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Hilbert's 16th problem for classical Liénard equations of even degree

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Abstract

Classical Liénard equations are two-dimensional vector fields, on the phase plane or on the Liénard plane, related to scalar differential equations $\ddot{x} + f(x)\dot{x} + x = 0$. In this paper, we consider f to be a polynomial of degree $2l - 1$, with l a fixed but arbitrary natural number. The related Liénard equation is of degree $2l$. We prove that the number of limit cycles of such an equation is uniformly bounded, if we restrict f to some compact set of polynomials of degree exactly $2l - 1$. The main problem consists in studying the large amplitude limit cycles, of which we show that there are at most l.

MSC: 34C07; 34C37; 34D10

Keywords: Classical Liénard equation; Limit cycle; Heteroclinic connection; Cyclicity

1. Introduction

Hilbert's 16th problem [4] asks for the maximum number of limit cycles that a polynomial vector field, for a given degree, in the plane can have. Although the problem is more than 100 years old it is not even known whether a uniform upper bound, only depending on the degree of the vector field, might exist, even not when the degree is two. In the year 2000, S. Smale added the question to his list of problems for the 21st century [8], but restricting it to the classical (polynomial) Liénard equations.

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Classical (polynomial) Liénard equations are planar differential equations associated to the second order scalar differential equations

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{1}$$

with f a polynomial of degree n.

In the phase plane the representation of the Liénard equation is given by

$$y\frac{\partial}{\partial x} - \big(x + f(x)y\big)\frac{\partial}{\partial y}. \tag{2}$$

In the so-called Liénard plane it is given by

$$\big(y - F(x)\big)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \tag{3}$$

with $F(x) = \int_0^x f(u)\,\mathrm{d}u$. In both cases the Liénard equation is of degree $n+1$ when f is of degree n.

In this paper we will focus on the problem of how to prove the existence of a finite upper bound on the number of limit cycles that a Liénard equation of degree $n+1$ can have, only depending on the degree n.

In [7] it has been proven that for even n, hence for Liénard equations of odd degree, a positive answer to the question can be given, if the same result is true for the slow–fast Liénard systems $(y - F(x))\frac{\partial}{\partial x} - \varepsilon x\frac{\partial}{\partial y}$, with $\varepsilon > 0$, ε sufficiently small. These systems, small perturbations of the layer equations $(y - F(x))\frac{\partial}{\partial x}$, are related to the scalar differential equations $\varepsilon\ddot{x} + f(x)\dot{x} + x = 0$, for similar $\varepsilon > 0$. We can even limit $f(x)$ to

$$f(x) = x^{2l} + \sum_{i=0}^{2l-1}\lambda_i x^i \quad \text{with} \sum_{i=0}^{2l-1}\lambda_i^2 = 1.$$

In this paper we want to prove a similar result for Liénard equations of even degree. For Liénard equations of odd degree the result is not so hard to obtain, due to the fact that—except near the layer equations—the circle at infinity in that case is a uniform repellor. The rest is merely a consequence of the analyticity of the return map around the unique singularity.

In case n is odd, hence for Liénard equations of even degree, the situation is more complicated, since now limit cycles with an amplitude near infinity can be created and this process does not seem to be subject to an analytic description.

The best that one seems able to do, in case $n = 2l - 1$, with $l \geqslant 1$, is to compactify the plane to the appropriate Poincaré–Lyapunov disc, which in this case is the disc $D^{(1,2l)}$ (see [2,3] and also [1]).

By this compactification there is a possibility of encountering a heteroclinic connection Γ between two semi-hyperbolic saddles at infinity. Together with part of the circle at infinity, this gives rise to a (non-hyperbolic) two-saddle cycle. From it, large amplitude limit cycles can be perturbed.

In this paper we make a complete study of these limit cycles, providing precise cyclicity results for these two-saddle cycles.

The precise results on these non-hyperbolic two-saddle cycles will be stated in Theorems 9 and 10, that can be found in Section 8. We prefer not to state these results now, to avoid having to introduce extra notation for the moment. In any case we can formulate a precise result, that we call our 'main theorem,' and that is an immediate consequence of Theorems 10 and 9.

We will state it in terms of the vector space $\mathcal{P}_{2l-1}(\mathbb{R})$ of polynomials $f(x)$ of degree at most $2l-1$, on which we consider, as usual, the coefficient topology. We also state the theorem for Liénard systems on the phase plane, in terms of expression (2), knowing that it would make no difference to work on the Liénard plane with systems written in expression (3). $B_R(0)$ will denote the ball around the origin having radius R.

Theorem 1 *(Main Theorem). Let $K \subset \mathcal{P}_{2l-1}(\mathbb{R})$ be compact and consisting of polynomials of degree exactly $2l-1$, then there exists $R > 0$ such that any system X having an expression* (2) *with $f \in K$ has at most l limit cycles having intersection with $\mathbb{R}^2 \setminus B_R(0)$.*

We call such limit cycles, having an intersection with $\mathbb{R}^2 \setminus B_R(0)$, 'large amplitude' limit cycles. In a less precise way of stating the main theorem, we might say that in a uniform way (as long as we keep the polynomials f, of degree exactly $2l-1$, in a compact region of the space of polynomials) the number of 'large amplitude' limit cycles is bounded by l. This bound is presumably not a sharp one. It is however not so bad since from [5] it can be expected that it will be at least $l-1$. Of course [5] does not deal with large amplitude limit cycles, but rather with small amplitude limit cycles.

An interesting consequence of Theorem 1, is the following result, which we again state in terms of expression (2), but which is equally valid for expression (3).

Theorem 2. *Let $K \subset \mathcal{P}_{2l-1}(\mathbb{R})$ be compact and consisting of polynomials of degree exactly $2l-1$, then there exists $N \in \mathbb{N}$ such that any system X having an expression* (2) *with $f \in K$ has at most N limit cycles.*

As shown in [7] a similar statement holds for systems (2) with f of even degree. Theorem 2 follows from Theorem 1 in exactly the same way as the similar result in [7] has been proven, namely by observing that in the finite plane the return map around the origin, with respect to the positive x-axis, is analytic both in x and f.

The paper is organized as follows. In Sections 2 and 3, we specify the Liénard system we work with, and its compactification, respectively. In Section 4, if the non-hyperbolic 2-saddle cycle Γ of the compactification exists, we introduce the difference map Δ. Small (isolated) positive zeroes w of Δ correspond to large amplitude limit cycles of the Liénard system, in such a way that $w = 0$ corresponds to Γ. The difference map is described by two corner transitions near the semi-hyperbolic saddles and two regular transitions, that are studied separately in Sections 5 and 6, respectively. In Section 7, we study the asymptotics of the difference map Δ; using the results of Sections 5 and 6, we find that Δ is exponentially flat at $w = 0$, and we introduce a so-called reduced difference map $\bar{\Delta}$; $\bar{\Delta}$ is obtained from Δ after some algebraic manipulations and a derivation with respect to w. In this way, zeroes of $\bar{\Delta}$ represent zeroes of $\frac{\partial \Delta}{\partial w}$. Hence, an upper bound of the cyclicity near Γ can be found by studying the asymptotics of $\bar{\Delta}$, as we do in Section 8.

2. Settings

As announced in the introduction, we suppose that the Liénard equation is of even degree. Hence, the function f in Eq. (1) or the equivalent planar system (2), is of odd degree, say $2l-1$, with $l \geqslant 1$. Without loss of generality, we can suppose that the function f is given by

$$f(x,a) = 2lx^{2l-1} + (2l-1)a_1x^{2l-2} + \cdots + ia_{2l-i}x^{i-1} + \cdots + 2a_{2l-2}x + a_{2l-1}, \tag{4}$$

with real parameters $a_1, a_2, \ldots, a_{2l-1} \in \mathbb{R}$. Indeed, if $f(x,\alpha) = \sum_{i=1}^{2l} \alpha_{2l-i}x^{i-1}$ with $\alpha_0 \neq 0$, then one can perform the linear transformation on $\mathbb{R}^{2l+1}$:

$$\begin{cases} \bar{t} = \operatorname{sgn}(\alpha_0)t, \\ \bar{x} = \sqrt[2l]{\dfrac{|\alpha_0|}{2l}}x, \\ \bar{\alpha}_i = \dfrac{\alpha_i}{2l-i}\left(\sqrt[2l]{\dfrac{2l}{|\alpha_0|}}\right)^{2l-i-1}, \quad i = 1, \ldots, 2l-1, \end{cases}$$

in order that $f(\bar{x}, \bar{\alpha})$ takes the form (4). Equivalently, the function F in (3) can be supposed to be given by

$$F(x,a) = x^{2l} + a_1x^{2l-1} + \cdots + a_{2l-2}x^2 + a_{2l-1}x \equiv x^{2l} + g(x,a), \tag{5}$$

where $a = (a_1, \ldots, a_{2l-1}) \in \mathbb{R}^{2l-1}$. In this paper, system (3) will systematically be written as

$$X_a^{(2l)} \leftrightarrow \begin{cases} \dot{x} = y - (x^{2l} + g(x,a)), \\ \dot{y} = -x, \end{cases} \tag{6}$$

with $l \geqslant 1$ and where g is the polynomial defined in (5). In this paper, we will refer to system (6) as $X_a^{(2l)}$.

In case that $a_{2i+1} = 0, \forall 0 \leqslant i \leqslant l-1$, system (6) represents a center. Indeed, in that case the system is symmetric with respect to $(x,y,t) \mapsto (-x,y,-t)$; hence it is time-reversible. It is also easy to see, and it will in fact also follow from our calculations that in all other cases system (6) is not a center.

Let us finally remark that the study of the Liénard equation

$$\ddot{x} + (\beta_0x + \beta_1) + \left(\alpha_0x^{2l-1} + \alpha_1x^{2l} + \cdots + \alpha_{2l-i}x^{i-1} + \cdots + \alpha_{2l-2}x + \alpha_{2l-1}\right)\dot{x} = 0,$$

with $\alpha_i, \beta_j \in \mathbb{R}$, $\forall 0 \leqslant i \leqslant 2l-1$, $j = 0, 1$, and $\alpha_0\beta_0 \neq 0$, can be reduced to the study of (1), by performing a parameter-dependent translation in the Liénard plane.

3. Compactification

As we are interested in large amplitude limit cycles, we use a Poincaré–Lyapunov compactification of type $(1, 2l)$. Since this construction plays an essential role in this paper, including some interesting peculiarities, we will briefly recall the construction.

The main idea consists in extending $X_a^{(2l)}$ in an analytic way by defining an interesting extension on the circle at infinity. This is done by changing, near infinity, the coordinates (x, y) as

$$\begin{cases} x = \bar{x}/S, \\ y = \bar{y}/S^{2l}, \end{cases}$$

with $S > 0$, $\bar{x}^2 + \bar{y}^2 = 1$. The resulting vector field is multiplied by S^{2l-1}, and is denoted by $\bar{X}_a^{(2l)}$. It is defined on a disc that we denote by $D^{(1,2l)}$, to specify the construction that we used. As usual when working with an abstractly defined manifold, it is better to perform the calculations in well chosen simple charts.

More precisely, for the problem under consideration, we could study system (6) near infinity in the positive y-direction by introducing the variables (u, s) with

$$\begin{cases} x = u/s, \\ y = 1/s^{2l}, \end{cases} \quad u \in \mathbb{R},\ s > 0. \tag{7}$$

Multiplying the result with a factor s^{2l-1}, this yields the family

$$\hat{X}_a^{(2l)} \leftrightarrow \begin{cases} \dot{u} = 1 - u^{2l} - s^{2l} g(\frac{u}{s}, a) + (2l)^{-1} u^2 s^{4l-2}, \\ \dot{s} = (2l)^{-1} u s^{4l-1}. \end{cases} \tag{8}$$

We note that, in the equation for $\dot{u}$,

$$s^{2l} g\left(\frac{u}{s}, a\right) = a_1 u^{2l-1} s + a_2 u^{2l-2} s^2 + \cdots + a_{2l-1} u s^{2l-1}.$$

We remark that (8) does not exactly describe $\bar{X}_a^{(2l)}$ in the chart under consideration, given by (7), but it expresses $\zeta \cdot \bar{X}a^{(2l)}$, where $\zeta(u, s) > 0$.

Recall that $(u_0, 0)$ corresponds to a point at ∞; u_0 gives the 'direction' at ∞, defined by the curve $x^{2l} = u_0^{2l} y$.

System (8) has two singularities at ∞: $(-1, 0)$ and $(1, 0)$. Let us denote the corresponding singularities at ∞ by $\mathbf{s}_-$ and $\mathbf{s}_+$, respectively. Both singularities at ∞ are semi-hyperbolic, since the corresponding linear parts at $(u_0, 0)$ with $u_0 = \pm 1$ are given by

$$\begin{bmatrix} -2lu_0 & * \\ 0 & 0 \end{bmatrix}, \quad \text{where } * = \left.\frac{\partial \dot{u}}{\partial s}\right|_{(u_0, 0)}.$$

To understand the topological behaviour of the semi-hyperbolic singularities, we need to calculate the behaviour on a center manifold. The topological behaviour of the singularities is not influenced by use of another chart at infinity.

Therefore, to keep the calculations as clear as possible, we can use the chart in the positive x-direction to understand the behaviour near $\mathbf{s}_+$ and analogously, the chart in the negative x-direction to understand the behaviour near $\mathbf{s}_-$. Furthermore, in this way, all possible singularities at ∞ will be detected, since in a chart in the negative y-direction, there are no singularities at ∞.

To study the behaviour of $\bar{X}_a^{(2l)}$, and hence of system (6) in the positive x-direction we use the chart

$$\begin{cases} x = 1/r, \\ y = \upsilon/r^{2l}, \end{cases} \quad \upsilon \in \mathbb{R},\ r > 0. \tag{9}$$

in which system (6) is expressed by

$$\begin{cases} \dot{\upsilon} = -r^{4l-2} - 2l\upsilon\big((\upsilon - 1) - r^{2l} g\big(r^{-1}, a\big)\big), \\ \dot{r} = -r\big((\upsilon - 1) - r^{2l} g\big(r^{-1}, a\big)\big). \end{cases} \tag{10}$$

Again (10) does not exactly describe $\bar{X}_a^{(2l)}$ in the chart under consideration, given by (9), but it expresses $\xi \cdot \bar{X}_a^{(2l)}$, where $\xi(\upsilon, r) > 0$.

