Existence of heteroclinic orbits of the Shil’nikov type in a 3D quadratic autonomous chaotic system

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Abstract

In this paper, the existence of heteroclinic orbits of Shil’nikov type in a three-dimensional quadratic autonomous system is proved. Four heteroclinic orbits and four critical points together constitute two cycles simultaneously. The dynamical behaviors of the system are also studied.

Keywords: Heteroclinic orbit; Invariant set; Behavior of trajectory; Shil’nikov map

1. Introduction

Consider the following 3-dimensional system of quadratic autonomous differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= c_1x + d_1yz, \\
\frac{dy}{dt} &= c_2y + d_2xz, \\
\frac{dz}{dt} &= c_3z + d_3xy,
\end{align*}
\]  

(1.1)
where $c_i$ and $d_i$, $i = 1, 2, 3$, are real parameters. Recently, Liu and Chen [1,2] have shown that system (1.1) has two double-scroll chaotic attractors, which are separated only by one point (the origin), as can be seen from Figs. 1–4, when the system parameters satisfy some conditions.

In this paper, we prove that heteroclinic orbits exist in this system, with the typical parameter setting $c_1 = 0.4$, $c_2 = -12$, $c_3 = -5$, $d_1 = -1$, $d_2 = 1$ and $d_3 = 1$. More precisely, we prove that the system has four heteroclinic orbits of the Shil’nikov type, and that these four heteroclinic orbits and four critical points constitute two cycles simultaneously. The main mathematical tool used here is the undetermined coefficients method, introduced by Zhou et al. [3,4].

The paper is organized as follows. In Section 2, the construction of critical points of the system will be considered. In Section 3, the behaviors of trajectories of the system will be studied. In Section 4, some preliminary knowledge of the Shil’nikov criterion will be first reviewed; then, by using the undetermined coefficient method and the Shil’nikov criterion, the existence of heteroclinic orbits will be investigated. Conclusions are drawn in Section 5.

2. Construction of critical points

Let $d_1 = -1$, $d_2 = d_3 = 1$, $c_1 > 0$, $c_2 < 0$ and $c_3 < 0$. Then, system (1.1) takes the following form:
where $a > 0$, $b > 0$ and $c > 0$ are parameters. System (2.1) has five critical points: $A_1(0, 0, 0)$, $A_2(\sqrt{bc}, \sqrt{ac}, \sqrt{ab})$, $A_3(-\sqrt{bc}, -\sqrt{ac}, \sqrt{ab})$, $A_4(\sqrt{bc}, -\sqrt{ac}, -\sqrt{ab})$ and $A_5(-\sqrt{bc}, \sqrt{ac}, -\sqrt{ab})$, in which $A_1(0, 0, 0)$ is a saddle node.

The characteristic equation of $A_2$ is

$$
\lambda^3 + (b + c - a)\lambda^2 + 4abc = 0.
$$

Let $F(\lambda) = \lambda^3 + (b + c - a)\lambda^2 + 4abc$. Then, $F(0) = 4abc > 0$ and

$$
\frac{dF}{d\lambda} = \lambda(3\lambda + 2(b + c - a)).
$$

Let $\lambda_0 = \min_{\lambda \in R} \{\lambda \ | \ F(\lambda) = 0\}$.

1. If $b + c > a$, then since $F(0) = F(a - b - c) = 4abc > 0$, we have $\lambda_0 < -(b + c - a)$. Let $F(\lambda) = (\lambda - \lambda_0)(\lambda^2 + a_1\lambda + b_1)$. Then, $a_1 = \lambda_0 + (b + c - a) < 0$ and $b_1 = \lambda_0^2 + \lambda_0(b + c - a)$. Thus

$$
\Delta = a_1^2 - 4b_1 = -3\lambda_0^2 + (b + c - a)^2 - 2\lambda_0(b + c - a) < -2\lambda_0(\lambda_0 + b + c - a) < 0.
$$

Hence, the characteristic equation (2.2) has a negative real root, $\lambda_0$, and two imaginary roots with positive real parts:

$$
\lambda_0 + \frac{(b + c - a)}{2} + \frac{\sqrt{3\lambda_0^2 - (b + c - a)^2 + 2\lambda_0(b + c - a)}}{2}i
$$

and

$$
\lambda_0 + \frac{(b + c - a)}{2} - \frac{\sqrt{3\lambda_0^2 - (b + c - a)^2 + 2\lambda_0(b + c - a)}}{2}i.
$$

So, $A_2$ is a saddle-focus.

