On complete system of invariants for the binary form of degree 7

Leonid Bedratyuk*

Khmelnytsky National University, Applied Mathematics, Instytut'ska st. 11, Khmel’nytsky, 29016, Ukraine

Received 29 November 2006; accepted 25 July 2007
Available online 6 August 2007

Abstract

A minimal system of homogeneous generating elements of the algebra of invariants for the binary form of degree 7 is calculated.

Keywords: Classical invariant theory; Invariants of binary form; Derivations

1. Introduction

Let $V_n$ be the vector $\mathbb{C}$-space of the binary forms of degree $d$ equipped with the natural action of the group $G = SL(2, \mathbb{C})$. Consider the corresponding action of the group $G$ on the coordinate rings $\mathbb{C}[V_d]$ and $\mathbb{C}[V_d \oplus \mathbb{C}^2]$. Denote by $I_d = \mathbb{C}[V_d]^G$ and by $C_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^G$ the subalgebras of $G$-invariant polynomial functions. In the language of classical invariant theory the algebras $I_d$ and $C_d$ are called the algebra of invariants and the algebra of covariants for the binary form of degree $d$, respectively. Let $I_d^+$ be an ideal of $I_d$ generated by all homogeneous elements of positive degree. Denote by $\tilde{I}_d$ a set of homogeneous elements of $I_d^+$ such that their images in $I_d^+/ (I_d^+)^2$ form a basis of the vector space. The set $\tilde{I}_d$ is called the complete system of invariants for the binary form of degree $d$. Denote by $n_d$ the cardinality of the set $\tilde{I}_d$. The complete system $\bar{C}_d$ of covariants for the binary form may be similarly defined.

The complete systems of invariants and covariants were a major object of research in classical invariant theory in the 19th century. It can be readily shown that $n_1 = 0$, $n_2 = 1$ and $n_3 = 1$. 

*Tel.: +380 3822 49043.
E-mail address: bedratyuk@ief.tup.km.ua.
The complete systems of invariants and covariants in the case \( d = 4 \) were calculated by Boole, Cayley, Eisenstein. The case \( d = 5 \) was calculated by the efforts of Cayley and Hermite, see Dixmier (1990). In particular, they showed that \( n_4 = 2, n_5 = 4 \). The complete systems of invariants and covariants in the case \( d = 6 \) were calculated by Gordan, see Gordan (1885). He showed that \( n_6 = 5 \).

A complete system of 9 invariants in the case \( d = 8 \) was computed by Gall (1888) and Shioda (1967). The case \( d = 7 \) was considered by Gall (1880). Gall established that \( n_7 \leq 33 \). Almost a century later Dixmier and Lazard (1986) found that \( n_7 = 30 \). However, the question: what elements, exactly, form the complete system of invariants, has remained open to this day, see survey Dixmier (1990). Gall found a non-minimal set of 33 generators of invariants for binary forms of degree 7. The symbolic presentation of invariants, which Gall used, makes it very difficult to decide which are irreducible. The current paper proposes a different symbolic presentation of invariants and presents for the first time a minimal set of 30 generating invariants.

To solve the computation problem we offer a representation of the invariants, which is an intermediate between the highly unwieldy explicit representation and the highly “compressed” symbolic representation. Also, we find a new symbolic representation of the fundamental intermediate between the highly unwieldy explicit representation and the highly “compressed” symbolic representation. Also, we find a new symbolic representation of the fundamental invariants, which is different from Gall’s representation. The representation uses transvectants of low orders. The first step of the simplification is calculation of semi-invariants instead of covariants. Now, consider a covariant as a polynomial of generating functions of the polynomial algebra \( \mathbb{C}[V_d \oplus \mathbb{C}^2] \). Then, a semi-invariant is just the leading coefficient of a polynomial with respect to usual lexicographical ordering. A semi-invariant is an invariant of upper unipotent matrix subalgebra of Lie algebra \( \mathfrak{sl}_2 \).

Let us identify the algebra \( \mathbb{C}[V_d] \) with the algebra \( \mathbb{C}[X_d] := \mathbb{C}[t, x_1, x_2, \ldots, x_d] \). Also, identify the algebra \( \mathbb{C}[V_d \oplus \mathbb{C}^2] \) with the polynomial algebra \( \mathbb{C}[t, x_1, x_2, \ldots, x_d, Y_1, Y_2] \).

The generating elements \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) of the tangent Lie algebra \( \mathfrak{sl}_2 \) act on \( \mathbb{C}[V_d] \) by derivations
\[
D_1 := t \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + \cdots + dx_{d-1} \frac{\partial}{\partial x_d},
\]
\[
D_2 := dx_1 \frac{\partial}{\partial t} + (d-1)x_2 \frac{\partial}{\partial x_1} + \cdots + x_d \frac{\partial}{\partial x_{d-1}}.
\]

It follows that the algebra \( I_d \) coincides with the algebra of polynomial solutions of the following first order PDE system, see Hilbert (1993), Glenn (1915):
\[
\begin{align*}
& t \frac{\partial u}{\partial x_1} + 2x_1 \frac{\partial u}{\partial x_2} + \cdots + nx_{d-1} \frac{\partial u}{\partial x_d} = 0, \\
& dx_1 \frac{\partial u}{\partial t} + (d-1)x_2 \frac{\partial u}{\partial x_1} + \cdots + x_d \frac{\partial u}{\partial x_{d-1}} = 0,
\end{align*}
\]

i.e. \( I_d = \mathbb{C}[X_d]^{D_1} \cap \mathbb{C}[X_d]^{D_2} \), where \( u \in \mathbb{C}[X_d] \), and
\[
\mathbb{C}[X_d]^{D_i} := \{ f \in \mathbb{C}[X_d] | D_i(f) = 0 \}, \quad i = 1, 2.
\]

For the computation of semi-invariants we introduce the semi-transvectant that is an analogue of the transvectant. In the current paper, we find an effective formula for computation of the semi-transvectants. For the case \( d = 7 \), we calculate all irreducible semi-invariants up to 13th degree. An invariant of even degree \( n \) may be considered as a semi-transvectant of the form
$[u, v]^I$, where $u, v$ are semi-invariants of degree $\frac{n}{2}$. Then, using the obtained semi-invariants we compute all irreducible invariants up to 26th degree.

The invariant of degree 30 was taken from Gall’s paper, see Gall (1880). We compute the invariant and check that it is irreducible. In this way, here we explicitly compute a complete system of 30 invariants of the binary form of degree 7.

All calculations were done with Maple.

2. Preliminaries

To begin with, we afford a simplification of the representation of covariants. Let

$$\kappa : C_d \longrightarrow \mathbb{C}[X_d]^{D_1}$$

be the $\mathbb{C}$-linear map that takes each homogeneous covariant of order $k$ to its leading coefficient, i.e. to a coefficient of $Y_1^k$. In the terminology of the classical invariant theory, an element of the algebra $\mathbb{C}[X_d]^{D_1}$ is called a semi-invariant, the degree of a homogeneous covariant with respect to the variables set $X_d$ is called the degree of the covariant and its degree with respect to the variables set $Y_1, Y_2$ is called the order of the covariant.