System (10) has two singularities at ∞: $(0, 0)$ and $(1, 0)$. Their topological type is easily studied: $(0, 0)$ is a repelling node and $(1, 0)$ is semi-hyperbolic. We are interested in the last one, since it corresponds to $\mathbf{s}_+$. This singularity is brought into the origin by performing the translation $(V, r) = (\upsilon - 1, r)$, resulting in the translated system

$$\begin{cases} \dot{V} = -r^{4l-2} - 2l(V + 1)\big(V - r^{2l} g\big(r^{-1}, a\big)\big), \\ \dot{r} = -r\big(V - r^{2l} g\big(r^{-1}, a\big)\big), \end{cases}$$

with corresponding linear part

$$\begin{bmatrix} -2l & * \\ 0 & 0 \end{bmatrix}, \quad \text{where } * = \frac{\partial \dot{V}}{\partial r}\bigg|_{(1,0)}.$$

To understand the behaviour of this semi-hyperbolic singularity $\mathbf{s}_+$, we reduce the study to the one on the center manifold, that locally can be written as a graphic $W^c = \{(V(r), r)\colon r$ small and positive$\}$ with

$$V(r) = r^{2l} g\big(r^{-1}, a\big) - \frac{1}{2l} r^{4l-2} + O\big(r^{4l}\big), \quad r \to 0.$$

Hence, the behaviour on the center manifold is given by

$$\dot{r} = \frac{1}{2l} r^{4l-1} + O\big(r^{4l}\big), \quad r \to 0. \tag{11}$$

An analogous result will lead to the behaviour around $\mathbf{s}_-$. In fact it is interesting, and it will even be essential in the further calculation, to observe that the study near $\mathbf{s}_-$ can be done near $\mathbf{s}_+$.

We indeed observe that instead of looking in the negative x-direction by considering the chart

$$\begin{cases} x = -1/r, \\ y = \upsilon/r^{2l}, \end{cases} \quad \upsilon \in \mathbb{R},\ r > 0,$$

we can as well use (9), followed by $r \mapsto -r$. It means that the behaviour of $\bar{X}_a^{(2l)}$ near $\mathbf{s}_-$ on the side $r > 0$ is hence exactly given by the behaviour of $-\bar{X}_a^{(2l)}$ (since we have multiplied by r^{2l-1}) near $\mathbf{s}_+$ on the side $r < 0$. The phase portrait of $\bar{X}_a^{(2l)}$ is shown in Fig. 1.

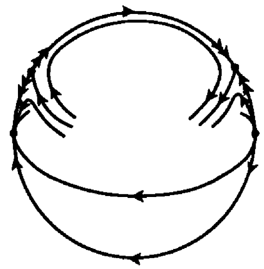

Fig. 1. Behaviour near infinity.

In the coordinates (u,s), leading to the expression (8), we will denote the vector field by $\hat{X}_a^{(2l)}$; recall that it is equal to $\zeta(u,s)\cdot\bar{X}_a^{(2l)}$ with $\zeta(u,s)>0$.

The phase portrait of $\hat{X}_a^{(2l)}$ covers the semi-hyperbolic saddle points $\mathbf{s}_+$ and $\mathbf{s}_-$, and the connection at infinity in between them. In the rest of the paper, we will often work with system $\hat{X}_a^{(2l)}$, given by (8).

From the equations in (8), it is clear that the flow of $\hat{X}_a^{(2l)}$ is invariant under the transformation

$$(t,u,s)\mapsto(-t,-u,-s).$$

So also in these coordinates it is clear that the behaviour of the flow of $-\hat{X}_a^{(2l)}$ near $\mathbf{s}_-$ in the region $\{(s,u)\colon\ s>0,\ -1<u<-u_0\}$, is found precisely in the behaviour of the flow of $\hat{X}_a^{(2l)}$ near $\mathbf{s}_+$ in the region $\{(s,u)\colon\ s<0,\ u_0<u<1\}$.

4. Difference map

From Section 3, we conclude that system $\bar{X}_a^{(2l)}$ has four singularities at ∞: an attracting node, a repelling node and two semi-hyperbolic saddle points (see Fig. 1). Furthermore, $\mathbf{s}_-$ is connected to $\mathbf{s}_+$ by an invariant manifold at ∞, that we denote by Γ_1. In coordinates (u,s), in which we prefer to consider system $\bar{X}_a^{(2l)}$, the connection Γ_1 is given by the set

$$\big\{(u,0)\colon\ -1\leqslant u\leqslant 1\big\}.$$

The only possibility to find large amplitude limit cycles is to investigate perturbations of limit periodic sets Γ of system $\bar{X}_a^{(2l)}$, that contain the semi-hyperbolic saddle points $s_\pm$ and the connection Γ_1 in between them. System (6) has only one singularity, viz. the Hopf singularity situated at the origin. Therefore, the only possibility for Γ to be a limit periodic set containing $\Gamma_1\cup\{\mathbf{s}_+,\mathbf{s}_-\}$, is that Γ would be a 2-saddle cycle, surrounding the origin.

We denote by $\Gamma_{2,a}^+$ the local center manifold at $\mathbf{s}_+$ and $\Gamma_{2,a}^-$ the local center manifold at $\mathbf{s}_-$. In order to permit the creation of limit cycles we suppose, that, for certain parameter values $a=a^0$, Γ_{2,a^0}^- and Γ_{2,a^0}^+ belong to a heteroclinic connection between $\mathbf{s}_-$ and $\mathbf{s}_+$; we call it Γ_{2,a^0}.

Let us stress that the invariant manifold Γ_1, connecting $\mathbf{s}_-$ and $\mathbf{s}_+$ exists and remains fixed in the phase portrait of $\bar{X}_a^{(2l)}$, as well as $\hat{X}_a^{(2l)}$, for all parameter values a; the connection $\Gamma_{2,a}$, if it exists, depends on the parameter a, also the invariant manifolds $\Gamma_{2,a}^+$ and $\Gamma_{2,a}^-$ change in general, when the parameter a is varied.

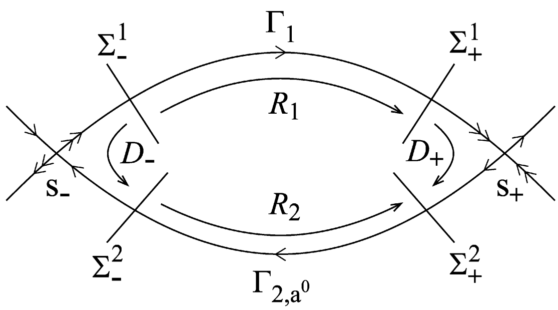

Fig. 2. Non-hyperbolic two-saddle cycle.

Remark 3. In the rest of the paper, we suppose that a^0 is an arbitrary but fixed parameter with the property that Γ_{2,a^0} exists as a heteroclinic connection, and we write $\Gamma_2 \equiv \Gamma_{2,a^0}$, $\Gamma \equiv \Gamma_{a^0}$.

We would like to investigate limit cycles that can perturb from $\Gamma_{a^0} = \{\mathbf{s}_+, \mathbf{s}_-\} \cup \Gamma_1 \cup \Gamma_{2,a^0}$. Therefore, we introduce an analogous tool as the traditional Poincaré map near Γ and for $a \sim a^0$. Since we would like to describe the transition along Γ, Γ being an elementary graphic containing two (semi-hyperbolic) singularities, it is more convenient to split up this transition into four parts: corner passages $\mathcal{D}_\pm$ near $s_\pm$ and regular transitions $\mathcal{R}_1$ and $\mathcal{R}_2$ in between the corners.

To be more precise, suppose that sections $\Sigma^i_\pm$ are transverse to Γ_i near $s_\pm$, $i = 1, 2$, and that these sections are parametrized by a given regular parameter. Furthermore, if the regular parameter on Σ^1_- is denoted by w, $w > 0$, then we suppose that the intersection $\Gamma_1 \cap \Sigma^1_-$ corresponds to $w = 0$. Let the map $\mathcal{D}_\pm$ describe the corner passage near $s_\pm$ from $\Sigma^1_\pm$ to $\Sigma^2_\pm$, defined by the flow of $\pm\bar{X}^{(2l)}_a$. Let the map $\mathcal{R}_1$ (respectively $\mathcal{R}_2$) describe the regular transition near Γ_i from Σ^i_- to Σ^i_+, $i = 1$ (respectively $i = 2$), defined by the flow of $\bar{X}^{(2l)}_a$ (respectively $-\bar{X}^{(2l)}_a$); see Fig. 2. Let us remark that the definition of the map $\mathcal{R}_2$ does only make sense near values a^0 for which the connection Γ_{2,a^0} exists. Suppose that these transition maps are expressed in the regular parameter, that is given for the corresponding sections $\Sigma^i_\pm$, $i = 1, 2$. Then, a so-called difference map Δ, can be expressed, using the regular parameter chosen on Σ^1_-, as follows:

$$\Delta^a : \Sigma^1_- \to \mathbb{R} : w \mapsto \Delta(w, a) = \Delta^a(w) = \left(\mathcal{R}^a_2 \circ \mathcal{D}^a_- - \mathcal{D}^a_+ \circ \mathcal{R}^a_1\right)(w). \tag{12}$$

We stress the dependence on the parameter a, by adding it in the notation as super-index: e.g., $\Delta^a \equiv \Delta(\cdot, a)$, $\mathcal{D}^a_\pm$ and $\mathcal{R}^a_i$ $(i = 1, 2)$.

It is clear that small zeroes w of Δ^a, $w \downarrow 0$, represent large amplitude limit cycles of system $X^{(2l)}_a$, and their number near any value $a = a^0$, for which a connection Γ_2 exists, is given by $\mathrm{Cycl}(\bar{X}^{(2l)}_a, (\Gamma, a^0))$, which is equal to the least upper bound for the number of isolated zeroes w of Δ^a, for $w \downarrow 0$, $a \to a^0$.

To compute or estimate this cyclicity, it is convenient to apply a division-derivation algorithm to Δ^a, based on Rolle's theorem, see e.g. [6]. Therefore, we start by investigating the differentiability properties and asymptotics of the respective transition maps: the corner passage is studied in Section 5 and the regular transition along the connection Γ_1 in Section 6. There is no need to pay special attention to the regular transition along the connection Γ_2, as will become clear in Section 7.

5. Corner passage near the semi-hyperbolic saddle points

The corner passages near the semi-hyperbolic saddle points $s_\pm$ of (8) are rather difficult to calculate in coordinates (u,s). To describe the corner passage along a semi-hyperbolic saddle point, we introduce smooth normal form coordinates as well as a smooth time change, bringing system (8) into the C^∞ normal form of type

$$X^{\mathrm{norm}}_{(\alpha,2l)} \leftrightarrow \begin{cases} \dot{z} = -z, \\ \dot{w} = w^{4l-1}(1+\alpha w^{4l-2})^{-1}, \end{cases} \quad \alpha \in \mathbb{R}. \tag{13}$$

A clear and detailed study of normal forms near semi-hyperbolic saddles can be found in [1]. Although the study in [1] is presented there for individual vector fields, the result can be applied here for the family (8), since the semi-hyperbolic saddle points remain fixed and their codimension is unaltered by perturbation.

By the symmetry property of $\hat{X}_a^{(2l)}$, described in Section 3, we only need to introduce local coordinates near $\mathbf{s}_+$ bringing the family $\hat{X}_a^{(2l)}$ into the normal form $X^{\mathrm{norm}}_{(\alpha(a),2l)}$. Indeed, by the invariance of the phase portrait of the vector field $\hat{X}_a^{(2l)}$ with respect to the map $(t,u,s) \mapsto (-t,-u,-s)$, the local unstable manifold of $\mathbf{s}_+$ inside the half plane $\{s<0\}$ is given by

$$\tilde{\Gamma}^+_{2,a} = \{(u,s)\colon (-u,-s) \in \Gamma^-_{2,a}\};$$

furthermore, for $i=1,2$, the set $\tilde{\Sigma}^i_+ = \{(u,s)\colon (-u,-s) \in \Sigma^i_-\}$ is transverse to $\tilde{\Gamma}^+_{i,a}$ (where $\tilde{\Gamma}^+_{1,a} = \Gamma_1$). Suppose now that the transition map $\tilde{\mathcal{D}}$ is expressed in the local coordinates (u,s), and that it describes the corner passage near $\mathbf{s}_+$ defined by the flow of $\hat{X}_a^{(2l)}$, inside the half plane $\{s<0\}$, intersecting subsequently the transverse sections $\tilde{\Sigma}^1_+$ and $\tilde{\Sigma}^2_+$. Then, again by this symmetry property of $\hat{X}_a^{(2l)}$, the corner passage $\mathcal{D}_-$ near $\mathbf{s}_-$ in the half plane $\{s>0\}$ is given by

$$\mathcal{D}_- : \Sigma^1_- \to \Sigma^2_- : (u,s) \mapsto -\tilde{\mathcal{D}}(-u,-s). \tag{14}$$

From (11), we see that the semi-hyperbolic singularity $\mathbf{s}_+$ is of codimension $4l-2$. Therefore, there exist a compact neighbourhood $\mathcal{W}$ of $a^0 \in \mathbb{R}^{2l-1}$, a neighbourhood $\mathcal{U}$ of $\mathbf{s}_+ \in \mathbb{R}^2$, a smooth family $(\varphi^a : \mathcal{U} \to \varphi^a(\mathcal{U}))_{a\in\mathcal{W}}$ of coordinate transformations, a smooth family $(h^a : \mathcal{U} \to \mathbb{R})_{a\in\mathcal{W}}$ of strictly positive functions, and a polynomial $\alpha : \mathbb{R}^{2l-1} \to \mathbb{R}$ such that

$$\forall a \in \mathcal{W}\colon \varphi^a(\mathbf{s}_+) = (0,0)$$

and such that the system $h^a \cdot \hat{X}_a^{(2l)}|_{\mathcal{U}}$ is transformed by φ^a into the normal form $X^{\mathrm{norm}}_{(\alpha(a),2l)}|_{\varphi^a(\mathcal{U})}$, preserving the direction of the flow. The polynomial dependence of α on the parameter a can be checked by repeating the proofs in [1] for the specific family of planar vector fields, given by (8). We will often omit the dependence of φ^a on a in the notation, and simply write φ instead of φ^a. Furthermore, without loss of generality we can suppose that $[0,1]\times[0,1] \subset \varphi(\mathcal{U})$ (by performing a dilatation in the phase plane, if necessary).

Recall that in the normal form coordinates (z,w), defined by φ, the semi-hyperbolic saddle point $\mathbf{s}_+$ is located at the origin. Near the origin, in the coordinates (z,w), the positive z-axis corresponds to the connection Γ_1, while the positive (respectively negative) w-axis corresponds to the connection $\Gamma^+_{2,a}$ (respectively $\tilde{\Gamma}^+_{2,a}$).