2. If $b + c = a$, then $\lambda_0 = -\sqrt[3]{4abc}$, $a_1 = -\sqrt[3]{4abc}$, $b_1 = \sqrt[3]{(4abc)^2}$ and $\Delta = -3\sqrt[3]{(4abc)^2} < 0$. Therefore, the characteristic equation (2.2) has a negative real root $\lambda_0 = -\sqrt[3]{4abc}$, and two imaginary roots with positive real parts:

$$
\frac{\sqrt[3]{4abc}}{2} + \frac{\sqrt[3]{3}}{2} \sqrt[3]{4abc}i \quad \text{and} \quad \frac{\sqrt[3]{4abc}}{2} - \frac{\sqrt[3]{3}}{2} \sqrt[3]{4abc}i.
$$

So, $A_2$ is a saddle-focus.

3. If $b + c < a$, then $\frac{dF}{dx} > 0$, $\lambda \in (-\infty, 0)$. So, $\lambda_0 < 0$.

(a) If $(a - b - c)^3 > 27abc$, then

$$
F\left(\frac{2}{3}(a - b - c)\right) = \frac{8}{27}(a - b - c)^3 - \frac{4}{9}(a - b - c)^3 + 4abc
$$

$$
= -\frac{4}{27}(a - b - c)^3 - 27abc < 0.
$$

Thus, the characteristic equation (2.2) has a negative real root, $\lambda_0$, and two positive real roots, $\lambda_1$ and $\lambda_2$ with $\lambda_1 \in (0, \frac{2}{3}(a - b - c))$ and $\lambda_2 > \frac{2}{3}(a - b - c)$. So, $A_2$ is a saddle-node.
(b) If \((a - b - c)^3 < 27abc\), then

\[
F\left(\frac{1}{3}(b + c - a)\right) = -\frac{1}{27}(a - b - c)^3 - \frac{1}{9}(a - b - c)^3 + 4abc
\]

\[
= -\frac{4}{27}(a - b - c)^3 - 27abc
\]

\[> 0.\]

Thus \(\lambda_0 < -\frac{1}{3}(a - b - c)\), \(a_1 = \lambda_0 + (b + c - a) < 0\) and \(b_1 = \lambda_0^2 + \lambda_0(b + c - a)\). Consequently,

\[
\Delta = a_1^2 - 4b_1
\]

\[
= -3\lambda_0^2 + (b + c - a)^2 - 2\lambda_0(b + c - a)
\]

\[< -\frac{3}{9}(a - b - c)^2 - \frac{2}{3}(a - b - c)^2 + (a - b - c)^2
\]

\[= 0.\]

Therefore, the characteristic equation (2.2) has a negative real root, \(\lambda_0 = -\frac{1}{3}(a - b - c)\), and two imaginary roots with positive real part:

\[
-\frac{\lambda_0 + (b + c - a)}{2} + \frac{\sqrt{3\lambda_0^2 - (b + c - a)^2 + 2\lambda_0(b + c - a)}}{2}i
\]

and

\[
-\frac{\lambda_0 + (b + c - a)}{2} - \frac{\sqrt{3\lambda_0^2 - (b + c - a)^2 + 2\lambda_0(b + c - a)}}{2}i.
\]

So, \(A_2\) is a saddle-focus.

(c) If \((a - b - c)^3 = 27abc\), then \(F\left(\frac{1}{3}(b + c - a)\right) = 0\). So, \(\lambda_0 = -\frac{1}{3}(a - b - c)\), \(a_1 = \lambda_0 + (b + c - a) = -\frac{4}{3}(a - b - c)\) and \(b_1 = \lambda_0^2 + \lambda_0(b + c - a) = \frac{4}{3}(a - b - c)^2\).

Thus, the characteristic equation (2.2) has a negative real root, \(\lambda_0 = -\frac{1}{3}(a - b - c)\), and two positive real roots, \(\lambda_1 = \lambda_2 = \frac{2}{3}(a - b - c)\). So, \(A_2\) is a saddle-node.

