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Solvability of Rado systems in D-sets ☆

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This paper is dedicated to Neil Hindman on the occasion of his 65th birthday

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ABSTRACT

Rado's Theorem characterizes the systems of homogeneous linear equations having the property that for any finite partition of the positive integers one cell contains a solution to these equations. Furstenberg and Weiss proved that solutions to those systems can in fact be found in every central set. (Since one cell of any finite partition is central, this generalizes Rado's Theorem.) We show that the same holds true for the larger class of D-sets. Moreover we will see that the conclusion of Furstenberg's Central Sets Theorem is true for all sets in this class.

1. Introduction

Schur's Theorem [20] states that for any finite partition of the positive integers one cell contains solutions to the equation $x_1 + x_2 = x_3$.

Another classical result of partition Ramsey theory is van der Waerden's Theorem [21] which states that arithmetic progressions of arbitrary finite length can be found in one cell of any finite partition. This follows from the fact that a solution to the equations $x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1}$ can always be found in one cell.[1]

Both statements are special instances of Rado's Theorem which provides necessary and sufficient conditions for the system $A(x_1, \ldots, x_q)^T = 0$, $A \in \mathbb{Z}^{p\times q}$ to be partition regular in the sense that for every finite partition of the positive integers one cell contains $x_1, \ldots, x_q$ satisfying $A(x_1, \ldots, x_q)^T = 0$. Each such system of linear equations is called a *Rado system*.

Theorem 1 *(Rado's Theorem). ([18]) A system of linear equations of the form $A(x_1, \ldots, x_q)^T = 0$, $A = (a_{ij}) \in \mathbb{Z}^{p\times q}$ is a Rado system iff the index set $\{1, 2, \ldots, q\}$ can be divided into disjoint subsets $I_1, I_2, \ldots, I_l$ and for all $r \in \{1, \ldots, l\}$, $j \in I_1 \cup \cdots \cup I_r$ rational numbers c_j^r may be found such that the following relations are satisfied*:

$$\sum_{j\in I_1} a_{ij} = 0,$$

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[1] This somewhat stronger result which asserts that the arithmetic progression and the increment can be forced to lie in the same cell is due to Brauer [6].

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$$\sum_{j\in I_2} a_{ij} = \sum_{j\in I_1} c_j^1 a_{ij},$$
$$\dots,$$
$$\sum_{j\in I_l} a_{ij} = \sum_{j\in I_1\cup I_2\cup\cdots\cup I_{l-1}} c_j^{l-1} a_{ij}.$$

We want to mention a corollary[2] of Rado's Theorem extending Schur's Theorem. It is possible to find arbitrarily many numbers $x_1,\dots,x_n$, together with all finite sums $x_{k_1}+\cdots+x_{k_l}$, $k_1<\cdots<k_l\leqslant n$ in one cell of any finite partition. Only some forty years after the publication of Rado's result, Hindman [13, Theorem 3.1] established that one can actually find an infinite sequence together with all finite sums from its elements in one cell. (See [8,17,15] for more information on infinite partition regular systems of equations.)

Sometimes a deeper understanding of results in Partition Ramsey Theory is achieved by finding the proper notion of *largeness* which guarantees that one cell of a finite partition contains rich combinatorial structure. The theorems of van der Waerden and Szemerédi provide an example of this principle: While the first one states that every finite partition of the integers has one cell which contains arbitrarily long arithmetic progressions, the latter reveals that those can be found in every set S of positive upper Banach density $d^*(S)=\overline{\lim}_{(m-n)\to\infty}|S\cap\{n,\dots,m\}|/(m-n+1)$. Clearly, at least one cell of each finite partition of $\mathbb{N}$ has positive upper Banach density and thus van der Waerden's Theorem is a corollary of Szemerédi's Theorem.

Furstenberg and Weiss improved Rado's result by showing that solutions to Rado systems can be found in every *central set*.[3] One can give a definition of central sets using ultrafilters on $\mathbb{N}$, but since we want to postpone dealing with these somewhat esoteric objects to Section 4, we will give here Furstenberg's original definition [11, Definition 8.3].

