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Characteristic numbers and invariant subvarieties for projective webs

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Abstract

We define the characteristic numbers of a holomorphic k-distribution of any dimension on $\mathbb{P}^n$ and obtain relations between these numbers and the characteristic numbers of an invariant subvariety. As an application we bound the degree of a smooth invariant hypersurface.

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1. Introduction

The aim of this work is to relate the characteristic numbers of projective k-webs, or more generally, k-distributions of arbitrary dimension to those of invariant subvarieties. Loosely speaking, a k-distribution $\mathcal{W}$ of dimension p on $\mathbb{P}^n$ is locally given by k holomorphic fields of p-planes on the complement of a Zariski closed set. The most basic invariants attached to it are its characteristic numbers $d_0, \ldots, d_p$ where d_i is defined as the degree of the tancency locus of the distribution with a generic $\mathbb{P}^{n-p+i-1}$ linearly embedded in $\mathbb{P}^n$. Suppose now $V \hookrightarrow \mathbb{P}^n$ is a subvariety invariant by $\mathcal{W}$. Our goal is to obtain inequalities involving the characteristic numbers of V and $\mathcal{W}$. As a corollary we give some bounds for the degree of a smooth invariant hypersurface.

The question of bounding the degree of an algebraic curve which is a solution of a foliation on $\mathbb{P}^2$ in terms of the degree of the foliation was treated by H. Poincaré in [12]. Versions of

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this problem have been considered in a number of recent works, see for example [4,3,2,14,15,5] placed in chronological order.

In this paper we associate to any k-distribution $\mathcal{W}$ a subvariety $S_{\mathcal{W}}$ of $\mathbb{P}(T^*\mathbb{P}^n)$. First we show that when we write its cohomology class, the characteristic numbers $d_0, \ldots, d_p$ appear naturally:

$$[S_{\mathcal{W}}] = d_p h^p + \cdots + d_1 h \check{h}^{p-1} + d_0 \check{h}^p,$$

where h and $\check{h}$ are the pullbacks of the hyperplane classes on $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$ respectively. We also define, as in the case of foliations, the polar classes $P_s^{\mathcal{W}}$ of a k-distribution and get theirs degrees in terms of the characteristic numbers:

$$\deg\bigl(P_s^{\mathcal{W}}\bigr) = d_s + d_{s-1}.$$

Then we consider a subvariety which is invariant by a k-distribution and relate the polar classes of them obtaining more relations than the known for distributions, see Theorem 3.1. As a consequence we obtain as many bounds for the degree of a smooth invariant hypersurface as the dimension of the k-distribution, see Corollary 3.1.

2. Characteristic numbers of projective webs

Let $\mathbb{P}^n$ be the n-dimensional complex projective space and $M = \mathbb{P}(T^*\mathbb{P}^n)$ the projectivization of its cotangent bundle. Since M can be identified with the incidence variety of points and hyperplanes in $\mathbb{P}^n$, one has two natural projections

$$\begin{array}{ccccc} & & M = \mathbb{P}(T^*\mathbb{P}^n) & & \subseteq \mathbb{P}^n \times \check{\mathbb{P}}^n \\ & \swarrow^{\pi} & & \searrow^{\check{\pi}} & \\ \mathbb{P}^n & & & & \check{\mathbb{P}}^n \end{array}$$

Let us denote by $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and $\check{h} = c_1(\mathcal{O}_{\check{\mathbb{P}}^n}(1))$ the hyperplane classes on $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$ respectively. We still denote by h and $\check{h}$ the respective pullbacks to M by π and $\check{\pi}$. Note that the cohomology ring $H^*(M)$ is, via the pullback map $\pi^* : H^*(\mathbb{P}^n) \to H^*(M)$, an algebra over the ring $H^*(\mathbb{P}^n)$, which is generated by $\xi = c_1(\mathcal{O}_M(-1))$, the Chern class of the tautological bundle $\mathcal{O}_M(-1)$, with the relation $\sum_{i=0}^n \binom{n+1}{i+1} h^{n-i} \xi^i = 0$ (see [7, p. 606]).

