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ScienceDirect
Journal of Functional Analysis 249 (2007) 477–504
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Maximal function and multiplier theorem for weighted space on the unit sphere ☆

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Received 3 December 2006; accepted 26 March 2007

Available online 4 May 2007

Communicated by G. Pisier

Abstract

For a family of weight functions invariant under a finite reflection group, the boundedness of a maximal function on the unit sphere is established and used to prove a multiplier theorem for the orthogonal expansions with respect to the weight function on the unit sphere. Similar results are also established for the weighted space on the unit ball and on the standard simplex.

Keywords: Maximal function; Multiplier; *h*-Harmonics; Sphere; Orthogonal polynomials; Ball; Simplex

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1. Introduction

The purpose of this paper is to study the maximal function in the weighted spaces on the unit sphere and the related domains. Let $S^d = \{x\colon \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$, where $\|x\|$ denotes the usual Euclidean norm. Let $\langle x, y\rangle$ denote the usual Euclidean inner product. We

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☆ The first author was partially supported by the NSERC Canada under grant G121211001. The second author was partially supported by the National Science Foundation under Grant DMS-0604056.

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doi:10.1016/j.jfa.2007.03.023

consider the weighted space on S^d with respect to the measure $h_\kappa^2\,d\omega$, where $d\omega$ is the surface (Lebesgue) measure on S^d and the weight function h_κ is defined by

$$h_\kappa(x) = \prod_{v\in R_+} \bigl|\langle x, v\rangle\bigr|^{\kappa_v}, \quad x \in \mathbb{R}^{d+1}, \tag{1.1}$$

in which R_+ is a fixed positive root system of $\mathbb{R}^{d+1}$, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, and κ is a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u, the reflection with respect to the hyperplane perpendicular to u, is conjugate to σ_v in the reflection group G generated by the reflections $\{\sigma_v\colon v \in R_+\}$. The function h_κ is invariant under the reflection group G. The simplest example is given by the case $G = \mathbb{Z}_2^{d+1}$ for which h_κ is just the product weight function

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \quad \kappa_i \geqslant 0,\ x = (x_1, \ldots, x_{d+1}). \tag{1.2}$$

Denote by a_κ the normalization constant, $a_\kappa^{-1} = \int_{S^d} h_\kappa^2(y)\,d\omega(y)$. We consider the weighted space $L^p(h_\kappa^2; S^d)$ of functions on S^d with the finite norm

$$\|f\|_{\kappa,p} := \biggl(a_\kappa \int_{S^d} \bigl|f(y)\bigr|^p h_\kappa^2(y)\,d\omega(y)\biggr)^{1/p}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(S^d)$, the space of continuous functions on S^d with the usual uniform norm $\|f\|_\infty$.

The weight function (1.1) was first studied by Dunkl in the context of h-harmonics, which are orthogonal polynomials with respect to h_κ^2. A homogeneous polynomial is called an h-spherical harmonics if it is orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2; S^d)$. The theory of h-harmonics is in many ways parallel to that of ordinary harmonics (see [5]). In particular, many results on the spherical harmonics expansions have been extended to h-harmonics expansions, see [3–5,8,12,13] and the references therein. Much of the analysis of h-harmonics depends on the intertwining operator V_κ that intertwines between Dunkl operators, which are a commuting family of first order differential–difference operators, and the usual partial derivatives. The operator V_κ is a uniquely determined positive linear operator. To see the importance of this operator, let $\mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the space of h-harmonics of degree n; the reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ can be written in terms of V_κ as

$$P_n^h(x, y) = \frac{n + \lambda_k}{\lambda_\kappa} V_\kappa\bigl[C_n^{\lambda_k}\bigl(\langle x, \cdot\rangle\bigr)\bigr](y), \quad x, y \in S^d, \tag{1.3}$$

where C_n^λ is the nth Gegenbauer polynomial, which is orthogonal with respect to the weight function $w_\lambda(t) := (1 - t^2)^{\lambda - 1/2}$ on $[-1, 1]$, and

$$\lambda_\kappa = \gamma_\kappa + \frac{d-1}{2} \quad \text{with } \gamma_\kappa = \sum_{v\in R_+} \kappa_v. \tag{1.4}$$

Furthermore, using V_κ, a maximal function that is particularly suitable for studying the h-harmonic expansion is defined in [13] by

$$\mathcal{M}_\kappa f(x) := \sup_{0<\theta\leqslant\pi} \frac{\int_{S^d} |f(y)| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y)}{\int_{S^d} V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y)}, \tag{1.5}$$

where $B(x,\theta) := \{y \in B^{d+1}\colon \langle x, y\rangle \geqslant \cos\theta\}$, $B^{d+1} := \{x\colon \|x\| \leqslant 1\} \subset \mathbb{R}^{d+1}$, and χ_E denotes the characteristic function of the set E. A weak type $(1,1)$ inequality was established for $\mathcal{M}_\kappa f$ in [13]. The result, however, is weaker than the usual weak type $(1,1)$ inequality and it does not imply the strong (p,p) inequality. One of our main results in this paper is to establish a genuine weak type $(1,1)$ result, for which we rely on the general result of [9] on semi-groups of operators. Furthermore, the Fefferman–Stein type result

$$\left\| \left(\sum_j |\mathcal{M}_\kappa f_j|^2 \right)^{1/2} \right\|_{\kappa,p} \leqslant c \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{\kappa,p}$$

also holds, which can be used to derive a multiplier theorem for h-harmonic expansions, following the approach of [1]. These results are presented in Section 2.

In the case of $\mathbb{Z}_2^{d+1}$, the explicit formula of V_κ as an integral operator is known, which allows us to link the maximal function $\mathcal{M}_\kappa f$ with the weighted Hardy–Littlewood maximal function defined by

$$M_\kappa f(x) := \sup_{0<\theta\leqslant\pi} \frac{\int_{c(x,\theta)} |f(y)| h_\kappa^2(y)\, d\omega(y)}{\int_{c(x,\theta)} h_\kappa^2(y)\, d\omega(y)}, \tag{1.6}$$

where $c(x,\theta) := \{y \in S^d\colon \langle x, y\rangle \geqslant \cos\theta\}$ is the spherical cap. We will show that the maximal function $\mathcal{M}_\kappa f$ is bounded by a sum of the Hardy–Littlewood maximal function $M_\kappa f$. As a consequence, we establish a weighted weak $(1,1)$ result for $\mathcal{M}_k f(x)$, in which the weight is also of the form (1.2) but with different parameters. Furthermore, we show that the Fefferman–Stein type inequality holds in the weighted L^p norm. These results are discussed in Section 3.

The analysis on the sphere is closely related to the analysis on the unit ball B^d and on the standard simplex T^d. In fact, much of the results on the later two cases can be deduced from those on the sphere (see [5,12,13] and the references therein). In particular, maximal functions are also defined on B^d and T^d in terms of the generalized translation operators [13]. We will extend our results on the sphere in Section 2 to these two domains, including a multiplier theorem for the orthogonal expansions in the weighted space on B^d and T^d, in Sections 4 and 5, respectively.

Throughout this paper, the constant c denotes a generic constant, which depends only on the values of d, κ and other fixed parameters and whose value may be different from line to line. Furthermore, we write $A \sim B$ if $A \leqslant cB$ and $B \leqslant cA$.

2. Maximal function and multiplier theorem on S^d

2.1. Background

In this subsection we give a brief account of what will be needed later on in the paper. For more background and details, we refer to [5,12,13].

h-Harmonic expansion. Let $\mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the space of spherical h-harmonics of degree n. It is known that $\dim \mathcal{H}_n^{d+1}(h_\kappa^2) = \binom{n+d+1}{n} - \binom{n+d-1}{n-2}$. The usual Hilbert space theory shows that

$$L^2\big(h_k^2; S^d\big) = \sum_{n=0}^{\infty} \mathcal{H}_n^{d+1}\big(h_\kappa^2\big): \quad f = \sum_{n=0}^{\infty} \operatorname{proj}_n^\kappa f,$$

where $\operatorname{proj}_n^\kappa : L^2(h_\kappa^2; S^d) \mapsto \mathcal{H}_n^{d+1}(h_\kappa^2)$ is the projection operator, which can be written as an integral operator

$$\operatorname{proj}_n^\kappa f(x) = a_\kappa \int\limits_{S^d} f(y) P_n^h(x, y) h_\kappa^2(y)\, d\omega(y), \tag{2.1}$$

where P_n^h is the reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$, which satisfies the compact representation (1.3).

Intertwining operator. For a general reflection group, the explicit formula of V_κ is not known. In the case of $\mathbb{Z}_2^{d+1}$, it is an integral operator given by

$$V_\kappa f(x) = c_\kappa \int\limits_{[-1,1]^{d+1}} f(x_1 t_1, \ldots, x_{d+1} t_{d+1}) \prod_{i=1}^{d+1} (1 + t_i)\big(1 - t_i^2\big)^{\kappa_i - 1}\, dt, \tag{2.2}$$

where c_κ is the normalization constant determined by $V_\kappa 1 = 1$. If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int\limits_{-1}^{1} f(t)(1-t)^{\lambda-1}\, dt = \big[f(1) + f(-1)\big]/2.$$

Convolution. For $f \in L^1(h_\kappa^2; S^d)$ and $g \in L^1(w_{\lambda_\kappa}; [-1, 1])$, define [12, Definition 2.1, p. 6]

$$f \star_\kappa g(x) := a_\kappa \int\limits_{S^d} f(y) V_\kappa\big[g\big(\langle x, \cdot\rangle\big)\big](y) h_\kappa^2(y)\, d\omega(y). \tag{2.3}$$

This convolution satisfies the usual Young's inequality (see [12, Proposition 2.2, p. 6]): for $f \in L^q(h_\kappa^2; S^d)$ and $g \in L^r(w_{\lambda_\kappa}; [-1, 1])$, $\|f \star_\kappa g\|_{k,p} \leqslant \|f\|_{k,q} \|g\|_{w_{\lambda_\kappa}, r}$, where $p, q, r \geqslant 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $\kappa = 0$, $V_\kappa = id$, this becomes the classical convolution on the sphere [2]. Notice that by (1.3) and (2.1), we can write $\operatorname{proj}_n^\kappa f$ as a convolution.

Cesàro (C, δ) means. For $\delta > 0$, the (C, δ) means, s_n^δ, of a sequence $\{c_n\}$ are defined by

$$s_n^\delta = \big(A_n^\delta\big)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta c_k, \quad A_{n-k}^\delta = \binom{n-k+\delta}{n-k}.$$

We denote the nth (C,δ) means of the h-harmonic expansion by $S_n^\delta(h_\kappa^2; f)$. These means can be written as

$$S_n^\delta\big(h_\kappa^2; f\big) = \big(f \star_\kappa q_n^\delta\big)(x), \quad q_n^\delta(t) = \big(A_n^\delta\big)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \frac{(k+\lambda)}{\lambda} C_k^\lambda(t),$$

where $\lambda = \lambda_\kappa$. The function $q_n^\delta(t)$ is the kernel of the (C,δ) means of the Gegenbauer expansions at $x = 1$.