A set $S\subseteq\mathbb{N}$ is central iff there exist a dynamical system (X,T) (i.e. a compact metric space (X,ρ) and a continuous transformation T of X), a point $x\in X$, a uniformly recurrent point y which is proximal to x, and an open neighborhood U of y such that

$$S=\{n\in\mathbb{N}:\ T^n x\in U\}.$$

(A point $y\in X$ is called uniformly recurrent if for each neighborhood U of y the set $\{n\in\mathbb{N}:\ T^n y\in U\}$ is syndetic, i.e. has bounded gaps. Points $x, y\in X$ (which are not necessarily distinct) are proximal if $\inf_{n\geqslant 0}\rho(T^n x, T^n y)=0$.)

For our purposes the following characterization of central sets via product systems will also be of interest.

Proposition 2. *([2, Theorem 2.3]) A set $S\subseteq\mathbb{N}$ is central iff there exist a dynamical system (X,T), a pair $(x,y)\in X\times X$ where y is uniformly recurrent in (X,T) and such that (y,y) belongs to the orbit closure of (x,y) in the product system $(X\times X, T\times T)$, and an open neighborhood U of (y,y) such that*

$$S=\{n\in\mathbb{N}:\ (T^n x, T^n y)\in U\}.$$

For the sake of completeness we include a short sketch of the proof.

Sketch of proof. First assume that S is central and that it is obtained via (X,T), $x,y\in X$ and an open $U\subseteq X$. Utilizing the fact that y is uniformly recurrent and that x, y are proximal one easily checks that (y,y) belongs to the orbit closure of (x,y) in the product system $(X\times X, T\times T)$. Clearly $S=\{n\in\mathbb{N}:\ (T^n x, T^n y)\in U\times X\}$.

Conversely, let $S=\{n\in\mathbb{N}:\ (T^n x, T^n y)\in U\}$, where (X,T), x, y and $U\subseteq X$ satisfy the assumptions of the proposition. Notice that (y,y) is uniformly recurrent and (x,y), (y,y) are proximal in the product system. Hence one gets that S is central by using (x,y) and (y,y) as a pair of proximal points in the system $(X\times X, T\times T)$. □

One can prove (and this is in fact apparent from the ultrafilter description given in Section 4) that one cell of each finite partition of the positive integers is central, that every set containing a central set is central itself and that central sets remain central after removing finitely many points.

Central sets have positive upper Banach density. In fact, if S is central, it possesses the strictly stronger property that there exists $k\in\mathbb{N}$ such that $S\cup(S-1)\cup\cdots\cup(S-k)$ contains arbitrarily long intervals, i.e. S is *piecewise syndetic*.

While it is merely an exercise to derive van der Waerden's Theorem from the fact that every piecewise syndetic set contains arbitrarily long arithmetic progressions, Szemerédi's Theorem which guarantees the existence of arbitrarily long arithmetic progressions in sets of positive upper Banach density is highly nontrivial. Analogously, one might search for a class of not necessarily piecewise syndetic sets which contain solutions to Rado systems. Clearly positive upper Banach density is not the appropriate notion (for instance the set of all odd numbers contains no configuration of the form x_1, x_2,

[2] This was proved independently (but much later then Rado's Theorem) by Folkman (unpublished) and Sanders [19].

[3] The theorem in this form was spelled out in [11, Theorem 8.22] but can also be deduced from [12, Theorem 4.4] which was published before the introduction of central sets.

$x_1 + x_2$). In fact it is possible to find for each $\varepsilon > 0$ a set S of density bigger than $1 - \varepsilon$ such that for no $t \in \mathbb{Z}$, $S - t$ contains solutions to all Rado systems.[4] (In contrast to this it is always possible to shift a piecewise syndetic set such that it becomes central (see [16, Theorem 4.40]) and then contains solutions to all Rado systems.)

In this paper we prove that solutions of Rado systems are contained in any member of a class of sets which is larger than the class of central sets. This class is comprised of D-sets defined in [2]. The main distinction of D-sets from the class of central sets is that in the definition of central sets instead of a uniformly recurrent point, one considers an *essentially recurrent point* y, meaning that the set $\{n \in \mathbb{N}\colon\ T^n y \in U\}$ has positive upper Banach density for every neighborhood U of y. Note that since every syndetic set has positive upper density, every uniformly recurrent point is an essentially recurrent point.