Suppose that $F = \sum_{i=0}^{m} f_i^{(m)} Y_1^{m-i} Y_2^i$ is a covariant of order $m$, $\kappa(F) = f_0 \in \mathbb{C}[X_d]^{D_1}$. The classical Robert’s theorem, Roberts (1861), states that the covariant $F$ is completely and uniquely determined by its leading coefficient $f_0$, namely,

$$F = \sum_{i=0}^{m} \frac{D_2^i(f_0)}{i!} Y_1^{m-i} Y_2^i.$$

On the other hand, every semi-invariant is a leading coefficient of some covariant, Glenn (1915). This give us a well-defined explicit form of the inverse map

$$\kappa^{-1} : \mathbb{C}[X]^{D_1} \longrightarrow C_d,$$

namely,

$$\kappa^{-1}(a) = \sum_{i=0}^{\text{ord}(a)} \frac{D_2^i(a)}{i!} Y_1^{\text{ord}(a)-i} Y_2^i,$$

where $a \in \mathbb{C}[X]^{D_1}$ and $\text{ord}(a)$ is the order of the element $a$ with respect to the locally nilpotent derivation $D_2$, i.e. $\text{ord}(a) = \max\{s, D_2^s(a) \neq 0\}$. For instance, since $\text{ord}(t) = d$, we have

$$\kappa^{-1}(t) = \sum_{i=0}^{\text{ord}(t)} \frac{D_2^i(t)}{i!} Y_1^{\text{ord}(t)-i} Y_2^i = t Y_1^d + \sum_{i=1}^{d} \binom{d}{i} x_i Y_1^{d-i} Y_2^i.$$

As we see $\kappa^{-1}(t)$ is just the basic binary form. From polynomial function point of view the covariant $\kappa^{-1}(t)$ is the evaluation map.

It is clear that an invariant is a semi-invariant of order zero. Thus, the problem of finding a complete system of the algebra $\overline{C}_d$ is equivalent to the problem of finding a complete system of semi-covariants of the algebra $\mathbb{C}[X]^{D_1}$. On the other hand, the problem of finding a complete system of the algebra $\overline{I}_d$ is equivalent to the problem of finding a subsystem in $\overline{C}_d$ such that it is generated by elements of order zero, these being well-known classical results.
A structure of algebras of constants for such locally nilpotent derivations can be easily
determined, see, for example, van den Essen (2000). In particular, for the derivation $D_1$ we get

$$\mathbb{C}[X_d]^{D_1} = \mathbb{C}[t_1, \sigma(x_2), \ldots, \sigma(x_d)] \left[ \frac{1}{t_1} \right] \cap \mathbb{C}[X_d],$$

where $\sigma : \mathbb{C}[X_d] \to \mathbb{C}[X_d]^{D_1}$ is a ring homomorphism defined by

$$\sigma(a) = \sum_{i=0}^{\infty} D_1^i(a) \frac{\lambda^i}{i!}, \quad \lambda = -\frac{x_1}{t_1}, \quad a \in \mathbb{C}[X_d].$$

After an uncomplicated simplification, we obtain $\sigma(x_i) = \frac{z_i + 1}{t^n}$, where $z_i \in \mathbb{C}[X_d]^{D_1}$ and

$$z_i := \sum_{k=0}^{i-2} (-1)^k \binom{i}{k} x_{i-k} x_1^k t^{i-k-1} + (i-1)(-1)^{i+1} x_1^i, \quad i = 2, \ldots, d.$$

In particular,

\begin{align*}
    z_2 &= x_2 t - x_1^2, \\
    z_3 &= x_3 t^2 + 2x_1^3 - 3x_1 x_2 t, \\
    z_4 &= x_4 t^3 - 3x_1^4 + 6x_1^2 x_2 t - 4x_1 x_3 t^2, \\
    z_5 &= x_5 t^4 + 4x_1^5 - 10x_1^3 x_2 t + 10x_1^2 x_3 t^2 - 5x_1 x_4 t^3, \\
    z_6 &= x_6 t^5 - 5x_1^6 + 15x_1^4 x_2 t - 20x_1^3 x_3 t^2 + 15x_1^2 x_4 t^3 - 6x_1 x_5 t^4, \\
    z_7 &= x_7 t^6 + 6x_1^7 - 21x_1^5 x_2 t + 35x_1^4 x_3 t^2 - 35x_1^3 x_4 t^3 + 21x_1^2 x_5 t^4 - 7x_1 x_6 t^5.
\end{align*}

Thus, we obtain

$$\mathbb{C}[X_d]^{D_1} = \mathbb{C}[t_1, z_2, \ldots, z_d] \left[ \frac{1}{t_1} \right] \cap \mathbb{C}[X_d].$$

Hence, the generating elements of the algebra $\mathbb{C}[X_d]^{D_1}$ may be regarded as the fraction

$$\frac{f(z_2, \ldots, z_d)}{t^n}, \quad f \in \mathbb{C}[Z_d] := \mathbb{C}[t_1, z_2, \ldots, z_d], \quad s \in \mathbb{Z}_+.$$

To make a calculation with invariants in such a representation we should specify an action of the operator $D_2$ in the new coordinates $t, z_2, \ldots, z_d$. Denote by $D$ the extension of the derivation $D_2$ to the algebra $\mathbb{C}[Z_d] \left[ \frac{1}{t_1} \right]$

$$D := D_2(t) \frac{\partial}{\partial t} + D_2(z_2) \frac{\partial}{\partial z_2} + \cdots + D_2(z_d) \frac{\partial}{\partial z_d}.$$ 

In Bedratyuk (2006) we showed that

$$D(t) = -nt \lambda,$$

$$D(\sigma(x_2)) = (n-2)\sigma(x_3) - (n-4)\sigma(x_2)\lambda,$$

$$D(\sigma(x_i)) = (n-i)\sigma(x_{i+1}) - (n-2i)\sigma(x_i)\lambda - i(n-1) \frac{\sigma(x_i)\sigma(x_{i-1})}{i}, \quad i > 2.$$
Taking into account $\sigma(x_i) = \frac{z_{i+1}}{t}$, $\lambda = -\frac{a_1}{t}$ we can obtain an expression for $D(z_i), i = 2, \ldots, d$. In particular, for $d = 7$ we get:

$$D = 7x_1 \frac{\partial}{\partial t} - \left( -15x_1z_3 + 18z_2^2 - 4z_4 \right) \frac{\partial}{\partial z_3} + \frac{(20x_1z_4 - 24z_2z_3 + 3z_5)}{t} \frac{\partial}{\partial z_4} + \frac{(2z_6 + 25x_1z_5 - 30z_2z_4)}{t} \frac{\partial}{\partial z_5} + \frac{(z_7 + 30x_1z_6 - 36z_2z_5)}{t} \frac{\partial}{\partial z_6} + \frac{7(5x_1z_7 - 6z_2z_6)}{t} \frac{\partial}{\partial z_7} + \frac{5(2x_1z_8 + z_3)}{t} \frac{\partial}{\partial z_8}.$$ 