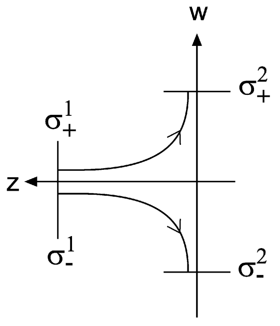

Fig. 3. Corner passage in normal form.

In the (z,w)-plane, we choose transverse sections $\sigma^i_\pm$ with respect to $\Gamma_i^\pm$, for $i=1,2$, as follows (see Fig. 3). By compactness of $\mathcal{W}$, we can take a neighbourhood $\mathcal{V}$ of $(0,0)$ in $\mathbb{R}^2$ such that $\varphi^a(\mathcal{U}) \subset \mathcal{V}$, $\forall a \in \mathcal{W}$; then we define

$$\sigma^1_\pm = \{(1, \pm w)\colon\ w \geqslant 0\} \cap \mathcal{V} \quad \text{and} \quad \sigma^2_\pm = \{(z, \pm 1)\colon\ z \geqslant 0\} \cap \mathcal{V}. \tag{15}$$

Fix an arbitrary $a \in \mathcal{W}$. Then, in normal form coordinates (z,w), the corner passage near $\mathbf{s}_+$ in the half plane $\{s>0\}$, writes

$$D_+ = D_+^{\alpha(a)} : \sigma^1_+ \to \sigma^2_+ : (1, w) \mapsto \big(D(w), 1\big),$$

where $D = D^{\alpha(a)}$ is defined by the following integral equation:

$$\int_w^1 \frac{(1+\alpha v^{4l-2})}{v^{4l-1}}\,\mathrm{d}v = -\int_1^{D(w)} \frac{1}{z}\,\mathrm{d}z,$$

where $\alpha = \alpha(a)$. This elementary integral can easily be solved, and one finds

$$\frac{1}{-4l+2} - \frac{w^{-4l+2}}{-4l+2} - \alpha \ln w = -\ln D(w).$$

Hence, we have the following explicit expression for $D(w)$,

$$D(w) = w^\alpha \exp\left(\frac{1 - w^{-4l+2}}{4l-2}\right). \tag{16}$$

From the equations in (13), it follows that the flow of $X^{\mathrm{norm}}_{(\alpha,2l)}$ is invariant under the reflections $(t,z,w) \mapsto (t,z,-w)$ and $(t,z,w) \mapsto (t,-z,w)$. As a consequence, in normal form coordinates, the corner passage near $\mathbf{s}_+$ in the half plane $\{w<0\}$, is given by

$$D_- = D_-^\alpha : \sigma^1_- \to \sigma^2_- : (1, w) \mapsto \big(D(-w), -1\big).$$

In the rest of the paper, we suppose that the transverse sections $\Sigma^i_\pm$ are chosen as follows:

$$\Sigma^i_+ = \varphi^{-1}\big(\sigma^i_+\big), \quad i = 1, 2, \tag{17}$$

and

$$\Sigma^i_- = \big\{(u, s)\colon \varphi(-u, -s) \in \sigma^i_-\big\}, \quad i = 1, 2. \tag{18}$$

Notice that by this definition the sections $\Sigma^i_\pm = \Sigma^i_{\pm a}$ depend on a. Clearly, we can also choose the absolute value of the w-coordinate as regular parameter on the sections $\Sigma^1_\pm$, and the z-coordinate as regular parameter on the sections $\Sigma^2_\pm$. Now, the corner passage $\mathcal{D}_+ : \Sigma^1_+ \to \Sigma^2_+$ (respectively $\mathcal{D}_- : \Sigma^1_- \to \Sigma^2_-$) can be described by the (one-dimensional) Dulac map D, since

$$\mathcal{D}_+ : \Sigma^1_+ \to \Sigma^2_+ : (u, s) \mapsto \varphi^{-1}\big(D_+\big(\varphi(u, s)\big)\big)$$

and respectively, by the remark above in (14),

$$\mathcal{D}_- : \Sigma^1_- \to \Sigma^2_- : (u, s) \mapsto -\varphi^{-1}\big(D_-\big(\varphi(-u, -s)\big)\big).$$

Remark that the maps $\mathcal{D}_\pm$ depend smoothly on a through the smooth dependence of $\varphi = \varphi^a$ and $D_\pm = D^{\alpha(a)}_\pm$.

6. Regular transition along the connection Γ_1

In this section we will study the asymptotics of the regular transition $\mathcal{R}_1$ along Γ_1 in terms of the normalizing coordinates introduced in Section 5.

Therefore, let us recall some essential notations. There exist a compact neighbourhood $\mathcal{W}$ of a^0 in $\mathbb{R}^{2l-1}$, and a smooth family of coordinate transformations $(\varphi^a : \mathcal{U} \subset \mathbb{R}^2 \to \varphi^a(\mathcal{U}))_{a\in\mathcal{W}}$ bringing the system $h^a \cdot \hat{X}^{(2l)}_a|_{\mathcal{U}}$ into the normal form $X^{\text{norm}}_{(\alpha(a),2l)}|_{\varphi(\mathcal{U})}$, for a certain smooth family of strictly positive functions $(h^a : \mathcal{U} \to \mathbb{R})_{a\in\mathcal{W}}$. Let $\Sigma^1_{\pm,a}$ be the transverse sections defined by the normalizing coordinates in (15), (17) and (18). In this section, we work with an arbitrary but fixed parameter $a \in \mathcal{W}$; therefore, we will often omit the dependence of the transition map on the parameter and we write $\Sigma^1_\pm = \Sigma^1_{\pm,a}$.

Using the normalizing coordinates (z, w), the transition map $\mathcal{R}_1$ can also be described by a 1-dimensional map $z = R_1(w)$, as is shown by the following diagram:

$$\begin{tikzcd}
(1, -w) \in \sigma^1_- \arrow[r] \arrow[d, "-\varphi^{-1}"'] & (1, R_1(w)) \in \sigma^1_+ \\
\Sigma^1_- \arrow[r, "\mathcal{R}_1"'] & \Sigma^1_+ \arrow[u, "\varphi"']
\end{tikzcd}$$

Equivalently, the maps R_1 and $\mathcal{R}_1$ are related by

$$\varphi\big(\mathcal{R}_1\big(-\varphi^{-1}(1, -w)\big)\big) = \big(1, R_1(w)\big), \quad (1, -w) \in \sigma^1_-. \tag{19}$$

Since the map $R_1 = R_1^a$ is C^∞ in (w, a), we can consider the infinite jet of R_1 with respect to w at $w = 0$:

$$j_\infty\big(R_1^a\big)_0(w) = \sum_{j=1}^{\infty} \frac{1}{j!} \frac{\partial^j R_1^a}{\partial w^j}(0) w^j.$$

In Proposition 4 below, we state an expression for the derivative $\frac{\partial^j R_1^a}{\partial w^j}(0)$ in terms of the parameter a. Since the calculations are rather lenghtly and technical, we leave it out of this section, and place it in a separate section at the end of the paper (Section 9).

Proposition 4. *Consider the map R_1, as introduced in* (19)*, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, and that is expressed in terms of the normalizing parameter $w > 0$. Then,*

$$R_1^a(0) = 0, \qquad \frac{\partial R_1^a}{\partial w}(0) = 1, \qquad \frac{\partial^2 R_1^a}{\partial w^2}(0) = \frac{-2\sqrt[2l-1]{2l}}{l(4l-3)} a_1,$$

and if $1 \leqslant k \leqslant l-1$ is such that

$$a_{2i+1} = 0, \quad \forall 0 \leqslant i \leqslant k-1,$$

then

$$\begin{cases} \dfrac{\partial^n R_1}{\partial w^n}(0) = 0, \quad \forall 2 \leqslant n \leqslant 2k+1, \\ \dfrac{1}{(2k+2)!} \dfrac{\partial^{2k+2} R_1}{\partial w^{2k+2}}(0) = C_k \cdot a_{2k+1} \quad \textit{and} \quad \dfrac{\partial^{2k+3} R_1}{\partial w^{2k+3}}(0) = 0, \end{cases}$$

for

$$C_k = \frac{(2k-1)}{l(4l-2k-3)} \sqrt[2l-1]{(2l)^{2k+1}}, \quad \forall 1 \leqslant k \leqslant l-1. \tag{20}$$

7. Reduced difference map $\bar{\Delta}$

Now we can study the asymptotics of the difference map Δ, defined in Section 4, in terms of the normalizing coordinates introduced in Section 5. Therefore, let us again recall some essential notations from Section 5. Recall that $\mathcal{W} \subset \mathbb{R}^{2l-1}$ is a compact neighbourhood of a^0 and that $(\varphi^a : \mathcal{U} \subset \mathbb{R}^2 \to \varphi^a(\mathcal{U}))_{a \in \mathcal{W}}$ is the smooth family of coordinate transformations bringing the system $h^a \cdot \hat{X}_a^{(2l)}|_{\mathcal{U}}$ into the normal form $X^{\text{norm}}_{(\alpha(a),2l)}|_{\varphi(\mathcal{U})}$, for a certain smooth family of strictly positive functions $(h^a : \mathcal{U} \to \mathbb{R})_{a \in \mathcal{W}}$. Let $\Sigma^1_{\pm,a}$ be the transverse sections defined by the normalizing coordinates in (15), (17) and (18).

In a natural way, the normalizing coordinates can be used to parametrize these sections. Let $D = D^{\alpha(a)}$ be the Dulac map, as introduced in Section 5; let $R_1 = R_1^a$ be the map that describes the regular transition from $\Sigma^1_{-,a}$ to $\Sigma^1_{+,a}$, as considered in Section 6. Let $R_2 = R_2^a$ be the map that describes the regular transition from $\Sigma^2_{-,a}$ to $\Sigma^2_{+,a}$, expressed using the normalizing coordinates.

Then, the difference map Δ, that measures the difference between the first intersection points with $\Sigma^2_{+,a}$, when one follows the flow of $X_a^{(2l)}$ in both positive and negative directions starting at a point on $\Sigma^1_{-,a}$, is in normalizing coordinates expressed by

$$\Delta(w,a) = \Delta^a(w) = \left(R_2^a \circ D^{\alpha(a)} - D^{\alpha(a)} \circ R_1^a\right)(w),$$

where $(w,a) \in \mathrm{dom}(\Delta) \equiv \{(w,a) \in \mathbb{R}\times\mathcal{W}\colon\ w \geqslant 0, -\varphi^a(1,-w) \in \Sigma^1_{-,a}\}$; recall that $-\varphi^a(1,-w) \in \Sigma^1_{-,a}$ if and only if $(1,-w) \in \sigma^1_-$. As follows from Sections 5 and 6, the difference map Δ is C^∞ in $\mathrm{dom}(\Delta)$.

As usual in establishing a bound for the cyclicity, we would like to apply Rolle's division-derivation algorithm on Δ. By Rolle's theorem, system (8) has cyclicity at most $N+1$ if $\frac{\partial}{\partial w}\Delta$ has at most N zeroes in a small neighbourhood of $w=0$, multiplicity taken into account; equivalently,

$$\mathrm{Cycl}\left(\bar{X}_a^{(2l)}, \left(\Gamma, a^0\right)\right) \leqslant N+1,$$

if the equation

$$R_2'\left(D(w)\right)\cdot D'(w) = D'\left(R_1(w)\right)\cdot R_1'(w) \tag{21}$$

has at most N solutions.

Remark that here, in the writing of the maps Δ, $\bar{\Delta}$, D, R_1, $\bar{R}_1$, R_2, we will omit the dependence on the parameter a, and accordingly write α instead of $\alpha(a)$. From a direct calculation, it follows that

$$D'(w) = w^{\alpha-4l+1}\exp\left(\frac{1}{4l-2}\right)\exp\left(\frac{w^{-4l+2}}{-4l+2}\right)\left(1+\alpha w^{4l-2}\right). \tag{22}$$

As one can notice from (22), both sides of (21) are exponentially flat at $w=0$. Therefore, Rolle's division-derivation algorithm cannot be applied in a straightforward way. By removing the exponentially flatness, we introduce here a so-called reduced difference map $\bar{\Delta}$, in such a way that its zeroes represent the roots of (21), and hence the zeroes of $\frac{\partial\Delta}{\partial w}$.

To get rid of the exponentially flatness, one takes the logarithm on both sides of Eq. (21), after suppressing the factor $\exp(-\frac{1}{4l+2})$ on both sides of the equation. In this way, the left-hand side of Eq. (21) is reduced to

$$\log\left(R_2'\left(D(w)\right)\right) + (\alpha-4l+1)\log w + \frac{w^{-4l+2}}{-4l+2} + \log\left(1+\alpha w^{4l-2}\right), \tag{23}$$

and the right-hand side is reduced to

$$(\alpha-4l+1)\log\left(R_1(w)\right) + \frac{(R_1(w))^{-4l+2}}{-4l+2} + \log\left(1+\alpha\left(R_1(w)\right)^{4l-2}\right) + \log\left(R_1'(w)\right). \tag{24}$$

For simplicity of writing, we introduce a smooth function $\bar{R}_1$ by the relation

$$R_1(w) = w\bar{R}_1(w). \tag{25}$$

Notice that $R_2'(0) > 0$, and from Proposition 4, we know that

$$\bar{R}_1(w) = 1 + O(w), \quad w \to 0, \tag{26}$$

and hence,

$$\frac{1}{-4l+2}\bigl(1 - \bigl(\bar{R}_1(w)\bigr)^{-4l+2}\bigr) = O(w), \quad w \to 0. \tag{27}$$

Now we define the C^∞ map $\bar{\Delta}^a$ by

$$\begin{aligned}\bar{\Delta}^a(w) = {} & \frac{1}{-4l+2} \cdot \left(\frac{1 - (\bar{R}_1(w))^{-4l+2}}{w}\right) + w^{4l-3} \log\bigl(R_2'\bigl(D(w)\bigr)\bigr) \\ & - (\alpha - 4l + 1) w^{4l-3} \log\bigl(\bar{R}_1(w)\bigr) - w^{4l-3} \log\bigl(R_1'(w)\bigr) \\ & + w^{4l-3} \log\left(\frac{1 + \alpha w^{4l-2}}{1 + \alpha (R_1(w))^{4l-2}}\right).\end{aligned}$$

Recall that $\bar{\Delta}^a$ depends on the parameter a through the maps $\bar{R}_1$, R_2, D; we also write

$$\bar{\Delta}(w, a) = \bar{\Delta}^a(w). \tag{28}$$

By (27), it follows that there exists $W_0 > 0$ with $[0, W_0[\times\mathcal{W} \subset \mathrm{dom}(\Delta)$ such that the map $\bar{\Delta}$ is well defined on $[0, W_0[\times\mathcal{W}$. By construction, the zeroes of $\bar{\Delta}$ represent the roots of Eq. (21). Therefore, the map $\bar{\Delta}^a$ is obtained from Δ by performing one derivation with respect to w, in a way that zeroes of $\bar{\Delta}^a$ correspond to zeroes of $\frac{\partial \Delta}{\partial w}$. By Rolle's theorem, it follows that the maximum number of zeroes of Δ is at most one more than this number for $\bar{\Delta}$. In order to obtain a good upper bound on the number of limit cycles near Γ for a near a^0, it suffices to count small positive zeroes of $\bar{\Delta}$; we call $\bar{\Delta}$ the *reduced difference map* for the family $(\bar{X}_a^{(2l)})_a$.