Observe that h^n is the class of a fiber of π and the restriction of $\mathcal{O}_M(-1)$ to each fiber is the universal bundle, so that $\int_M \xi^{n-1} h^n = (-1)^{n-1}$ and $\int_M \xi^n h^{n-1} = (-1)^n(n+1)$, where the last equation follows from the previous relation. Then if we write $\check{h} = ah + b\xi$ it is easy to see that $b = -1$ and therefore we get the following description of $H^*(M)$

$$H^*(M) = \frac{\mathbb{Z}[h, \check{h}]}{\langle h^{n+1}, h^n - h^{n-1}\check{h} + \cdots + (-1)^n \check{h}^n \rangle}.$$

Clearly we also have the relations $\check{h}^{n+1} = 0$, $\int_M h^n \check{h}^{n-1} = \int_M h^{n-1} \check{h}^n = 1$.

Let $V \subseteq \mathbb{P}^n$ be an irreducible projective subvariety, the **conormal variety** of V is defined as $\mathrm{Con}(V) = \overline{\mathbb{P}(N^* V_{sm})}$, where V_{sm} denotes the smooth part of V and $N^* V_{sm}$ its conormal bundle. We note that via the identification $M \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$, $\mathrm{Con}(V)$ is the closure of the set of pairs (x, H) such that x is a smooth point of V and H is a hyperplane containing the tangent plane $T_x V$. For example, the conormal variety of a point $\mathbb{P}^0 \subseteq \mathbb{P}^n$ is all the fiber $\pi^{-1}(\mathbb{P}^0)$, so its class is h^n. More generally one has the following lemma.

Lemma 2.1. *The conormal variety of a linearly embedded $\mathbb{P}^j \subseteq \mathbb{P}^n$ is a trivial $\mathbb{P}^{n-j-1}$ bundle over $\mathbb{P}^j$ which class is given by*

$$[\mathrm{Con}(\mathbb{P}^j)] = (-1)^j h^n + \cdots + h^{n-j+2}\check{h}^{j-2} - h^{n-j+1}\check{h}^{j-1} + h^{n-j}\check{h}^j.$$

Proof. Recall that $\mathrm{Con}(\mathbb{P}^j) = \{(p, H) \in M\colon\ p \in \mathbb{P}^j,\ H \supseteq \mathbb{P}^j\}$ is an irreducible subvariety of M of codimension n, so we can write

$$[\mathrm{Con}(\mathbb{P}^j)] = a_n h^n + a_{n-1}h^{n-1}\check{h} + \cdots + a_1 h\check{h}^{n-1}$$

and use the above relations to get

$$1 = \int_M [\mathrm{Con}(\mathbb{P}^j)] \cdot h^j \cdot \check{h}^{n-j-1} = a_{n-j} + a_{n-j-1}$$

and

$$0 = \int_M [\mathrm{Con}(\mathbb{P}^j)] \cdot h^k \cdot \check{h}^{n-k-1} = a_{n-k} + a_{n-k-1}$$

for $k \in \{0, 1, \ldots, j-1, j+1, \ldots, n-1\}$ (here $a_0 = 0$). The lemma follows from the previous equalities. □

For any projective subvariety $V \subseteq \mathbb{P}^n$ of dimension q we define its **characteristic numbers** as the integers a_is such that

$$[\mathrm{Con}(V)] = a_n h^n + a_{n-1}h^{n-1}\check{h} + \cdots + a_1 h\check{h}^{n-1}.$$

For convenience we fix $a_0 = 0$ and in particular we have

$$\deg(V) = \int_M [\mathrm{Con}(V)] \cdot h^q \cdot \check{h}^{n-q-1} = a_{n-q} + a_{n-q-1}.$$

Now we refer to [10, Section 1.3] for more details on the following definitions. Fix $k, p \in \mathbb{N}$ with $1 \leqslant p < n$. Roughly speaking, to give a k-distribution of dimension p is the same to give, over a generic point, a set of k various p-dimensional planes, varying holomorphically. More precisely, a **k-distribution $\mathcal{W}$ of dimension p on $\mathbb{P}^n$** is given by an open covering $\mathcal{U} = \{U_i\}$ of $\mathbb{P}^n$ and k-symmetric $(n-p)$-forms $\omega_i \in \mathrm{Sym}^k\, \Omega^{n-p}_{\mathbb{P}^n}(U_i)$ subject to the conditions:

