_{-\infty < t < \infty} |z(t, \mu) - \zeta(t)| = O(\mu)$$

and the variational equation

$$\dot{z} = \left[ g_z(z(t, \mu)) + \mu h_z(t, z(t, \mu), \mu) \right]z \quad (41)$$

has an exponential dichotomy on $(-\infty, \infty)$.

Proof: Introducing the new variable $x$ through

$$z = \zeta(t) + x,$$
(40) can be written in the form
\[ \dot{x} = g(\zeta(t) + x) - g(\zeta(t)) + \mu h(t, \zeta(t) + x, \mu). \]

Now we want to apply Theorem 4.1 with \( \mathcal{B} = C^1(\mathbb{R}, \mathbb{R}^n) \), \( \mathcal{F} = C^0(\mathbb{R}, \mathbb{R}^n) \) and
\[ [f(x, \mu)](t) = \dot{x}(t) - \{ g(\zeta(t) + x(t)) - g(\zeta(t)) + \mu h(t, \zeta(t) + x(t), \mu) \}. \]

It is clear that \( f \) is \( C^2 \) in a neighborhood of \((0, 0)\) in \( \mathcal{B} \times \mathbb{R} \) and that for all \( y \) in \( \mathcal{B} \)
\[ [f_x(x, \mu)y](t) = \dot{y}(t) - \{ g_z(\zeta(t) + x(t)) + \mu h_z(t, \zeta(t) + x(t), \mu) \} y(t). \]

In particular,
\[ (Ly)(t) = [f_x(0, 0)y](t) = \dot{y}(t) - g_z(\zeta(t)) y(t). \]

The hypotheses imply that the kernel of \( L \) is the one-dimensional subspace of \( \mathcal{B} \) spanned by \( \zeta \). It follows from Lemma 4.2 and its proof that (up to a scalar multiple) the system adjoint to (38) has one bounded solution \( \psi(t) \) and also that \( L \) is Fredholm with index zero.

Now note that \( f(x, 0) = 0 \) for \( x \) in
\[ \mathcal{M} = \{ x_\alpha : x_\alpha(t) = \zeta(t + \alpha) - \zeta(t) \}. \]

This is a one-dimensional submanifold of \( \mathcal{B} \) containing 0 whose tangent space there is the one-dimensional subspace generated by \( \zeta \), that is, the kernel of \( L \).

We calculate
\[ [f_\mu(0, 0)](t) = -h(t, \zeta(t), 0) \]
and it follows from the hypotheses and Lemma 4.2 that there exists \( p \) in \( \mathcal{B} \) satisfying \( Lp = -f_\mu(0, 0) \). So \( p(t) \) is a bounded solution of the equation
\[ \dot{p} = g_z(\zeta(t)) p(t) + h(t, \zeta(t), 0). \]

Then if we define
\[ w(t) = \{ [f_{xx}(0, 0)p + f_{x\mu}(0, 0)] \zeta \}(t), \]
we see, using the symmetry of second order derivatives, that
\[ w(t) = -\{ g_{zz}(\zeta(t)) \zeta(t) - h_z(t, \zeta(t), 0) \} \zeta(t) \]
\[ = -\{ g_{zz}(\zeta(t)) \zeta(t) \} p(t) - h_z(t, \zeta(t), 0) \zeta(t). \]
Differentiating (43), we find that $\dot{p}(t)$ is a bounded solution of the equation

$$\dot{x} = g_z(\zeta(t))x + [-w(t) + h_t(t, \zeta(t), 0)].$$

Hence $-w + h_t(\cdot, \zeta(\cdot), 0)$ is in $\mathcal{R}(L)$. But by Lemma 4.2 and the hypotheses, $h_t(\cdot, \zeta(\cdot), 0)$ is not in $\mathcal{R}(L)$. Hence $w$ is not in $\mathcal{R}(L)$ also. Thus we have shown that $[f_{xx}(0, 0)p + f_{xu}(0, 0)]y$ is not in $\mathcal{R}(L)$ when $y$ is in $\mathcal{N}(L)$ and $y \neq 0$.

So the conditions of Theorem 4.1 hold and all the assertions of the corollary, except perhaps for the last one, follow at once with $z(t, \mu) = \zeta(t) + x(t, \mu)$ (where $x(\cdot, \mu)$ is the solution found in Theorem 4.1). To verify the last assertion, note that Theorem 4.1 says that $f_z(x(\mu), \mu)$ is invertible. Referring to Eq. (42), this means that for all $f$ in $\mathcal{F}$ the differential equation

$$\dot{y} = [g_z(z(t, \mu)) + \mu h_z(t, z(t, \mu), \mu)]y + f(t)$$

has a unique bounded solution. It follows from Coppel [10, Chap. 3] and Proposition 2.1 that (41) has an exponential dichotomy on $(-\infty, \infty)$. This completes the proof of the corollary.