Proposition 5. *Consider $X_a^{(2l)}$ as in* (6)*, and let a^0 be a value for which $\bar{X}_{a^0}^{(2l)}$ has a* (*non-hyperbolic*) *2-saddle cycle Γ as defined in Section* 4*. Let $\bar{\Delta} : [0, W_0[\times\mathcal{W} \to \mathbb{R}$ be the reduced difference map as defined in* (28)*. If there exists $0 \leqslant k \leqslant l-1$ such that*

$$a_{2i-1} = 0, \quad \forall 1 \leqslant i \leqslant k, \tag{29}$$

then

$$\bar{\Delta}(w, a) = -C_k \cdot a_{2k+1} w^{2k} + O\bigl(w^{2k+1}\bigr), \quad w \to 0,$$

where C_k is the non-zero constant, defined in (20)*.*

Proof. By (25), Proposition 4 and (29), it follows that

$$\bar{R}_1(w) = 1 + C_k a_{2k+1} w^{2k+1} + O\bigl(w^{2k+2}\bigr), \quad w \to 0.$$

As a consequence,

$$\log\big(\bar{R}_1(w)\big) = O\big(w^{2k+1}\big), \quad w \to 0, \quad \text{and} \quad \log\big(R_1'(w)\big) = O\big(w^{2k+1}\big), \quad w \to 0,$$

and

$$\begin{aligned}\log\left(\frac{1+\alpha w^{4l-2}}{1+\alpha(R_1(w))^{4l-2}}\right) &= -\log\left(1+\alpha w^{4l-2}\left(\frac{(\bar{R}_1(w))^{4l-2}-1}{1+\alpha w^{4l-2}}\right)\right)\\ &= O\big(w^{4l+2k-1}\big), \quad w \to 0.\end{aligned}$$

Furthermore, since the Dulac map D is exponentially flat at $w=0$, it follows that, $\forall N \in \mathbb{N}$,

$$\log R_2'\big(D(w)\big) = \log R_2'(0) + O\big(w^N\big), \quad w \to 0.$$

As a consequence, for $w \to 0$,

$$\begin{aligned}\bar{\Delta}(w,a) &= \frac{1}{-4l+2}\cdot\left(\frac{1-(\bar{R}_1(w))^{-4l+2}}{w}\right) + O\big(w^{4l-3}\big)\\ &= -C_k \cdot a_{2k+1} w^{2k} + O\big(w^{2k+1}\big). \qquad \square\end{aligned}$$

Now, the center case is characterized by one of the conditions in the following proposition:

Proposition 6. *Consider $X_a^{(2l)}$ as in* (6)*, and let a^0 be a value for which $\bar{X}_{a^0}^{(2l)}$ has a* (*non-hyperbolic*) 2*-saddle cycle Γ as defined in Section* 4*. Let $\bar{\Delta}: [0, W_0[\times\mathcal{W} \to \mathbb{R}$ be the reduced difference map as defined in* (28)*. Then the following conditions all are equivalent*:

1. *$X_{a^0}^{(2l)}$ has a center in the origin.*
2. $\bar{\Delta}(w, a^0) \equiv 0$.
3. $a_1^0 = a_3^0 = \cdots = a_{2l-1}^0 = 0$.

Proof. Condition 1 implies condition 2. Indeed, if $X_{a^0}^{(2l)}$ has a center at the origin, then $\Delta(w,a^0) \equiv 0$, and also $\frac{\partial}{\partial w}\Delta(w,a^0) \equiv 0$. From this, it follows that

$$R_2'\big(D\big(w,a^0\big)\big)\cdot D'\big(w,a^0\big) \equiv D'\big(R_1\big(w,a^0\big),a^0\big)\cdot R_1'\big(w,a^0\big),$$

and hence,

$$\bar{\Delta}\big(w,a^0\big) \equiv 0.$$

From Proposition 5 it is clear that condition 2 implies condition 3. In Section 2, we already noticed that condition 3 implies that $X_{a^0}^{(2l)}$ has a symmetric center at the origin. Hence, all conditions are equivalent. $\square$

For $m \in \mathbb{N}$, we define the natural projections π_i as follows:

$$\pi_i : \mathbb{R}^m \to \mathbb{R}^{m-i+1} : (b_1, \ldots, b_m) \mapsto (b_i, b_{i+1}, \ldots, b_m). \tag{30}$$

Then, the following lemma is a useful specification of Taylor's theorem:

Lemma 7. *Let $\mathcal{P} : W \times V \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a C^∞ map defined on an open set $W \times V$ in $\mathbb{R}^m \times \mathbb{R}^n$ such that $\mathcal{P}(0, c) \equiv 0$, $\forall c \in V$. Then there exist C^∞ functions $p_i : W_i \subset \mathbb{R}^{m-i+1} \times \mathbb{R}^n \to \mathbb{R}$, that are defined on an open set W_i in $\mathbb{R}^{m-i+1}$ such that $\pi_i^{-1}(W_i) \subset W$, $1 \leqslant i \leqslant m$, such that $\forall (b, c) \in \bigcap_{i=1}^m \pi_i^{-1}(W_i) \times V$:*

$$\mathcal{P}(b, c) = \sum_{i=1}^m b_i p_i(b_i, \dots, b_m, c).$$

Let $\kappa : \mathbb{R}^{2l-1} \to \mathbb{R}^{2l-1}$ denote the permutation of the parameter variables, putting the variables with odd sub-index in front; hence,

$$\begin{aligned} \pi_i\big(\kappa(a_1, a_2, \dots, a_{2l-1})\big) &= \pi_i(a_1, a_3, \dots, a_{2l-1}, a_2, a_4, \dots, a_{2l-2}) \\ &= (a_{2i-1}, a_{2i+1}, \dots, a_{2l-1}, a_2, a_4, \dots, a_{2l-2}), \end{aligned} \tag{31}$$

where π_i defines the natural projection defined above in (30) with $m = l, n = l - 1$.

Corollary 8. *Let $\bar{\Delta} : [0, W_0[\times \mathcal{W} \to \mathbb{R}$ be the reduced difference map as defined in* (28)*. Then, there exist $0 < W_1 < W_0$ and C^∞ functions $\Phi_i : [0, W_1[\times \mathcal{W}_i \to \mathbb{R}, 1 \leqslant i \leqslant l$, with $\forall 1 \leqslant i \leqslant l, (\pi_i \circ \kappa)^{-1}(\mathcal{W}_i) \subset \mathcal{W}_{i-1} \subset \mathcal{W}$ such that $\forall (w, a) \in [0, W_0[\times \mathcal{W}$:*

$$\bar{\Delta}(w, a) = \sum_{i=1}^l a_{2i-1} \Phi_i\big(w, \pi_i \circ \kappa(a)\big), \tag{32}$$

with

$$\Phi_i\big(w, \pi_i \circ \kappa(a)\big) = -C_{i-1} w^{2i-2} + o\big(w^{2i-2}\big), \quad w \to 0, \tag{33}$$

where $C_{i-1}, 1 \leqslant i \leqslant l$, are the non-zero constants defined in (20)*, and $\pi_i \circ \kappa$ is the permuted projection defined in* (31)*.*

Proof. By Lemma 7 and Proposition 6, we find C^∞ functions Φ_i such that the reduced difference map can be written as proposed in (32). By Proposition 5, for $a \in \mathbb{R}^{2l-1}$ with $a_{2j-1} = 0, \forall 1 \leqslant j \leqslant k$, the right-hand side in (32) has the following asymptotics for $w \downarrow 0$:

$$a_{2k+1}\big(\Phi_{k+1}\big(w, \pi_{k+1}\big(\kappa(a)\big)\big) + C_k w^{2k}\big) + \sum_{i=k+2}^l a_{2i-1} \Phi_i\big(w, \pi_i\big(\kappa(a)\big)\big) = O\big(w^{2k+1}\big).$$

Since the terms behind the summation sign are independent of a_{2k+1}, it follows that

$$\Phi_{k+1}\big(w, \pi_{k+1}\big(\kappa(a)\big)\big) = -C_k w^{2k} + o\big(w^{2k}\big), \quad w \to 0. \qquad \square$$

8. Cyclicity result

Now we can prove the following cyclicity result, only depending on the degree of the Liénard system.

Theorem 9. *Consider $X_a^{(2l)}$ as in* (6)*, and let a^0 be a value for which $\bar{X}_{a^0}^{(2l)}$ has a* (*non-hyperbolic*) 2*-saddle cycle Γ as defined in Section* 4*, then*

$$\mathrm{Cycl}\big(\bar{X}_a^{(2l)}, \big(\Gamma, a^0\big)\big) \leqslant l.$$

Before proving this theorem, we first formulate and prove a more precise cyclicity result, depending on the limiting parameter $a = a^0 \in \mathbb{R}^{2l-1}$; here we distinguish between the '*regular case*' and the '*center case*.' Recall that the center case is characterized by one of the conditions in Proposition 6.

We now start with the so-called *regular case*, i.e. when the limiting vector field does not represent a center. The case that the limiting vector field $X_{a^0}^{(2l)}$ represents a center will be treated afterwards. If $X_{a^0}^{(2l)}$ does not represent a center, then by Proposition 6, there is at least one $a_{2k+1}^0 \neq 0$; take $0 \leqslant k \leqslant l-1$ with

$$a_{2j-1}^0 = 0, \quad \forall j \leqslant k, \quad \text{and} \quad a_{2k+1}^0 \neq 0.$$

Theorem 10. *Consider $X_a^{(2l)}$ as in* (6)*, and let a^0 be a value for which $\bar{X}_{a^0}^{(2l)}$ has a* (*non-hyperbolic*) 2*-saddle cycle Γ as defined in Section* 4*. Furthermore, suppose that there exists $0 \leqslant k \leqslant l-1$ with*

$$a_{2j-1}^0 = 0, \quad \forall j \leqslant k, \quad \textit{and} \quad a_{2k+1}^0 \neq 0,$$

then

$$\mathrm{Cycl}\big(\bar{X}_a^{(2l)}, \big(\Gamma, a^0\big)\big) \leqslant k+1.$$

Proof. By Corollary 8, there exist C^∞ functions $\Phi_i : [0, W_0[\times\mathcal{W} \to \mathbb{R}$, $1 \leqslant i \leqslant l-1$, satisfying (33) such that $\forall(w,a) \in [0, W_0[\times\mathcal{W}$,

$$\bar{\Delta}(w,a) = \sum_{i=1}^{l} a_{2i-1}\Phi_i(w,a).$$

Now, we divide $\bar{\Delta}$ by the non-zero function Φ_1, $w > 0$, and then we derive with respect to w to obtain C^∞ functions $\bar{\Delta}^1, \Phi_1^1, \ldots, \Phi_{l-1}^1$ such that, for $(w,a) \in [0, W_0[\times\mathcal{W}$,

$$\bar{\Delta}^1(w,a) = \sum_{i=1}^{l-1} a_{2i+1}\Phi_i^1(w,a), \quad w \to 0,$$

with $\bar{\Delta}^1 = \frac{\partial}{\partial w}(\frac{\bar{\Delta}}{\Phi_1})$, $\Phi_i^1 = \frac{\partial}{\partial w}(\frac{\Phi_{i+1}}{\Phi_1})$. Then,

$$\Phi_i^1(w,a) = C_i^1 w^{2i-1} + o\big(w^{2i-1}\big), \quad w \to 0,$$

where $C_i^1 = -2iC_i/C_0 \neq 0$. Continuing the division-derivation procedure, after k derivations and divisions by a non-zero function for $w > 0$, we find C^∞ functions $\bar{\Delta}^k, \Phi_1^k, \dots, \Phi_{l-k}^k$ and non-zero constants $C_1^k, \dots, C_{l-k}^k$ such that

$$\bar{\Delta}^k(w,a) = \sum_{i=1}^{l-k} a_{2(k+i)-1} \Phi_i^k(w,a), \quad w \to 0,$$

with $\forall 1 \leqslant i \leqslant l-k$,

$$\Phi_i^k(w,a) = C_i^k w^{2i-1} + o\big(w^{2i-1}\big), \quad w \to 0,$$

where $C_i^k > 0$ is a non-zero constant. As a consequence, since $a_{2k+1}^0 \neq 0$,

$$\bar{\Delta}^k\big(w,a^0\big) = a_{2k+1}^0 C_1^k\big(1 + o(1)\big), \quad w \to 0.$$

Then by continuity, there exist a constant $0 < W_1 < W_0$ and a neighbourhood $\mathcal{W}_0 \subset \mathcal{W}$ of a^0 in $\mathbb{R}^{2l-1}$ such that, $\forall a \in \mathcal{W}_0$, the map $\bar{\Delta}(\cdot, a)$ has at most k zeroes w on $[0, W_1[$; or,

$$\mathrm{Cycl}\big(\bar{X}_a^{(2l)}, \big(\Gamma, a^0\big)\big) \leqslant k + 1. \qquad \square$$

Corollary 11. *Consider $X_a^{(2l)}$ as in* (6)*, and let a^0 be a value for which $\bar{X}_{a^0}^{(2l)}$ has a* (*non-hyperbolic*) 2*-saddle cycle Γ as defined in Section* 4*. Furthermore, suppose that $X_{a^0}^{(2l)}$ does not represent a center, then*

$$\mathrm{Cycl}\big(\bar{X}_a^{(2l)}, \big(\Gamma, a^0\big)\big) \leqslant l.$$

Proof. By Proposition 6, there is at least one $a_{2k+1}^0 \neq 0$; take $0 \leqslant k \leqslant l-1$ with

$$a_{2j-1}^0 = 0, \quad \forall j \leqslant k, \quad \text{and} \quad a_{2k+1}^0 \neq 0.$$

Now the corollary follows immediately from Theorem 10. $\square$

Theorem 12. *Consider $X_a^{(2l)}$ as in* (6)*, and let a^0 be a value for which $\bar{X}_{a^0}^{(2l)}$ has a* (*non-hyperbolic*) 2*-saddle cycle Γ as defined in Section* 4*. Suppose that $X_{a^0}^{(2l)}$ represents a center, then*

$$\mathrm{Cycl}\big(\bar{X}_a^{(2l)}, \big(\Gamma, a^0\big)\big) \leqslant l. \tag{34}$$