A set $S \subseteq \mathbb{N}$ is a *D-set* iff there exist a dynamical system (X, T) (i.e. a compact metric space X and a continuous transformation T of X), a pair of points $x, y \in X$ where y is essentially recurrent, and such that (y, y) belongs to the orbit closure of (x, y) in the product system $(X \times X, T \times T)$, and an open neighborhood U of (y, y) such that

$$S = \{n \in \mathbb{N}\colon\ (T^n x, T^n y) \in U\}.$$

By Proposition 2 a set $S \subseteq \mathbb{N}$ is central iff it satisfies the above definition with the twist that y is not just essentially recurrent, but a uniformly recurrent point. Hence every central set is a D-set. Similarly to central sets, the family of D-sets is closed under forming supersets and every D-set has positive upper Banach density. But D-sets do not need to be piecewise syndetic. (See [5].) In particular the class of D-sets is strictly larger than the class of central sets.

So our main result is:

Theorem 3. *Rado systems are solvable in D-sets.*

We will give two proofs of Theorem 3. The first one, given in Section 3 is formulated in the language of topological dynamics, while the second one, presented in Section 4 makes use of the algebraic structure on the set of ultrafilters on $\mathbb{N}$. This second proof actually establishes that Furstenberg's Central Sets Theorem [11, Proposition 8.21] can be extended to D-sets. In Section 2 we collect some tools which will be used in both proofs of Theorem 3.

Note that in [2] D-sets are defined as subsets of the group $\mathbb{Z}$ and also the transformations considered there are invertible. However it is more traditional to work with subsets of the positive integers when the focus of interest lies on combinatorial applications. The proofs of the statements in [2] work in this modified setting without any significant changes and the connection between D-sets in $\mathbb{N}$ and D-sets in $\mathbb{Z}$ is rather natural: Every D-set in $\mathbb{N}$ is a D-set in $\mathbb{Z}$ and $S \subseteq \mathbb{Z}$ is a D-set iff $S \cap \mathbb{N}$ or $(-S) \cap \mathbb{N}$ is a D-set in $\mathbb{N}$. (The analogous statement holds true for central sets.)

One might wonder whether any set satisfying the conclusion of the Central Sets Theorem must have positive Banach density. It is shown in [14] that this is not the case.

2. Preliminaries

The following concept is due to Deuber ([7], see also [16, Chapter 15]). Given positive integers m, p, c, the (m, p, c)-*system* generated by the $(m+1)$-tuple $(s^{(0)}, \ldots, s^{(m)})$ is the following array of numbers:

$$cs^{(0)},$$
$$cs^{(1)} + i_0 s^{(0)}, \quad |i_0| \leqslant p,$$
$$\ldots,$$
$$cs^{(m)} + i_{m-1} s^{(m-1)} + \cdots + i_0 s^{(0)}, \quad |i_{m-1}|, \ldots, |i_0| \leqslant p.$$

Deuber [7, Satz 2.1] proved that every Rado system is solvable within a set S of positive integers iff for any triple (m, p, c) of positive integers S contains an (m, p, c)-system. Thus for our purposes it is sufficient to prove the following result.

Proposition 4. *Let S be a D-set and m, p, c positive integers. Then S contains an (m, p, c)-system.*

Since we are going to prove the existence of structures extending arithmetic progressions in sets which need not be piecewise syndetic, it is no surprise that we will employ some version of Szemerédi's Theorem. In fact we shall use the Furstenberg and Katznelson's deep *multiple IP-recurrence theorem* and its combinatorial corollary, the *IP Szemerédi Theorem*.

To formulate these theorems we introduce some notation: By $\mathcal{F}$ we denote the set of all finite nonempty sets of positive integers. For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ iff $\max\alpha < \min\beta$. Given a sequence $s_1, s_2, \ldots$ in $\mathbb{Z}$ or $\mathbb{Z}^m$ and $\alpha = \{k_1, \ldots, k_l\} \in \mathcal{F}$,

[4] Following Ernst Strauss (see [1, Theorem 2.20]) one can construct a set S with density arbitrarily close to 1 such that there does not exist $t \in \mathbb{Z}$ such that $(S - t) \cap \mathbb{N}n \neq \emptyset$ for every $n \in \mathbb{N}$. Given positive integers $x_1, \ldots, x_m, m = n^2$ there exist $i_1 < \cdots < i_k \leqslant m$ such that $x_{i_1} + \cdots + x_{i_k} \in n\mathbb{N}$. Hence if $(S - t) \cap \mathbb{N}n = \emptyset$, $S - t$ cannot contain positive integers $x_1, \ldots, x_m$ and all finite sums from these numbers. In particular, no shifted copy of S contains solutions to all Rado systems.