To calculate semi-invariants we should have an analogue of the transvectants. Suppose $F$ is such that $D(F) = 0$. Then we get

$$F = \sum_{i=0}^{m} f_i \binom{m}{i} y_1^{m-i} y_2^i, \quad G = \sum_{i=0}^{k} g_i \binom{k}{i} y_1^{k-i} y_2^i, \quad f_i, g_i \in \mathbb{C}[Z_d] \left[ \frac{1}{t} \right],$$

are two covariants of degrees $m$ and $k$ respectively. Let $(F, G)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r F}{\partial y_1^{r-i} y_2^i} \frac{\partial^r G}{\partial y_1^{r-i} y_2^i}$, be their $r$th transvectant (Hilbert, 1993; Glenn, 1915). The following lemma gives us the rule how to find the semi-invariant $\chi((F, G)^r)$ without direct computing of the covariant $(F, G)^r$.

**Lemma 1.** The leading coefficient $\chi((F, G)^r)$ of the covariant $(F, G)^r$, $0 \leq r \leq \min(m, k)$ is calculated by the formula

$$\chi((F, G)^r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{D^i(\chi(F))}{[m]_i} \bigg|_{x_1=0, \ldots, x_r=0} \frac{D^{r-i}(\chi(G))}{[k]_{r-i}} \bigg|_{x_1=0, \ldots, x_r=0},$$

where $[a]_i := a(a - 1) \ldots (a - (i - 1)), a \in \mathbb{Z}.$

**Proof.** In Hilbert (1993), p. 87, one may find that

$$\chi((F, G)^r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f_i g_{r-i}. $$

By comparing two different forms of the covariant $F$, namely

$$F = \sum_{i=0}^{m} f_i \binom{m}{i} y_1^{m-i} y_2^i \quad \text{and} \quad F = \sum_{i=0}^{m} \frac{D^i_2(f_0)}{i!} y_1^{m-i} y_2^i,$$

we get $f_i \binom{m}{i} = \frac{D^i_2(f_0)}{i!}$. Thus, $f_i = \frac{D^i_2(f_0)}{[m]_i} = \frac{D^i(\chi(F))}{[m]_i}$. Similarly $g_i = \frac{D^i(\chi(G))}{[k]_{r-i}}$. Therefore,

$$\chi((F, G)^r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{D^i(\chi(F))}{[m]_i} \frac{D^{r-i}(\chi(G))}{[k]_{r-i}}.$$

The derivation $D$ is such that $D(\mathbb{C}[Z_d][\frac{1}{t}]) \subset \mathbb{C}[Z_d, x_1][\frac{1}{t}]$. Further, we have

$$D^2 \left( \mathbb{C}[Z_d] \left[ \frac{1}{t} \right] \right) \subset \mathbb{C}[Z_d, x_1, x_2] \left[ \frac{1}{t} \right].$$
and for allowed \( r \) we get
\[
D' \left( \mathbb{C}[Z_d] \left[ \frac{1}{r} \right] \right) \subset \mathbb{C}[Z_d, x_1, x_2, \ldots, x_r] \left[ \frac{1}{r} \right].
\]
Therefore,
\[
\kappa((F, G)'') \subset \mathbb{C}[Z_d, x_1, x_2, \ldots, x_r] \left[ \frac{1}{r} \right].
\]

On the other hand, since \( \kappa((F, G)') \) is always a semi-invariant, we see that the inclusion \( \kappa((F, G)') \subset \mathbb{C}[Z_d] \left[ \frac{1}{r} \right] \) is correct. Thus, in the expression for \( \kappa((F, G)') \) after cancellation, all coefficients of \( x_1, x_2, \ldots, x_r \) must be equal to zero. Hence,
\[
\kappa((F, G)') = \kappa((F, G)') \big|_{x_1=0, \ldots, x_r=0}
\]
\[
= \sum_{i=0}^{r} (-1)^i \left( \begin{array}{c} r \\ i \end{array} \right) \frac{D_i(\kappa(F)) \ D_{r-i}^\prime(\kappa(G))}{[m]_i \ [k]_{r-i}} \bigg|_{x_1=0, \ldots, x_r=0}
\]
\[
= \sum_{i=0}^{r} (-1)^i \left( \begin{array}{c} r \\ i \end{array} \right) \frac{D_i(\kappa(F)) \ D_{r-i}^\prime(\kappa(G))}{[m]_i \ [k]_{r-i}} \bigg|_{x_1=0, \ldots, x_r=0}, \quad \Box
\]

Let \( f, g \) be two semi-invariants. Their numerators are polynomials of \( z_2, \ldots, z_n \) with rational coefficients. Then, the semi-invariant \( \kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r) \) is a fraction and its numerator is a polynomial of \( z_2, \ldots, z_n \) with rational coefficients. Therefore, we may multiply \( \kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r) \) by some rational number \( q_r(f, g) \in \mathbb{Q} \) such that the numerator of the expression \( q_r(f, g)\kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r) \) is then a polynomial with an integer coprime coefficients. Put
\[
[f, g]^r := q_r(f, g)\kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r), \quad 0 \leq r \leq \min(\text{ord}(f), \text{ord}(g)).
\]
The expression \( [f, g]^r \) is said to be the \( r \)th semi-transvectant of the semi-invariants \( f \) and \( g \).

The following statements are direct consequences of corresponding properties of transvectants, see Glenn (1915):

**Lemma 2.** Let \( f, g \) be two semi-invariants. Then, the following conditions hold:

(i) the semi-transvectant \([t, fg]^1\) is reducible for \( 0 \leq i \leq \min(d, \max(\text{ord}(f), \text{ord}(g)))\);
(ii) if \( \text{ord}(f) = 0 \), then \([t, fg]^1 = f[t, g]^1\);
(iii) \( \text{ord}([f, g]^i) = \text{ord}(f) + \text{ord}(g) - 2i \);
(iv) \( \text{ord}(z_2^i z_3^j \cdots z_d^k) = d(i_2 + i_3 + \cdots + i_d) - 2(2i_2 + 3i_3 + \cdots + di_d) \).