Proof. By Corollary 8, there exist C^∞ functions $\Phi_i : [0, W_0[\times \mathcal{W} \to \mathbb{R}$ and non-zero constants $C_i > 0$, $1 \leqslant i \leqslant l$, such that $\forall (w,a) \in [0, W_0[\times \mathcal{W}$,

$$\bar{\Delta}(w,a) = \sum_{i=1}^{l} a_{2i-1} \Phi_i(w,a),$$

with $\forall 1 \leqslant i \leqslant k, \forall (w,a) \in [0, W_0[\times \mathcal{W}$,

$$\Phi_i(w,a) = -C_i w^{2i-2} + o\big(w^{2i-2}\big), \quad w \to 0.$$

By Proposition 6, we know that $a^0_{2i-1} = 0, \forall 1 \leqslant i \leqslant l$; parameter values a for which $a_{2j-1} = 0$, $\forall 1 \leqslant j \leqslant l$, correspond to a vector field $\bar{X}^{(2l)}_a$ of center type, and hence, it has no limit cycles near Γ; let us denote the set of these parameter values by $\mathcal{C}$:

$$\mathcal{C} = \big\{a \in \mathbb{R}^{2l-1}\colon\ a_{2j-1} = 0,\ \forall 1 \leqslant j \leqslant l\big\}.$$

The set $\mathcal{W}$ is a neighbourhood of a^0 in $\mathbb{R}^{2l-1}$; hence there exists $\rho_0 > 0$ with

$$\mathcal{W}_0 = \left\{a \in \mathbb{R}^{2l-1}\colon\ \big|a_{2j} - a^0_{2j}\big| \leqslant \rho_0,\ \forall 1 \leqslant j \leqslant l-1,\ \text{and}\ \sum_{j=1}^{l} a^2_{2j-1} \leqslant \rho_0^2\right\},$$

such that $\mathcal{W}_0 \subset \mathcal{W}$. For all parameter values $a \in \mathcal{W}_0 \setminus \mathcal{C}$, we can write

$$a_{2j-1} = \rho \bar{a}_{2j-1}, \quad \forall 1 \leqslant j \leqslant l, \quad \text{and} \quad \sum_{j=1}^{l} \bar{a}^2_{2j-1} = 1, \tag{35}$$

in a unique way, for $0 < \rho < \rho_0$. Furthermore, on $[0, W_1[\times(\mathcal{W}_0 \setminus \mathcal{C})$, we can express the map $\bar{\Delta}$ in terms of $(\rho, \bar{a})$ as follows:

$$\bar{\Delta}\big(w, \chi(\rho,\bar{a})\big) = \rho \sum_{j=1}^{l} \bar{a}_{2i-1} \bar{\Phi}_i\big(w, \chi(\rho,\bar{a})\big) \equiv \rho \cdot \Psi\big(w, (\rho,\bar{a})\big),$$

where $a = \chi(\rho,\bar{a})$ is defined by (35) and $a_{2j} = \bar{a}_{2j}, \forall 1 \leqslant j \leqslant l-1$. As a consequence, for $a \in \mathcal{W}_0 \setminus \mathcal{C}$, zeroes of $\bar{\Delta}(\cdot, a)$ correspond to isolated zeroes of the map $\Psi(\cdot, (\rho,\bar{a}))$, where $a = \chi(\rho,\bar{a})$.

For all $b^0 = (\bar{a}^0_1, \ldots, \bar{a}^0_{2l-1}) \in \mathbb{S}^{l-1}$, we can take $k = k(b^0) \leqslant l-1$ such that $\bar{a}^0_{2j-1} = 0$, $1 \leqslant j \leqslant k$, $\bar{a}^0_{2k+1} \neq 0$; as a consequence, we find, with a straightforward derivation-division argument, a neighbourhood $\mathcal{W}_{b^0}$ of b^0 in $\mathbb{S}^{l-1}$ and a constant $0 < W_{b^0} < W_1$ such that the map $\Psi(\cdot, (\rho,\bar{a}))$ has at most k zeroes in $[0, W_{b^0}[$, for all $(\rho,\bar{a})$ with $\chi(\rho,\bar{a}) \in \mathcal{W}_0$ and $(\bar{a}_1, \ldots, \bar{a}_{2l-1}) \in \mathcal{W}_{b^0}$. By compactness of the sphere $\mathbb{S}^{l-1}$, we can now take a constant $0 < W < W_1$, independent of b^0, such that $\bar{\Delta}(\cdot, a)$ has at most $l-1$ zeroes in $[0, W[, \forall a \in \mathcal{W}_0$. As a consequence, we obtain the proposed cyclicity result (34). $\square$

9. Proof of Proposition 4

In this section, we present the technical calculations necessary to prove Proposition 4, expressing the first $4l-1$ higher order derivatives of the transition map R_1. Recall that R_1 is introduced in (19); it describes the transition of the flow of $\bar{X}^{(2l)}_a$ from σ^1_- to σ^1_+ along the unbroken connection Γ_1, and is expressed in terms of the normalizing parameter $w > 0$.

The normal form $X^{\mathrm{norm}}_{(\alpha(a),2l)}$ is only valid in a neighbourhood of $\mathbf{s}_+$. By a symmetry argument on $\hat{X}^{(2l)}_a$, this normal form can also be used in a neighbourhood of $\mathbf{s}_-$. Not knowing φ, it is not possible to calculate the transition R_1, defined above. However, it reveals to be possible to calculate the derivatives of R_1 at 0. This relies on calculating the transition map along Γ_1 in terms of the coordinates (u,s), when the transverse sections with respect to Γ_1 near s_- and s_+ are chosen parallel to and symmetric with respect to the s-axis.

Let $0 < u_0, s_0 < 1$ be fixed and (u_0, s_0) sufficiently close to $\mathbf{s}_+$ such that $\Pi_{u_0} \equiv \{(u,s)\colon\ u_0 < u < 1,\ |s| < s_0\}$ is contained in $\mathcal{U}$. Let

$$\Pi_u^{\pm} \equiv \big\{(\pm u, s)\colon\ 0 \leqslant s < s_0\big\}$$

and consider the transition from Π_u^- to Π_u^+, defined by the flow of $\hat{X}^{(2l)}_a$:

$$\Pi_u^- \to \Pi_u^+ : (-u, s) \mapsto \big(u, H_u^a(s)\big). \tag{36}$$

Put $\sigma = \sigma_+^1 \cup \sigma_-^1$; we choose the w-coordinate to define a regular parameter on σ. Put $\pi_{u,a} = \varphi^a(\Pi_u^+)$, and consider the map

$$\Psi_u = \Psi_u^a : \sigma \to \mathbb{R}^2 \tag{37}$$

defined by the flow of $X^{\mathrm{norm}}_{(\alpha(a),2l)}$ with a choice of time such that σ is mapped onto $\pi_{u,a}$. Let the inverse map of φ^a be denoted by ψ^a and $\mathcal{V}^a \equiv \varphi^a(\mathcal{U}) \subset (\mathbb{R}^2, (0,0))$:

$$\psi = \psi^a = (\psi_1, \psi_2) \equiv \big(\varphi^a\big)^{-1} : \mathcal{V}^a \to \mathbb{R}^2,$$

then we can write, for $(1, w) \in \sigma$,

$$\psi\big(\Psi_u^a(w)\big) = \big(u, \psi_2\big(\Psi_u^a(w)\big)\big) \in \Pi_u^+. \tag{38}$$

By construction, the relation between R_1^a and H_u^a is schematically shown in the following commutative diagram:

$$\begin{tikzcd}
\sigma_-^1 \arrow[r, "R_1^a \circ (-\mathrm{Id})"] \arrow[d, "\Psi_u^a"'] & \sigma_+^1 \arrow[r, "\Psi_u^a"] & \pi_{u,a}^+ \\
\pi_{u,a}^- \arrow[r, "\psi_2^a"'] & \Pi_u \cap \{s<0\} \arrow[r, "H_u^a \circ (-\mathrm{Id})"'] & \Pi_u \cap \{s>0\} \arrow[u, "\varphi^a"']
\end{tikzcd} \tag{39}$$

Equivalently, we can write, using the normalizing parameter w that corresponds to the point $(1, w) \in \sigma_-^1$ (hence, $w < 0$):

$$\Psi_u^a\big(R_1^a(-w)\big) = \varphi^a\big(u, H_u^a\big(-\psi_2\big(\Psi_u^a(w)\big)\big)\big). \tag{40}$$

In order to derive an expression for the derivatives of R_1 with respect to w, at $w = 0$, in function of the parameter a, we first derive such an expression for the derivatives of H_u^a with respect to s at $s = 0$; next, we use the relation in (40) to obtain the required derivatives of R_1 by taking the limit for $u \uparrow 1$.

In a first step, to keep the calculations more transparent, we will use the fact that there exists a C^∞ family of transformations that brings $\hat{X}_a^{(2l)}$, on a neighbourhood of $\mathbf{s}_+$, into an intermediate normal form $X_{(a,2l)}^{\text{int.norm}}$:

$$X_{(a,2l)}^{\text{int.norm}} \leftrightarrow \begin{cases} \dot{V} = -\bar{h}(S)V, \\ \dot{S} = \bar{g}(S)S^{4l-1}, \end{cases} \tag{41}$$

where $\bar{h}, \bar{g}$ are C^∞ functions. Again, by symmetry arguments, this normal form can also be used in the neighbourhood of $\mathbf{s}_-$. We denote the transition map R_1, when it is expressed in the intermediate normalizing coordinate $S > 0$, by L_1. Then, the map L_1 satisfies a similar scheme as R_1. In fact, if we denote the conjugation that transforms $\hat{X}_a^{(2l)}$ into $X_{(a,2l)}^{\text{int.norm}}$, again by φ, and if we introduce the transition Ψ_u, as defined in (37), in the coordinates (V, S), then L_1 satisfies the scheme (39) where R_1 is replaced by L_1. We will rely on this scheme for L_1, in order to calculate the derivatives of L_1 using the corresponding derivatives for H_u. In a second step, we will calculate the derivatives of R_1, using the derivatives of L_1 (Section 9.4).

9.1. Derivatives of the transition map H_u

Lemma 13. *Let $u_0 < u_1 < 1$ be arbitrary but fixed. Consider the map H_{u_1}, as introduced above in* (36)*, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from $\Pi_{u_1}^-$ to $\Pi_{u_1}^+$ along the unbroken connection $\Gamma_1 \equiv \{s = 0\}$ in terms of the parameter s. Then, the map H_{u_1} is C^ω in (u_1, s, a). Furthermore, we have*

$$H_{u_1}^a(0) = 0, \qquad \frac{\partial H_{u_1}^a}{\partial s}(0) = 1, \quad \text{and} \quad \frac{\partial^k H_{u_1}^a}{\partial s^k}(0) = 0, \quad \forall 2 \leqslant k \leqslant 4l-1,$$

or equivalently,

$$H_{u_1}^a(s) = s + O\left(s^{4l}\right), \quad s \to 0.$$

Proof. System (8) can be reduced to the scalar differential equation

$$\frac{\mathrm{d}s}{\mathrm{d}u} = \frac{(2l)^{-1}us^{4l-1}}{1 - u^{2l} - s^{2l}g(\frac{u}{s}, a) + (2l)^{-1}u^2 s^{4l-2}}.$$

This differential equation can be written as

$$\frac{\mathrm{d}s}{\mathrm{d}u} = s^{4l-1}\frac{(2l)^{-1}u}{1-u^{2l}} \cdot \left(1 - \frac{s^{2l}g(\frac{u}{s}, a) - (2l)^{-1}u^2 s^{4l-2}}{1-u^{2l}}\right)^{-1}. \tag{42}$$

Denote by $s(-u_1, s_1, \cdot, a)$ the solution of (42) with $s(-u_1, s_1, -u_1, a) = s_1$; then the transition map $H_{u_1}^a$ is defined by

$$H_{u_1}^a(s_1) = s(-u_1, s_1, u_1, a).$$

Remark that the map T, defined by $T(s_1, a, u_1) = H^a_{u_1}(s_1)$ is analytic at $(s_1, a, u_1) \in]-1, 1[\times \mathbb{R}^{2l-1} \times]-1, 1[$, since s is the global solution of (42), being an analytic differential equation. Furthermore, we can write an expansion of s up to any order k in terms of s_1:

$$s(-u_1, s_1, u, a) = h_1(u, a)s_1 + h_2(u, a)s_1^2 + \cdots + h_k(u, a)s_1^k + o\big(s_1^k\big), \tag{43}$$

for $s_1 \to 0$. Clearly, one has that

$$\frac{\partial^k H^a_{u_1}}{\partial s^k}(s) = k!h_k(u_1, a), \quad \forall k \geqslant 1. \tag{44}$$

By substitution of expression (43) for the global solution s in (42), it follows that the functions $h_j(\cdot, a)$, $1 \leqslant j \leqslant 4l-2$, satisfy the following differential equations:

$$\begin{cases} h_j'(u, a) = 0, \\ h_j(-u_1, a) = \delta_{1j}, \end{cases}$$

where δ_{1j} denotes the Kronecker delta ($\delta_{1j} = 1$ if $j = 1$ and $\delta_{1j} = 0$ if $j \neq 1$). As a consequence, $h_j(u, a) \equiv \delta_{1j}$, for $1 \leqslant j \leqslant 4l-2$. Next,

$$\begin{cases} h'_{4l-1}(u, a) = \dfrac{(2l)^{-1}u}{1-u^{2l}}, \\ h_{4l-1}(-u_1, a) = 0 \end{cases}$$

and thus

$$h_{4l-1}(u_1, a) = \int_{-u_1}^{u_1} \frac{(2l)^{-1}u}{1-u^{2l}}\, \mathrm{d}u = 0.$$

The required result now follows from (44). $\square$

Remark 14. In case that $a_1 = a_3 = \cdots = a_{2l-1} = 0$, system (8) is time-reversible with respect to $(u, s, t) \mapsto (-u, s, -t)$, and hence $H^a_{u_1}(s) \equiv s$.