Generalized translation operator T_θ^κ. This operator is defined implicitly by [12, p. 7]

$$c_\lambda \int_0^\pi T_\theta^\kappa f(x) g(\cos\theta)(\sin\theta)^{2\lambda}\, d\theta = (f \star_\kappa g)(x), \tag{2.4}$$

where g is any $L^1(w_\lambda)$ function and $\lambda = \lambda_\kappa$. The operator T_θ^κ is well defined and becomes the classical spherical means

$$T_\theta f(x) = \frac{1}{\sigma_d (\sin\theta)^{d-1}} \int_{\langle x,y\rangle = \cos\theta} f(y)\, d\omega(y),$$

when $\kappa = 0$, where $\sigma_d = \int_{S^{d-1}} d\omega = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of S^{d-1}. Furthermore, T_θ^κ satisfies similar properties as those satisfied by T_θ, as shown in [12,13]. In particular, if $f(x) = 1$, then $T_\theta^\kappa f(x) = 1$.

Spherical caps. Let $d(x,y) := \arccos\langle x,y\rangle$ denote the geodesic distance of $x, y \in S^d$. For $0 \leqslant \theta \leqslant \pi$, the set

$$c(x,\theta) := \big\{y \in S^d\colon\ d(x,y) \leqslant \theta\big\} = \big\{y \in S^d\colon\ \langle x,y\rangle \geqslant \cos\theta\big\}$$

is called the spherical cap centered at x. Sometimes we need to consider the solid set under the spherical cap, which we denote by $B(x,\theta)$ to distinguish it from $c(x,\theta)$; that is,

$$B(x,\theta) := \big\{y \in B^{d+1}\colon\ \langle x,y\rangle \geqslant \cos\theta\big\},$$

where $B^{d+1} = \{y \in \mathbb{R}^{d+1}\colon\ \|y\| \leqslant 1\}$.

Maximal function. For $f \in L^1(h_\kappa^2)$, define [13]

$$\mathcal{M}_\kappa f(x) := \sup_{0<\theta\leqslant\pi} \frac{\int_0^\theta T_\phi^\kappa |f|(x)(\sin\phi)^{2\lambda_\kappa}\, d\phi}{\int_0^\theta (\sin\phi)^{2\lambda_\kappa}\, d\phi}.$$

This maximal function can be used to study the h-harmonic expansions, since we can often prove $|(f \star_\kappa g)(x)| \leqslant c\mathcal{M}_\kappa f(x)$. Using (2.4) it is shown in [13] that an equivalent definition for $\mathcal{M}_\kappa f$ is (1.5); that is,

$$\mathcal{M}_\kappa f(x) = \sup_{0<\theta\leqslant\pi} \frac{\int_{S^d} |f(y)| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y)}{\int_{S^d} V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y)}. \tag{2.5}$$

We note that setting $f(x)=1$ and $g(t)=\chi_{[\cos\theta,1]}(t)$ in (2.4) leads to

$$a_\kappa \int\limits_{S^d} V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\,d\omega(y) = c_{\lambda_\kappa} \int\limits_0^\theta (\sin\phi)^{2\lambda_\kappa}\,d\phi \sim \theta^{2\lambda_\kappa+1}. \tag{2.6}$$

2.2. *Maximal function*

To state the weak type inequality, we define, for any measurable subset E of S^d, the measure with respect to h_κ^2 as

$$\mathrm{meas}_\kappa E := \int\limits_E h_\kappa^2(y)\,d\omega(y).$$

Our main result in this section is the boundeness of $\mathcal{M}_\kappa f$.

Theorem 2.1. *If $f \in L^1(h_\kappa^2; S^d)$, then $\mathcal{M}_\kappa f$ satisfies*

$$\mathrm{meas}_\kappa \bigl\{x\colon \mathcal{M}_\kappa f(x) \geqslant \alpha\bigr\} \leqslant c\frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{2.7}$$

Furthermore, if $f \in L^p(h_\kappa^2; S^d)$ for $1 < p \leqslant \infty$, then $\|\mathcal{M}_k f\|_{\kappa,p} \leqslant c\|f\|_{\kappa,p}$.

The inequality (2.7) is usually refereed to as weak type $(1,1)$ inequality. In order to prove this theorem, we follow the approach of [9] on general diffusion semi-groups of operators on a measure space. For this we need the Poisson integral with respect to h_κ^2, which can be written as [5, Theorem 5.3.3, p. 190]

$$P_r^\kappa f(x) = f \star_\kappa p_r^\kappa, \quad \text{where } p_r^\kappa(s) = \frac{1-r^2}{(1-2rs+r^2)^{\lambda_k+1}}. \tag{2.8}$$

The kernel p_r^κ is one of the generating function of the Gegenbauer polynomials of parameter λ_κ. Hence, by (1.3), we can write $P_r^\kappa f$ as

$$P_r^\kappa f(x) = \sum_{n=0}^\infty r^n \operatorname{proj}_n^\kappa f(x), \quad 0 \leqslant r < 1,$$

from which it follows easily that $T^t := P_r^\kappa f$ with $r = e^{-t}$ defines a semi-group. Since V_κ is positive and p_r^κ is clearly non-negative, $P_r^\kappa f \geqslant 0$ if $f \geqslant 0$. We see that the semi-group $P_{e^{-t}}^\kappa f$ is positive. We will need another semi-group, which is the discrete analog of the heat operator:

$$H_t^\kappa f := f \star_\kappa q_t^\kappa, \quad q_t^\kappa(s) := \sum_{n=0}^\infty e^{-n(n+2\lambda_\kappa)t} \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(s). \tag{2.9}$$

In fact, the h-harmonics in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ are the eigenfunctions of an operator $\Delta_{h,0}$, which is the spherical part of a second order differential–difference operator analogous to the ordinary Laplacian, the eigenvalues are $-n(n+2\lambda_k)$. It follows immediately from (2.9) that $\{H_t^\kappa\}_{t\geqslant 0}$ is a semi-group. The following result is the key for the proof of the theorem.

Lemma 2.2. *The Poisson and the heat semi-groups are connected by*

$$P_{e^{-t}}^\kappa f(x) = \int_0^\infty \phi_t(s) H_s^\kappa f(x)\,ds, \tag{2.10}$$

where

$$\phi_t(s) := \frac{t}{2\sqrt{\pi}} s^{-3/2} e^{-(\frac{t}{2\sqrt{s}}-\lambda_\kappa\sqrt{s})^2}.$$

Furthermore, assume that $f(x)\geqslant 0$ for all x, then for all $t>0$,

$$P_*^\kappa f(x) := \sup_{0<r<1} P_r^\kappa f(x) \leqslant c \sup_{s>0} \frac{1}{s}\int_0^s H_u^\kappa f(x)\,du. \tag{2.11}$$

Consequently, $P_*^\kappa f$ is bounded on $L^p(h_\kappa^2; S^d)$ for $1<p\leqslant\infty$ and of weak type $(1,1)$.

Proof. That $\{H_t^\kappa\}_{t\geqslant 0}$ is a semi-group is obvious. Moreover, since V_κ is positive and q_t^κ is known to be non-negative [7], it follows that $H_t^\kappa f$ is positive. The positivity shows that $\|q_t^\kappa\|_{w_{\lambda_\kappa},1}=1$, so that $\|H_t^\kappa f\|_{\kappa,p}\leqslant\|f\|_{\kappa,p}$, $1\leqslant p\leqslant\infty$, by applying Young's inequality on $f \star_\kappa q_t^\kappa$. Thus, using the Hopf–Dunford–Schwarz ergodic theorem [9, p. 48], we conclude that the maximal operator $\sup_{s>0}(\frac{1}{s}\int_0^s H_u^\kappa f(x)\,du)$ is bounded on $L^p(h_\kappa^2; S^d)$ for $1<p\leqslant\infty$ and of week type $(1,1)$. Therefore, it is sufficient to prove (2.10) and (2.11).

First we prove (2.10). Applying the well-known identity [9, p. 46]

$$e^{-v} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{v^2}{4u}}\,du, \qquad v>0,$$

with $v=(n+\lambda_\kappa)t$ and making a change of variable $s=t^2/4u$, we conclude that

$$\begin{aligned} e^{-nt} &= e^{\lambda_\kappa t}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{n(n+2\lambda_\kappa)t^2}{4u}} e^{-\frac{\lambda_\kappa^2 t^2}{4u}}\,du \\ &= \frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-n(n+2\lambda_\kappa)s} s^{-3/2} e^{-(\frac{t}{2\sqrt{s}}-\lambda_\kappa\sqrt{s})^2}\,ds \\ &= \int_0^\infty e^{-n(n+2\lambda_\kappa)s}\phi_t(s)\,ds. \end{aligned}$$

Multiplying by $\operatorname{proj}_n^\kappa f$ and summing up over n proves the integral relation (2.10).

For the proof of (2.11), we use (2.10) and integration by parts to obtain

$$\begin{aligned}P^{\kappa}_{e^{-t}}f(x) &= -\int_0^\infty\biggl(\int_0^s H^{\kappa}_u f(x)\,du\biggr)\phi_t'(s)\,ds\\ &\leqslant \sup_{s>0}\biggl(\frac{1}{s}\int_0^s H^{\kappa}_u f(x)\,du\biggr)\int_0^\infty s\bigl|\phi_t'(s)\bigr|\,ds,\end{aligned}$$

where the derivative of $\phi_t'(s)$ is taken with respect to s. Also, we note that by (2.8) and (2.3),

$$\sup_{0<r\leqslant e^{-1}} P^{\kappa}_r f(x) \leqslant c\|f\|_{1,\kappa} = c\lim_{s\to\infty}\frac{1}{s}\int_0^s H^{\kappa}_u\bigl(|f|\bigr)(x)\,du.$$

Therefore, to finish the proof of (2.11), it suffices to show that $\sup_{0<t\leqslant 1}\int_0^\infty s|\phi_t'(s)|\,ds$ is bounded by a constant.

A quick computation shows that $\phi_t'(s)>0$ if $s<\alpha_t$ and $\phi_t'(s)<0$ if $s>\alpha_t$, where

$$\alpha_t := \frac{t^2}{3+\sqrt{9+4\lambda_\kappa^2 t^2}} \sim t^2, \quad 0\leqslant t\leqslant 1.$$

Since the integral of $\phi_t(s)$ over $[0,\infty)$ is 1 and $\phi_t(s)\geqslant 0$, integration by parts gives

$$\begin{aligned}\int_0^\infty s\bigl|\phi_t'(s)\bigr|\,ds &= 2\alpha_t\phi_t(\alpha_t) - \int_0^{\alpha_t}\phi_t(s)\,ds + \int_{\alpha_t}^\infty \phi_t(s)\,ds\\ &\leqslant 2\alpha_t\phi_t(\alpha_t)+1 = \frac{t}{\sqrt{\pi\alpha_t}}e^{-\frac{(t-2\lambda_\kappa\alpha_t)^2}{4\alpha_t}}+1\leqslant c\end{aligned}$$

as desired. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. From the definition of p^{κ}_r, it follows easily that if $1-r\sim\theta$, then

$$\begin{aligned}p^{\kappa}_r(\cos\theta) &= \frac{1-r^2}{((1-r)^2+4r\sin^2\frac{\theta}{2})^{\lambda_\kappa+1}}\\ &\geqslant c\frac{1-r^2}{((1-r)^2+r\theta^2)^{\lambda_\kappa+1}} \geqslant c(1-r)^{-(2\lambda_\kappa+1)}.\end{aligned}$$

For $j\geqslant 0$ define $r_j := 1-2^{-j}\theta$ and set $B_j := \{y\in B^{d+1}\colon\ 2^{-j-1}\theta\leqslant d(x,y)\leqslant 2^{-j}\theta\}$. The lower bound of p^{κ}_r proved above shows that

$$\chi_{B_j}(y)\leqslant c\bigl(2^{-j}\theta\bigr)^{2\lambda_k+1}p^{\kappa}_{r_j}\bigl(\langle x,y\rangle\bigr),$$

which implies immediately that

$$\chi_{B(x,\theta)}(y) \leqslant \sum_{j=0}^{\infty} \chi_{B_j}(y) \leqslant c\theta^{2\lambda_k+1} \sum_{j=0}^{\infty} 2^{-j(2\lambda_\kappa+1)} p_{r_j}^{\kappa}\big(\langle x, y\rangle\big).$$