1. For each non-empty intersection $U_i \cap U_j$ there exists a non-vanishing function $g_{ij} \in \mathcal{O}_{U_i \cap U_j}$ such that $\omega_i = g_{ij}\omega_j$.
2. The zero set of ω_i has codimension at least two for every i.
3. For every i and a generic $x \in U_i$, the germ of ω_i at x seen as homogeneous polynomial of degree k in the ring $\mathcal{O}_x[\ldots, dx_{i_1} \wedge \cdots \wedge dx_{i_{n-p}}, \ldots]$ is square-free.
4. For every i and a generic $x \in U_i$, the germ of ω_i at x is a product of k various $(n-p)$-forms $\beta_1, \ldots, \beta_k$, where each β_i is a wedge product of $(n-p)$ linear forms.

If in addition the forms β_i are integrable we will say that the distribution is a **k-web of dimension p on $\mathbb{P}^n$**.

The k-symmetric $(n-p)$-forms $\{\omega_i\}$ patch together to form a global section $\omega = \{\omega_i\} \in \mathrm{H}^0(\mathbb{P}^n, \mathrm{Sym}^k\, \Omega^{n-p}_{\mathbb{P}^n} \otimes \mathcal{L})$ where $\mathcal{L}$ is the line bundle over $\mathbb{P}^n$ determined by the cocycle $\{g_{ij}\}$.

The **singular set** of $\mathcal{W}$, denoted by $\mathrm{Sing}(\mathcal{W})$, is the zero set of the twisted k-symmetric $(n-p)$-form ω. The **degree** of $\mathcal{W}$, denoted by $\deg(\mathcal{W})$, is geometrically defined as the degree of the tangency locus between $\mathcal{W}$ and a generic $\mathbb{P}^{n-p}$ linearly embedded in $\mathbb{P}^n$. If $i : \mathbb{P}^{n-p} \hookrightarrow \mathbb{P}^n$ is the inclusion then the degree of $\mathcal{W}$ is the degree of the zero divisor of the twisted k-symmetric $(n-p)$-form $i^*\omega \in \mathrm{H}^0(\mathbb{P}^{n-p}, \mathrm{Sym}^k\, \Omega^{n-p}_{\mathbb{P}^{n-p}} \otimes \mathcal{L}|_{\mathbb{P}^{n-p}})$. Since $\Omega^{n-p}_{\mathbb{P}^{n-p}} = \mathcal{O}_{\mathbb{P}^{n-p}}(-n+p-1)$ it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathcal{W}) + k(n-p) + k)$.

We say that $x \in \mathbb{P}^n$ is a **smooth point** of $\mathcal{W}$, for short $x \in \mathcal{W}_{sm}$, if $x \notin \mathrm{Sing}(\mathcal{W})$ and the germ of ω at x satisfies the conditions described in (3) and (4) above. For any smooth point x of $\mathcal{W}$ we have k distinct (not necessarily in general position) linearly embedded subspaces of dimension p passing through x. Each one of these subspaces will be called p-plane tangent to $\mathcal{W}$ at x and denoted by $T^1_x\mathcal{W}, \ldots, T^k_x\mathcal{W}$.

To any k-distribution $\mathcal{W}$ of dimension p we can associate the subvariety $S_{\mathcal{W}}$ of codimension p of M defined as

$$S_{\mathcal{W}} = \overline{\big\{(x, H) \in M\colon\ x \in \mathcal{W}_{sm} \text{ and } \exists 1 \leqslant i \leqslant k,\ H \supset T^i_x\mathcal{W}\big\}},$$

where the overline in the right side means the Zariski closure in M.

The **characteristic numbers of** $\mathcal{W}$ are by definition the $p+1$ integers

$$d_i = \int_M [S_{\mathcal{W}}] \cdot \big[\mathrm{Con}\big(\mathbb{P}^{n-p-1+i}\big)\big] \cdot h^{n-p-1}$$

with i ranging from 0 to p. We note that d_i is the degree of the tangency locus between $\mathcal{W}$ and a generic $\mathbb{P}^{n-p+i-1}$. In particular $d_0 = k$ and d_1 is the degree of $\mathcal{W}$, that is $d_1 = \deg(\mathcal{W})$. We remark that in the case $p = n-1$ we arrive in the same definition of [10, Section 1.4.1].