We want to consider a particular case of Corollary 4.3, where the autonomous system (37) has equilibrium solutions $z_0, z_1$ (which may coincide) such that $\zeta(t) \to z_0$ as $t \to -\infty$ and $\zeta(t) \to z_1$ as $t \to \infty$ and such that the eigenvalues of the matrices $g_z(z_0), g_z(z_1)$ have nonzero real parts and the same number (counting multiplicities) with negative real parts. It follows from the roughness theorem for exponential dichotomies that the variational equation (38) has an exponential dichotomy on both half-lines and that the dimensions of the stable and unstable subspaces have sum $n$.

We assume the other conditions in Corollary 4.3 hold. It is a consequence of standard perturbation theory (see, for example, Coppel [9, p. 137]) that for sufficiently small $\mu$ the perturbed system (40) has unique bounded solutions $z_i(t, \mu)$ ($i = 0, 1$) such that

$$\sup_{-\infty < t < \infty} |z_i(t, \mu) - z_i| \to 0 \quad \text{as} \quad \mu \to 0.$$ 

(Also when $h$ has period $T$ in $t$, the $z_i(t, \mu)$ also have period $T$.) Moreover from integral manifold theory (Hale [12], Knobloch and Kappel [16], Palmer [21]) there exists $\delta > 0$ independent of $\mu$ such that if $z(t)$ is a solution of (40) satisfying $|z(t) - z_i(t, \mu)| \leq \delta$ for sufficiently large $t$ (positive when $i = 1$, negative when $i = 0$), then

$$|z(t) - z_i(t, \mu)| \to 0 \quad \text{as} \quad t \to \infty \quad (i = 1), \quad t \to -\infty \quad (i = 0).$$
Now according to Corollary 4.3 (40) has a solution $z(t, \mu)$ satisfying
\[
\sup_{-\infty < t < \infty} |z(t, \mu) - \zeta(t)| \to 0 \text{ as } \mu \to 0.
\]
Then for $i = 0, 1$
\[
|z(t, \mu) - z_i(t, \mu)| \leq |z(t, \mu) - \zeta(t)| + |\zeta(t) - z_i| + |z_i(t, \mu) - z_i| \leq \delta
\]
provided $\mu$ is sufficiently small and $t$ is sufficiently large (positive when $i = 1$, negative when $i = 0$). It follows that
\[
|z(t, \mu) - z_i(t, \mu)| \to 0 \quad \text{as} \quad t \to \infty \quad (i = 1), \quad t \to -\infty \quad (i = 0).
\]

Summing up, we have proved the following corollary.

**Corollary 4.4.** Let the conditions of Corollary 4.3 hold but assume, in addition, that system (31) has hyperbolic equilibrium points $z_0, z_1$ such that the matrices $g_z(z_0), g_z(z_1)$ have the same number (counting multiplicities) of eigenvalues with negative real parts and such that $\zeta(t) \to z_0$ as $t \to -\infty$, $\zeta(t) \to z_1$ as $t \to \infty$. Suppose also that $h$ has period $T$ in $t$.

Then for sufficiently small $\mu$ system (40) has unique $T$-periodic solutions $z_0(t, \mu), z_1(t, \mu)$ such that for $i = 0, 1$
\[
\sup_{-\infty < t < \infty} |z_i(t, \mu) - z_i| \to 0 \quad \text{as} \quad \mu \to 0.
\]

Moreover, (40) has a solution $z(t, \mu)$ such that
\[
|z(t, \mu) - z_0(t, \mu)| \to 0 \quad \text{as} \quad t \to -\infty
\]
\[
|z(t, \mu) - z_1(t, \mu)| \to 0 \quad \text{as} \quad t \to \infty
\]
and when $\mu \neq 0$ the variational equation (41) has an exponential dichotomy on $(-\infty, \infty)$.

**Remark.** In view of Proposition 2.2, the period map for system (40) has, under the conditions of Corollary 4.4, a transversal homoclinic point when $z_0 = z_1$. Note also, as remarked above, that the condition on the eigenvalues of $g_z(z_0)$ and $g_z(z_1)$ implies that the variational equation (38) has an exponential dichotomy on both half-lines and that the dimensions of the stable and unstable subspaces have sum $n$.