$k_1 < \cdots < k_l$, we let $s_\alpha = s_{k_1} + \cdots + s_{k_l}$ and call the family $(s_\alpha)_{\alpha\in\mathcal{F}}$ an *IP-system*. Similarly, for a sequence $T_1, T_2, \ldots$ of commuting transformations of a space, we assign to α the transformation $T_\alpha = T_{k_1} \circ \cdots \circ T_{k_l}$ and call $(T_\alpha)_{\alpha\in\mathcal{F}}$ an *IP-system of transformations*.

Theorem 5 *(Multiple IP-recurrence theorem). ([10, Theorem A]) Let $(X, \mathcal{B}, \mu)$ be a probability measure space. Let $(T_\alpha^{(1)})_{\alpha\in\mathcal{F}}, \ldots, (T_\alpha^{(p)})_{\alpha\in\mathcal{F}}$ be commuting IP-systems of transformations which preserve μ. Then for every $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $\alpha \in \mathcal{F}$ such that*

$$\mu\big(A \cap \big(T_\alpha^{(1)}\big)^{-1} A \cap \cdots \cap \big(T_\alpha^{(p)}\big)^{-1} A\big) > 0.$$

We plan to apply Theorem 5 in the dynamical proof of Proposition 4. The link to D-sets will be established using the following result:

Theorem 6. *([2, Theorem 2.6]) Let (X, T) be a dynamical system, let $y \in X$ be an essentially recurrent point and U a neighborhood of y. Then there exists a probability Borel measure μ on X preserved by the action T such that $\mu(U) > 0$.*

In the ultrafilter proof of Proposition 4 we use the above mentioned combinatorial corollary of Theorem 5.

Theorem 7 *(IP Szemerédi Theorem). Assume that $S \subseteq \mathbb{N}$ has positive upper Banach density and let $(s_\alpha^{(1)})_{\alpha\in\mathcal{F}}, \ldots, (s_\alpha^{(p)})_{\alpha\in\mathcal{F}}$ be IP-systems of integers. Then there exist $\alpha \in \mathcal{F}$ and $a \in S$ such that $a + s_\alpha^{(1)}, \ldots, a + s_\alpha^{(p)} \in S$.*

We will also need the following lemma on *IP*-systems. (The proof is left as an exercise.)

Lemma 8. *Let $(s_\alpha)_{\alpha\in\mathcal{F}}$ be an IP-system of integers and let $c \in \mathbb{N}$. There exist $\alpha_1 < \alpha_2 < \cdots$ in $\mathcal{F}$ such that for every $n \in \mathbb{N}$, s_{α_n} is divisible by c.*

3. A proof via topological dynamics

The proof of Proposition 4 is more transparent for $c = 1$. Therefore we will first restrict ourselves to this special case and make some remarks on what needs to be changed to achieve the result in full generality later.

For the rest of this section, fix a dynamical system (X, T) and $x, y \in X$ such that y is essentially recurrent, and such that (y, y) belongs to the orbit closure of (x, y) in the product system $(X \times X, T \times T)$. Given an open neighborhood U of (y, y), we let $S_U = \{n \in \mathbb{N}:\ (T^n x, T^n y) \in U\}$. In this setting Proposition 4 (for $c = 1$) translates to:

Proposition 9. *Let $m, p \in \mathbb{N}$ and let U be an open neighborhood of (x, y). Then S_U contains an $(m, p, 1)$-system.*

The proof of Proposition 9 is based on the following lemma:

Lemma 10. *Fix some $p \in \mathbb{N}$. If $m \geqslant 0$ is such that for every open $U \ni (y, y)$, S_U contains an $(m, p, 1)$-system, then for every such U, S_U contains a family of $(m, p, 1)$-systems such that their generating $(m+1)$-tuples $(s_\alpha^{(0)}, \ldots, s_\alpha^{(m)})_{\alpha\in\mathcal{F}}$ form an IP-system in $\mathbb{N}^{m+1}$.*

Proof. Clearly, it suffices to consider *symmetric* sets U, i.e. sets of the form $V \times V$, where V is an open set containing y.