Lemma 2.2. *The class of* $S_{\mathcal{W}} \subseteq M$ *is given by*

$$[S_{\mathcal{W}}] = d_p h^p + \cdots + d_1 h \check{h}^{p-1} + d_0 \check{h}^p.$$

Proof. It follows from Lemma 2.1 and from the definition of the characteristic numbers. □

3. Relations on the characteristic numbers for k-distributions and invariant subvarieties

Let $\mathcal{D} : \emptyset = L_{n+1} \subseteq L_n \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathbb{P}^n$ be a flag of linearly embedded subspaces, where L_i has codimension i. For each $i \in \{0, \ldots, n+1\}$ we fix the set $\mathcal{H}_i$ of hyperplanes containing L_i; it corresponds to a $(i-1)$-dimensional linear subspace of $\check{\mathbb{P}}^n$. Therefore the class of its associated variety $S_{\mathcal{H}_i} = \check{\pi}^{-1}(\mathcal{H}_i) \subseteq M$ is

$$[S_{\mathcal{H}_i}] = \check{h}^{n-i+1}.$$

Now for a projective subvariety $V \subseteq \mathbb{P}^n$ of dimension q and $j \in \{0, \ldots, q\}$ we denote by $P^V_j = \mathrm{tang}(V, \mathcal{H}_{q-j+2})$, where $\mathrm{tang}(V, \mathcal{H}_i) := \pi(\mathrm{Con}(V) \cap S_{\mathcal{H}_i})$. On the other hand P^V_j can be seen as pre-image of a Schubert cycle in the Grassmannian by the Gauss map of V. To be more precise let $\mathbb{G}(q, n)$ be the Grassmannian of q-dimensional linear spaces of $\mathbb{P}^n$ and consider the Schubert cycle of codimension j

$$\sigma^q_j = \sigma^q_j(L_{q-j+2}) = \big\{\Gamma \in \mathbb{G}(q, n)\colon\ \dim(\Gamma \cap L_{q-j+2}) \geqslant j-1\big\}.$$

If $\mathcal{G}_V : V \dashrightarrow \mathbb{G}(q, n)$ is the natural Gauss map associated to V which sends a smooth point $x \in V_{sm}$ to the tangent space T_xV then $P^V_j = \overline{\mathcal{G}_V{}^{-1}_{|V_{sm}}(\sigma^q_j)}$. These are the polar classes of the variety V defined in [11]. It follows from the transversality of a general translate (cf. [8]) that for a generic flag, P_j^V is equidimensional and its dimension is $q-j$. See [11] for details.

In the same spirit, for a k-distribution $\mathcal{W}$ of dimension p and j a natural number with $1 \leqslant j \leqslant p+1$, we set $P_j^{\mathcal{W}} := \mathrm{tang}(\mathcal{W}, \mathcal{H}_{p-j+2})$ where $\mathrm{tang}(\mathcal{W}, \mathcal{H}_i) := \pi(S_{\mathcal{W}} \cap S_{\mathcal{H}_i})$. When $k=1$ we obtain the polar classes of the distribution $\mathcal{W}$ given in [9] and also in [6].

In order to define the Gauss map associated to the distribution we consider $X = \mathbb{G}(p,n)^k/S_k$ the quotient of $\mathbb{G}(p,n)^k = \mathbb{G}(p,n) \times \cdots \times \mathbb{G}(p,n)$ by the equivalence relation which identifies $(\Lambda_1, \ldots, \Lambda_k)$ and $(\Lambda_{\tau(1)}, \ldots, \Lambda_{\tau(k)})$, where $\tau \in S_k$ (the symmetric group in k elements). Then we define the Gauss map

$$\begin{aligned} \mathcal{G}_{\mathcal{W}} : \mathbb{P}^n &\dashrightarrow X \\ x &\mapsto \left[T_x^1\mathcal{W}, \ldots, T_x^k\mathcal{W}\right]. \end{aligned}$$

Since $\mathcal{W}$ is given locally by k holomorphic distributions of dimension p on the complement of a Zariski closed set, each coordinate of $\mathcal{G}_{\mathcal{W}}$ is locally the Gauss map associated to one of these distributions. Therefore $\mathcal{G}_{\mathcal{W}}$ is a rational map.