Let us consider an example. Suppose \( d = 7, f = z_2, g = z_3 \). Then, \( \text{ord}(z_2) = 7 : 2 = 2:2 = 10 \), and \( \text{ord}(z_3) = 15 \). We have
\[
D(z_2) = \frac{5(2x_1z_2 + z_3)}{t}, \quad [10]_0 = 1, \quad [10]_1 = 1,
\]
\[
D^2(z_2) = \frac{10(3x_1^2z_2 + 9x_1z_3 + 6x_2z_2t - 9z_2^2 + 2z_4)}{t^2}, \quad [10]_2 = 10,
\]
\[
D(z_3) = \frac{15x_1z_3 - 18z_2^2 + 4z_4}{t}, \quad [15]_0 = 1, \quad [15]_1 = 1,
\]
\[
D^2(z_3) = \frac{2(60x_1^2z_3 - 252x_1z_2^2 + 56x_1z_4 + 45x_2z_3t - 138z_2z_3 + 6z_5)}{t^2}, \quad [15]_2 = 15,
\]
and
\[
D(z_2)_{x_1=0,x_2=0} = \frac{5(z_2)}{r}, \quad D^2(z_2)_{x_1=0,x_2=0} = \frac{10(-9z_2^2+2z_4)}{r^2}, \\
D(z_3)_{x_1=0,x_2=0} = \frac{-18z_2^2+4z_4}{r}, \quad D^2(z_3)_{x_1=0,x_2=0} = \frac{2(-138z_2+6z_4)}{r^2}.
\]

We get
\[
\kappa((\kappa^{-1}(z_2), \kappa^{-1}(z_3))^2) = \sum_{r=0}^{2} (-1)^r \binom{2}{r} \frac{d^r(z_2)}{[10]_r} \bigg|_{x_1=0,x_2=0} \frac{d^{2-r}(z_3)}{[15]_{2-r}} \bigg|_{x_1=0,x_2=0}.
\]

After simplification we obtain
\[
\kappa((\kappa^{-1}(z_2), \kappa^{-1}(z_3))^2) = -\frac{2}{315} \frac{3z_2^2z_3 - 9z_2z_5 + 7z_3z_4}{t^2}. 
\]

It is obviously that \(q_2(z_2, z_3) = -\frac{315}{2}\). Therefore,
\[
[z_2, z_3]^2 = q_2(z_2, z_3)\kappa((\kappa^{-1}(z_2), \kappa^{-1}(z_3))^2) = \frac{3z_2^2z_3 - 9z_2z_5 + 7z_3z_4}{t^2}
\]
\[
= -31x_1^3x_4t + 16x_1^4x_3 + 9x_1^2x_5t^2 + 7x_3^3x_4 + 30x_3tx_1^2x_2 + 24x_1x_2t^2x_4
\]
\[
- 28x_3^2t^2x_1 - 12x_1^3x_2^2 + 3x_2^2t^2x_3 - 9x_2t^3x_5 - 9x_2^3tx_1.
\]

In the same way one may get
\[
[t, [z_2, z_3]^2] = \frac{27z_2^4 - 78z_2^2z_4 - 14z_4^2 + 69z_2z_3^2 + 12z_3z_5 + 9z_2z_6}{t^2}
\]
\[
= -78x_2^2t^3x_4 - 126x_2^2x_1^4 + 45x_2^2tx_1^2 + 12x_3^3t^5x_5 + 69x_2t^3x_3^2 - 173x_1^2x_3^2t^2
\]
\[
+ 168x_1^5x_3 - 249x_1^4x_4t - 9x_1^2x_6t^3 + 9x_2t^4x_6 + 78x_1^3x_5t^2 + 27x_2^4t^2 - 14x_4^2t^4
\]
\[
- 102x_2^2t^2x_1x_3 + 303x_1^2x_2t^2x_4 + 78x_1^3x_2tx_3 + 52x_4t^3x_1x_3 - 90x_1x_2t^3x_5.
\]

It can be seen from these examples that a representation of the semi-invariants as elements of the algebra \(\mathbb{C}[Z_d]\) is more compact than their standard representation as elements of the algebra \(\mathbb{C}[X_d]\). A very rough empirical estimate would be that a semi-invariant \(\kappa^{-1}(F)\) has terms in \((\deg(F)/d) + 2)(\text{ord}(F) + 1)\) times less than the corresponding covariant \(F\). Moreover, from computing point of view, the semi-transvectant formula is more effective than the transvectant formula. These two favorable circumstances, coupled with utilizing the great practical powers of Maple software, allow us to compute a complete system of invariants for the binary form of degree 7.

3. Computation of auxiliary semi-invariants

Prior to any calculation of invariants, we find all irreducible covariants up to 13th degree. To do so, we use an analogue of well-known, Glenn (1915), \(\Omega\)-process. Let \(\overline{C}_{7,i}\) be a subset of elements of \(\kappa(\overline{C}_7)\) whose degree is \(i\). We are seeking the elements of the set \(\overline{C}_{7,i+1}\) as irreducible elements of a basis of the vector space generated by semi-transvectants of the form \([t, uv]^r, u \in \overline{C}_{7,i}, v \in C_{7,k}, l + k = i, \max(\text{ord}(u), \text{ord}(v)) \leq r \leq 7\). It is a standard linear algebra problem.

The unique semi-invariant of degree one obviously is \(t, \text{ord}(t) = 7\). The semi-transvectants \([t, t]^r, i = 1 \ldots 7\) are equal to zero for odd \(i\). Put
The polynomials $t^2$, $dv_1$, $dv_2$, $dv_3$ are linearly independent. Therefore, the set $\overline{C}_{7,2}$ consists of the irreducible semi-invariants $dv_1$, $dv_2$, $dv_3$ of 2nd degree.

To define the set $\overline{C}_{7,3}$ consider the following polynomials $t^3$, $tdv_1$, $tdv_2$, $tdv_3$, $[t, dv_1]$, $i = 1 \ldots 6$, $[t, dv_2]$, $i = 1, 2$ $[t, dv_3]$, $i = 1 \ldots 7$. By direct calculation we select six irreducible semi-invariants of degree 3:

$$
\begin{align*}
tr_1 &= [t, dv_1]^4, \quad \text{ord}(tr_1) = 5, \\
tr_2 &= [t, dv_3], \quad \text{ord}(tr_2) = 15, \\
tr_3 &= [t, dv_3]^3, \quad \text{ord}(tr_3) = 11, \\
tr_4 &= [t, dv_3]^4, \quad \text{ord}(tr_4) = 9, \\
tr_5 &= [t, dv_3]^5, \quad \text{ord}(tr_5) = 7, \\
tr_6 &= [t, dv_3]^7, \quad \text{ord}(tr_6) = 3.
\end{align*}
$$

In the same way, by using Lemma 2, we may calculate all sets $\overline{C}_{7,i}$, $i \leq 13$. Let us present the lists of the generating elements.