The characteristic equations of \(A_2\), \(A_3\), \(A_4\) and \(A_5\) are all identical.

3. Dynamical behaviors of the system

In this section, the dynamical behaviors of system (2.1) are studied.

Let

\[
H_1 = \{(x, y, z) \in \mathbb{R}^3 \mid |z| \geq |y|\},
\]

\[
H_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, \ z > 0, -y > 0, \ |z| \geq |y|\},
\]

\[
H_3 = \{(x, 0, z) \in \mathbb{R}^3 \mid x > 0, \ z > 0\},
\]

\[
H_4 = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, \ z \geq y \geq 0\},
\]

\[
H_5 = \{(0, y, z) \in \mathbb{R}^3 \mid z \geq y > 0\},
\]

\[
H_6 = \{(x, y, z) \in \mathbb{R}^3 \mid -x > 0, \ z \geq y > 0\},
\]

\[
H_7 = \{(x, 0, z) \in \mathbb{R}^3 \mid -x > 0, \ z > 0\}.
\]
Solving Eq. (3.4) yields
\[
\begin{aligned}
H_8 &= \{(x, y, z) \in \mathbb{R}^3 \mid -x > 0, \quad -y > 0, \quad z > 0, \quad z \geq |y|\}, \\
H_9 &= \{(0, y, z) \in \mathbb{R}^3 \mid -y > 0, \quad z \geq |y|\},
\end{aligned}
\]
when \( b > c > 0 \), it has been proved [2] that \( H_1 \) is a positive invariant set, and if \( q = (x_0, y_0, z_0) \in H_1, z_0 \neq 0 \), then \( z_0 y(t, q) > 0, \forall t \in [0, +\infty) \).

We have the following new results.

**Proposition 3.1.** Suppose that \( b > c > 0 \) and \( a > 0 \). Then, the following statements hold:

1. If \( q \in \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\} \), then \( \gamma(R, q) \subset \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\} \) and \( \lim_{t \to +\infty} \gamma(t, q) = A_1 \).
2. If \( q = (x_0, y_0, z_0) \in \mathbb{R}^3 \setminus (H_1 \cup \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}) \), then:
   (a) there exist \( T > 0 \) such that \( \gamma([T, +\infty), q) \subset H_1 \); or
   (b) if \( y_0 y(t, q) > 0, \forall t \in [0, +\infty) \), then
      \[ \lim_{t \to +\infty} y(t, q) = \lim_{t \to +\infty} z(t, q) = 0, \lim_{t \to +\infty} x(t, q) = +\infty \] or
      \[ \lim_{t \to +\infty} x(t, q) = -\infty, \text{ there exist } T_1 > 0 \text{ such that } x(t, q)y(t, q)z(t, q) < 0, \forall t \in [T_1, +\infty) \].

**Proof.** Part (1) easily follows from system (2.1).

Suppose that \( q = (x_0, y_0, z_0) \in \mathbb{R}^3 \setminus (H_1 \cup \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}) \). Then, \( \gamma(R, q) \cap \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\} = \emptyset \), followed from system (2.1). If there exists a \( T > 0 \) such that \( \gamma(T, q) \in H_1 \), then
\[
\gamma([T, +\infty), q) \subset H_1, \tag{3.1}
\]
followed from the fact that \( H_1 \) is a positive invariant set of system (2.1).

If \( \gamma([0, +\infty), q) \cap H_1 = \emptyset \), then
\[
y_0 y(t, q) > 0, \quad \forall t \in [0, +\infty). \tag{3.2}
\]

From system (2.1),
\[
\frac{d(z^2 - y^2)}{dt} = 2(-cz^2 + by^2). \tag{3.3}
\]
Let \( b = c + m \). Because \( b > c > 0 \), we have \( m > 0 \). So, Eq. (3.3) becomes
\[
\frac{d(z^2 - y^2)}{dt} = -2c(z^2 - y^2) + 2my^2. \tag{3.4}
\]
Solving Eq. (3.4) yields
\[
z^2(t, q) - y^2(t, q) = e^{-2ct} \left[ \int_0^t 2my^2(u, q)e^{2cu} du + (z_0^2 - y_0^2) \right]. \tag{3.5}
\]
\[z^2(t, q) - y^2(t, q) < 0, \forall t \in [0, +\infty) \], followed from the fact that \( \gamma([0, +\infty), q) \cap H_1 = \emptyset \).