Fix $m \geqslant 0$ for which the assumption holds. Fix a symmetric open set $U_1 \ni (y, y)$. We know that S_{U_1} contains an $(m, p, 1)$-system D_1 generated by some $(m+1)$-tuple $(s_1^{(0)}, \ldots, s_1^{(m)})$. In particular, the set

$$U_2' = \bigcap_{n\in D_1} (T \times T)^{-n}(U_1)$$

contains (x, y) and, since U_1 was symmetric, also (y, y). Let now U_2 be a symmetric neighborhood of (y, y) contained in $U_1 \cap U_2'$. By assumption, the set S_{U_2} also contains an $(m, p, 1)$-system D_2 generated by $(s_2^{(0)}, \ldots, s_2^{(m)})$. Clearly $S_{U_2} \subseteq S_{U_1}$, so S_{U_1} contains both D_1 and D_2. Moreover, it contains the algebraic sum of D_1 and D_2. In particular, it contains the Deuber system $D_{1,2}$ generated by $(s_1^{(0)} + s_2^{(0)}, \ldots, s_1^{(m)} + s_2^{(m)})$. Continuing by an obvious induction we construct a family of Deuber systems as in the assertion. □

Proof of Proposition 9. We proceed by induction on m.

For $m = 0$ the $(m, p, 1)$-system reduces to a single number s_0, such that $(T \times T)^{s_0}(x, y) \in U$. The set of such numbers s_0 is nonempty for every open $U \ni (y, y)$, because $(y, y) \in \overline{O}(x, y)$.

Suppose the assertion holds for some m and all open sets $U \ni (y, y)$. Fix U. By Lemma 10, S_U contains many such systems indexed by $\alpha \in \mathcal{F}$, where the generating $(m+1)$-tuples $(s_\alpha^{(0)}, \ldots, s_\alpha^{(m)})$ form an *IP*-system in $\mathbb{N}^{m+1}$. Then for any fixed integers $i_0, \dots, i_m$ the numbers $i_0 s_\alpha^{(0)} + \cdots + i_m s_\alpha^{(m)}$ (with varying α) form an *IP*-system and $(R^{i_0 s_\alpha^{(0)} + \cdots + i_m s_\alpha^{(m)}})_{\alpha \in \mathcal{F}}$ is an *IP*-system of transformations for any given transformation R. Let $(i_0, \dots, i_m)$ range over the integers in $[-p, p]^{m+1}$ and consider the following $(2p+1)^{m+1}$ commuting *IP*-systems of transformations on $X \times X$:

$$T_\alpha^{(i_0,\dots,i_m)} = (T \times T)^{i_0 s_\alpha^{(0)} + \cdots + i_m s_\alpha^{(m)}}.$$

Apply Theorem 6 to $(y, y) \in \overline{O}(y, y)(\subseteq (X \times X, T \times T))$ to get a $T \times T$-invariant measure μ which assigns positive measure to U. Since μ is preserved by all the above transformations, Theorem 5 asserts that there exists an $\alpha \in \mathcal{F}$ such that

$$V = \bigcap_{(i_0,\dots,i_m) \in [-p,p]^{m+1}} \bigl(T_\alpha^{(i_0,\dots,i_m)}\bigr)^{-1} U$$

has positive measure. Since μ is supported by $\overline{O}(y, y) \subseteq \overline{O}(x, y)$, these facts imply that there exists an integer s, such that $(T \times T)^s(x, y) \in V$. This, in turn, implies that the numbers $s + i_0 s_\alpha^{(0)} + \cdots + i_m s_\alpha^{(m)}$ belong to S_U for all $(i_0, \dots, i_m) \in [-p, p]^{m+1}$. Because S_U already contains the $(m, p, 1)$-system generated by the $(m+1)$-tuple $(s_\alpha^{(0)}, \dots, s_\alpha^{(m)})$, we have proved that S_U also contains the $(m+1, p, 1)$-system generated by the $(m+2)$-tuple $(s_\alpha^{(0)}, \dots, s_\alpha^{(m)}, s)$. The proof of Proposition 9 is now complete. □

Finally we explain what has to be changed if $c \neq 1$. Lemma 10 is valid without any significant changes in the proof, if we just replace every appearance of "$(m, p, 1)$-system" with "(m, p, c)-system". The same holds true for the inductive step in the proof of Proposition 9 up to the point where s is chosen. In order to achieve that S_U contains an $(m+1, p, c)$-system, we would need that s is divisible by c but at this point it is not obvious why this should be the case. Thus we end up with S_U containing an $(m+1, p, c)$-system generated by $(s_\alpha^{(0)}, \dots, s_\alpha^{(m)}, s) = (t^{(0)}, \dots, t^{(m+1)})$ which is flawed in the sense that $t^{(m+1)}$ is multiplied by 1 instead of c. However we can apply Lemma 10 to see that S_U actually contains such structures generated by $(m+2)$-tuples which form an *IP*-system $(t_\alpha^{(0)}, \dots, t_\alpha^{(m+1)})_{\alpha \in \mathcal{F}}$. Hence we can apply Lemma 8 to get that $t_\alpha^{(m+1)}$ is divisible by c if $\alpha \in \mathcal{F}$ is properly chosen. Therefore S_U contains the $(m+1, p, c)$-system generated by $(t_\alpha^{(0)}, \dots, t_\alpha^{(m)}, t_\alpha^{(m+1)}/c)$ and thus the $c \neq 1$ version of Proposition 9 also holds for $m+1$.