The set $\overline{C}_{7,4}$ consists of the following 8 irreducible semi-invariants:

$$
\begin{align*}
ch_1 &= [t, tr_5]^7, \quad \text{ord}(ch_1) = 0, & ch_2 &= [t, tr_3]^7, \quad \text{ord}(ch_2) = 4, \\
ch_3 &= [t, tr_3]^2, \quad \text{ord}(ch_3) = 14, & ch_4 &= [t, tr_3]^4, \quad \text{ord}(ch_4) = 10, \\
ch_5 &= [t, tr_3]^3, \quad \text{ord}(ch_5) = 8, & ch_6 &= [t, tr_1]^2, \quad \text{ord}(ch_6) = 8, \\
ch_7 &= [t, tr_1]^3, \quad \text{ord}(ch_7) = 6, & ch_8 &= [t, tr_1]^4, \quad \text{ord}(ch_8) = 4.
\end{align*}
$$

The set $\overline{C}_{7,5}$ consists of the following 10 irreducible semi-invariants:

$$
\begin{align*}
pt_1 &= [t, ch_6]^5, \quad \text{ord}(pt_1) = 5, & pt_2 &= [t, ch_6]^6, \quad \text{ord}(pt_2) = 3, \\
pt_3 &= [t, ch_7]^2, \quad \text{ord}(pt_3) = 9, & pt_4 &= [t, ch_7]^3, \quad \text{ord}(pt_4) = 7, \\
pt_5 &= [t, ch_7]^5, \quad \text{ord}(pt_5) = 3, & pt_6 &= [t, ch_6]^3, \quad \text{ord}(pt_6) = 9, \\
pt_7 &= [t, ch_4]^2, \quad \text{ord}(pt_7) = 13, & pt_8 &= [t, ch_4]^5, \quad \text{ord}(pt_8) = 7, \\
pt_9 &= [t, dv_2]^7, \quad \text{ord}(pt_9) = 5, & pt_{10} &= [t, dv_1 dv_2]^7, \quad \text{ord}(pt_{10}) = 1.
\end{align*}
$$

The set $\overline{C}_{7,6}$ consists of the following 10 irreducible semi-invariants:

$$
\begin{align*}
sh_1 &= [t, pt_5]^5, \quad \text{ord}(sh_1) = 6, & sh_2 &= [t, pt_7]^6, \quad \text{ord}(sh_2) = 8, \\
sh_3 &= [t, pt_4]^5, \quad \text{ord}(sh_3) = 4, & sh_4 &= [t, pt_4]^6, \quad \text{ord}(sh_4) = 2, \\
sh_5 &= [t, pt_3]^2, \quad \text{ord}(sh_5) = 12, & sh_6 &= [t, pt_3]^4, \quad \text{ord}(sh_6) = 8, \\
sh_7 &= [t, pt_4]^4, \quad \text{ord}(sh_7) = 6, & sh_8 &= [t, tr_1 dv_1]^7, \quad \text{ord}(sh_8) = 4, \\
sh_9 &= [t, tr_1 dv_2]^6, \quad \text{ord}(sh_9) = 2, & sh_{10} &= [t, tr_6 dv_1]^7, \quad \text{ord}(sh_{10}) = 2.
\end{align*}
$$

The set $\overline{C}_{7,7}$ consists of the following 12 irreducible semi-invariants:

$$
\begin{align*}
si_1 &= [t, sh_5]^4, \quad \text{ord}(si_1) = 11, & si_2 &= [t, sh_7]^4, \quad \text{ord}(si_2) = 5, \\
si_3 &= [t, tr_2]^7, \quad \text{ord}(si_3) = 3, & si_4 &= [t, sh_1]^3, \quad \text{ord}(si_4) = 7, \\
si_5 &= [t, ch_7 dv_1]^7, \quad \text{ord}(si_5) = 5, & si_6 &= [t, ch_7 dv_2]^7, \quad \text{ord}(si_6) = 1, \\
si_7 &= [t, tr_2]^4, \quad \text{ord}(si_7) = 5, & si_8 &= [t, tr_2]^6, \quad \text{ord}(si_8) = 6, \\
si_9 &= [t, tr_6 tr_1]^6, \quad \text{ord}(si_9) = 3, & si_{10} &= [t, tr_6 tr_1]^7, \quad \text{ord}(si_{10}) = 1, \\
si_{11} &= [t, tr_1^2]^6, \quad \text{ord}(si_{11}) = 5, & si_{12} &= [t, sh_{10}], \quad \text{ord}(si_{12}) = 7.
\end{align*}
$$
The set $C_{7,8}$ consists of the following 13 irreducible semi-invariants:

$$
\begin{align*}
vi_1 &= [t, si_1]^3, & \text{ord}(vi_1) &= 6, & vi_2 &= [t, si_1]^4, & \text{ord}(vi_2) &= 4, \\
vi_3 &= [t, ch_8 tr_6]^7, & \text{ord}(vi_3) &= 0, & vi_4 &= [t, ch_8 tr_1]^6, & \text{ord}(vi_4) &= 4, \\
vi_5 &= [t, ch_8 tr_1]^7, & \text{ord}(vi_5) &= 2, & vi_6 &= [t, ch_7 tr_6]^7, & \text{ord}(vi_6) &= 2, \\
vi_7 &= [t, ch_7 tr_1]^7, & \text{ord}(vi_7) &= 4, & vi_8 &= [t, ch_8 tr_6]^6, & \text{ord}(vi_8) &= 2, \\
vi_9 &= [t, tr_6 dv_2]^2, & \text{ord}(vi_9) &= 0, & vi_{10} &= [t, si_4]^2, & \text{ord}(vi_{10}) &= 10, \\
vi_{11} &= [t, si_2]^4, & \text{ord}(vi_{11}) &= 6, & vi_{12} &= [t, si_{11}]^3, & \text{ord}(vi_{12}) &= 6, \\
vi_{13} &= [t, pt_9 dv_2]^7, & \text{ord}(vi_{13}) &= 0.
\end{align*}
$$

The set $C_{7,9}$ consists of the following 11 irreducible semi-invariants:

$$
\begin{align*}
d e_1 &= [t, sh_3 dv_1]^7, & \text{ord}(de_1) &= 3, & de_2 &= [t, ch_7 ch_8]^7, & \text{ord}(de_2) &= 3, \\
d e_3 &= [t, pt_5 tr_6]^5, & \text{ord}(de_3) &= 3, & de_4 &= [t, pt_5 tr]^6, & \text{ord}(de_4) &= 3, \\
d e_5 &= [t, pt_5 tr_1]^7, & \text{ord}(de_5) &= 1, & de_6 &= [t, sh_9 dv_1]^7, & \text{ord}(de_6) &= 1, \\
d e_7 &= [t, sh_10 dv_1]^7, & \text{ord}(de_7) &= 1, & de_8 &= [t, sh_10 dv_2]^3, & \text{ord}(de_8) &= 5, \\
de_9 &= [t, vi_3]^2, & \text{ord}(de_9) &= 5, & de_{10} &= [t, vi_2]^4, & \text{ord}(de_{10}) &= 3, \\
d e_{11} &= [t, vi_{11}]^2, & \text{ord}(de_{11}) &= 9.
\end{align*}
$$