Since V_κ is a positive linear operator, applying V_κ to the above inequality gives

$$\begin{aligned}
&\int_{S^{d-1}} \big|f(y)\big| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y) \\
&\quad \leqslant c\theta^{2\lambda_\kappa+1} \sum_{j=0}^{\infty} 2^{-j(2\lambda_\kappa+1)} \int_{S^{d-1}} \big|f(y)\big| V_\kappa\big[p_{r_j}\big(\langle x, y\rangle\big)\big](y) h_\kappa^2(y)\, d\omega(y) \\
&\quad = c\theta^{2\lambda_\kappa+1} \sum_{j=0}^{\infty} 2^{-j(2\lambda_\kappa+1)} P_{r_j}^{\kappa}\big(|f|; x\big) \\
&\quad \leqslant c\theta^{2\lambda_\kappa+1} \sup_{0<r<1} P_r^{\kappa}\big(|f|; x\big).
\end{aligned}$$

Dividing by $\theta^{2\lambda_\kappa+1}$ and using (2.6), we have proved that $\mathcal{M}_\kappa f(x) \leqslant cP_*^{\kappa}|f|(x)$. The desired result now follows from Lemma 2.2. □

A weighted maximal function, call it $M_\kappa f$, on $\mathbb{R}^d$ is defined in [11] in terms of a translation that is defined via Dunkl transform, the analogue of Fourier transform for the weighted $L^2(h_\kappa^2; \mathbb{R}^d)$. The translation can be expressed in term of V_κ when acting on radial functions. The boundedness of the maximal function $M_\kappa f$ was established in [11]. Although the relation between the maximal function $M_\kappa f$ and $\mathcal{M}_\kappa f$ is not known at this moment, it should be pointed out that our proof of Theorem 2.1 follows the line of argument used in the proof of [11].

2.3. *A multiplier theorem*

As an application of the above result we state a multiplier theorem. Let $\Delta g(t) = g(t+1) - g(t)$ and $\Delta^k = \Delta^{k-1}\Delta$.

Theorem 2.3. *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of real numbers that satisfies*

(1) $\sup_j |\mu_j| \leqslant c < \infty$,
(2) $\sup_j 2^{j(k-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^k u_l| \leqslant c < \infty$,

where k is the smallest integer $\geqslant \lambda_\kappa + 1$. Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2; S^d)$, $1 < p < \infty$, multiplier; that is,

$$\Bigg\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j^{\kappa} f \Bigg\|_{\kappa,p} \leqslant c\|f\|_{\kappa,p}, \quad 1 < p < \infty,$$

where c is independent of μ_j and f.

When $\kappa = 0$, the theorem becomes part of [1, Theorem 4.9] on the ordinary spherical harmonic expansions. The proof of this theorem follows that of the theorem in [1]. One of the main ingredient is the Littlewood–Paley function

$$g(f) = \left(\int_0^1 (1-r) \left| \frac{\partial}{\partial r} P_r^\kappa f \right|^2 dr \right)^{1/2}, \tag{2.12}$$

where $P_r^\kappa f$ is the Poisson integral with respect to h_κ^2 defined in (2.8). A general Littlewood-Paley theory was established in [9] for a family of diffusion semi-group of operators $\{T^t\}_{t \geqslant 0}$ on a measure space, in which the g function is defined as

$$g_1(f) = \left(\int_0^\infty t \left| \frac{\partial}{\partial t} T^t f \right|^2 dt \right)^{1/2}.$$

Applying the general theory to $T^t := P_r^\kappa f$ with $r = e^{-t}$ and using the fact that the crucial point in the definition of $g(f)$ is when r close to 1, it follows that

$$c^{-1} \|f\|_{\kappa,p} \leqslant \big\| g(f) \big\|_{\kappa,p} \leqslant c \|f\|_{\kappa,p}, \quad 1 < p < \infty, \tag{2.13}$$

for $f \in L^p(h_\kappa^2; S^d)$, where the inequality in the left-hand side holds under the additional assumption that $\int_{S^d} f(y) h_\kappa^2(y)\, dy = 0$. Another ingredient of the proof is the Cesàro means. Recall that the (C, δ) means are denoted by $S_n^\delta(h_\kappa^2; f)$. What is needed is the following result.

Theorem 2.4. *For $\delta > \lambda_\kappa$, $1 < p < \infty$ and any sequence $\{n_j\}$ of positive integers,*

$$\left\| \left(\sum_{j=0}^\infty \left| S_{n_j}^\delta (h_\kappa^2; f_j) \right|^2 \right)^{1/2} \right\|_{\kappa,p} \leqslant c \left\| \left(\sum_{j=0}^\infty |f_j|^2 \right)^{1/2} \right\|_{\kappa,p}. \tag{2.14}$$

Proof. The proof of (2.14) follows the approach of [9, pp. 104–105] that uses a generalization of the Riesz convexity theorem for sequences of functions. Let $L^p(\ell^q)$ denote the space of all sequences $\{f_k\}$ of functions for which the norm

$$\big\| (f_k) \big\|_{L^p(\ell^q)} := \left(\int_{S^d} \left(\sum_{j=0}^\infty |f_j(x)|^q \right)^{p/q} h_\kappa^2(x)\, d\omega(x) \right)^{1/p}$$

is finite. If T is bounded as operator on $L^{p_0}(\ell^{q_0})$ and on $L^{p_1}(\ell^{q_1})$, then the Riesz convexity theorem states that T is also bounded on $L^{p_t}(\ell^{q_t})$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad 0 \leqslant t \leqslant 1.$$

We apply this theorem on the operator T that maps the sequence $\{f_j\}$ to the sequence $\{S_{n_j}^\delta(h_\kappa^2; f_j)\}$. It is shown in [13, pp. 76 and 78] that $\sup_n |S_n^\delta(h_\kappa^2; f)(x)| \leqslant c\mathcal{M}_\kappa f(x)$ for all $x \in S^d$ if $\delta > \lambda_\kappa$. Consequently, T is bounded on $L^p(\ell^p)$. It is also bounded on $L^p(\ell^\infty)$ since

$$\Bigl\| \sup_{j\geqslant 0} \bigl|S_{n_j}^\delta\bigl(h_\kappa^2; f_j\bigr)\bigr| \Bigr\|_{\kappa,p} \leqslant c \Bigl\| \mathcal{M}_\kappa\Bigl(\sup_{j\geqslant 0} |f_j|\Bigr) \Bigr\|_{\kappa,p} \leqslant c \Bigl\| \sup_{j\geqslant 0} |f_j| \Bigr\|_{\kappa,p}.$$

Hence, the Riesz convexity theorem shows that T is bounded on $L^p(\ell^q)$ if $1 < p \leqslant q \leqslant \infty$. In particular, T is bounded on $L^p(\ell^2)$ if $1 < p \leqslant 2$. The case $2 < p < \infty$ follows by the standard duality argument, since the dual space of $L^p(\ell^2)$ is $L^{p'}(\ell^2)$, where $1/p + 1/p' = 1$, under the paring

$$\bigl\langle (f_j), (g_j) \bigr\rangle := \int_{S^d} \sum_j f_j(x) g_j(x) h_\kappa^2(x)\, d\omega(x)$$

and T is self-adjoint under this paring as $S_n^\delta(h_\kappa^2)$ is self-adjoint in $L^2(h_\kappa^2; S^d)$. □

Using the two ingredients, (2.13) and (2.14), the proof of Theorem 2.3 follows from the corresponding proof in [1] almost verbatim.

Remark 2.1. In the case of $\kappa = 0$, the condition $\delta > \lambda_\kappa = (d-1)/2$ is the critical index for the convergence of (C, δ) means in $L^p(h_\kappa^2; S^d)$ for all $1 \leqslant p \leqslant \infty$. For h_κ^2 given in (1.2) and $G = \mathbb{Z}_2^d$, this remains true if at least one κ_i is zero. However, if $\kappa_i \neq 0$ for all i, then the critical index is $\lambda_\kappa - \min_{1\leqslant i\leqslant d+1} \kappa_i$ [8]. It remains to be seen if the condition $k \geqslant \lambda_\kappa + 1$ in Theorem 2.3 can be improved to $k \geqslant \lambda_\kappa - \min_{1\leqslant i\leqslant d+1} \kappa_i + 1$.

The proof of Theorem 2.4 actually yields the following Fefferman–Stein type inequality [6] for the maximal function $\mathcal{M}_\kappa f$.

Corollary 2.5. *Let $1 < p \leqslant 2$ and f_j be a sequence of functions. Then*

$$\Bigl\| \Bigl(\sum_j (\mathcal{M}_\kappa f_j)^2 \Bigr)^{1/2} \Bigr\|_{\kappa,p} \leqslant c \Bigl\| \Bigl(\sum_j |f_j|^2 \Bigr)^{1/2} \Bigr\|_{\kappa,p}. \tag{2.15}$$

We do not know if the inequality (2.15) holds for $2 < p < \infty$ under a general finite reflection group. However, it will be shown in the next section that (2.15) is true for all $1 < p < \infty$ in the case of $G = \mathbb{Z}_2^{d+1}$.

3. Maximal function for product weight

The result on the maximal function in the previous section is established for every finite reflection group. In the case of $G = \mathbb{Z}_2^{d+1}$, the weight function becomes (1.2); that is,

$$h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \quad \kappa_i \geqslant 0.$$

We know the explicit formula of the intertwining operator V_κ, as shown in (2.2). This additional information turns out to offer more insight into the maximal function $\mathcal{M}_k f$. The main result in this section relates $\mathcal{M}_\kappa f$ to the weighted Hardy–Littlewood maximal function.

Definition 3.1. For $f \in L^1(h_\kappa^2; S^d)$, the weighted Hardy–Littlewood maximal function is defined by

$$M_\kappa f(x) := \sup_{0<\theta\leqslant\pi} \frac{\int_{c(x,\theta)} |f(y)| h_\kappa^2(y)\, d\omega(y)}{\int_{c(x,\theta)} h_\kappa^2(y)\, d\omega(y)}. \tag{3.1}$$

Since h_κ is a doubling weight [3], $M_\kappa f$ enjoys the classical properties of maximal functions. We will show that the maximal function $\mathcal{M}_\kappa f$ is bounded by a sum of $M_\kappa f$, so that the properties of $\mathcal{M}_k f$ can be deduced from those of $M_\kappa f$. We shall need several lemmas. The first lemma is an observation made in [13, p. 72], which we state as a lemma to emphasize its importance in the development below.

Lemma 3.2. *For $x \in S^d$ let $\bar{x} := (|x_1|, \ldots, |x_d|)$. Then the support set of the function $V_\kappa[\chi_{B(x,\theta)}](y)$ is $\{y\colon d(\bar{x}, \bar{y}) \leqslant \theta\}$; in other words,*

$$V_\kappa[\chi_{B(x,\theta)}](y) = 0 \quad \textit{if } \langle \bar{x}, \bar{y}\rangle < \cos\theta.$$

Proof. The explicit formula (2.2) of V_κ shows that if $V_\kappa[\chi_{B(x,\theta)}](y) = 0$ if

$$\chi_{B(x,\theta)}(t_1 y_1, t_2 y_2, \ldots, t_{d+1} y_{d+1}) = 0$$

for every $t \in [-1,1]^{d+1}$ or if $x_1 y_1 t_1 + \cdots + x_{d+1} y_{d+1} t_{d+1} < \cos\theta$, which clearly holds if $\langle \bar{x}, \bar{y}\rangle < \cos\theta$. □

Our second lemma contains the essential estimate for an upper bound of $\mathcal{M}_\kappa f$.