9.2. Derivatives of the transition map L_1

Consider the map H_{u_1}, as introduced above in (36), that describes the transition of the flow of $\bar{X}^{(2l)}_a$ from $\Pi^-_{u_1}$ to $\Pi^+_{u_1}$ along the unbroken connection $\Gamma_1 \equiv \{s = 0\}$ in terms of the parameter s. By the chain rule and implicit differentiation with respect to S, we derive the following vectorial equation from (40), for $(1, -S) \in \sigma^1_-$:

$$D(\Psi_{u_1})_{L_1(-S)}\big(-L_1'(-S)\big) = D\varphi_{(u_1, H_{u_1}(-s))}\big(0, -H'_{u_1}(-s) \cdot D(\psi_2)_{\Psi_{u_1}(S)}\big(D(\Psi_{u_1})_S(1)\big)\big),$$

where $s = \psi_2(\Psi_{u_1}(S))$. By linearity, the preceding equation can be rewritten as

$$L_1'(-S)\cdot D(\Psi_{u_1})_{L_1(-S)}(1)=H_{u_1}'(-s)\cdot D\varphi_{(u_1,H_{u_1}(-s))}\big(0,D(\psi_2)_{\Psi_{u_1}(S)}\big(D(\Psi_{u_1})_S(1)\big)\big),\tag{45}$$

and we also have

$$\varphi(u_1,s)=\Psi_{u_1}(S).\tag{46}$$

Proof. The vector $D(\Psi_{u_1})_S(1)$ (as well as $D(\Psi_{u_1})_{L_1(-S)}(1)$), appearing in (45), is a tangent vector of the regular curve π_{u_1} at the point $\Psi_{u_1}(S)$ (and $\Psi_{u_1}(L_1(-S))$ respectively). Indeed, the vector $(0,1)$ is tangent to the curve σ^1 at the point $(1,-S)$; since the restriction $\Psi_{u_1}|_{\sigma^1\to\pi_{u_1}}$ is a diffeomorphism, expressed by the S-coordinate, $D(\Psi_{u_1})_S$ sends 1 to a non-zero tangent vector on π_{u_1}.

On the other hand, $(0,1)$ is a tangent vector of Π_{u_1} at the point (u_1,s) that, by construction, is sent to the non-zero tangent vector $D\varphi_{(u_1,s)}(0,1)$ of $\pi_{u_1}=\varphi(\Pi_{u_1})$ at the point $\Psi_{u_1}(S)$.

Hence, for each S, both non-zero vectors $D(\Psi_{u_1})_S(1)$ and $D\varphi_{(u_1,s)}(0,1)$ are parallel, and hence, can be related by a non-zero factor $A_{u_1}(S)=A^a_{u_1}(S)$, defined by

$$D\big(\Psi^a_{u_1}\big)_S(1)=A^a_{u_1}(S)D\varphi^a_{(u_1,s)}(0,1).\tag{47}$$

By C^∞ smoothness of the maps $\Psi^a_{u_1}$ and φ^a, this equation defines a C^∞ function A_{u_1} in (S,a) for S on an interval around $S=0$ and $a\in\mathcal{W}$. From now on, we will not explicitly write the dependence on a anymore, since we will only use the fact that $A^a_{u_1}(0)$ is a non-zero constant, for any $a\in\mathcal{W}$.

Now the vectorial equation (45) can be rewritten as

$$\begin{aligned}&L_1'(-S)\cdot A_{u_1}\big(L_1(-S)\big)D\varphi_{(u_1,\psi_2(\Psi_{u_1}(L_1(-S))))}(0,1)\\&\quad=H_{u_1}'(-s)\cdot A_{u_1}(S)\cdot D\varphi_{(u_1,H_{u_1}(-s))}\big(0,D(\psi_2)_{\Psi_{u_1}(S)}\big(D\varphi_{(u_1,s)}(0,1)\big)\big).\end{aligned}\tag{48}$$

By the chain rule, the following identities hold:

$$\begin{aligned}\mathrm{Id}&=D(\psi\circ\varphi)_{(u_1,s)}\\&=D\psi_{\varphi(u_1,s)}\circ D\varphi_{(u_1,s)}.\end{aligned}$$

As a consequence, by (46),

$$D\psi_{\Psi_{u_1}(S)}=\big(D\varphi_{(u_1,s)}\big)^{-1}$$

and

$$D(\psi_2)_{\Psi_{u_1}(S)}\circ D\varphi_{(u_1,s)}(0,1)=1.\tag{49}$$

Combining (49) and (47), we can also define the function A_{u_1} explicitly as

$$A_{u_1}(S)=D(\psi_2\circ\Psi_{u_1})_S(1).\tag{50}$$

Then, by using (49), the vectorial equation (48) can now be rewritten as

$$L_1'(-S)\cdot A_{u_1}\big(L_1(-S)\big)\cdot D\varphi_{(u_1,\psi_2(\Psi_{u_1}(L_1(-S))))}(0,1)$$
$$= A_{u_1}(S)\cdot H_{u_1}'(-s)\cdot D\varphi_{(u_1,H_{u_1}(-s))}(0,1),$$

and by (40),

$$L_1'(-S)\cdot A_{u_1}\big(L_1(-S)\big)\cdot D\varphi_{(u_1,H_{u_1}(-s))}(0,1) = A_{u_1}(S)\cdot H_{u_1}'(-s)\cdot D\varphi_{(u_1,H_{u_1}(-s))}(0,1).$$

As we already noticed, the vector $D\varphi_{(u_1,H_{u_1}(-s))}(0,1)$ is non-zero; therefore, the preceding vectorial equation can be reduced to a scalar equation, relating the first derivative of L_1 with respect to S and the first derivative of H_{u_1} with respect to s:

$$L_1'(-S)\cdot A_{u_1}\big(L_1(-S)\big) = A_{u_1}(S)\cdot H_{u_1}'\big(-\psi_2\big(\Psi_{u_1}(S)\big)\big). \qquad \square \tag{51}$$

Lemma 15. *Consider the map L_1, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, using the normalizing parameter S. Then,*

$$L_1'(0) = 1.$$

Proof. By substitution of $S=0$ in (51) and division by the non-zero factor $A_{u_1}(0)$, we find $L_1'(0) = H_{u_1}'(0)$, since $L_1(0)=0$. The claim of the lemma now follows from Lemma 13, with $H_{u_1}'(0) = \frac{\partial H_{u_1}^a}{\partial s}(0) = 1$. $\square$

Lemma 16. *Consider the map H_{u_1}, as introduced above in* (36)*, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from $\Pi_{u_1}^-$ to $\Pi_{u_1}^+$ along the unbroken connection $\Gamma_1 \equiv \{s=0\}$ in terms of the parameter s. Consider the map L_1, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, using the normalizing parameter S. If $k \geqslant 2$ is such that*

$$\frac{\partial^n L_1}{\partial S^n}(0) = \frac{\partial^n H_{u_1}}{\partial s^n}(0), \quad \forall 1 \leqslant n \leqslant k-1,$$

then

$$\frac{\partial^k L_1}{\partial S^k}(0) = \big(A_{u_1}(0)\big)^{k-1}\frac{\partial^k H_{u_1}}{\partial s^k}(0) - \big((-1)^k+1\big)\frac{A_{u_1}^{(k-1)}(0)}{A_{u_1}(0)},$$

where $A_{u_1}(0)\colon \mathcal{W}\subset\mathbb{R}^{2l-1}\to\mathbb{R}\setminus\{0\}$ is the function of a as described in (47) *and* (50).

Proof. Using (50), the claim is clear for $k=2$, by deriving (51) with respect to s. Furthermore, by induction on k, using (50), one proves that, after $k-1$ derivations with respect to S, Eq. (51) reduces to

$$(-1)^{k-1}L_1^{(k)}(-S)\cdot A_{u_1}\big(L_1(-S)\big) + G_k(u_1,S) + (-1)^{k-1}\big(L_1'(-S)\big)^k\cdot A_{u_1}^{(k-1)}\big(L_1(-S)\big)$$
$$= (-1)^{k-1}\big(A_{u_1}(S)\big)^k\cdot H_{u_1}^{(k)}\big(-\psi_2\big(\Psi_{u_1}(S)\big)\big) + F_k(u_1,S)$$
$$+ H_{u_1}'\big(-\psi_2\big(\Psi_{u_1}(S)\big)\big)\cdot (A_{u_1})^{(k-1)}(S),$$

which can also be written as

$$
\begin{aligned}
&(-1)^{k-1}L_1^{(k)}(-S)\cdot A_{u_1}\big(L_1(-S)\big)+(-1)^{k-1}\big(L_1'(-S)\big)^k\cdot A_{u_1}^{(k-1)}\big(L_1(-S)\big)\\
&\quad=(-1)^{k-1}\big(A_{u_1}(S)\big)^k\cdot H_{u_1}^{(k)}\big(-\psi_2\big(\Psi_{u_1}(S)\big)\big)+H_{u_1}'\big(-\psi_2\big(\Psi_{u_1}(S)\big)\big)\cdot(A_{u_1})^{(k-1)}(S)\\
&\qquad+G_k(u_1,S)+F_k(u_1,S),
\end{aligned}
$$

where the functions F_k and G_k are C^∞. Moreover, for $k\geqslant 3$, F_k is the sum of C^∞ functions, of which each function is divisible by

$$H_{u_1}^{(n)}\big(-\psi_2\big(\Psi_{u_1}(S)\big)\big)\quad\text{for a certain } 2\leqslant n\leqslant k-1,$$

and G_k is the sum of C^∞ functions of which each function is divisible by

$$L_1^{(n)}(-S)\quad\text{for a certain } 2\leqslant n\leqslant k-1.$$

As a consequence, the result now follows by substitution of $S=0$ and other algebraic manipulations. □

Combining Lemmas 13 and 16, we obtain the following corollary.

Corollary 17. *Consider the map L_1, as introduced in* (19)*, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, and that is expressed in terms of the normalizing parameter S. If $1\leqslant k\leqslant 2l-1$ is such that*

$$\frac{\partial^n L_1}{\partial S^n}(0)=0,\quad \forall 2\leqslant n\leqslant 2k-1,$$

then

$$\frac{\partial^{2k} L_1}{\partial S^{2k}}(0)=-2\lim_{u_1\uparrow 1}\frac{A_{u_1}^{(2k-1)}(0)}{A_{u_1}(0)}\quad\textit{and}\quad \frac{\partial^{2k+1} L_1}{\partial S^{2k+1}}(0)=0,$$

where $A_{u_1}(0):\mathcal{W}\subset\mathbb{R}^{2l-1}\to\mathbb{R}\setminus\{0\}$ is the function of a as described in (47) *and* (50).

Proof. If $1\leqslant k\leqslant 2l-1$ such that

$$\frac{\partial^n L_1}{\partial S^n}(0)=0,\quad \forall 2\leqslant n\leqslant 2k-1,$$

it follows from Lemma 16, that

$$\frac{\partial^{2k} L_1}{\partial S^{2k}}(0)=-2\frac{A_{u_1}^{(2k-1)}(0)}{A_{u_1}(0)}\quad\text{and}\quad \frac{\partial^{2k+1} L_1}{\partial S^{2k+1}}(0)=0.$$

Since $\frac{\partial^{2k}L_1}{\partial S^{2k}}(0)$ is independent of u_1, and the equation holds for all $u_1\uparrow 1$, the assertion of the corollary follows by taking the limit. □

As we are interested in the relation between the first $2l-1$ even order derivatives of L_1 with respect to S at $S=0$ and the coefficients a_{2k-1}, $1 \leqslant k \leqslant l$, we need to calculate the odd derivatives $A_{u_1}^{(2k-1)}(0)$, $1 \leqslant k \leqslant 2l-1$. Using the identity (50), we can derive expressions for $A_{u_1}^{(2k-1)}(0)$ in terms of $\frac{\partial^{2k}\varphi_2(u_1,0)}{\partial s^{2k}}$, $0 \leqslant k \leqslant l-1$, where φ_2 is defined by $\varphi = (\varphi_1, \varphi_2)$. To start, we take $u_0 < u_1 < 1$ arbitrary close to 1 but fixed. Finally, we take the limit for $u_1 \uparrow 1$, and calculate the relevant derivatives of φ_2 with respect to s in $(1,0)$ in terms of the coefficients a_{2k-1}, $0 \leqslant k \leqslant l$.

Lemma 18. *For the notations introduced above, we have*

$$A_{u_1}(0) = \left(\frac{\partial \varphi_2}{\partial s}(u_1, 0)\right)^{-1}. \tag{52}$$

Furthermore, if $1 \leqslant k \leqslant 2l-2$ *is such that*

$$\frac{\partial^{2i}\varphi_2}{\partial s^{2i}}(u_1, 0) = 0, \quad \forall 1 \leqslant i \leqslant k,$$

then

$$\begin{cases} A_{u_1}^{(2i+1)}(0) = 0, \quad \forall 0 \leqslant i \leqslant k-1, \\ A_{u_1}^{(2k+1)}(0) = -\left(\frac{\partial \varphi_2}{\partial s}(u_1, 0)\right)^{-2k-3} \cdot \frac{\partial^{2k+2}\varphi_2}{\partial s^{2k+2}}(u_1, 0). \end{cases} \tag{53}$$

Proof. By (50), it follows that the function A_{u_1} is the derivative of $\psi_2 \circ \Psi_{u_1}$, where $\psi_2 = (\varphi^{-1})_2$. If

$$s = \psi_2\big(\Psi_{u_1}(S)\big) \quad \text{and} \quad \varphi_2(u_1, s) \equiv s\bar{\varphi}_2(s), \tag{54}$$

for a certain C^∞ mapping $\bar{\varphi}_2$, then we have on the one hand,

$$\Psi_{u_1}(S) = \varphi(u_1, s) = \big(\varphi_1(u_1, s), s\bar{\varphi}_2(s)\big).$$

On the other hand, Ψ_{u_1} describes the transition from $(1, S)$ to $\Psi_{u_1}(S)$, and hence it satisfies the integral equation

$$\int_1^{\varphi_1(u_1,s)} \frac{\mathrm{d}V}{V} = -\int_S^{s\bar{\varphi}_2(s)} \frac{\bar{h}(S')}{\bar{g}(S')} S'^{-4l+1}\,\mathrm{d}S'. \tag{55}$$

By Taylor's theorem, there exist a C^∞ function $\bar{G}$ and real constants e_j, $0 \leqslant j \leqslant 4l-2$, such that

$$-\bar{h}(S)/\bar{g}(S) = \sum_{j=0}^{4l-2} e_j S^j + S^{4l-1}\bar{G}(S),$$

where $e_0 = -4l^2$; furthermore, there exists a C^∞ function $\bar{F}$ such that

$$s\bar{\varphi}_2(s) = \psi_2\big(\Psi_{u_1}(S)\big)\bar{\varphi}_2\big(\psi_2\big(\Psi_{u_1}(S)\big)\big) = S\bar{F}(S). \tag{56}$$

Then, Eq. (55) can be rewritten as

$$\varphi_1\big(u_1, \psi_2\big(\Psi_{u_1}(S)\big)\big) = \exp\left[S^{-4l+2}\sum_{j=0}^{4l-3}\frac{e_j}{j-4l+2}S^j\big(\big(\bar{F}(S)\big)^{j-4l+2}-1\big)\right] \cdot \bar{F}(S)^{e_{4l-2}} \cdot \exp\left[\int_S^{S\bar{F}(S)} \bar{G}(S')\,\mathrm{d}S'\right].$$