Let us return to Corollary 4.3 again. Note that for any $t_0$, $\zeta(t - t_0)$ is also a solution of (37) to which we can apply the corollary. In this case conditions (39) become
\[
\Delta(t_0) = \int_{-\infty}^{\infty} \psi^*(t - t_0) h(t, \zeta(t - t_0), 0) \, dt = 0,
\]
\[
\int_{-\infty}^{\infty} \psi^*(t - t_0) h_i(t, \zeta(t - t_0), 0) \, dt \neq 0.
\]
Note that
\[ \Delta(t_0) = \int_{-\infty}^{\infty} \psi^*(t) h(t + t_0, \zeta(t), 0) \, dt \]
so that
\[ \Delta'(t_0) = \int_{-\infty}^{\infty} \psi^*(t) h_1(t + t_0, \zeta(t), 0) \, dt \]
\[ = \int_{-\infty}^{\infty} \psi^*(t - t_0) h_1(t, \zeta(t - t_0), 0) \, dt. \]
Hence the conditions reduce to
\[ \Delta(t_0) = 0, \Delta'(t_0) \neq 0. \]

Now in a couple of special cases $\psi(t)$ can be determined easily in terms of $\zeta(t)$. For example, suppose (37) and (40) are in the Hamiltonian forms
\[ \dot{x} = H_y(x, y), \]
\[ \dot{y} = -H_x(x, y) \]
and
\[ \dot{x} = H_y(x, y) + \mu \tilde{H}_y(t, x, y, \mu), \]
\[ \dot{y} = -H_x(x, y) - \mu \tilde{H}_x(t, x, y, \mu), \]
respectively, where $x$ and $y$ are $m$-vectors. Then
\[ \zeta(t) = (H_y(\zeta(t)), -H_x(\zeta(t))) \]
and one verifies that
\[ \psi(t) = (H_x(\zeta(t)), H_y(\zeta(t))) \]
is a bounded solution of the system adjoint to the variational equation for $\zeta(t)$. So, in this case, $\Delta(t_0)$ has the form
\[ \int_{-\infty}^{\infty} \langle H_x^*(\zeta(t)) \tilde{H}_y(t + t_0, \zeta(t), 0) - H_y^*(\zeta(t)) \tilde{H}_x(t + t_0, \zeta(t), 0) \rangle \, dt, \]
where the integrand is just the Poisson bracket of $H$ and $\tilde{H}$ evaluated at $(t + t_0, \zeta(t), 0)$ (cf. Holmes [13, p. 71], Arnol’d [2]).
In the special case $n = 2$ (not necessarily Hamiltonian) we can write (37) and (40) in the forms
\[
\dot{x} = p_0(x, y),
\]
\[
\dot{y} = q_0(x, y)
\]
and
\[
\dot{x} = p_0(x, y) + \mu p_1(t, x, y, \mu),
\]
\[
\dot{y} = q_0(x, y) + \mu q_1(t, x, y, \mu),
\]
respectively. If $\zeta(t) = (x_0(t), y_0(t))$ so that $\dot{\zeta}(t) = (p_0(\zeta(t)), q_0(\zeta(t)))$, one verifies that
\[
\psi(t) = \exp \left[ -\int_0^t \left\{ \frac{\partial p_0}{\partial x}(\zeta(s)) + \frac{\partial q_0}{\partial y}(\zeta(s)) \right\} ds \right] (y_0(t), -\dot{x}_0(t))
\]
is a bounded solution of the system adjoint to the variational equation for $\zeta(t)$. So, in this case, $\Delta(t_0)$ is
\[
\int_{-\infty}^{\infty} [q_0(\zeta(t)) p_1(t + t_0, \zeta(t), 0) - p_0(\zeta(t)) q_1(t + t_0, \zeta(t), 0)]
\times \exp \left[ -\int_0^{t + t_0} \left\{ \frac{\partial p_0}{\partial x}(\zeta(s)) + \frac{\partial q_0}{\partial y}(\zeta(s)) \right\} ds \right] dt
\]
(cf. Melnikov [17], Holmes [14, p. 427]).

5. APPLICATION TO THE FORCED PENDULUM EQUATION

Let us now examine Kirchgraber's example [15], which stimulated the present work. He considers the sinusoidally forced pendulum equation
\[
\ddot{x} + \sin x = \mu \sin t,
\]
(46)
which as a system has the Hamiltonian form (44) with
\[
y = \dot{x}, \quad H(x, y) = \frac{1}{2} y^2 - \cos x, \quad \tilde{H}(t, x, y, \mu) = -\mu x \sin t.
\]
The unforced system has saddle points $z_0 = (-\pi, 0), z_1 = (\pi, 0)$ and there is a solution (the upper separatrix)
\[
\zeta(t) = (\pi - 4 \tan^{-1}(e^{-t}), 2 \sech t)
\]
such that \( \zeta(t) \to z_0 \) as \( t \to -\infty \), \( \zeta(t) \to z_1 \) as \( t \to \infty \). Since the stable and unstable subspaces for the variational equation corresponding to \( \zeta(t) \) must have dimension one, \( \zeta(t) \) must be the unique (up to a scalar multiple) bounded solution of it.