Consequently,
\[
\int_0^{+\infty} 2my^2(u, q)e^{2cu} du \leq y_0^2 - z_0^2.
\]
Thus,
\[
\lim_{t \to +\infty} y^2(t, q) = 0, \tag{3.6}
\]
and
\[
\lim_{t \to +\infty} z^2(t, q) = 0.
\] (3.7)

\(\lim_{t \to +\infty} x(t, q) = +\infty\) or \(\lim_{t \to +\infty} x(t, q) = -\infty\), followed from the facts that \(\frac{dx}{dt} = ax - yz\), \(A_1\) is a saddle-node, and (3.6), (3.7). Without loss of generality, assume that \(\lim_{t \to +\infty} x(t, q) = +\infty\). Then, there exists a \(T_1 > 0\) such that
\[
x(t, q) > 2c, \quad \forall t \in [T_1, +\infty).
\] (3.8)

If \(y_0 < 0\), then \(\forall t \in [T_1, +\infty)\)
\[
\frac{dz(t, q)}{dt} = -cz(t, q) + x(t, q)y(t, q)
\[
\leq -cz(t, q) + 2cy(t, q)
\[
= cy(t, q) + c(y(t, q) - z(t, q))
\[
< 0.
\]
So, \(z(t, q)\) is strictly decreasing on \([T_1, +\infty)\). Hence,
\[
z(t, q) > 0, \quad \forall t \in [T_1, +\infty)
\] (3.9)
followed from (3.7). So
\[
x(t, q)y(t, q)z(t, q) < 0, \quad \forall t \in [T_1, +\infty).
\] (3.10)
If \(y_0 > 0\), then \(\forall t \in [T_1, +\infty)\)
\[
\frac{dz(t, q)}{dt} = -cz(t, q) + x(t, q)y(t, q)
\[
> -cz(t, q) + 2cy(t, q)
\[
= cy(t, q) + c(y(t, q) - z(t, q))
\[
> 0.
\]
So, \(z(t, q)\) is strictly increasing on \([T_1, +\infty)\). Hence,
\[
z(t, q) < 0, \quad \forall t \in [T_1, +\infty),
\] (3.11)
followed from (3.7). Then,
\[
x(t, q)y(t, q)z(t, q) < 0, \quad \forall t \in [T_1, +\infty).
\] (3.12)
Proposition 3.1 is thus proven. \(\square\)

Let
\[
H_{10} = \{(x, y, z) \in \mathbb{R}^3 \mid z > |y|\},
\]
\[
H_{11} = \{(x, y, z) \in \mathbb{R}^3 \mid z < -|y|\},
\]
\[
H_{12} = \{q \in \mathbb{R}^3 \setminus (H_1 \cup \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}) \mid \exists T > 0 \text{ such that } y(T, q) \in H_{10}\},
\]
\[
H_{13} = \{q \in \mathbb{R}^3 \setminus (H_1 \cup \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}) \mid \exists T > 0 \text{ such that } y(T, q) \in H_{11}\},
\]
\[
H_{14} = \{q \in \mathbb{R}^3 \setminus (H_1 \cup \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}) \mid y([0, +\infty), q) \cap H_1 = \emptyset\}.
\]
Proposition 3.2. If \( b > c > 0 \) and \( a > 0 \), then \( H_{12} \neq \emptyset \), \( H_{13} \neq \emptyset \) and \( H_{14} \neq \emptyset \).

Proof. It follows from (3.5) and system (2.1) that \( H_{12} \neq \emptyset \) and \( H_{13} \neq \emptyset \).

Let \( q_1 = (1, 1, 1) \), \( q_2 = (1, 1, -1) \), and let \( \overline{q_1 q_2} \) be the line segment connecting points \( q_1 \) and \( q_2 \). Then, it follows from (3.5) that \( \forall q \in (\overline{q_1 q_2} \setminus \{q_1, q_2\}) \). If \( \rho(q, q_1) \) is small enough, then \( q \in H_{12} \); if \( \rho(q, q_2) \) is small enough, then \( q \in H_{13} \).