Lemma 3.3. *For $0 \leqslant \theta \leqslant \pi$, $x = (x_1, \ldots, x_{d+1}) \in S^d$ and $y \in S^d$,*

$$\left|V_\kappa[\chi_{B(x,\theta)}](y)\right| \leqslant c \prod_{j=1}^{d+1} \frac{\theta^{2\kappa_j}}{(|x_j| + \theta)^{2\kappa_j}} \chi_{c(\bar{x},\theta)}(\bar{y}). \tag{3.2}$$

Proof. The presence of $\chi_{c(\bar{x},\theta)}(\bar{y})$ in the right-hand side of the stated estimate comes from Lemma 3.2. Hence, we only need to derive the upper bound of $V_\kappa[\chi_{B(x,\theta)}](y)$ for $d(\bar{x}, \bar{y}) \leqslant \theta$, which we assume in the rest of the proof. If $\pi/2 \leqslant \theta \leqslant \pi$, then $\theta/(|x_j| + \theta) \geqslant c$ and the inequality (3.2) is trivial. So we can assume $0 \leqslant \theta \leqslant \pi/2$ below. By the definition of V_κ,

$$V_\kappa[\chi_{B(x,\theta)}](y) = c_\kappa \int\limits_{\sum_{i=1}^{d+1} t_i x_i y_i \geqslant \cos\theta} \prod_{i=1}^{d+1} (1 + t_i)\left(1 - t_i^2\right)^{\kappa_i - 1} dt$$

where t also satisfies $t \in [-1,1]^{d+1}$. We first enlarge the domain of integration to $\{t \in [-1,1]^{d+1}\colon \sum_{i=1}^{d+1} |t_i x_i y_i| \geqslant \cos\theta\}$ and replace $(1 + t_i)$ by 2, so that we can use the $\mathbb{Z}_2^{d+1}$ symmetry of the resulted integrant to replace the integral to the one on $[0,1]^{d+1}$,

$$V_\kappa[\chi_{B(x,\theta)}](y) \leqslant c \int\limits_{\sum_{i=1}^{d+1} |t_i x_i y_i| \geqslant \cos\theta} \prod_{i=1}^{d+1} \left(1-t_i^2\right)^{\kappa_i - 1}\, dt$$
$$\leqslant c \int\limits_{t\in[0,1]^{d+1},\, \sum_{i=1}^{d+1} t_i |x_i y_i| \geqslant \cos\theta} \prod_{i=1}^{d+1} \left(1-t_i^2\right)^{\kappa_i - 1}\, dt.$$

To continue, we enlarge the domain of the integral to $\{t \in [0,1]^{d+1}\colon\ t_j|x_j y_j| + \sum_{i\neq j} |x_i y_i| \geqslant \cos\theta\}$ for each fixed j to obtain

$$V_\kappa[\chi_{B(x,\theta)}](y) \leqslant c \prod_{j=1}^{d+1} \int\limits_{0\leqslant t_j\leqslant 1,\, t_j|x_j y_j| + \sum_{i\neq j}|x_i y_i| \geqslant \cos\theta} (1-t_j)^{\kappa_j - 1}\, dt_j.$$

For each j we denote the last integral by I_j. First of all, there is the trivial estimate $I_j \leqslant \int_0^1 (1-t_j)^{\kappa_j - 1}\, dt_j = \kappa_j^{-1}$. Secondly, for $\langle \bar{x}, \bar{y}\rangle \geqslant \cos\theta$, we have the estimate

$$I_j \leqslant \int\limits_{\frac{\cos\theta - \sum_{i\neq j}|x_i y_i|}{|x_j y_j|}}^{1} (1-t_j)^{\kappa_j - 1}\, dt_j = \kappa_j^{-1} \frac{(\langle \bar{x}, \bar{y}\rangle - \cos\theta)^{\kappa_j}}{|x_j y_j|^{\kappa_j}}.$$

Together, we have established the estimate

$$I_j \leqslant \kappa_j^{-1} \min\left\{1, \frac{(\langle \bar{x}, \bar{y}\rangle - \cos\theta)^{\kappa_j}}{|x_j y_j|^{\kappa_j}}\right\}.$$

Recall that $d(\bar{x}, \bar{y}) \leqslant \theta$. Assume first that $|x_j| \geqslant 2\theta$. Then $|x_j| \geqslant (|x_j| + \theta)/2$. The inequality $||x_j| - |y_j|| \leqslant \|\bar{x} - \bar{y}\| \leqslant d(\bar{x}, \bar{y}) \leqslant \theta$ implies that $|y_j| \geqslant |x_j| - \theta \geqslant |x_j|/2$, so that $|y_j| \geqslant (|x_j| + \theta)/4$. Furthermore, write $t := d(\bar{x}, \bar{y}) \leqslant \theta$ and recall that $\theta \leqslant \pi/2$. We have then

$$\langle \bar{x}, \bar{y}\rangle - \cos\theta = \cos t - \cos\theta = 2 \sin\frac{\theta - t}{2} \sin\frac{t+\theta}{2} \leqslant (\theta - t)\theta \leqslant \theta^2.$$

Putting these ingredients together, we arrive at an upper bound for I_j,

$$I_j \leqslant c \frac{\theta^{2\kappa_j}}{(|x_j| + \theta)^{2\kappa_j}},$$

under the assumption that $|x_j| \geqslant 2\theta$. This estimate also holds for $|x_j| \leqslant 2\theta$, since in that case $\theta/(|x_j| + \theta) \geqslant 1/3$. Thus, the last inequality holds for all x and for all j, from which the stated inequality follows immediately. □

Our next lemma gives the order of the denominator in $M_\kappa f$, which was proved in [3, (5.3), p. 157] in the case when $\min_{1\leqslant j\leqslant d+1} \tau_j \geqslant 0$.

Lemma 3.4. *Let* $\tau = (\tau_1, \dots, \tau_{d+1}) \in \mathbb{R}^{d+1}$ *with* $\min_{1 \leqslant j \leqslant d+1} \tau_j > -1$. *Then for* $0 \leqslant \theta \leqslant \pi$ *and* $x = (x_1, \dots, x_{d+1}) \in S^d$,

$$\int\limits_{c(x,\theta)} \prod_{j=1}^{d+1} |y_j|^{\tau_j}\, d\omega(y) \sim \theta^d \prod_{j=1}^{d+1} \big(|x_j| + \theta\big)^{\tau_j},$$

where the constant of equivalence depends only on d and τ.

Proof. Without loss of generality we may assume that $x_j \geqslant 0$ for all $1 \leqslant j \leqslant d+1$ and $x_{d+1} = \max_{1 \leqslant j \leqslant d+1} x_j$, as well as $0 < \theta < \frac{1}{2\sqrt{d+1}}$. Since $x_{d+1} = \max_{1 \leqslant j \leqslant d+1} x_j \geqslant \frac{1}{\sqrt{d+1}}$, it follows that

$$y_{d+1} \geqslant x_{d+1} - \theta \geqslant \frac{1}{2\sqrt{d+1}}, \quad \forall y = (y_1, \dots, y_{d+1}) \in c(x, \theta). \tag{3.3}$$

Using (3.3) and the fact that $d\omega(y) = c_d (1 - \|\tilde{y}\|^2)^{-\frac{1}{2}}\, d\tilde{y}$ for $y = (\tilde{y}, y_{d+1})$ and $y_{d+1} = \sqrt{1 - \|\tilde{y}\|^2} \geqslant 0$, as well as the fact that $|x_j - y_j| \leqslant \|x - y\| \leqslant d(x, y)$, we conclude

$$\begin{aligned} \int\limits_{c(x,\theta)} \prod_{j=1}^{d+1} |y_j|^{\tau_j}\, d\omega(y) &\sim \int\limits_{d(x,y) \leqslant \theta} \prod_{j=1}^{d} |y_j|^{\tau_j}\, dy_1\, dy_2 \dots dy_d \\ &\leqslant c \prod_{j=1}^{d} \int\limits_{x_j - \theta}^{x_j + \theta} |y_j|^{\tau_j}\, dy_j \sim \theta^d \prod_{j=1}^{d+1} \big(|x_j| + \theta\big)^{\tau_j}, \end{aligned}$$

where in the last step, if $\tau_j < 0$, consider the cases $x_j \geqslant 2\theta_j$ and $x_j < 2\theta_j$ separately. This gives the desired upper estimate.

For the proof of the lower estimate, let $z = (z_1, \dots, z_{d+1}) \in S^d$ be defined by $z_j = x_j + \varepsilon\theta$ for $j = 1, 2, \dots, d$ and $z_{d+1} = (1 - z_1^2 - \dots - z_d^2)^{\frac{1}{2}}$, where $\varepsilon > 0$ is a sufficiently small constant depending only on d. Using (3.3), a quick computation shows that

$$\|x - z\|^2 = d(\varepsilon\theta)^2 + \frac{|z_{d+1}^2 - x_{d+1}^2|^2}{(z_{d+1} + x_{d+1})^2} \leqslant d(\varepsilon\theta)^2 + (d+1)d\big(2\varepsilon\theta + \varepsilon^2\theta^2\big)^2,$$

from which and the fact that $2 \sin \frac{d(x,z)}{2} = \|x - z\|$, it follows that we can choose ε small enough so that $z \in c(x, \frac{\theta}{2})$. Consequently, $c(z, \frac{\varepsilon\theta}{2}) \subset c(x, \theta)$ and, for any $y = (y_1, \dots, y_{d+1}) \in c(z, \frac{\varepsilon\theta}{2})$,

$$x_j + \frac{\varepsilon\theta}{2} = z_j - \frac{\varepsilon\theta}{2} \leqslant y_j \leqslant z_j + \frac{\varepsilon\theta}{2} = x_j + \frac{3\varepsilon\theta}{2}, \quad j = 1, 2, \dots, d+1,$$

which implies immediately that

$$\prod_{j=1}^{d+1} |y_j|^{\tau_j} \sim \prod_{j=1}^{d+1} \big(|x_j| + \theta\big)^{\tau_j}, \quad \forall y \in c\left(z, \frac{\varepsilon\theta}{2}\right),$$

and, as a consequence,

$$\int_{c(x,\theta)} \prod_{j=1}^{d+1} |y_j|^{\tau_j}\, d\omega(y) \geqslant \int_{c(z,\frac{\theta\varepsilon}{2})} \prod_{j=1}^{d+1} |y_j|^{\tau_j}\, d\omega(y) \geqslant c\theta^d \prod_{j=1}^{d+1} \bigl(|x_j|+\theta\bigr)^{\tau_j},$$

proving the desired lower estimate. □

In particular, Lemma 3.4 shows that h_κ^2 is a doubling weight in the sense that

$$\operatorname{meas}_\kappa c(x,2\theta) \leqslant c \operatorname{meas}_\kappa c(x,\theta), \quad \forall x \in S^d,\ \theta > 0.$$

We are now ready to prove our first main result. For $x \in \mathbb{R}^{d+1}$ and $\varepsilon \in \mathbb{Z}_2^{d+1}$, we write $x\varepsilon := (x_1\varepsilon_1, \dots, x_{d+1}\varepsilon_{d+1})$.