4. A proof via ultrafilters

In this section we use the algebraic structure of the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to give a proof to Proposition 4. We start with a brief description of the concepts required in this proof, see [3] for a short "self contained" or [16] for an exhaustive treatment of the algebraic structure on $\beta\mathbb{N}$.

Take $\beta\mathbb{N}$ to be the set of all ultrafilters on $\mathbb{N}$. A nonempty system of sets $q \subsetneq \mathcal{P}(\mathbb{N})$ is called a *filter* if it is closed under forming supersets and finite intersections. It is an *ultrafilter* if it is a filter with the additional property that whenever $C_1 \cup \cdots \cup C_n = \mathbb{N}$, some C_i lies in D. Using the axiom of choice, it is possible to show that $|\beta\mathbb{N}| = 2^{2^{|\mathbb{N}|}}$ but the only elements of $\beta\mathbb{N}$ which can be explicitly constructed are the *principle ultrafilters* $q(n) = \{S \subseteq \mathbb{N}\colon\ n \in S\}$ where $n \in \mathbb{N}$. While not being overly exciting, they allow us to view $\mathbb{N}$ as a subset of $\beta\mathbb{N}$ by identifying each $n \in \mathbb{N}$ with $q(n) \in \beta\mathbb{N}$. Using standard properties of the Stone–Čech compactification one obtains that there exists a unique extension of the addition on $\mathbb{N}$ to $\beta\mathbb{N}$ such that the map $q \mapsto r + q$ is continuous for every $r \in \beta\mathbb{N}$ and $q \mapsto q + n$ is continuous for every $n \in \mathbb{N}$. (Note that it is not possible to have both $q \mapsto q + r$ *and* $q \mapsto r + q$ continuous for all $r \in \beta\mathbb{N}$. In particular $+$ is highly noncommutative on $\beta\mathbb{N}$.) An explicit description of the addition on $\beta\mathbb{N}$ is given by

$$S \in q + r \quad \Leftrightarrow \quad \{n \in \mathbb{N}\colon\ S - n \in q\} \in r. \tag{1}$$

(Here $S - n = \{k \in \mathbb{N}\colon\ k + n \in \mathbb{N}\}$.) Another interpretation of $+$ is obtained if we interpret ultrafilters as $\{0, 1\}$-valued finitely additive measures. Then the addition turns out to be just the convolution of measures. Thus it is not very surprising that $+$ is associative. In fact, the compactness of $\beta\mathbb{N}$ together with continuity of $r + q$ in the right argument guarantees that $(\beta\mathbb{N}, +)$ is a semigroup with quite rich algebraic structure. Moreover, algebraic properties of ultrafilters are nicely linked with combinatorial properties of their elements as is exemplified by the following facts.

- Similar to finite semigroups, $\beta\mathbb{N}$ contains *idempotents*, that is elements q such that $q + q = q$. A set $S \subseteq \mathbb{N}$ is contained in an idempotent ultrafilter iff there exists an *IP*-system $(s_\alpha)_{\alpha \in \mathcal{F}}$ such that all s_α lie in S.
- A subset I of a semigroup $(G, +)$ is a two sided ideal if $I + G, G + I \subseteq I$. It can be shown that $\beta\mathbb{N}$ has a smallest two sided ideal $K(\beta\mathbb{N})$ (with respect to inclusion). A set $S \subseteq \mathbb{N}$ is piecewise syndetic iff there exists $q \in K(\beta\mathbb{N})$ such that $S \in q$.

An ultrafilter which is idempotent and lies in the smallest ideal of $\beta\mathbb{N}$ is called a *minimal idempotent*. It was established in [4, Corollary 6.12] that $S \subseteq \mathbb{N}$ is central iff there exists a minimal idempotent q such that $S \in q$.