Then, since the left-hand side, $\bar{F}(S)^{e_{4l-2}}$, and the second exponential term in the right-hand side are bounded and bounded away from zero, it is necessary that

$$\sum_{j=0}^{4l-3}\frac{e_j}{j-4l+2}S^j\big(\big(\bar{F}(S)\big)^{j-4l+2}-1\big) = O\big(S^{4l-2}\big), \quad S \to 0.$$

From this equation, we find by recurrence that

$$\bar{F}(S) = 1 + O\big(S^{4l-2}\big), \quad S \to 0.$$

Then, using (56), we find the following asymptotics:

$$(\psi_2 \circ \Psi_{u_1})(S) \cdot \bar{\varphi}_2\big((\psi_2 \circ \Psi_{u_1})(S)\big) = S + O\big(S^{4l-1}\big), \quad S \to 0. \tag{57}$$

Recall that $\psi_2 \circ \Psi_{u_1}$ is the antiderivative of A_{u_1}; therefore, (57) relates the derivatives of A_{u_1} at $S = 0$ to the ones of φ_2. If we write

$$(\psi_2 \circ \Psi_{u_1})(S) = S\big(\alpha_0 + \alpha_1 S + \cdots + \alpha_{4l-2}S^{4l-2} + O\big(S^{4l-1}\big)\big), \quad S \to 0,$$

then for $i \geqslant 1$,

$$\alpha_i = \frac{A_{u_1}^{(i)}(0)}{(i+1)!}.$$

Write

$$\bar{\varphi}_2(s) = \beta_0 + \beta_1 s + \cdots + \beta_{4l-2}s^{4l-2} + O\big(s^{4l-1}\big), \quad s \to 0.$$

Then, it follows, by taking $S = 0$ in (57) that $\alpha_0\beta_0 = 1$; from this equation the identity in (52) follows. By comparing the coefficients corresponding to first order terms in S in (57), we find $\alpha_1 = -\alpha_0^3\beta_1$. We continue by induction on $1 \leqslant k \leqslant 2l-2$, assuming that $\beta_1 = \beta_3 = \cdots = \beta_{2k-1} = 0$; by the induction hypothesis, it then follows that $\alpha_1 = \alpha_3 = \cdots = \alpha_{2k-1} = 0$. Hence, the relation in (57) is reduced to

$$\sum_{j=0}^{k}\beta_{2j}S^{2j}\big(\alpha_0+\alpha_2S^2+\cdots+\alpha_{2k}S^{2k}+\alpha_{2k+1}S^{2k+1}+\cdots\big)^{2j+1}$$
$$+\beta_{2k+1}S^{2k+1}\big(\alpha_0+O(S)\big)^{2k+2}=1+O\big(S^{4l-1}\big),\quad S\to 0.$$

By comparing coefficients corresponding to S^{2k+1}, we find the relation

$$\beta_0\alpha_{2k+1}+\beta_{2k+1}\alpha_0^{2k+2}=0.$$

Now the assertion in (53) follows since by (54), we have

$$(2k+2)!\beta_{2k+1}=\frac{\partial^{2k+2}}{\partial s^{2k+2}}\varphi_2(u_1,0).\qquad\square$$

Now, with the help of Lemma 18, Corollary 17 can be rewritten as

Corollary 19. *Consider the map L_1, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, using the normalizing parameter S. If $1\leqslant k\leqslant 2l-1$ such that*

$$\frac{\partial^{2i}\varphi_2}{\partial s^{2i}}(1,0)=0,\quad \forall 1\leqslant i\leqslant k-1, \tag{58}$$

then

$$\begin{cases}\dfrac{\partial^n L_1}{\partial S^n}(0)=0,\quad \forall 2\leqslant n\leqslant 2k-1,\\[2mm] \dfrac{\partial^{2k}L_1}{\partial S^{2k}}(0)=2\left(\dfrac{\partial\varphi_2}{\partial s}(1,0)\right)^{-2k}\dfrac{\partial^{2k}\varphi_2}{\partial s^{2k}}(1,0),\\[2mm] \dfrac{\partial^{2k+1}L_1}{\partial S^{2k+1}}(0)=0.\end{cases} \tag{59}$$

9.3. Derivatives of φ_2 and conclusions concerning L_1

By Corollary 19, the asymptotic expansion of L_1 with respect to $s=0$, can be expressed in terms of the even order derivatives of φ_2 with respect to s at $(1,0)$. We now use calculations on the center manifold at $(1,0)$ to express these derivatives of φ_2.

There exists a unique center manifold at the semi-hyperbolic saddle point $\mathbf{s}_+$ that can be written as the graph

$$\big\{\big(U(s,a),s\big):\ s\geqslant 0\big\},$$

for a C^∞ function U, where the C^∞ function U satisfies the equation

$$U'(s,a)U(s,a)s^{4l-1}=s^{4l-2}\big(U(s,a)\big)^2+2l-2l\big(U(s,a)\big)^{2l}-2ls^{2l}g\left(\frac{U(s,a)}{s},a\right), \tag{60}$$

with $U'(s,a) = \frac{\partial U}{\partial s}(s,a)$ and $U(0,a) = 1$. Let (v,s) denote the C^∞ coordinates in which the center manifold is straightened and positioned at the origin:

$$(v,s) = \bigl(-u + U(s,a), s\bigr), \tag{61}$$

and let ϕ be the transformation, that puts the vector field $\hat{X}_a^{(2l)}$, written in (v,s)-coordinates, into the intermediate normal form $X_{(a,2l)}^{\text{int.norm}}$:

$$\phi(v,s) = \bigl(\phi_1(v,s), \phi_2(v,s)\bigr) = \varphi\bigl(-v + U(s,a), s\bigr). \tag{62}$$

Since ϕ respects the local invariant manifolds at $\mathbf{s}_+$ ($\{v=0\}$ corresponds to the center manifold and $\{s=0\}$ corresponds to the stable manifold), we have

$$\phi_1(0,\cdot) \equiv 0 \quad \text{and} \quad \phi_2(\cdot,0) \equiv 0.$$

In particular, with the help of next lemma, we prove that the derivatives of ϕ_2 with respect to s at $(0,0)$ coincide with the ones of φ_2 at $(1,0)$.

Lemma 20. *In the notations introduced above, we have, $\forall j \in \mathbb{N}$, $\forall 0 \leqslant i \leqslant 4l-2$,*

$$\frac{\partial^{i+j+1}\phi_2}{\partial v^{j+1}\partial s^i}(0,0) = 0.$$

Proof. By (60), the vector field (8) in the coordinates (v,s), defined in (61) has the following asymptotics for $(v,s) \to 0$:

$$\begin{cases} v' = -2lv\bigl(1 + O\bigl(\|(v,s)\|\bigr)\bigr), \\ s' = (2l)^{-1}\bigl(U(s) - v\bigr)s^{4l-1}. \end{cases} \tag{63}$$

By definition of ϕ_2, the normal form coordinate S is given by $S = \phi_2(v,s)$. Furthermore, from (62), it follows that $\phi_2(v,0) \equiv 0$; hence,

$$\frac{\partial^k \phi_2(v,0)}{\partial v^k} \equiv 0, \quad \forall k \in \mathbb{N}. \tag{64}$$

Hence, from (41) and (63), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_2(v,s) = \bigl(\phi_2(v,s)\bigr)^{4l-1} \cdot \bar{g}\bigl(\phi_2(v,s)\bigr) = O\bigl(s^{4l-1}\bigr), \quad s \to 0. \tag{65}$$

By Eqs. (63), it follows that the left-hand side of (65) expands asymptotically of order $O(s^{4l-1})$, for $s \to 0$, as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi_2(v,s) &= \frac{\partial}{\partial v}\phi_2(v,s) \cdot v' + \frac{\partial}{\partial s}\phi_2(v,s) \cdot s' \\ &= -2l\frac{\partial}{\partial v}\phi_2(v,s)v\bigl(1 + O\bigl(\|(v,s)\|\bigr)\bigr) + O\bigl(s^{4l-1}\bigr), \end{aligned}$$

since $\frac{\partial}{\partial s}\phi_2(v,s)$ is bounded in a neighbourhood of $(v,s)=(0,0)$. As a consequence, by comparing the asymptotic expansions of right- and left-hand sides in (65), we find on a neighbourhood of $(v,s)=(0,0)$:

$$v\frac{\partial}{\partial v}\phi_2(v,s)=O\big(s^{4l-1}\big),\quad s\to 0.$$

In particular, it follows that, $\forall 0\leqslant i\leqslant 4l-2, \forall j\geqslant 1$,

$$\frac{\partial^{i+j+1}\phi_2}{\partial v^{j+1}\partial s^i}(0,0)\equiv 0. \qquad \square$$

Lemma 21. *In the notations introduced above, we have*:

1. $\frac{\partial\phi_2}{\partial s}(0,0)=1$.
2. *For* $1\leqslant k\leqslant 4l-2$: $\frac{\partial^k\phi_2}{\partial s^k}(0,0)=0$.

Proof. Restricting to the center manifold, i.e., for $v=0$, the equation s' in (8) is reduced to

$$s'=(2l)^{-1}U(s,a)s^{4l-1},$$

and the governing equation (13) for the coordinate $S=\phi_2(0,s)$, is then reduced to

$$\frac{\partial\phi_2}{\partial s}(0,s)(2l)^{-1}U(s,a)s^{4l-1}=\big(\phi_2(0,s)\big)^{4l-1}\bar{g}\big(\phi_2(0,s)\big),\quad s\to 0, \tag{66}$$

where

$$\bar{g}\big(\phi_2(0,s)\big)=(2l)^{-1}U\big(\phi_2(0,s),a\big)+O\big(s^{4l-1}\big),\quad s\to 0.$$

For certain real-valued C^∞ functions β_i and γ_i, $1\leqslant i\leqslant 4l-2$, depending on the parameter a, we can write

$$\phi_2(0,s)=s\big(\beta_1+\beta_2 s+\cdots+\beta_{4l-2}s^{4l-3}+O\big(s^{4l-2}\big)\big),\quad s\to 0, \tag{67}$$

and

$$U(s,a)=1+\gamma_1 s+\gamma_2 s^2+\gamma_3 s^3+\cdots+\gamma_{4l-2}s^{4l-2}+O\big(s^{4l-1}\big),\quad s\to 0. \tag{68}$$

Then, by substitution of the asymptotics (67) and (68) in Eq. (66), this equation has, after division by s^{4l-1}, the following asymptotics for $s\to 0$:

$$\begin{aligned}&\big[\beta_1+2\beta_2 s+\cdots+(4l-2)\beta_{4l-2}s^{4l-3}+O\big(s^{4l-2}\big)\big]\big[1+\gamma_1 s+\gamma_2 s^2+\cdots+O\big(s^{4l-1}\big)\big]\\&\quad=\big[\beta_1+\beta_2 s+\cdots+\beta_{4l-2}s^{4l-3}+O\big(s^{4l-2}\big)\big]^{4l-1}\\&\qquad\cdot\big[1+\gamma_1\phi_2(0,s)+\gamma_2\big(\phi_2(0,s)\big)^2+\cdots+O\big(s^{4l-1}\big)\big],\quad s\to 0.\end{aligned}$$

Therefore, $\beta_1^{4l-2} = 1$; since $\beta_1 = \frac{\partial \phi_2}{\partial s}(0,0) = \frac{\partial \varphi_2}{\partial s}(1,0) > 0$, it follows that $\beta_1 = 1$. Taking this into account, we can identify coefficients in the left- and right-hand sides corresponding to equal powers of s, to find inductively on $1 \leqslant k \leqslant 4l-2$:

$$k\beta_k + \gamma_{k-1} = (4l-1)\beta_k + \gamma_{k-1}.$$

As a consequence,

$$\frac{\partial^k \phi_2}{\partial s^k}(0,0) = k!\beta_k = 0. \qquad \square$$

Lemma 22. *In the notations introduced above, we have,* $\forall 0 \leqslant j \leqslant 4l-2$,

$$\frac{\partial^j \varphi_2}{\partial s^j}(1,0) = \frac{\partial^j \phi_2}{\partial s^j}(0,0). \tag{69}$$

Furthermore, $\forall k \in \mathbb{N}$, $\forall 0 \leqslant j \leqslant 4l-2$,

$$\frac{\partial^{k+j} \varphi_2}{\partial u^k \partial s^j}(1,0) = 0. \tag{70}$$

Proof. Identity (69) follows from Lemma 20, and the following observation, that is based on the chain rule:

$$\frac{\partial \varphi_2}{\partial s}(u,s) = \frac{\partial \phi_2}{\partial s}\big(-u + U(s,a), s\big) + \frac{\partial \phi_2}{\partial v}\big(-u + U(s,a), s\big) \cdot U'(s,a),$$

and, by induction on $k \in \mathbb{N}$, one finds C^∞ functions f_{kri}, $1 \leqslant r \leqslant k$, $1 \leqslant i \leqslant r$, such that

$$\frac{\partial^k \varphi_2}{\partial s^k}(u,s) = \frac{\partial^k \phi_2}{\partial s^k}\big(-u + U(s,a), s\big) + \sum_{r=1}^{k}\sum_{i=1}^{r} \frac{\partial^r \phi_2}{\partial v^i \partial s^{r-i}}\big(-u + U(s,a), s\big) \cdot f_{kri}(s,a).$$

In an analogous way, one finds identity (70). $\square$

By Lemmas 21 and 22, we can write down the following asymptotics of φ_2.

Corollary 23. *In the notations introduced above, we have*

$$\varphi_2(u,s) = s + O\big(s^{4l-1}\big), \quad s \to 0. \tag{71}$$

Combining Corollaries 19 and 23, we have proven the following asymptotic expansion for L_1.