Theorem 3.5. *Let $f \in L^1(h_\kappa^2; S^d)$. Then for every $x \in S^d$,*

$$\mathcal{M}_\kappa f(x) \leqslant c \sum_{\varepsilon\in\mathbb{Z}_2^{d+1}} M_\kappa f(x\varepsilon). \tag{3.4}$$

Proof. Since

$$\bigl\{y \in S^d\colon d(\bar{x},\bar{y}) \leqslant \theta\bigr\} = \bigcup_{\varepsilon\in\mathbb{Z}_2^{d+1}} \bigl\{y \in S^d\colon d(x\varepsilon, y) \leqslant \theta\bigr\},$$

it follows from Lemmas 3.2 that

$$\begin{aligned} J_\theta f(x) &:= \int_{S^d} \bigl|f(y)\bigr| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y) \\ &= \int_{\langle \bar{x},\bar{y}\rangle \geqslant \cos\theta} \bigl|f(y)\bigr| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y) \\ &\leqslant \sum_{\varepsilon\in\mathbb{Z}_2^{d+1}} \int_{\langle x\varepsilon,y\rangle \geqslant \cos\theta} \bigl|f(y)\bigr| V_\kappa[\chi_{B(x,\theta)}](y) h_\kappa^2(y)\, d\omega(y). \end{aligned}$$

Consequently, using Lemmas 3.3 and 3.4, we conclude that

$$\begin{aligned} J_\theta f(x) &\leqslant c \sum_{\varepsilon\in\mathbb{Z}_2^{d+1}} \prod_{j=1}^{d+1} \frac{\theta^{2\kappa_j}}{(|x_j|+\theta)^{2\kappa_j}} \int_{\langle x\varepsilon,y\rangle \geqslant \cos\theta} \bigl|f(y)\bigr| h_\kappa^2(y)\, d\omega(y) \\ &\leqslant c\theta^{2|\kappa|+d} \sum_{\varepsilon\in\mathbb{Z}_2^{d+1}} M_\kappa f(x\varepsilon). \end{aligned}$$

Dividing the above inequality by $\theta^{2|\kappa|+d} = \theta^{2\lambda_\kappa+1}$ and, recall (2.6), taking the supremum over θ lead to (3.4). $\square$

There are several applications of Theorem 3.5. First we need several notations. For $x = (x_1, \dots, x_{d+1})$, $y = (y_1, \dots, y_{d+1}) \in \mathbb{R}^{d+1}$, we write $x < y$ if $x_j < y_j$ for all $1 \leqslant j \leqslant d+1$. We denote by $\mathbb{1}$ the vector $\mathbb{1} := (1, 1, \dots, 1) \in \mathbb{R}^{d+1}$. Moreover, we extend the definitions of h_τ, meas_τ, $\|\cdot\|_{\tau,p}$, $L^p(h_\tau^2; S^d)$ and M_τ to the full range of $\tau = (\tau_1, \dots, \tau_{d+1}) > -\frac{\mathbb{1}}{2}$. Thus,

$$h_\tau(x) = \prod_{j=1}^{d+1} |x_j|^{\tau_j}, \qquad \|f\|_{\tau,p} = \left(\int_{S^d} \left|f(x)\right|^p h_\tau^2(x)\, d\omega(x)\right)^{1/p}$$

and M_τ denotes the Hardy–Littlewood maximal function with respect to the measure $h_\tau^2(x)\, d\omega(x)$, as defined in (3.1).

As an application of Theorem 3.5, we can prove the boundedness of $\mathcal{M}_\kappa f$ on the spaces $L^p(h_\tau^2; S^d)$ for a wider range of τ without using the Hopf–Dunford–Schwarz ergodic theorem.

Theorem 3.6. *If* $-\frac{\mathbb{1}}{2} < \tau \leqslant \kappa$ *and* $f \in L^1(h_\tau^2; S^d)$*, then* $\mathcal{M}_\kappa f$ *satisfies*

$$\mathrm{meas}_\tau\left\{x\colon\ \mathcal{M}_\kappa f(x) \geqslant \alpha\right\} \leqslant c\frac{\|f\|_{\tau,1}}{\alpha}, \quad \forall \alpha > 0. \tag{3.5}$$

Furthermore, if $1 < p \leqslant \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$ *and* $f \in L^p(h_\tau^2; S^d)$*, then*

$$\|\mathcal{M}_k f\|_{\tau,p} \leqslant c\|f\|_{\tau,p}. \tag{3.6}$$

Proof. We start with the proof of (3.5). Note that if $\tau = (\tau_1, \dots, \tau_{d+1}) \leqslant \kappa$, then

$$\int_{c(x,\theta)} \left|f(y)\right| h_\kappa^2(y)\, d\omega(y) \leqslant c\left(\prod_{j=1}^{d+1}\big(|x_j| + \theta\big)^{2(\kappa_j-\tau_j)}\right) \int_{c(x,\theta)} \left|f(y)\right| h_\tau^2(y)\, d\omega(y),$$

which, together with Lemma 3.4, implies

$$M_\kappa f(x) \leqslant c M_\tau f(x), \quad x \in S^d,\ \tau \leqslant \kappa.$$

Hence, using the inequality (3.4), we obtain that, for $-\frac{\mathbb{1}}{2} < \tau \leqslant \kappa$,

$$\begin{aligned}
\mathrm{meas}_\tau\left\{x\colon\ \mathcal{M}_k f(x) \geqslant \alpha\right\} &\leqslant \sum_{\varepsilon \in \mathbb{Z}_2^{d+1}} \mathrm{meas}_\tau\left\{x\colon\ M_\kappa f(x\varepsilon) \geqslant c\alpha/2^{d+1}\right\} \\
&\leqslant \sum_{\varepsilon \in \mathbb{Z}_2^{d+1}} \mathrm{meas}_\tau\left\{x\colon\ M_\tau f(x\varepsilon) \geqslant c'\alpha\right\} \\
&= \sum_{\varepsilon \in \mathbb{Z}_2^{d+1}} \int_{\{y\colon\ M_\tau f(y\varepsilon) \geqslant c'\alpha\}} h_\tau^2(y)\, d\omega(y)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{d+1} \int\limits_{\{x:\ M_\tau f(x) \geqslant c'\alpha\}} h_\tau^2(y)\, d\omega(y) \\
&\leqslant c \frac{\|f\|_{\tau,1}}{\alpha},
\end{aligned}
$$

where we have used the $\mathbb{Z}_2^{d+1}$-invariance of h_τ in the fourth step, and the fact that M_τ is of weak type $(1,1)$ with respect to the doubling measure $h_\tau^2(y)\, d\omega(y)$ in the last step. This proves (3.5).

For the proof of (3.6), we choose a number $q \in (1, p)$ such that $\tau < q\kappa + \frac{q-1}{2}\mathbb{1}$ and claim that it is sufficient to prove

$$
M_\kappa f(x) \leqslant c\big(M_\tau\big(|f|^q\big)(x)\big)^{1/q}. \tag{3.7}
$$

Indeed, using (3.7), the inequality (3.6) will follow from (3.4), the $\mathbb{Z}_2^{d+1}$ invariance of h_τ and the boundedness of the maximal function M_τ on the space $L^{p/q}(h_\tau^2; S^d)$.

To prove (3.7), we use Hölder's inequality with $q' = \frac{q}{q-1}$ and Lemma 3.4 to obtain

$$
\begin{aligned}
&\int\limits_{c(x,\theta)} \big|f(y)\big| h_\kappa^2(y)\, d\omega(y) \\
&\quad \leqslant \Bigg(\int\limits_{c(x,\theta)} \big|f(y)\big|^q h_\tau^2(y)\, d\omega(y)\Bigg)^{1/q} \Bigg(\int\limits_{c(x,\theta)} h^2_{q'\kappa - \frac{q'}{q}\tau}(y)\, d\omega(y)\Bigg)^{1/q'} \\
&\quad \sim \Bigg(\int\limits_{c(x,\theta)} \big|f(y)\big|^q h_\tau^2(y)\, d\omega(y)\Bigg)^{1/q} \Bigg(\prod_{j=1}^{d+1} \big(|x_j| + \theta\big)^{2\kappa_j - \frac{2\tau_j}{q}}\Bigg) \theta^{d/q'} \\
&\quad \sim \operatorname{meas}_\kappa\big(c(x,\theta)\big) \Bigg(\frac{1}{\operatorname{meas}_\tau(c(x,\theta))} \int\limits_{c(x,\theta)} \big|f(y)\big|^q h_\tau^2(y)\, dy \Bigg)^{1/q},
\end{aligned}
$$

where we have also used the fact that the assumption $\tau < q\kappa + \frac{q-1}{2}\mathbb{1}$ is equivalent to $q'\kappa - \frac{q'}{q}\tau > -\frac{\mathbb{1}}{2}$. This proves (3.7) and completes the proof. □

For our next application of Theorem 3.5 we will need the following result.

Lemma 3.7. *Let $1 < p < \infty$ and let W be a non-negative, local integrable function on S^d. Then*

$$
\int\limits_{S^d} \big|M_\kappa f(x)\big|^p W(x) h_\kappa^2(x)\, d\omega(x) \leqslant c_p \int\limits_{S^d} \big|f(x)\big|^p M_\kappa W(x) h_\kappa^2(x)\, d\omega(x). \tag{3.8}
$$

Such a result was first proved in [6] for maximal function on $\mathbb{R}^d$. The proof can be adopted easily to yield Lemma 3.7. Indeed, the fact that h_κ^2 is a doubling weight shows that the Hardy–Littlewood maximal function defined by (3.1) satisfies

$$M_\kappa f(x) \sim \sup_{x \in E \in \mathcal{C}} \frac{\int_E |f(y)| h_\kappa^2(y)\, d\omega(y)}{\int_E h_\kappa^2(y)\, d\omega(y)},$$

where $\mathcal{C}$ is the collection of all spherical caps in S^d, which implies that

$$\int\limits_{c(x,\theta)} \left|f(y)\right| h_\kappa^2(y)\, dy \leqslant c\big(\operatorname{meas}_\kappa c(x,\theta)\big) \inf_{z \in c(x,\theta)} M_\kappa f(z)$$

for any spherical cap $c(x,\theta)$. As a consequence, we can prove the key inequality

$$\operatorname{meas}_\kappa(E) \leqslant \frac{c}{\alpha} \int\limits_{S^d} \left|f(y)\right| M_\kappa W(y) h_\kappa^2(y)\, d\omega(y)$$

for any compact set E in $\{x \in S^d\colon\ M_k f(x) > \alpha\}$, as in the proof for the maximal function on $\mathbb{R}^d$ in [10, pp. 54–55]. In fact, (3.8) holds with $h_\kappa^2(y)\, d\omega$ replaced by any doubling measure $d\mu$ on the sphere.

An important tool in harmonic analysis is the Fefferman–Stein type inequality [6], which we established in Corollary 2.5 for $\mathcal{M}_\kappa f$ in the case of $1 < p \leqslant 2$ and a general reflection group. In the current setting of $G = \mathbb{Z}_2^{d+1}$, we can use Theorem 3.5 to prove a weighted version of this inequality for $1 < p < \infty$.