Here \( A(t_0) \), as defined in (45), is

\[
2 \int_{-\infty}^{\infty} \text{sech} \ t \sin (t + t_0) \ dt.
\]

So \( A(0) = 0 \) and \( A'(0) = 2 \int_{-\infty}^{\infty} \text{sech} \ t \cos t \ dt = 2\pi \text{sech} (\pi/2) \neq 0 \). Hence the conditions of Corollary 4.4 are satisfied and we deduce, first, that for \( \mu \) sufficiently small the system corresponding to (46) has unique \( 2\pi \)-periodic solutions \( z_0(t), z_1(t) \) (we suppress the dependence on \( \mu \)) near \( z_0 \) and \( z_1 \) respectively. Note, however, that \( z_0(t) + (2\pi, 0) \) is a solution satisfying the same conditions as \( z_1(t) \) and so, by uniqueness,

\[
z_1(t) = z_0(t) + (2\pi, 0). \tag{47}
\]

Similarly,

\[
z_0(t) = -z_0(t + \pi) - (2\pi, 0). \tag{48}
\]

Second, we deduce from Corollary 4.4 that for \( \mu \neq 0 \) sufficiently small there is a solution \( z^+(t) \) near \( \zeta(t) \) such that

\[
|z^+(t) - z_0(t)| \to 0 \quad \text{as } t \to -\infty, \quad |z^+(t) - z_1(t)| \to 0 \quad \text{as } t \to \infty
\]

and the variational equation for \( z^+(t) \) has an exponential dichotomy on \((-\infty, \infty)\). We see also that \( z^-(t) = -z^+(t + \pi) \) is a solution near the lower separatrix \( -\zeta(t + \pi) \) satisfying

\[
|z^-(t) - z_1(t)| \to 0 \quad \text{as } t \to -\infty, \quad |z^-(t) - z_0(t)| \to 0 \quad \text{as } t \to \infty,
\]

where we have used (47) and (48). Moreover, the corresponding variational equation is the same as for \( z^+(t) \) but with \( t \) replaced by \( t + \pi \) and hence also has an exponential dichotomy on \((-\infty, \infty)\).

Now let

\[
a = (...) a_{-1}, a_0, a_1, ...
\]

be in the Cantor set \( \mathcal{S}_2 \), the set of doubly infinite sequences of zeros and ones, and write for \( k \) in \( \mathbb{Z} \)

\[
b_k = \sum_{i=0}^{k} (2a_i - 1).
\]
Then for each integer \( k \)

\[
v_k(t) = \begin{cases} 
  z^+(t) + (2b_{k-1} \pi, 0) & \text{if } a_k = 1 \\
  z^-(t) + (2b_k \pi, 0) & \text{if } a_k = 0 
\end{cases}
\]

is a solution with variational equation the same as for \( z^+(t) \) or \( z^-(t) \) such that

\[
|v_k(t) - u_{k-1}(t)| \to 0 \quad \text{as } t \to -\infty, \quad |v_k(t) - u_k(t)| \to 0 \quad \text{as } t \to \infty
\]

uniformly with respect to \( k \), where

\[
u_k(t) = z_0(t) + (2b_k \pi, 0)
\]

is a \( 2\pi \)-periodic solution.

So we can apply Theorem 3.5 to deduce the existence of a number \( \varepsilon_0 > 0 \) and a function \( M(\varepsilon) \) (independent of \( a \)) such that if \( 0 < \varepsilon \leq \varepsilon_0 \) and \( m \geq M(\varepsilon) \) there is a unique solution \( z_a(t) \) of the system corresponding to (46) such that for all integers \( k \)

\[
|z_a(t + (2k - 1)\pi(t)) - v_k(t)| \leq \varepsilon \quad \text{if } -m\pi \leq t \leq m\pi.
\]

Physically this means that the pendulum rotates counterclockwise during the time segment \( [(2k - 2)m\pi, 2km\pi] \) if \( a_k = 1 \) and clockwise if \( a_k = 0 \).

One could prove as in Corollary 3.6 that the mapping \( a \to z_a(0) \) is a homeomorphism of \( S^2 \) onto a subset of \( \mathbb{R}^2 \) on which the time \( 2m\pi \)-map is invariant and isomorphic to the Bernoulli shift on \( S^2_2 \).

On the cylinder, the natural phase space for (46), the period map has a saddle point and two distinct transversal homoclinic points associated with it. We have shown the existence of orbits of the period map which switch randomly from following the orbit of one homoclinic point to following the orbit of the other.

**References**