The set $C_{7,10}$ consists of the following 9 irreducible semi-invariants:

$$
\begin{align*}
\text{odn}_1 &= [t, sh_9 tr_1]^6, & \text{ord}(\text{odn}_1) &= 2, & \text{odn}_2 &= [t, sh_4 tr_6]^4, & \text{ord}(\text{odn}_2) &= 4, \\
\text{odn}_3 &= [t, sh_4 tr]^6, & \text{ord}(\text{odn}_3) &= 2, & \text{odn}_4 &= [t, sh_1 tr_1]^7, & \text{ord}(\text{odn}_4) &= 4, \\
\text{odn}_5 &= [t, sh_3 tr_5]^5, & \text{ord}(\text{odn}_5) &= 4, & \text{odn}_6 &= [t, sh_9 dv_1]^7, & \text{ord}(\text{odn}_6) &= 8, \\
\text{odn}_7 &= [t, tr_6]^3, & \text{ord}(\text{odn}_7) &= 2, & \text{odn}_8 &= [t, sh_10 tr_1]^6, & \text{ord}(\text{odn}_8) &= 2, \\
\text{odn}_9 &= [t, pt_1 ch_7]^7, & \text{ord}(\text{odn}_9) &= 4.
\end{align*}
$$

The set $C_{7,11}$ consists of the following 9 irreducible semi-invariants:

$$
\begin{align*}
\text{odn}_1 &= [t, vi_2 dv_1]^7, & \text{ord}(\text{odn}_1) &= 3, & \text{odn}_2 &= [t, vi_2, dv_2]^6, & \text{ord}(\text{odn}_2) &= 1, \\
\text{odn}_3 &= [t, vi_4 dv_2]^6, & \text{ord}(\text{odn}_3) &= 1, & \text{odn}_4 &= [t, vi_2 dv_1]^7, & \text{ord}(\text{odn}_4) &= 1, \\
\text{odn}_5 &= [t, vi_6 dv_1]^7, & \text{ord}(\text{odn}_5) &= 1, & \text{odn}_6 &= [t, vi_2 dv_2]^5, & \text{ord}(\text{odn}_6) &= 3, \\
\text{odn}_7 &= [t, des_6]^4, & \text{ord}(\text{odn}_7) &= 7, & \text{odn}_8 &= [t, des_6]^6, & \text{ord}(\text{odn}_8) &= 3, \\
\text{odn}_9 &= [t, vi_1 dv_2]^7, & \text{ord}(\text{odn}_9) &= 1.
\end{align*}
$$

The set $C_{7,12}$ consists of the following 13 irreducible semi-invariants:

$$
\begin{align*}
\text{dvan}_1 &= [t, sh_1 pt_2]^7, & \text{ord}(\text{dvan}_1) &= 2, & \text{dvan}_2 &= [t, sh_1 pt_5]^7, & \text{ord}(\text{dvan}_2) &= 2, \\
\text{dvan}_3 &= [sh_9, sh_{10}]^2, & \text{ord}(\text{dvan}_3) &= 0, & \text{dvan}_4 &= [t, odn_7]^6, & \text{ord}(\text{dvan}_4) &= 2, \\
\text{dvan}_5 &= [t, de_8 dv_2]^6, & \text{ord}(\text{dvan}_5) &= 2, & \text{dvan}_6 &= [sh_{10}, sh_{10}]^2, & \text{ord}(\text{dvan}_6) &= 0, \\
\text{dvan}_7 &= [t, de_4 dv_1]^7, & \text{ord}(\text{dvan}_7) &= 2, & \text{dvan}_8 &= [t, de_10 dv_1]^7, & \text{ord}(\text{dvan}_8) &= 6, \\
\text{dvan}_9 &= [t, odn_7]^4, & \text{ord}(\text{dvan}_9) &= 6, & \text{dvan}_{10} &= [sh_1, sh_1]^2, & \text{ord}(\text{dvan}_{10}) &= 0, \\
\text{dvan}_{11} &= [sh_8, sh_2]^2, & \text{ord}(\text{dvan}_{11}) &= 0, & \text{dvan}_{12} &= [sh_4, sh_9]^2, & \text{ord}(\text{dvan}_{12}) &= 0, \\
\text{dvan}_{13} &= [sh_4, sh_2]^2, & \text{ord}(\text{dvan}_{13}) &= 0.
\end{align*}
$$

The set $C_{7,13}$ consists of the following 9 irreducible semi-invariants:

$$
\begin{align*}
\text{tryn}_1 &= [t, dv_9]^6, & \text{ord}(\text{tryn}_1) &= 1, & \text{tryn}_2 &= [t, vi_1 ch_7]^7, & \text{ord}(\text{tryn}_2) &= 5, \\
\text{tryn}_3 &= [t, vi_2 ch_8]^7, & \text{ord}(\text{tryn}_3) &= 1, & \text{tryn}_4 &= [t, vi_2 ch_7]^7, & \text{ord}(\text{tryn}_4) &= 1, \\
\text{tryn}_5 &= [t, vi_1 ch_8]^7, & \text{ord}(\text{tryn}_5) &= 3, & \text{tryn}_6 &= [t, vi_5 ch_2]^6, & \text{ord}(\text{tryn}_6) &= 1, \\
\text{tryn}_7 &= [t, vi_5 ch_8]^6, & \text{ord}(\text{tryn}_7) &= 1, & \text{tryn}_8 &= [t, vi_8 ch_7]^7, & \text{ord}(\text{tryn}_8) &= 1, \\
\text{tryn}_9 &= [t, vi_4 ch_8]^7, & \text{ord}(\text{tryn}_9) &= 1.
\end{align*}
$$
A number of elements of $\overline{C}_{7,i}$, $i = 1, \ldots, 13$ and orders of the elements so far coincide completely with Gall’s results, see Gall (1888).

4. Computation of the invariants

Put $I_i := \overline{I}_7 \cap \overline{C}_{7,i}$, $I_+ := I_7^+$. Let $(I_+^2)_i$ be a subset of $(I_7^+)^2$, whose elements have degree $i$. Denote by $\delta_i$ the number of irreducible invariants of degree $i$. It is clear that $\delta_i = \dim I_i - \dim (I_+^2)_i$, see Dixmier and Lazard (1986). The dimension of the vector space $I_i$ is calculated by Cayley–Sylvester formula, see, for example, Hilbert (1993) and Springer (1977).

The dimension of the vector space $(I_+^2)_i$ is calculated by the formula $\dim(I_+^2)_i = \sigma_i - \dim S_i$. Here $\sigma_i$ is the coefficient of $x^i$ in the series expansion $\left(\prod_{k<i} (1 - x^k)^{\delta_k}\right)^{-1}$, and $S_i$ is a vector subspace of $(I_+^2)_i$ generated by syzygies.