Theorem 3.8. *Let* $1 < p < \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$, *and let* $\{f_j\}_{j=1}^\infty$ *be a sequence of functions. Then*

$$\left\| \left(\sum_{j=1}^\infty (\mathcal{M}_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p} \leqslant c \left\| \left(\sum_{j=1}^\infty |f_j|^2 \right)^{1/2} \right\|_{\tau,p}. \tag{3.9}$$

Proof. Using Theorem 3.5 and the Minkowski inequality, we obtain

$$\begin{aligned} \left\| \left(\sum_j (\mathcal{M}_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p} &\leqslant c \left\| \left(\sum_j \left(\sum_{\varepsilon \in \mathbb{Z}_2^{d+1}} M_\kappa f_j(x\varepsilon) \right)^2 \right)^{1/2} \right\|_{\tau,p} \\ &\leqslant c \sum_{\varepsilon \in \mathbb{Z}_2^{d+1}} \left\| \left(\sum_j \big(M_\kappa f_j(x\varepsilon)\big)^2 \right)^{1/2} \right\|_{\tau,p} \leqslant c \left\| \left(\sum_j (M_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p}. \end{aligned}$$

Thus, it is sufficient to prove

$$\left\| \left(\sum_j (M_\kappa f_j)^2 \right)^{1/2} \right\|_{\tau,p} \leqslant c \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{\tau,p}. \tag{3.10}$$

We start with the case $1 < p \leqslant 2$. Let q be chosen such that $1 < q < p$ and $\tau < q\kappa + \frac{(q-1)\mathbb{1}}{2}$. We use the inequality (3.7) to obtain

$$\Bigl\| \Bigl(\sum_j (M_\kappa f_j)^2 \Bigr)^{1/2} \Bigr\|_{\tau,p} \leqslant c \Bigl\| \Bigl(\sum_j \bigl(M_\tau \bigl(|f_j|^q \bigr) \bigr)^{2/q} \Bigr)^{q/2} \Bigr\|_{\tau,p/q}^{1/q}$$
$$\leqslant c \Bigl\| \Bigl(\sum_j |f_j|^{q \cdot 2/q} \Bigr)^{q/2} \Bigr\|_{\tau,p/q}^{1/q} = \Bigl\| \Bigl(\sum_j |f_j|^2 \Bigr)^{1/2} \Bigr\|_{\tau,p},$$

where we have used the classical Fefferman–Stein inequality for the maximal function M_τ and the space $L^{p/q}(\ell^{2/q})$ in the second step. This proves (3.10) for $1 < p \leqslant 2$.

Next, we consider the case $2 < p < \infty$. Noticing that

$$-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{(p-1)\mathbb{1}}{2} \quad \Leftrightarrow \quad -\frac{\mathbb{1}}{2} < \frac{2}{p}\tau + \Bigl(\frac{1}{p} - \frac{1}{2} \Bigr)\mathbb{1} < 2\kappa + \frac{\mathbb{1}}{2},$$

we may choose a vector $\mu \in \mathbb{R}^{d+1}$ such that

$$-\frac{\mathbb{1}}{2} < \frac{2}{p}\tau + \Bigl(\frac{1}{p} - \frac{1}{2} \Bigr)\mathbb{1} < \mu < 2\kappa + \frac{\mathbb{1}}{2}, \tag{3.11}$$

and a number $1 < q < 2$ such that $\mu < q\kappa + \frac{q-1}{2}\mathbb{1}$. Let g be a non-negative function on S^d satisfying $\|g\|_{\tau,p/(p-2)} = 1$ and

$$\Bigl\| \Bigl(\sum_j |M_\kappa f_j|^2 \Bigr)^{1/2} \Bigr\|_{\tau,p}^2 = \int_{S^d} \Bigl(\sum_j \bigl| M_\kappa f_j(x) \bigr|^2 \Bigr) g(x) h_\tau^2(x)\, d\omega(x).$$

Then by the assumption $\mu < q\kappa + \frac{q-1}{2}\mathbb{1}$, (3.7), (3.8) with $p = 2/q > 1$ and Hölder's inequality, we obtain

$$\int_{S^d} \Bigl(\sum_j \bigl| M_\kappa f_j(x) \bigr|^2 \Bigr) g(x) h_\tau^2(x)\, d\omega(x) \leqslant c \sum_j \int_{S^d} \bigl(M_\mu \bigl(|f_j|^q \bigr)(x) \bigr)^{2/q} g(x) h_\tau^2(x)\, d\omega(x)$$
$$\leqslant c \int_{S^d} \Bigl(\sum_j \bigl| f_j(x) \bigr|^2 \Bigr) M_\mu \bigl(g h_{\tau-\mu}^2 \bigr)(x) h_\mu^2(x)\, d\omega(x)$$
$$\leqslant c \Bigl\| \Bigl(\sum_j |f_j|^2 \Bigr)^{1/2} \Bigr\|_{\tau,p}^2 \bigl\| M_\mu \bigl(g h_{\tau-\mu}^2 \bigr) h_{\mu-\tau}^2 \bigr\|_{\tau,p/(p-2)}.$$

Using the boundedness of $M_\kappa f$ and (3.11), we have

$$\bigl\| M_\mu \bigl(g h_{\tau-\mu}^2 \bigr) h_{\mu-\tau}^2 \bigr\|_{\tau,p/(p-2)} = \bigl\| M_\mu \bigl(g h_{\tau-\mu}^2 \bigr) \bigr\|_{p/(p-2)\mu - 2/(p-2)\tau,\, p/(p-2)}$$
$$\leqslant c \bigl\| g h_{\tau-\mu}^2 \bigr\|_{p/(p-2)\mu - 2/(p-2)\tau,\, p/(p-2)}$$
$$= c \|g\|_{\tau,p/(p-2)} = c.$$

Putting these two inequalities together, we have proved the inequality (3.10) for the case $2 < p < \infty$. $\square$

Remark 3.1. It is shown in [13] that, for $\delta > \lambda_\kappa$, the Cesàro (C, δ) means satisfy $|S_n^\delta(h_\kappa^2; f)| \leqslant c\mathcal{M}_\kappa f(x)$. Hence, we can get a weighted inequality for the Cesàro means by replacing $\mathcal{M}_\kappa f_j$ in (3.9) by $S_{n_j}^\delta(h_\kappa^2; f_j)$. This gives a $\|\cdot\|_{\tau,p}$ weighted version of Theorem 2.4 that holds under the condition $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$.

4. Maximal function and multiplier theorem on B^d

Analysis in weighted spaces on the unit ball $B^d = \{x \in \mathbb{R}^d \colon \|x\| \leqslant 1\}$ in $\mathbb{R}^d$ can often be deduced from the corresponding results on S^d; see [5,12,13] and the reference therein. Below we develop results analogous to those in the previous sections.

4.1. *Weight function invariant under a reflection group*

Let $\kappa = (\kappa', k_{d+1})$ with $\kappa' = (\kappa_1, \ldots, \kappa_d)$ and assume $\kappa_i \geqslant 0$ for $1 \leqslant i \leqslant d+1$. Let $h_{\kappa'}$ be the weight function (1.1), but defined on $\mathbb{R}^d$, that is invariant under a reflection group G. We consider the weight functions on B^d defined by

$$W_\kappa^B(x) := h_{\kappa'}^2(x)\bigl(1 - \|x\|^2\bigr)^{\kappa_{d+1}-1/2}, \quad x \in B^d, \tag{4.1}$$

which is invariant under the reflection group G. Under the mapping

$$\phi : x \in B^d \mapsto \bigl(x, \sqrt{1 - \|x\|^2}\bigr) \in S_+^d := \bigl\{y \in S^d \colon y_{d+1} \geqslant 0\bigr\} \tag{4.2}$$

and multiplying the Jacobian of this change of variables, the weight function W_κ^B comes exactly from h_κ^2 defined by

$$h_\kappa(x_1, \ldots, x_{d+1}) := h_{\kappa'}^2(x_1, \ldots, x_d)|x_{d+1}|^{2\kappa_{d+1}}.$$

The weight function h_κ is invariant under the reflection group $G \times \mathbb{Z}_2$. All of the results established in Section 2 holds for h_κ.

We denote the $L^p(W_\kappa^B; B^d)$ norm by $\|f\|_{W_\kappa^B,p}$. The norm of g on B^d and its extension on S^d are related by the identity

$$\int_{S^d} g(y)\,d\omega = \int_{B^d} \bigl[g\bigl(x, \sqrt{1 - \|x\|^2}\bigr) + g\bigl(x, -\sqrt{1 - \|x\|^2}\bigr)\bigr] \frac{dx}{\sqrt{1 - \|x\|^2}}. \tag{4.3}$$

The orthogonal structure is preserved under the mapping (4.2) and the study of orthogonal expansions for W_κ^B can be essentially reduced to that of h_κ^2. In fact, let $\mathcal{V}_n^d(W_\kappa^B)$ denote the space of orthogonal polynomials of degree n with respect to W_κ^B on B^d. The orthogonal projection, $\operatorname{proj}_n(W_\kappa^B; f)$, of $f \in L^2(W_\kappa^B; B^d)$ onto $\mathcal{V}_n^d(W_\kappa^B)$ can be expressed in terms of the orthogonal projection of $F(x, x_{d+1}) := f(x)$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$:

$$\operatorname{proj}_n\bigl(W_\kappa^B; f, x\bigr) = \operatorname{proj}_n^\kappa F(X), \quad X := \bigl(x, \sqrt{1 - \|x\|^2}\bigr). \tag{4.4}$$

Furthermore, a maximal function was defined in [13, p. 81] in terms of the generalized translation operator of the orthogonal expansion. More precisely, let

$$e(x,\theta) := \left\{(y, y_{d+1}) \in B^{d+1}\colon \langle x, y\rangle + \sqrt{1-\|x\|^2}\, y_{d+1} \geqslant \cos\theta,\ y_{d+1} \geqslant 0\right\}.$$

Then this maximal function, denoted by $\mathcal{M}_\kappa^B f(x)$, was shown to satisfy the relation

$$\mathcal{M}_\kappa^B f(x) = \sup_{0<\theta\leqslant\pi} \frac{\int_{B^d} |f(y)| V_\kappa^B[\chi_{e(x,\theta)}](Y) W_\kappa^B(y)\,dy}{\int_{B^d} V_\kappa^B[\chi_{e(x,\theta)}](Y) W_\kappa^B(y)\,dy},$$

where $Y = (y, \sqrt{1-\|y\|^2})$, and for $g : \mathbb{R}^{d+1} \mapsto \mathbb{R}$,

$$V_\kappa^B g(x, x_{d+1}) := \frac{1}{2}\left[V_\kappa g(x, x_{d+1}) + V_\kappa g(x, -x_{d+1})\right],$$

in which V_κ is the intertwining operator associate with h_κ. This maximal function can be written in terms of the maximal function $\mathcal{M}_\kappa f$ in (2.5). In fact, we have $\mathcal{M}_\kappa^B f(x) = \mathcal{M}_\kappa F(X)$. Our main result in this section states that $\mathcal{M}_\kappa^B f$ is of weak $(1,1)$. Let us define

$$\operatorname{meas}_\kappa^B E := \int_E W_\kappa^B(x)\,dx, \quad E \subset B^d.$$

Theorem 4.1. *If $f \in L^1(W_\kappa^B; B^d)$ then $\mathcal{M}_k^B$ satisfies*

$$\operatorname{meas}_\kappa^B\left\{x \in B^d\colon \mathcal{M}_\kappa^B f(x) \geqslant \alpha\right\} \leqslant c\frac{\|f\|_{W_\kappa^B,1}}{\alpha}, \quad \forall\alpha > 0.$$

Furthermore, if $f \in L^p(W_\kappa^B; B^d)$ for $1 < p \leqslant \infty$, then $\|\mathcal{M}_\kappa^B f\|_{W_\kappa^B,p} \leqslant c\|f\|_{W_\kappa^B,p}$.