Let us consider the Schubert cycle

$$\sigma_j^p = \sigma_j^p(L_{p-j+2}) = \left\{\Lambda \in \mathbb{G}(p,n)\colon \dim(\Lambda \cap L_{p-j+2}) \geqslant j-1\right\}$$

and the respective closed set in the quotient

$$\Sigma_j^p = \Sigma_j^p(L_{p-j+2}) = \sigma_j^p \times \mathbb{G}(p,n)^{k-1}/S_k \subset X.$$

If U is the maximal Zariski open set where $\mathcal{G}_{\mathcal{W}}$ is regular, it is not hard to see that $P_j^{\mathcal{W}} = \overline{\mathcal{G}_{\mathcal{W}|_U}^{-1}(\Sigma_j^p)}$.

Proposition 3.1. *If $a_0, \ldots, a_n$ and $d_0, \ldots, d_p$ are the characteristic numbers of the subvariety V and the k-distribution $\mathcal{W}$ respectively, then for any $j \in \{0, \ldots, q\}$ and any $s \in \{1, \ldots, p\}$ we have*

$$\deg\left(P_j^V\right) = a_{n-(q-j)} + a_{n-(q-j)-1}, \qquad \deg\left(P_s^{\mathcal{W}}\right) = d_s + d_{s-1}.$$

In particular $\deg(P_0^V) = \deg(V)$ *and* $\deg(P_1^{\mathcal{W}}) = k + \deg(\mathcal{W})$.

Proof. It follows from the facts

$$\deg\left(P_j^V\right) = \int_M \left[\mathrm{Con}(V)\right] \cdot [S_{\mathcal{H}_{q-j+2}}] \cdot h^{q-j}$$

and

$$\deg\left(P_j^{\mathcal{W}}\right) = \int_M [S_{\mathcal{W}}] \cdot [S_{\mathcal{H}_{p-j+2}}] \cdot h^{n-j}. \qquad \square$$

Let us assume now that the flag $\mathcal{D}$ is sufficiently generic. We state now our main result which relates the characteristic numbers of V and $\mathcal{W}$ when V is $\mathcal{W}$-invariant. We say that V is $\mathcal{W}$-invariant if $V \nsubseteq \operatorname{Sing}\mathcal{W}$ and $i^*\omega$ vanishes identically, where $i : V \hookrightarrow \mathbb{P}^n$ is the inclusion and ω is the twisted k-symmetric $(n-p)$-form defining $\mathcal{W}$.

Theorem 3.1. *Suppose that $\mathcal{W}$ is a k-distribution of dimension p on $\mathbb{P}^n$ admitting an invariant projective subvariety V of dimension $q \geqslant p$ and fix $m \in \{1, \ldots, p\}$. If j is a number between* 0 *and $q-p$ such that $P^V_{q-p-j+m} \subseteq P^{\mathcal{W}}_m$ then $P^V_{q-p-j} \nsubseteq P^{\mathcal{W}}_m$ and*

$$\frac{a_{n-(p-m+j)} + a_{n-(p-m+j)-1}}{a_{n-(p+j)} + a_{n-(p+j)-1}} \leqslant d_m + d_{m-1}.$$

In particular the inequality holds true for $j = 0$.

Proof. Let j be a number between 0 and $q-p$. To simplify the notation let us fix $\lambda_1 = p + j + 2$ and $\lambda_2 = p - m + 2$. Hence $P^V_{q-p-j} = \overline{\mathcal{G}_{V|_{V_{sm}}}^{-1}(\sigma^q_{q-p-j}(L_{\lambda_1}))}$ and $P^{\mathcal{W}}_m = \overline{\mathcal{G}_{\mathcal{W}|_U}^{-1}(\Sigma^p_m(L_{\lambda_2}))}$. We will first show that for a generic pair $(L_{\lambda_1}, L_{\lambda_2}) \in \mathbb{G}(n-\lambda_1, n) \times \mathbb{G}(n-\lambda_2, n)$ satisfying $L_{\lambda_1} \subset L_{\lambda_2}$, the dimension of $P^V_{q-p-j} \cap P^{\mathcal{W}}_m$ is at most $p + j - m$.