The invariants of 4th, 8th and 12th degrees were found above. We have

\[
I_4 = \langle p_4 \rangle, \quad \delta_4 = 1, \quad p_4 := ch_1 = [t, tr_3]^7,
I_8 = \langle p_{8,1}, p_{8,2}, p_{8,3} \rangle, \quad \delta_8 = 3,
\]

\[
p_{8,1} := vi_3 = [t, ch_4 tr_6]^7, \quad p_{8,2} := vi_9 = [t, tr_6 dv_2^2]^7, \quad p_{8,3} := vi_{13} = [t, pt_9 dv_2]^7,
I_{12} = \langle p_{12,1}, p_{12,2}, p_{12,3}, p_{12,4}, p_{12,5}, p_{12,6} \rangle, \quad \delta_{12} = 6,
\]

\[
p_{12,1} := dvan_3 = [sh_9, sh_{10}]^2, \quad p_{12,2} := dvan_6 = [sh_{10} sh_{10}]^2, \quad p_{12,3} := dvan_{10} = [sh_{10} sh_1]^2, \quad p_{12,4} := dvan_{11} = [sh_4 sh_1]^2, \quad p_{12,5} := dvan_{12} = [sh_4, sh_4]_2^2, \quad p_{12,6} := dvan_{13} = [sh_4 sh_2]^2.
\]

For $I_{14}$ we have $\dim I_{14} = 4, \sigma_{14} = 0$. Therefore $(I_+^2)_{14} = 0$, then, $\delta_{14} = 4$. It is enough to find 4 linearly independent invariants of degree 14. Given below is a typical instance of how these invariants are calculated. We are searching the Invariants of $I_{14}$ as semi-transvectants of the form $[u, v]^4$, where $u, v$ are semi-invariants of $\overline{C}_{7,7}, i < 4$, ord$(u) + $ ord$(v) - 2i = 0$. There are 9 such semi-transvectants:

\[
[s_{18}, s_{18}], [s_{18}, s_{10}], [s_{19}, s_{19}]^3, [s_{10}, s_{10}], [s_{16}, s_{16}], [s_{16}, s_{18}], [s_{16}, s_{10}],
[s_{13}, s_{13}]^3, [s_{13}, s_{19}]^3.
\]

Using Maple calculation we choose four linearly independent invariants:

\[
p_{14,1} := [s_{18}, s_{10}], \quad p_{14,2} := [s_{16}, s_{10}], \quad p_{14,3} := [s_{16}, s_{18}], \quad p_{14,4} := [s_{13}, s_{19}]^3.
\]

The invariants $p_{14,1}, p_{14,2}, p_{14,3}, p_{14,4}$ are fractions with the denominator $t^{35}$. The numerators of the fractions are polynomials of $\mathbb{Z}[z_2, \ldots, z_7]$ which consist of 937, 869, 978, 925 terms respectively.

For $I_{16}$ we have $\dim I_{16} = 18, \sigma_{16} = 16$. The vector space $(I_+^2)_{16}$ is generated by 16 elements and all of them are linearly independent. Thus $\delta_{16} = 2$. In order to calculate the invariants of $I_{16}$, consider a set of semi-transvectants of the form $[u, v]^4$, where $u, v \in \overline{C}_{7,8}, i < 5$. There are 12 such semi-transvectants:

\[
[v_i v_i]^4, [v_i vi_5]^2, [v_i v_i]^3, [v_i v_i v_i]^2, [v_i v_i v_i]^2, [v_i v_i v_i]^2, [v_i v_i v_i]^2, [v_i v_i v_i]^2, \]

\[
[v_i v_i v_i]^4, [v_i v_i v_i v_i]^4, [v_i v_i v_i v_i]^4, [v_i v_i v_i v_i]^4, [v_i v_i v_i v_i]^4, [v_i v_i v_i v_i]^4.
\]

In order to separate two linearly independent semi-transvectants consider the equality:

\[
\alpha_1 p_4^4 + \alpha_2 p_8^2 + \cdots + \alpha_7 [v_i v_i]^4 + \cdots + \alpha_{28} [v_i v_i v_i]^4 = 0.
\]
Substituting the values of the invariants into the equality we obtain an over-defined system of linear equations. After solving the system we get 18 elements of which 16 are basis elements of the vectors space \((I^2_+)_{16}\) and 2 invariants \(p_{16,1} := [v_2, v_4]_d\), and \(p_{16,2} := [v_4, v_7]_d\) that span the vector space \(I_6\).

The invariants \(p_{16,1}, p_{16,2}\) are fractions with the denominator \(t^{40}\). The numerators of the fractions are polynomials of \(\mathbb{Z}[z_2, \ldots, z_7]\) which consist of 1744 and 1698 terms.

For \(I_{18}\) we have \(\dim I_{18} = 13, \sigma_{18} = 4\). Since \((I^2_+)_{18} = p_4 I_{14}\), then \(\delta_{18} = 0\). Thus \(\delta_{18} = 9\).

The invariants of \(I_{18}\) we are searching as semi-transvectants of the form \([u, v]_i\), where \(u, v\) are semi-invariants of \(C_{7,9}, i < 6, \text{ord}(u) + \text{ord}(v) − 2i = 0\).

In the same way as above, we obtain the nine irreducible invariants:

\[
\begin{align*}
p_{18,1} &:= [de_4, de_3]^3, \quad p_{18,2} := [de_4, de_{10}]^3, \quad p_{18,3} := [de_5, de_6], \\
p_{18,4} &:= [de_1, de_{10}]^3, \quad p_{18,5} := [de_2, de_3]^3, \quad p_{18,6} := [de_2, de_{10}]^3, \\
p_{18,7} &:= [de_5, de_{10}]^3, \quad p_{18,8} := [de_6, de_7], \quad p_{18,9} := [de_8, de_9]^5.
\end{align*}
\]

The invariants \(p_{18,1}, p_{18,2}, \ldots, p_{18,9}\) are fractions with the denominator \(t^{45}\). The numerators of the fractions are polynomials of \(\mathbb{Z}[z_2, \ldots, z_7]\) which consist of 2674, 2758, 2645, 2800, 2718, 2772, 2769, 2661, 2739 terms respectively.

For \(I_{20}\) we have \(\dim I_{20} = 35, \sigma_{20} = 36\). The vector space \(S_{20}\) is spanned by the two syzygies:

\[
\begin{align*}
-142725300p_{12,4}p_4^2 &- 15449224200p_{8,1}p_{8,2}p_4 + 1320855600p_{8,3}p_{8,1}p_4 \\
-327281580p_{8,3}p_{8,2}p_4 &- 75375000p_{12,6}p_{8,1} + 13989210p_{8,3}^2p_4 - 691200p_{16,2}p_4 \\
+ 1530000p_{12,4}p_{8,3} &- 19980000p_{12,5}p_{8,2} + 3107575500p_{8,1}^2p_4 + 1890487920p_{8,2}^2p_4 \\
- 1687500p_{12,2}p_{8,1} - 2025000p_{12,1}p_{8,3} &+ 34290000p_{12,1}p_{8,2} - 151200000p_{12,1}p_{8,1} \\
+ 36045000p_{12,5}p_{8,1} &- 10800000p_{12,4}p_{8,2} + 197100000p_{12,4}p_{8,1} - 675000p_{16,1}p_4 \\
- 337500p_{12,2}p_{8,3} - 1507500p_{12,6}p_{8,3} &+ 2970000p_{12,5}p_{8,3} - 188556795p_{12,5}p_4^2 \\
+ 4050000p_{12,2}p_{8,2} &+ 19440000p_{12,6}p_{8,2} - 135229030p_{8,1}p_4^3 + 24000p_{12,2}p_4^2 \\
+ 35439332p_{8,2}p_4^3 &- 2797842p_{8,3}p_4^3 + 110336985p_{12,1}p_4^2 = 0.
\end{align*}
\]