Replacing piecewise syndetic with positive upper Banach density leads to the class of *essential idempotents*: $q \in \beta\mathbb{N}$ is an essential idempotent iff it is an idempotent ultrafilter, all of whose elements have positive upper Banach density. By [2, Theorem 2.8], $S \subseteq \mathbb{N}$ is a D-set iff it is contained in some essential idempotent.

We are now ready to employ these concepts to prove that D-sets satisfy the conclusion of Furstenberg's Central Sets Theorem [11, Proposition 8.21].

Theorem 11. [5] *Assume that S is a D-set and that $(s_\alpha^{(1)})_{\alpha\in\mathcal{F}}, \dots, (s_\alpha^{(p)})_{\alpha\in\mathcal{F}}$ are IP-systems. There exist sequences $a_1, a_2, \dots \in \mathbb{N}$ and $\alpha_1 < \alpha_2 < \cdots$ in $\mathcal{F}$ such that for all $k_1 < \cdots < k_l$ and $i \in \{1, \dots, p\}$,*

$$\big(a_{k_1} + s^{(i)}_{\alpha_{k_1}}\big) + \cdots + \big(a_{k_l} + s^{(i)}_{\alpha_{k_l}}\big) \in S. \tag{2}$$

The following standard lemma (cf. [16, Lemma 4.14]) nicely simplifies the inductive process used in the proof of Theorem 11.

Lemma 12. *Let q be an idempotent ultrafilter, let $S \in q$ and set $S^\star = \{n \in S\colon\ S - n \in q\}$. Then $S^\star \in q$ and $S^\star - n \in q$ for all $n \in S^\star$.*

Proof. $S^\star = S \cap \{m \in \mathbb{N}\colon\ S - m \in q\} \in q$ by (1) and since q is closed under finite intersections. Given $n \in S^\star$, we have $S^\star - n = (S - n) \cap \{m \in \mathbb{N}\colon\ S - m \in q\} - n = (S - n) \cap \{m \in \mathbb{N}\colon\ (S - n) - m \in q\}$. The first set lies in q since $n \in S^\star$ and the second set lies in q because $S - n \in q$ and we can substitute $S - n$ for S in (1). □

Proof of Theorem 11. Let q be an essential idempotent such that $S \in q$ and define $S^\star$ as in Lemma 12. We will inductively construct $a_1, a_2, \dots$ and $\alpha_1 < \alpha_2 < \cdots \in \mathcal{F}$ such that (2) is satisfied. To keep the induction going we will in fact demand that (2) is even true with S replaced by $S^\star$. To start the construction use the fact that $S^\star$ has positive upper Banach density together with Theorem 7 to find a_1 and α_1 such that $a_1 + s^{(i)}_{\alpha_1} \in S^\star$ for all $i \in \{1, \dots, p\}$.

Assume that after n steps we have found $a_1, \dots, a_n$ and $\alpha_1, \dots, \alpha_n$ such that all t which are of the form

$$t = \big(a_{k_1} + s^{(i)}_{\alpha_{k_1}}\big) + \cdots + \big(a_{k_l} + s^{(i)}_{\alpha_{k_l}}\big)$$

for some $k_1 < \cdots < k_l \leqslant n$ and $i \in \{1, \dots, p\}$ lie in $S^\star$. Then all sets $S^\star - t$ are in q and hence so is the intersection B of $S^\star$ with all the sets $S^\star - t$. Thus we may use Theorem 7 to find a_{n+1} and $\alpha_{n+1} > \alpha_n$ [6] such that $a_{n+1} + s^{(i)}_{\alpha_{n+1}} \in B$ for all $i \in \{1, \dots, p\}$. Then by the definition of B,

$$\big(a_{n+1} + s^{(i)}_{\alpha_{n+1}}\big),\, t + \big(a_{n+1} + s^{(i)}_{\alpha_{n+1}}\big) \in S^\star,$$

for all t as above and for all $i \in \{1, \dots, p\}$. Continuing in this fashion we arrive at the desired statement. □

Finally Proposition 4 follows from Theorem 11 using the following purely combinatorial fact.