Let $\mathbb{F} \subset \mathbb{G}(n-\lambda_1, n) \times \mathbb{G}(n-\lambda_2, n)$ be the closed set of pairs satisfying $L_{\lambda_1} \subset L_{\lambda_2}$ and consider

$$\mathcal{U} = \left\{(L_{\lambda_1}, L_{\lambda_2}, \Lambda, \Gamma) \in \mathbb{F} \times X \times \mathbb{G}(q,n)\colon\ \Gamma \in \sigma^q_{q-p-j}(L_{\lambda_1}),\ \Lambda \in \Sigma^p_m(L_{\lambda_2})\right\}.$$

If $\tilde{V} = V_{sm} \cap U$ then $P^V_{q-p-j} \cap P^{\mathcal{W}}_m \cap \tilde{V} = p_1(\psi^{-1}(L_{\lambda_1}, L_{\lambda_2}))$ where p_1 and ψ are the morphisms defined below

$$\begin{array}{ccccc}
 & & \overset{\psi}{\frown} & & \\
\tilde{V} \times_{X \times \mathbb{G}(q,n)} \mathcal{U} & \longrightarrow & \mathcal{U} & \longrightarrow & \mathbb{F} \\
\downarrow p_1 & & \downarrow & & \\
\tilde{V} & \xrightarrow{\mathcal{G}_{\mathcal{W}} \times \mathcal{G}_V} & X \times \mathbb{G}(q,n) & &
\end{array}$$

The unlabeled arrows are the corresponding natural projections. We note that $X \times \mathbb{G}(q,n)$ is an $\mathrm{aut}(\mathbb{P}^n)$-homogeneous space under the natural action. Since the vertical arrow $\mathcal{U} \to X \times \mathbb{G}(q,n)$ is an $\mathrm{aut}(\mathbb{P}^n)$-equivariant morphism the transversality of the general translate (cf. [8]) implies that

$$\begin{aligned}
\dim \tilde{V} \times_{X \times \mathbb{G}(q,n)} \mathcal{U} &= \dim \tilde{V} + \dim \mathcal{U} - \dim X \times \mathbb{G}(q,n) \\
&= q + \dim \mathcal{U} - k \dim \mathbb{G}(p,n) - \dim \mathbb{G}(q,n).
\end{aligned}$$

Since a fiber of the map $\mathcal{U} \longrightarrow \mathbb{F}$ is $\Sigma^p_m \times \sigma^q_{q-p-j}$ one obtains

$$\dim \mathcal{U} = k \dim \mathbb{G}(p,n) - m + \dim \mathbb{G}(q,n) - (q-p-j) + \dim \mathbb{F}.$$

The map ψ is dominant because by hypothesis given a generic pair $(L_{\lambda_1}, L_{\lambda_2}) \in \mathbb{F}$ we can take $x \in P^V_{q-p-j+m} \cap \tilde{V} \subseteq P^{\mathcal{W}}_m \cap P^V_{q-p-j} \cap \tilde{V}$. From this fact together with the above equalities we obtain $\dim \psi^{-1}(L_{\lambda_1}, L_{\lambda_2}) = p + j - m$ for generic pair in $\mathbb{F}$. Therefore

$$\dim P^V_{q-p-j} \cap P^{\mathcal{W}}_m \cap \tilde{V} \leqslant \dim \psi^{-1}(L_{\lambda_1}, L_{\lambda_2}) = p + j - m.$$

This shows that $P^V_{q-p-j} \nsubseteq P^{\mathcal{W}}_m$. Furthermore, from the fact that $P^V_{q-p-j+m} \cap \tilde{V}$ is dense in $P^V_{q-p-j+m}$ and $P^V_{q-p-j+m}$ has pure dimension $p + j - m$ one obtains that each irreducible component of $P^V_{q-p-j+m}$ is an irreducible component of $P^{\mathcal{W}}_m \cap P^V_{q-p-j}$. To conclude the proof of the theorem we have just to apply Bezout's theorem and Proposition 3.1. □