Proof. Since $\mathcal{M}_\kappa^B f(x) = \mathcal{M}_\kappa F(X)$, it follows from (4.3) that

$$\begin{aligned}\operatorname{meas}_\kappa^B\left\{x \in B^d\colon \mathcal{M}_\kappa^B f(x) \geqslant \alpha\right\} &= \int_{B^d} \chi_{\{\mathcal{M}_\kappa^B f(x)\geqslant\alpha\}}(x) W_\kappa^B(x)\,dx \\ &= \int_{S_+^d} \chi_{\{\mathcal{M}_\kappa F(y)\geqslant\alpha\}}(y) h_\kappa^2(y)\,d\omega(y).\end{aligned}$$

Enlarging the domain of the last integral to the entire S^d shows that

$$\operatorname{meas}_\kappa^B\left\{x \in B^d\colon \mathcal{M}_\kappa^B f(x) \geqslant \alpha\right\} \leqslant \operatorname{meas}_\kappa\left\{y \in S^d\colon \mathcal{M}_\kappa F(y) \geqslant \alpha\right\}.$$

Consequently, by Theorem 2.1, we obtain

$$\operatorname{meas}_\kappa^B\left\{x \in B^d\colon \mathcal{M}_\kappa^B f(x) \geqslant \alpha\right\} \leqslant c\frac{\|F\|_{\kappa,1}}{\alpha}$$

from which the weak (1, 1) inequality follows from $\|F\|_{\kappa,1} = \|f\|_{W_\kappa^B,1}$. Since $\mathcal{M}_\kappa^B f$ is evidently of strong type (∞, ∞), this completes the proof. □

The connection (4.4) and (4.3) allow us to deduce a multiplier theorem for orthogonal expansion with respect to W_κ^B from Theorem 2.3.

Theorem 4.2. *Let $\{\mu_j\}_{j=0}^\infty$ be a sequence of real numbers that satisfies*

(1) $\sup_j |\mu_j| \leqslant c < \infty$,
(2) $\sup_j 2^{j(k-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^k u_l| \leqslant c < \infty$,

where k is the smallest integer $\geqslant \lambda_\kappa + 1$, and $\lambda_\kappa = \frac{d-1}{2} + \sum_{j=1}^{d+1} \kappa_j$. Then $\{\mu_j\}$ defines an $L^p(W_\kappa^B; B^d)$, $1 < p < \infty$, multiplier; *that is,*

$$\left\| \sum_{j=0}^\infty \mu_j \operatorname{proj}_j^\kappa f \right\|_{W_\kappa^B,p} \leqslant c\|f\|_{W_\kappa^B,p}, \quad 1 < p < \infty,$$

where c is independent of $\{\mu_j\}$ and f.

4.2. Weight function invariant under $\mathbb{Z}_2^d$

In the case of $G = \mathbb{Z}_2^d$, the weight function becomes

$$W_\kappa^B(x) := \prod_{i=1}^d |x_i|^{2\kappa_i} \bigl(1 - \|x\|^2\bigr)^{\kappa_{d+1}-1/2}, \quad x \in B^d, \tag{4.5}$$

which corresponds to the product weight function $h_\kappa^2(x) = \prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$. Taking into the consideration of the boundary, an appropriate distance on B^d is defined by

$$d_B(x, y) = \arccos\bigl(\langle x, y\rangle + \sqrt{1 - \|x\|^2}\sqrt{1 - \|y\|^2}\bigr), \quad x, y \in B^d,$$

which is just the projection of the geodesic distance of S_+^d on B^d. Thus, one can define the weighted Hardy–Littlewood maximal function as

$$M_\kappa^B f(x) := \sup_{0<\theta\leqslant\pi} \frac{\int_{d_B(x,y)\leqslant\theta} |f(y)| W_\kappa^B(y)\, dy}{\int_{d_B(x,y)\leqslant\theta} W_\kappa^B(y)\, dy}, \quad x \in B^d.$$

We have the following analogue of Theorem 3.5.

Theorem 4.3. *Let $f \in L^1(W_\kappa^B; B^d)$. Then for any $x \in B^d$,*

$$\mathcal{M}_\kappa^B f(x) \leqslant c \sum_{\varepsilon\in\mathbb{Z}_2^d} M_\kappa^B f(x\varepsilon). \tag{4.6}$$

The proof of Theorem 4.3 replies on the following lemma, which implies, in particular, that $W_\kappa^B(y)$ is a doubling weight on B^d.

Lemma 4.4. *If* $\tau = (\tau_1, \dots, \tau_{d+1}) > -\frac{1}{2}\mathbb{1}$, *then for any* $x = (x_1, \dots, x_d) \in B^d$ *and* $0 \leqslant \theta \leqslant \pi$,

$$\int_{d_B(y,x)\leqslant\theta} W_\tau^B(y)\,dy \sim \theta^d \prod_{j=1}^{d+1} \bigl(|x_j| + \theta\bigr)^{2\tau_j},$$

where $x_{d+1} = \sqrt{1 - \|x\|^2}$ *and* $W_\tau^B(y)$ *is defined as in* (4.5).

Proof. Recall that $X = (x, x_{d+1})$, and $c(X, \theta) = \{z \in S^d\colon\ d(X, z) \leqslant \theta\}$. Set

$$c_+(X, \theta) = \bigl\{(y_1, \dots, y_{d+1}) \in c(X, \theta)\colon\ y_{d+1} \geqslant 0\bigr\}.$$

From (4.3) it follows that

$$\int_{d_B(y,x)\leqslant\theta} W_\tau^B(y)\,dy = \int_{c_+(X,\theta)} h_\tau^2(z)\,d\omega(z), \tag{4.7}$$

which, together with Lemma 3.4, implies the desired upper estimate

$$\int_{d_B(y,x)\leqslant\theta} W_\tau^B(y)\,dy \leqslant \int_{c(X,\theta)} h_\tau^2(z)\,d\omega(z) \leqslant c\theta^d \prod_{j=1}^{d+1} \bigl(|x_j| + \theta\bigr)^{2\tau_j}.$$

To prove the lower estimate, we choose a point $z = (z_1, \dots, z_{d+1}) \in c(X, \frac{\theta}{2})$ with $z_{d+1} \geqslant \varepsilon\theta$, where $\varepsilon > 0$ is a sufficiently small constant depending only on d. Clearly, $c(z, \frac{\varepsilon\theta}{2}) \subset c_+(X, \theta)$. Hence, by (4.7), we obtain

$$\begin{aligned}\int_{d_B(y,x)\leqslant\theta} W_\tau^B(y)\,dy &\geqslant \int_{c(z,\frac{\varepsilon\theta}{2})} h_\tau^2(y)\,d\omega(y)\\ &\sim \theta^d \prod_{j=1}^{d+1} \bigl(|z_j| + \theta\bigr)^{2\tau_j} \sim \theta^d \prod_{j=1}^{d+1} \bigl(|x_j| + \theta\bigr)^{2\tau_j},\end{aligned}$$

where we have used Lemma 3.4 in the second step, and the fact that $z \in c(X, \theta)$ in the last step. This gives the desired lower estimate. □

Now we are in a position to prove Theorem 4.3.

Proof of Theorem 4.3. It is shown in [13, p. 81] that

$$\int_{B^d} V_\kappa[\chi_{e(x,\theta)}](Y) W_\kappa^B(y)\,dy \sim \theta^{2\lambda_\kappa+1},$$

where $\lambda_\kappa = \frac{d-1}{2} + \sum_{j=1}^{d+1} \kappa_j$. The proof follows almost exactly as in the proof of Theorem 3.5, the main effort lies in the proof of the following inequality:

$$V_\kappa^B[\chi_{e(x,\theta)}](Y) \leqslant c \prod_{j=1}^{d+1} \frac{\theta^{2\kappa_j}}{(|x_j| + \theta)^{2\kappa_j}} \chi_{\{y \in B^d:\ d_B(\bar{x},\ \bar{y}) \leqslant \theta\}}(y), \tag{4.8}$$

where $x_{d+1} = \sqrt{1 - \|x\|^2}$ and $\bar{z} = (|z_1|, \dots, |z_d|)$ for $z = (z_1, \dots, z_d) \in B^d$. However, using (2.2) and the fact that $y_{d+1} = \sqrt{1 - \|y\|^2}$, we have

$$\begin{aligned} V_\kappa^B[\chi_{e(x,\theta)}](Y) &= \frac{1}{2}\big(V_\kappa[\chi_{e(x,\theta)}](y, y_{d+1}) + V_\kappa[\chi_{e(x,\theta)}](y, -y_{d+1})\big) \\ &= c_\kappa \int_D \prod_{j=1}^{d} (1 + t_j)\big(1 - t_j^2\big)^{\kappa_j - 1}\big(1 - t_{d+1}^2\big)^{\kappa_{d+1} - 1}\, dt, \end{aligned}$$

where

$$D = \left\{ (t_1, \dots, t_d, t_{d+1}) \in [-1, 1]^d \times [0, 1]:\ \sum_{j=1}^{d+1} t_j x_j y_j \geqslant \cos\theta \right\}.$$

This last integral can be estimated exactly as in the proof of Lemma 3.3, which yields the desired inequality (4.8). □

As a consequence of Theorem 4.3, we have the following analogues of Theorems 3.6 and 3.8.

Corollary 4.5. *If* $-\frac{\mathbb{1}}{2} < \tau \leqslant \kappa$ *and* $f \in L^1(W_\tau^B; B^d)$*, then* $\mathcal{M}_\kappa f$ *satisfies*

$$\operatorname{meas}_\tau^B\big\{x:\ \mathcal{M}_\kappa^B f(x) \geqslant \alpha\big\} \leqslant c \frac{\|f\|_{W_\tau^B,1}}{\alpha}, \quad \forall \alpha > 0.$$

Furthermore, if $1 < p < \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$ *and* $f \in L^p(W_\tau^B; B^d)$*, then*

$$\big\|\mathcal{M}_\kappa^B f\big\|_{W_\tau^B,p} \leqslant c\|f\|_{W_\tau^B,p}.$$

Corollary 4.6. *Let* $1 < p < \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$*, and let* $\{f_j\}_{j=1}^\infty$ *be a sequence of functions. Then*

$$\left\| \left(\sum_{j=1}^{\infty} \big(\mathcal{M}_\kappa^B f_j\big)^2 \right)^{1/2} \right\|_{W_\tau^B,p} \leqslant c \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{W_\tau^B,p}.$$

Using the formula $\mathcal{M}_\kappa^B f(x) = \mathcal{M}_\kappa F(X)$ and the method of [13], one can also deduce Corollaries 4.5 and 4.6 directly from Theorems 3.6 and 3.8.

5. Maximal function and multiplier theorem on T^d

Just like the connection between the structure of function spaces on S^d and B^d, analysis in weighted spaces on the simplex

$$T^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d\colon\ x_1 \geqslant 0, \dots, x_d \geqslant 0 \text{ and } x_1 + \cdots + x_d \leqslant 1\}$$

can often be deduced from the corresponding results on B^d; see [5,12,13] and the reference therein.