and

\[
\begin{align*}
144093300p_{12,4}p_4^2 &+ 15499144200p_{8,1}p_{8,2}p_4 - 1328535600p_{8,3}p_{8,1}p_4 \\
+ 329009580p_{8,3}p_{8,2}p_4 &- 68625000p_{12,6}p_{8,1} - 14085210p_{8,3}^2p_4 - 110768985p_{12,1}p_4^2 \\
- 4410000p_{12,4}p_{8,3} &- 49140000p_{12,5}p_{8,2} - 3122935500p_{8,1}^2p_4 - 1903639920p_{8,2}^2p_4 \\
+ 1687500p_{12,2}p_{8,1} + 2025000p_{12,1}p_{8,3} &+ 270000p_{12,1}p_{8,2} + 675000p_{16,1}p_4 \\
+ 67635000p_{12,5}p_{8,1} &+ 1080000p_{12,4}p_{8,2} - 110700000p_{12,4}p_{8,1} \\
+ 337500p_{12,2}p_{8,3} - 1372500p_{12,6}p_{8,3} &+ 14310000p_{12,5}p_{8,3} + 691200p_{16,2}p_4 \\
- 4050000p_{12,2}p_{8,2} + 15120000p_{12,6}p_{8,2} &+ 136157030p_{8,1}p_4^3 + 336000p_{12,6}p_4^2 \\
+ 21600000p_{12,1}p_{8,1} &- 35882532p_{8,2}p_4^3 + 2817042p_{8,3}p_4^3 \\
+ 186396795p_{12,5}p_4^2 &= 0.
\end{align*}
\]

Hence \(\dim S_{20} = 2\) and \(\delta_{20} = 1\). This confirms the result reported in the paper Dixmier and Lazard (1986).

We find the unique invariant \(I_{20}\) as semi-transvectant of the form \([u, v]_i\), where \(u, v \in C_{7,10}, i < 3, \text{ord}(u) + \text{ord}(v) − 2i = 0\). As a result of the calculation we get that the element
\( p_{20} := [des_7, des_7]^2 \) is the irreducible invariant of degree 20. The invariant \( p_{20} \) is a fraction with the denominator \( t^{50} \). The numerator of the fraction is a polynomial of \( \mathbb{Z}[z_2, \ldots, z_7] \) which consists of 4392 terms.

For \( I_{22} \) we have \( \dim I_{22} = 26, \sigma_{22} = 25 \). By direct calculation we find that those 25 elements of \( (I_{22}^2)_{22} \) satisfy the unique syzygy

\[
\begin{align*}
p_{14,3}p_{8,3} - 4p_{14,2}p_{8,3} + 40p_{14,3}p_{8,1} &+ 750p_{14,1}p_{8,3} + 150p_{14,4}p_{8,3} - 160p_{14,2}p_{8,1} \\
&\quad - 1275p_{14,4}p_{8,2} - 21p_{14,3}p_{8,2} - 1875p_{18,8}p_{4} + 84p_{14,2}p_{8,2} + 48750p_{14,1}p_{8,1} \\
&\quad + 9750p_{14,4}p_{8,1} + 1125p_{14,1}p_{8,2} = 0.
\end{align*}
\]

Hence \( \dim S_{22} = 1 \) and \( \delta_{22} = 26 - 25 + 1 = 2 \). It coincides with the results of Dixmier and Lazard (1986).

We find the invariants of \( I_{22} \) as semi-transvectants of the form \([u, v]^i\), where \( u, v \in \overline{C}_{7,11}, i < 4, \) \( \text{ord}(u) + \text{ord}(v) - 2i = 0 \). As result of the calculation we get that the elements

\[
p_{22,1} := [odn_6, odn_1]^3, \quad p_{22,2} := [odn_8, odn_1]^3
\]

are irreducible invariants of degree 22.

The invariants \( p_{22,1}, p_{22,2} \) are fractions with the denominator \( t^{55} \). The numerators of the fractions are polynomials of \( \mathbb{Z}[z_2, \ldots, z_7] \) which consist of 6569 and 6556 terms respectively.

For \( I_{24} \) we have \( \dim I_{24} = 62, \sigma_{24} = 74 \). By direct calculation we get that the vector space \( S_{24} \) is spanned by 12 syzygies. Hence \( \delta_{24} = 62 - 74 + 12 = 0 \).

For \( I_{26} \) we have \( \dim I_{26} = 52, \sigma_{26} = 78 \). By direct Maple calculation we obtain that the vector space \( S_{26} \) is spanned by 27 syzygies. Hence, \( \delta_{26} = 52 - 78 + 27 = 1 \), that coincides with the result of Dixmier and Lazard (1986). Invariants of \( I_{26} \) we are seeking as a semi-transvectants of the form \([u, v]^i\), where \( u, v \in \overline{C}_{7,13}, i = 1, \) \( \text{ord}(u) + \text{ord}(v) - 2i = 0 \). As result of the calculation we get that the element

\[
p_{26} = [\text{tryn}_4, \text{tryn}_3]
\]

is the unique irreducible invariant of degree 26. The invariant \( p_{26} \) is a fraction with the denominator \( t^{65} \). The numerator of the fraction is a polynomial of \( \mathbb{Z}[z_2, \ldots, z_7] \) which consists of 13 651 terms.

For \( I_{28} \) we have \( \dim I_{28} = 97, \sigma_{28} = 135 \). By direct calculation we get the vector space \( S_{28} \) spanned by 38 syzygies. Hence \( \delta_{28} = 97 - 135 + 38 = 0 \).

For \( I_{30} \) we have \( \dim I_{30} = 92, \sigma_{30} = 171 \). By direct Maple calculation we obtain that the vector space \( S_{30} \) is spanned by 80 syzygies. Hence, \( \delta_{30} = 92 - 171 + 80 = 1 \).

A unique irreducible invariant of \( I_{30} \) we take from Gall’s paper, Gall (1888). In the paper’s notation we find that \( p_{30} = (h, \alpha) \). After all calculations we obtain that \( p_{30} \) is a fraction with the denominator \( t^{75} \). The numerator of the fraction is a polynomial of \( \mathbb{Z}[z_2, \ldots, z_7] \) which consists of 25 868 terms.

Summarizing the above results we get

**Theorem.** The following system of 30 invariants is a complete system of the invariants for the binary form of degree 7
Acknowledgements

The author is grateful to Dr. Ivan Arzhantsev and Dr. Farogh Dovlatshahi for the useful discussions.

References