Corollary 3.1. *Let $\mathcal{W}$ be a k-distribution of dimension p on $\mathbb{P}^n$ and V a smooth invariant hypersurface of degree d. Then for each $m \in \{1, \ldots, p\}$ we obtain*

$$(d-1)^m \leqslant d_m + d_{m-1}.$$

In particular,

$$d \leqslant k + \deg(\mathcal{W}) + 1.$$

Proof. When V is a smooth hypersurface, it is well known that $\deg(P_j^V) = d(d-1)^j$ (cf. [13,16] or [14] for a modern approach). In addition, it follows from Theorem 3.1 that for each $m \in \{1, \ldots, p\}$ we have

$$\deg\bigl(P_{n-1-p+m}^V\bigr) \leqslant \deg\bigl(P_{n-1-p}^V\bigr)(d_m + d_{m-1}). \qquad \square$$

Remark 3.1. This corollary generalizes the bound obtained by M.G. Soares for one-dimensional foliations in [15], where it has been considered one-dimensional projective foliations and their tangency locus with a pencil of hyperplanes. This tangency locus is an analogous for foliations of the polar classes for projective varieties. For a variety V of dimension q invariant by a one-dimensional foliation $\mathcal{F}$, he compared their polar classes to get the relation

$$\deg\bigl(P_{q-j}^V\bigr) \leqslant \deg\bigl(P_{q-j-1}^V\bigr) \cdot \bigl(\deg(\mathcal{F}) + 1\bigr).$$

Where P_k^V is the kth polar class of V and j is some number between 0 and $q-1$, see [15, Theorem 1]. As a consequence the bound

$$\deg(V) \leqslant \deg(\mathcal{F}) + 2$$

was obtained for a smooth invariant hypersurface. Polar classes were also considered by R. Mol in [9] for holomorphic distributions of arbitrary dimension. He expressed these classes in terms of the Chern–Mather classes of the tangent sheaf of the distribution, moreover, Theorem 1 of [15] is generalized.

Also we remark that the bound

$$d \leqslant \deg(\mathcal{W}) + (n-p) + 1$$

has been proved by M. Brunella and L.G. Mendes in [2] for normal crossing hypersurfaces invariant by a p-dimensional foliation.

Remark 3.2. By the classical formulas for the polar classes of a smooth complete intersection V (see [13,16]), it is possible to obtains more explicit relations (similar to [9, Corollary 6.3]) between the degree of the homogeneous polynomials defining V and the characteristic numbers of $\mathcal{W}$.

Remark 3.3.[1] Unlike the case of foliations, we cannot expect to bound the degree of non-smooth invariant subvarieties in terms of the degree of the web, even in the case of nodal curves in dimension two. To see this let us take the elliptic curve $E = \mathbb{C}/\langle 1, \tau\rangle$ and consider the foliation $\mathcal{F}_\alpha$ induced by the 1-form $\omega = dy - \alpha\, dx$ on the complex torus $X = E \times E$, where $\alpha \in \mathbb{Q}$. Since X is smooth we have an embedding $X \hookrightarrow \mathbb{P}^5$ and if we fix a leaf C_α of $\mathcal{F}_\alpha$ one may take the

[1] The authors are grateful to Jorge Vitório Pereira for having pointed out this remark.

restriction of a generic linear projection to $\mathbb{P}^2$, $\pi_\alpha : X \to \mathbb{P}^2$, such that the image $D_\alpha = \pi(C_\alpha)$ would be an algebraic curve which has only nodal singularities. Projecting the foliation $\mathcal{F}_\alpha$ we obtain a d-web $\mathcal{W}_\alpha$, where $d = \deg(X) > 1$. Observe that

$$\deg(\mathcal{W}_\alpha) = \mathrm{tang}(\mathcal{W}_\alpha, L) = \mathrm{tang}(\mathcal{F}_\alpha, H) = T^*\mathcal{F}_\alpha.H + H^2$$

where the last equality follows from [1, Proposition 2, p. 23], L is a generic line in $\mathbb{P}^2$ and H is a hyperplane section in X; on the other hand the cotangent bundle $T^*\mathcal{F}_\alpha$ is the same for all these foliations, therefore $\deg(\mathcal{W}_\alpha)$ does not depend of α. Since that varying α we can grow the intersection number between C_α and the curve $C := \{0\} \times E \subseteq X$, and therefore also the intersection between D_α and the fixed curve $\pi_\alpha(C)$, we deduce that D_α is a $\mathcal{W}$-invariant nodal curve in which $\deg(D_\alpha)$ increases and cannot be bound by the fixed number $\deg(\mathcal{W}_\alpha)$.

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