5.1. *Weight function associated with a reflection group*

Let $\kappa' = (\kappa_1, \dots, \kappa_d)$ and $h_{\kappa'}$ be the weight function (1.1) on $\mathbb{R}^d$ invariant under the reflection group G. We further require that $h_{\kappa'}$ is even in all of its variables; in other words, we require that $h_{\kappa'}$ is invariant under the semi-direct product of G and $\mathbb{Z}_2^d$. Let $\kappa_{d+1} \geqslant 0$ and $\kappa = (\kappa', \kappa_{d+1})$. The weight functions on T^d we consider are

$$W_\kappa^T(x) := h_{\kappa'}(\sqrt{x_1}, \dots, \sqrt{x_d})\bigl(1 - |x|\bigr)^{\kappa_{d+1} - 1/2}, \quad x \in T^d, \tag{5.1}$$

where $|x| = x_1 + \cdots + x_d$. These weight functions are related to W_κ^B in (4.1). In fact, W_κ^T is exactly the weight function W_κ^B under the mapping

$$\psi : (x_1, \dots, x_d) \in T^d \mapsto \bigl(x_1^2, \dots, x_d^2\bigr) \in B^d \tag{5.2}$$

and upon multiplying the Jacobian of this change of variables. We denote the norm of $L^p(W_\kappa^T; T^d)$ by $\|\cdot\|_{W_\kappa^T, p}$. The norm of g on T^d and $g \circ \psi$ on B^d are related by

$$\int_{B^d} g\bigl(x_1^2, \dots, x_d^2\bigr)\, dx = \int_{T^d} g(x_1, \dots, x_d) \frac{dx}{\sqrt{x_1 \cdots x_d}}. \tag{5.3}$$

The orthogonal structure is preserved under the mapping (5.2). Let $\mathcal{V}_n^d(W_\kappa^T)$ denote the space of orthogonal polynomials of degree n with respect to W_κ^T on T^d. Then $R \in \mathcal{V}_n^d(W_\kappa^T)$ if and only if $R \circ \psi \in \mathcal{V}_{2n}^d(W_\kappa^B)$. The orthogonal projection, $\operatorname{proj}_n(W_\kappa^T; f)$, of $f \in L^2(W_\kappa^T; T^d)$ onto $\mathcal{V}_n^d(W_\kappa^T)$ can be expressed in terms of the orthogonal projection of $f \circ \psi$ onto $\mathcal{V}_{2n}(W_\kappa^B)$:

$$\bigl(\operatorname{proj}_n\bigl(W_\kappa^T; f\bigr) \circ \psi\bigr)(x) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} \operatorname{proj}_{2n}\bigl(W_\kappa^B; f \circ \psi, x\varepsilon\bigr). \tag{5.4}$$

The fact that $\operatorname{proj}_n(W_\kappa^T)$ of degree n is related to $\operatorname{proj}_{2n}(W_\kappa^B)$ of degree $2n$ suggests that some properties of the orthogonal expansions on B^d cannot be transformed directly to those on T^d.

A maximal function $\mathcal{M}_k^T f$ is defined in [13, Definition 4.5, p. 86] in terms of the generalized translation operator of the orthogonal expansion. It is closely related to the maximal function $\mathcal{M}_k^B f$ on B^d. It was shown in [13, Proposition 4.6] that

$$\bigl(\mathcal{M}_\kappa^T f\bigr) \circ \psi = \mathcal{M}_\kappa^B(f \circ \psi). \tag{5.5}$$

We show that this maximal function is of weak type $(1,1)$. Let us define

$$\operatorname{meas}_\kappa^T E := \int_E W_\kappa^T(x)\,dx, \quad E \subset T^d.$$

Theorem 5.1. *If $f \in L^1(W_\kappa^T; T^d)$, then $\mathcal{M}_k^T$ satisfies*

$$\operatorname{meas}_\kappa^T \bigl\{x \in T^d\colon \mathcal{M}_\kappa^T f(x) \geqslant \alpha\bigr\} \leqslant c \frac{\|f\|_{W_\kappa^T,1}}{\alpha}, \quad \forall \alpha > 0.$$

Furthermore, if $f \in L^p(W_\kappa^T; T^d)$ for $1 < p \leqslant \infty$, then $\|\mathcal{M}_\kappa^T f\|_{W_\kappa^T,p} \leqslant c\|f\|_{W_\kappa^T,p}$.

Proof. Using the relation (5.5) and (5.3), we obtain

$$\int_{T^d} \chi_{\{x\in T^d\colon \mathcal{M}_\kappa^T f(x)\geqslant\alpha\}}(x) W_\kappa^T(x)\,dx = \int_{B^d} \chi_{\{x\in B^d\colon \mathcal{M}_\kappa^T (f\circ\psi)(x)\geqslant\alpha\}}(x) W_\kappa^B(x)\,dx.$$

Hence, by Theorem 4.1, we conclude that

$$\operatorname{meas}_\kappa^B \bigl\{x \in B^d\colon \mathcal{M}_\kappa^B (f\circ\psi)(x) \geqslant \alpha\bigr\} \leqslant c \frac{\|f\circ\psi\|_{W_\kappa^B,1}}{\alpha} = c\frac{\|f\|_{W_\kappa^T,1}}{\alpha},$$

where the last step follows again from (5.3). □

The relation (5.4) shows that we cannot expect to deduce all results on orthogonal expansion with respect to W_κ^T on T^d from those on B_d. This applies to the multiplier theorem. On the other hand, as it is shown in [13, p. 85], we can introduce a convolution $\star_\kappa^T$ structure and write $\operatorname{proj}_n^\kappa(W_\kappa^T; f) = f \star_\kappa^T P_n$. Moreover, we often have the inequality $|f \star_\kappa^T g(x)| \leqslant c\mathcal{M}_\kappa^T(x)$. For example, for the Cesàro (C,δ) means $S_n^\delta(W_\kappa^T; f)$, we have

$$\sup_n \bigl|S_n^\delta\bigl(W_\kappa^T; f, x\bigr)\bigr| \leqslant c\mathcal{M}_\kappa^T(x), \quad \text{if } \delta > \lambda_\kappa = \sum_{j=1}^{d+1} \kappa_j + \frac{d-1}{2}.$$

Using this result, we can prove an analogue of Theorem 2.4 almost verbatim. Furthermore, the Poisson operator, $P_r^T f$, of the orthogonal expansion with respect to W_κ^T on T^d is still a semi-group when we define $T^t f = P_r^T f$ with $r = e^{-t}$ (see, for example, [13, p. 90]). So, the Littlewood–Paley function $g(f)$, defined as in (2.12), is bounded in $L^p(W_\kappa^T; T^d)$ for $1 < p < \infty$. Hence, all the essential ingredients of the proof of the multiplier theorem in [1] hold for the orthogonal expansion with respect to W_κ^T. As a consequence, we have the following multiplier theorem.

Theorem 5.2. *Let $\{\mu_j\}_{j=0}^\infty$ be a sequence that satisfies*

(1) $\sup_j |\mu_j| \leqslant c < \infty$,
(2) $\sup_j 2^{j(k-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^k u_l| \leqslant c < \infty$,

where k is the smallest integer $\geqslant \lambda_\kappa + 1$*. Then* $\{\mu_j\}$ *defines an* $L^p(W_\kappa^T; T^d)$*,* $1 < p < \infty$*, multiplier*; *that is,*

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j^\kappa f \right\|_{W_\kappa^T, p} \leqslant c \|f\|_{W_\kappa^T, p}, \quad 1 < p < \infty,$$

where c is independent of f and μ_j*.*

5.2. Weight function associated with $\mathbb{Z}_2^d$

In the case $G = \mathbb{Z}_2^d$, we are dealing with the classical weight function on T^d,

$$W_\kappa^T(x) := \prod_{i=1}^{d} |x_i|^{\kappa_i - 1/2} \big(1 - |x|\big)^{\kappa_{d+1} - 1/2}, \quad x \in T^d. \tag{5.6}$$

Under the mapping (5.2), this weight function corresponds to W_κ^B at (4.5). Taking into the consideration of the boundary, an appropriate distance on T^d is defined by

$$d_T(x, y) = \arccos\big(\langle x^{1/2}, y^{1/2}\rangle + \sqrt{1 - |x|}\sqrt{1 - |y|}\big), \quad x, y \in T^d,$$

where $x^{1/2} = (x_1^{1/2}, \ldots, x_d^{1/2})$ for $x \in T^d$. Evidently, we have $d_B(x, y) = d_T(\psi(x), \psi(y))$. Using this distance, one can define the weighted Hardy–Littlewood maximal function as

$$M_\kappa^T f(x) := \sup_{0 < \theta \leqslant \pi} \frac{\int_{d_T(x,y) \leqslant \theta} |f(y)| W_\kappa^T(y)\, dy}{\int_{d_T(x,y) \leqslant \theta} W_\kappa^T(y)\, dy}, \quad x \in T^d.$$

We have the following analogue of Theorem 4.3.

Theorem 5.3. *Let* $f \in L^1(W_\kappa^T; T^d)$*. Then for any* $x \in T^d$*,*

$$\mathcal{M}_\kappa^T f(x) \leqslant c M_\kappa^T f(x). \tag{5.7}$$

Proof. Using (5.3), it follows readily from the definitions of $M_\kappa^B f$ and $M_\kappa^T f$ that $(M_\kappa^T f) \circ \psi = M_\kappa^B (f \circ \psi)$. Hence, using the fact that if g is invariant under the sign changes, then $M_\kappa^B g(x\varepsilon) = M_\kappa^B g(x)$ by a simple change of variables, it follows from (5.5) and Theorem 4.3 that

$$\begin{aligned} \big(\mathcal{M}_\kappa^T f\big) \circ \psi(x) = \mathcal{M}_\kappa^B (f \circ \psi)(x) &\leqslant c \sum_{\varepsilon \in \mathbb{Z}_2^d} M_\kappa^B (f \circ \psi)(x\varepsilon) \\ &= c' M_\kappa^B (f \circ \psi)(x) = c' \big(M_\kappa^T f\big) \circ \psi(x) \end{aligned}$$

for $x \in T^d$, from which the stated result follows immediately. □

Although the proof of this theorem may look like a trivial consequence of the definition of $M_\kappa f$, we should mention that the definition of $\mathcal{M}_\kappa^T f$ in [13, Definition 4.5, p. 86] is given in terms of the general translation operator of the orthogonal expansions with respect to W_κ^T.

As a consequence of Theorem 5.3 or by (5.5) and (5.3), we have the following analogues of Corollaries 4.5 and 4.6.

Corollary 5.4. *If* $-\frac{\mathbb{1}}{2} < \tau \leqslant \kappa$ *and* $f \in L^1(W_\tau^T; T^d)$, *then* $\mathcal{M}_\kappa f$ *satisfies*

$$\operatorname{meas}_\tau^T \left\{x\colon \mathcal{M}_\kappa^T f(x) \geqslant \alpha\right\} \leqslant c \frac{\|f\|_{W_\tau^T,1}}{\alpha}, \quad \forall \alpha > 0.$$

Furthermore, if $1 < p < \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$ *and* $f \in L^p(W_\tau^T; T^d)$, *then*

$$\left\|\mathcal{M}_k^T f\right\|_{W_\tau^T,p} \leqslant c\|f\|_{W_\tau^T,p}.$$

Corollary 5.5. *Let* $1 < p < \infty$, $-\frac{\mathbb{1}}{2} < \tau < p\kappa + \frac{p-1}{2}\mathbb{1}$, *and let* $\{f_j\}_{j=1}^\infty$ *be a sequence of functions. Then*

$$\left\|\left(\sum_{j=1}^\infty \left(\mathcal{M}_\kappa^T f_j\right)^2\right)^{1/2}\right\|_{W_\tau^T,p} \leqslant c \left\|\left(\sum_{j=1}^\infty |f_j|^2\right)^{1/2}\right\|_{W_\tau^T,p}.$$

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