Proposition 13. *Let $S \subseteq \mathbb{N}$. Assume that for every $q \in \mathbb{N}$ and IP-systems $(s_\alpha^{(1)})_{\alpha\in\mathcal{F}}, \dots, (s_\alpha^{(q)})_{\alpha\in\mathcal{F}}$ there exist sequences $a_1, a_2, \dots \in \mathbb{N}$ and $\alpha_1 < \alpha_2 < \cdots$ in $\mathcal{F}$ such that for all $k_1 < \cdots < k_l$ and $i \in \{1, \dots, q\}$,*

$$\big(a_{k_1} + s^{(i)}_{\alpha_{k_1}}\big) + \cdots + \big(a_{k_l} + s^{(i)}_{\alpha_{k_l}}\big) \in S. \tag{3}$$

(*In short, let S be a set which satisfies the conclusion of the Central Sets Theorem, i.e. the conclusion of Theorem* 11 *above.*)

Then S contains an (m, p, c)-system for all positive integers m, p, c.

The proof of Proposition 13 is sketched in [11, p. 174] and fully carried out in [16, Theorem 15.5]. Therefore we refrain from giving a full proof, but try to explain the required ideas in the case $c = 1$.

Proof. To carry out an inductive argument one proves a stronger statement already familiar from Lemma 10. Fix $S \subseteq \mathbb{N}$ and $p \in \mathbb{N}$. We show that for each $m \geqslant 0$ there exists an *IP*-system $(s_\alpha^{(0)}, \dots, s_\alpha^{(m)})_{\alpha\in\mathcal{F}}$ in $\mathbb{N}^{m+1}$ such that for all $\alpha \in \mathcal{F}$ and all integers $i_0, \dots, i_{m-1} \in [-p, p]^{m+1}$

$$s_\alpha^{(0)} \in S, \quad s_\alpha^{(1)} + i_0 s_\alpha^{(0)} \in S, \quad \dots, \quad s_\alpha^{(m)} + i_{m-1} s_\alpha^{(m-1)} + \cdots + i_0 s_\alpha^{(0)} \in S. \tag{4}$$

The case $m = 0$ of our claim asserts precisely that S contains some *IP*-system. This is quite obvious by the assumption on the set S. Applying it to the trivial system consisting only of 0's, we find that there exists a sequence $a_1, a_2, \dots \in \mathbb{N}$ such that $a_{k_1} + \cdots + a_{k_l} \in S$ for all $k_1 < \cdots < k_l \in \mathbb{N}$. Setting $s_n^{(0)} = a_n$ for $n \in \mathbb{N}$, this means that

$$\big(s_\alpha^{(0)}\big) \in S \tag{5}$$

for all $\alpha \in \mathcal{F}$.

[5] While stronger versions of the Central Sets Theorem hold true (see in particular [9]), we chose to go with this version to keep the formulation simple.

[6] To see that one can in fact require that $\alpha_{n+1} > \alpha_n$, apply Theorem 7 to the *IP*-systems generated by the numbers the sequences $(s_k^{(i)})_{k > \max \alpha_n}$.

In order to prove the first nontrivial instance $m = 1$, we apply our assumption on S to the $q = (2p + 1)$ *IP*-systems

$$\big(i_0 s_\alpha^{(0)}\big)_{\alpha \in \mathcal{F}} \quad \big(|i_0| \leqslant p\big)$$

to find a_n, $n \in \mathbb{N}$, and $\alpha_1 < \alpha_2 < \cdots$ such that for all $k_1 < \cdots < k_l$

$$\big(a_{k_1} + i_0 s_{\alpha_{k_1}}^{(0)}\big) + \cdots + \big(a_{k_l} + i_0 s_{\alpha_{k_l}}^{(0)}\big) \in S. \tag{6}$$

Set $t_n^{(1)} = a_n$ and $t_n^{(0)} = s_{\alpha_n}^{(0)}$ for $n \in \mathbb{N}$. Then it is special case of (5) that $t_\alpha^{(0)} \in S$ for $\alpha \in \mathcal{F}$ and it follows from (6) that $t_\alpha^{(1)} + i_0 t_\alpha^{(0)} \in S$ for $\alpha \in \mathcal{F}$ and all integers $i_0 \in [-p, p]$. Hence the *IP*-system $(t_\alpha^{(0)}, t_\alpha^{(1)})$ witnesses that the case $m = 1$ of (4) is valid.

To prove the case $m = 2$, apply the assumption on S to the $q = (2p + 1)^2$ *IP*-systems

$$\big(i_1 t_\alpha^{(1)} + i_0 t_\alpha^{(0)}\big)_{\alpha \in \mathcal{F}} \quad \big(|i_0|, |i_1| \leqslant p\big).$$

The induction continues in the natural way. □

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