Proposition 24. *Consider the map L_1, as introduced in* (19)*, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, and that is expressed in terms of the normalizing parameter S. Then,*

$$L_1(S) = S + O\big(S^{4l}\big), \quad S \to 0.$$

9.4. Derivatives of R_1

There exists a C^∞ family of coordinate transformations $(\mathrm{Id}, T^a):(V,S) \mapsto (z,w) = (V, T^a(S))$, that puts the family $(\bar{h}(S))^{-1} \cdot X^{\text{int.norm}}_{(a,2l)}$ into the normal form $X^{\text{norm}}_{(\alpha(a),2l)}$, that was introduced in Section 5 (cf. [1]). Here, we will denote the C^∞ family of transformations that brings $(h \cdot \hat{X}^{(2l)}_a)_a$ into $(X^{\text{norm}}_{(\alpha(a),2l)})_a$ by $\tilde{\varphi}^a$ instead of φ^a, to avoid confusion with the notations used throughout this section (i.e. Section 9). The relation between the C^∞ coordinate transformations $\tilde{\varphi}^a$ and φ^a and the non-zero factors h and $\bar{h}$ is as follows:

$$\begin{aligned}\tilde{\varphi}^a(u,s) &= \bigl(\varphi^a_1(u,s), T^a\bigl(\varphi^a_2(u,s)\bigr)\bigr),\\ h(u,s) &= \bigl(\bar{h}\bigl(\varphi_2(u,s)\bigr)\bigr)^{-1}.\end{aligned}$$

Let $\Sigma^1_\pm$ and $\Sigma^2_\pm$ be the sections near $\mathbf{s}_\pm$, introduced in Section 5, that are transverse to Γ_1 and Γ_2, respectively. Then, we have in the intermediate normalizing coordinates $(V,S) = \varphi(u,s)$:

$$\Sigma^1_+ = \varphi^{-1}\bigl(\{V=1\}\bigr) \quad \text{and} \quad \Sigma^1_- = \bigl\{(u,s)\colon \varphi_1(-u,-s) = 1,\ s \geqslant 0\bigr\}.$$

Recall that L_1 is the transition map of $\hat{X}^{(2l)}_a$ along Γ_1 from sections Σ^1_- to Σ^1_+, expressed in terms of S. In a natural way, we define the transverse sections

$$\bar{\sigma}^2_+ = \varphi\bigl(\Sigma^2_+\bigr) \quad \text{and} \quad \bar{\sigma}^2_- = \bigl\{(V,S)\colon -\varphi^{-1}(V,S) \in \Sigma^2_-\bigr\}.$$

Now we denote the Dulac maps by D_1 and D_2, that describe the corner passages for the flow of $X^{\text{int.norm}}_{(a,2l)}$ near $\mathbf{s}_+$ and $\mathbf{s}_-$, respectively:

$$D_1 : \{V=1, S>0\} \to \bar{\sigma}^2_+ \quad \text{and} \quad D_2 \circ (-\mathrm{Id}) : \{V=1, S<0\} \to \bar{\sigma}^2_-,$$

and the regular transition along Γ_2 by $L_2 : \bar{\sigma}^2_- \to \bar{\sigma}^2_+$, choosing regular coordinates on $\bar{\sigma}^2_\pm$ such that

$$D_1|_{\{S>0\}} = D \circ T|_{\{S>0\}} \quad \text{and} \quad T \circ (D_2 \circ -\mathrm{Id})|_{\{S<0\}} = (D \circ -\mathrm{Id}) \circ T|_{\{S<0\}},$$

where $T = T^a$ denotes the transformation $w = T^a(S)$. Notice that D_1 and D_2 are likewise the corner passages for the flow of $(\bar{h}(S))^{-1} \cdot X^{\text{int.norm}}_{(a,2l)}$ near $\mathbf{s}_+$ and $\mathbf{s}_-$. Furthermore,

$$\begin{cases} D_1 \circ (L_1 \circ -\mathrm{Id})|_{\{S<0\}} = D \circ (R_1 \circ -\mathrm{Id}) \circ T|_{\{S<0\}},\\ T \circ L_2 \circ (D_2 \circ -\mathrm{Id})|_{\{S<0\}} = R_2 \circ (D \circ -\mathrm{Id}) \circ T|_{\{S<0\}}. \end{cases} \tag{72}$$

From the first equation in (72), we can write the following relation between L_1 and R_1:

$$T \circ (L_1 \circ -\mathrm{Id}) = (R_1 \circ -\mathrm{Id}) \circ T. \tag{73}$$

Using relation (73) and Proposition 24, we can relate the first $4l-1$ higher order derivatives of R_1 with respect to w at $w=0$ to the corresponding ones of T with respect to S at $S=0$.

Proposition 25. *Consider the map R_1, as introduced in* (19)*, that describes the transition of the flow of $\bar{X}_a^{(2l)}$ from σ_-^1 to σ_+^1 along the unbroken connection Γ_1, and that is expressed in terms of the normalizing parameter w. Let* (Id, T^a) *be the C^∞ family of transformations that transforms the vector fields $((\bar{h}(S))^{-1} \cdot X_{(a,2l)}^{\text{int.norm}})$ into the vector fields $(X_{(\alpha(a),2l)}^{\text{norm}})$. If we write $T = T^a$, then:*

1. $R_1(0) = 0$ *and* $R_1'(0) = 1$.
2. $R_1''(0) = \frac{2T''(0)}{(T'(0))^2}$ *and* $R_1^{(3)}(0) = -3T''(0)T'(0)R_1''(0)$.
3. *If* $2 \leqslant k \leqslant 2l-1$ *such that*

$$R_1^{(j)}(0) = 0, \quad \forall 2 \leqslant j \leqslant 2k-1,$$

then

$$R_1^{(2k)}(0) = 2\frac{T^{(2k)}(0)}{(T'(0))^{2k}} \quad \text{and} \quad R_1^{(2k+1)}(0) = 0.$$

Proposition 26. *Let* (Id, T^a) *be the C^∞ family of transformations that transforms the vector fields $((\bar{h}(S))^{-1} \cdot X_{(a,2l)}^{\text{int.norm}})$ into the vector fields $(X_{(\alpha(a),2l)}^{\text{norm}})$, and write $T^a = T$. Let $\bar{g}, \bar{h}$ be the functions that occur in the intermediate normal form $X_{(a,2l)}^{\text{int.norm}}$ in* (41)*, and write*

$$4l^2\bar{g}(S)/\bar{h}(S) = 1 + \sum_{i=1}^{4l-1} g_i S^i + O\left(S^{4l}\right), \quad S \to 0. \tag{74}$$

Then,

$$T(S) = \sqrt[2l-1]{\frac{1}{2l}}\, S\left(1 + \frac{g_1}{4l-3}S + O\left(S^2\right)\right), \quad S \to 0.$$

Furthermore, if $g_{2j-1} = 0$, $\forall 1 \leqslant j \leqslant k$, for a certain $1 \leqslant k \leqslant l-1$, then

$$\begin{cases} T^{(2j)}(0) = 0, \quad \forall 1 \leqslant j \leqslant k, \\ \dfrac{\sqrt[2l-1]{2l}}{(2k+2)!} T^{(2k+2)}(0) = \dfrac{g_{2k+1}}{4l-2k-3}. \end{cases} \tag{75}$$

Proof. The transformation $w = T(S)$ transforms the equation $\dot{S} = S^{4l-1}G(S)$ into $\dot{w} = w^{4l-1}(1 + \alpha w^{4l-2})^{-1}$, where $G(S) = \frac{\bar{g}(S)}{\bar{h}(S)}$. Then, we have for $S \to 0$,

$$T'(S)G(S)S^{4l-1} = \left(T(S)\right)^{4l-1}\left(1 + \alpha \cdot \left(T(S)\right)^{4l-2}\right)^{-1}, \quad S \to 0. \tag{76}$$

For certain C^∞ functions $\tilde{t}_i$, $1 \leqslant i \leqslant 4l-2$, depending on a, we can write

$$T(S) = t_1 S\left(1 + \sum_{i=1}^{4l-2} \tilde{t}_i S^i\right) + O\left(S^{4l}\right), \quad S \to 0; \tag{77}$$

notice that $\forall 2 \leqslant i \leqslant 4l-1$,

$$\tilde{t}_{i-1} = \frac{T^{(i)}(0)}{i!T'(0)}.$$

Then, for $S=0$, the identity in (76) displays the following relation:

$$t_1 = (2l)^{-\frac{1}{2l-1}}. \tag{78}$$

Using (74) and (77), we can rewrite (76) as follows:

$$\left(1+\sum_{i=1}^{4l-2}(i+1)\tilde{t}_i S^i\right)\left(1+\sum_{i=1}^{4l-3} g_i S^i\right) + O\big(S^{4l-1}\big)$$
$$= \left(1+\sum_{i=1}^{4l-2}\tilde{t}_i S^i\right)^{4l-1}\big(1+O\big(S^{4l-2}\big)\big), \quad S \to 0.$$

By identifying coefficients according to linear terms in (76), we find

$$\tilde{t}_1 = \frac{1}{4l-3} g_1. \tag{79}$$

Proceeding by induction, one finds (75). □

From Proposition 26, the formulas that express the derivatives of R_1 in terms of a_{2k+1}, $0 \leqslant k \leqslant l-1$, are obtained, once we have found such formulas for the coefficients g_{2k+1}, defined in (74). Next lemma, that gives similar expressions for the center manifold representation, will be useful as will be seen in Proposition 28.

Lemma 27. *Let $U(s,a)$ be the center manifold at $(1,0)$, that is defined by* (60) *with asymptotic expansion* (68) *for $s \to 0$. Then*:

1. $U(0,a)=1$ *and* $\frac{\partial U}{\partial s}(0,a) = -\frac{1}{2l}a_1$.
2. *If* $1 \leqslant k \leqslant l-1$ *such that* $a_1 = a_3 = \cdots = a_{2k-1} = 0$, *then*

$$\begin{cases} \dfrac{\partial^{2i+1}U}{\partial s^{2i+1}}(0,a) = 0, & \forall 0 \leqslant i \leqslant k-1, \\ \dfrac{1}{(2k+1)!}\dfrac{\partial^{2k+1}U}{\partial s^{2k+1}}(0,a) = -\dfrac{1}{2l}\cdot a_{2k+1}. \end{cases}$$

3. *If* $a_1 = a_3 = \cdots = a_{2l-1} = 0$, *then* $\frac{\partial^{2i+1}U}{\partial s^{2i+1}}(0,a) = 0$, $\forall i \in \mathbb{N}$.

Proof. Up to terms of order $O(s^{4l-2})$, the identity in (60) reduces to

$$1 - \big(U(s,a)\big)^{2l} - s^{2l} g\left(\frac{U(s,a)}{s}, a\right) = O\big(s^{4l-2}\big), \quad s \to 0. \tag{80}$$

With the notation introduced in (68), we have

$$\begin{cases} \big(U(s,a)\big)^{2l} = 1 + 2l\gamma_1 s + O\big(s^2\big), & s \to 0, \\ s^{2l} g\left(\dfrac{U(s,a)}{s}, a\right) = a_1 s + O\big(s^2\big), & s \to 0. \end{cases}$$

Substituting these asymptotics in (80), we find

$$\frac{\partial U}{\partial s}(0,a) = -\frac{1}{2l}a_1.$$

We continue by induction on $1 \leqslant k \leqslant l-1$, assuming that $a_{2i-1} = \gamma_{2i-1} = 0, \forall 1 \leqslant i \leqslant k$. Then,

$$\begin{aligned} s^{2l} g\left(\frac{U(s,a)}{s}, a\right) &= \sum_{i=1}^{k} a_{2i}\big[U(s,a)\big]^{2l-2i} s^{2i} \\ &\quad + a_{2k+1}\big[U(s,a)\big]^{2l-2k-1} s^{2k+1} + O\big(s^{2k+2}\big), \quad s \to 0, \end{aligned} \tag{81}$$

and, for $J \in \{0, 2, \dots, 2k, 2k+1\}$,

$$\big(U(s,a)\big)^{2l-J} = 1 + \sum_{i=1}^{k} \xi_{2i,j} s^{2i} + (2l-J)\gamma_{2k+1} s^{2k+1} + O\big(s^{2k+2}\big), \quad s \to 0, \tag{82}$$

for certain polynomials $\xi_{2i,J}$ in γ_{2n}, $1 \leqslant n \leqslant i$, $1 \leqslant i \leqslant k$, $J \in \{0, 2, \dots, 2k, 2k+1\}$. According to the asymptotics in (80), (81) and (82), the only relevant coefficient corresponds to the one of s^{2k+1} in $(U(s,a))^{2l}$. For $k \leqslant 2l-1$, the coefficient that corresponds to s^{2k+1} in (80), vanishes, and hence

$$-2l\gamma_{2k+1} - a_{2k+1} = 0.$$

This proves the induction. □

Proposition 28. *Let $\bar{g}, \bar{h}$ be the functions that occur in the intermediate normal form $(X^{\text{int.norm}}_{(a,2l)})$ in* (41)*. Write $G(S) = \bar{g}(S)(\bar{h}(S))^{-1}$, and let g_i, $1 \leqslant i \leqslant 2l-1$, be the coefficients defined in* (74)*. Then*

$$G(S) = \frac{1}{4l^2}\left(1 - \frac{1}{2l}a_1 S + \sum_{i=2}^{2l-1} g_i S^i + O\big(S^{2l}\big)\right), \quad S \to 0.$$

Furthermore, if $1 \leqslant k \leqslant l-1$ with $a_{2j-1} = 0$, $\forall 1 \leqslant j \leqslant k$, then

$$\begin{cases} g_{2j-1} = 0, & \forall 1 \leqslant j \leqslant k, \\ g_{2k+1} = \dfrac{2k-1}{2l} a_{2k+1}. \end{cases}$$

Proof. The functions $\bar{g}$ and $\bar{h}$ satisfy the following asymptotics:

$$\begin{cases} \bar{g}(S) = (2l)^{-1}U(S) + O\big(S^{4l-1}\big), & S \to 0, \\ \bar{h}(S) = 2l\big(U(S)\big)^{2l-1} + \sum_{k=1}^{2l-1}(2l-k)a_k\big(U(S)\big)^{2l-k-1}S^k + O\big(S^{4l-1}\big), & S \to 0. \end{cases}$$

From the first line, it follows that the asymptotics of $\bar{g}$ follow immediately from Lemma 27. From the second line, it follows that $\bar{h}(0) = 2l$ and $\bar{h}'(0) = 2l(2l-1)\gamma_1 + (2l-1)a_1 = 0$. This implies the following lower order asymptotics of $\bar{h}$ at $S = 0$:

$$\bar{h}(S) = 2l + O\big(S^2\big), \quad S \to 0.$$

Furthermore, by induction on $1 \leqslant k \leqslant l-1$, it follows by use of Lemma 27, that, if $a_{2j-1} = 0$, $\forall 1 \leqslant j \leqslant k$, then

$$\begin{aligned} \frac{\bar{h}^{(2k+1)}(0)}{(2k+1)!} &= 2l(2l-1)\gamma_{2k+1} + (2l-2k-1)a_{2k+1} \\ &= -(2k)a_{2k+1}. \end{aligned}$$

The statement in the proposition now follows by a straightforward induction on $1 \leqslant k \leqslant l-1$. $\square$

Combining Propositions 25, 26 and 28, we find the formulas for the derivatives of R_1, as stated in Proposition 4.

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