Consequently, there exists a \( q \in (\overline{q_1 q_2} \setminus \{q_1, q_2\}) \) such that \( q \in H_{14} \), which follows from the continuous dependence of the solution on the initial value. Proposition 3.2 is thus proven. \( \square \)

Remark. Propositions 3.1 and 3.2 show that when \( q \in R^3 \setminus H_1 \), there exist four trajectories \( \gamma(t, q) \) of system (2.1).

Proposition 3.3. Suppose that \( b > c > 0 \) and \( a > 0 \). Then, \( \forall q \in H_2 \):

1. there exists a \( T_1 > T > 0 \) such that \( \gamma([0, T), q) \subset H_2, \gamma(T, q) \subset H_3 \) and \( \gamma((T, T_1], q) \subset H_4 \);
2. \( \gamma([0, +\infty), q) \subset H_2, \gamma(t, q) \) is strictly decreasing on \([0, +\infty), and \( \gamma(t, q) \) and \( x(t, q) \) are both strictly increasing on \([0, +\infty), \lim_{t \to +\infty} x(t, q) = +\infty \) and \( \lim_{t \to +\infty} y(t, q) = \lim_{t \to +\infty} z(t, q) = 0 \).

Proof. For all \( q = (x_0, y_0, z_0) \in H_2 \), it follows from system (2.1) that

\[
\frac{dy}{dt} \bigg|_q = -by_0 + x_0z_0 > 0.
\] (3.13)

Then, (1) there exists a \( T_1 > T > 0 \) such that \( \gamma([0, T), q) \subset H_2, \gamma(T, q) \subset H_3 \) and \( \gamma((T, T_1], q) \subset H_4 \); or (2) \( \gamma([0, +\infty), q) \subset H_2 \); or (3) there exists a \( T_2 > 0 \) such that \( \gamma(T_2, q) \in H_9 \) and \( \gamma((0, T_2], q) \subset H_2 \).

For case (3), \( \frac{dx}{dt} l_q' > 0 \), \( \forall q' \in H_9 \), which is a contradiction. For case (2), \( z(t, q) \) is strictly decreasing on \([0, +\infty), y(t, q) \) and \( x(t, q) \) are both strictly increasing on \([0, +\infty) \) in view of system (2.1). So, \( \lim_{t \to +\infty} y(t, q) = 0 \) according to (3.13), and \( \lim_{t \to +\infty} z(t, q) = 0 \) since \( \frac{dz}{dt} = -cz + xy < 0 \). Also, \( \lim_{t \to +\infty} x(t, q) = +\infty \) because \( \frac{dx}{dt} = ax - yz > 0 \). Proposition 3.3 is thus proven. \( \square \)

Proposition 3.4. Suppose that \( b > c > 0 \) and \( a > 0 \). Then, \( \forall q \in H_2 \):

1. there exists a \( T_4 > T_3 > 0 \) such that \( \gamma((-T_3, 0), q) \subset H_2, \gamma(-T_3, q) \in H_9 \) and \( \gamma([-T_4, -T_3], q) \subset H_8 \); or
2. there exists a \( T_5 > 0 \) such that \( \gamma((-T_5, 0), q) \subset H_2 \) and \( \gamma((-\infty, -T_5), q) \cap H_1 = \emptyset \).

Proof. It follows from system (2.1) that \( \forall q = (x_0, y_0, z_0) \in H_2 \),

\[
\frac{dx}{dt} \bigg|_q = ax_0 - y_0z_0 > 0.
\] (3.14)

Then, (1) there exists a \( T_4 > T_3 > 0 \) such that \( \gamma((-T_3, 0), q) \subset H_2, \gamma(-T_3, q) \in H_9 \) and \( \gamma([-T_4, -T_3], q) \subset H_8 \); or (2) there exists a \( T_5 > 0 \) such that \( \gamma((-T_5, 0), q) \subset H_2 \) and \( \gamma((-\infty, -T_5), q) \cap H_1 = \emptyset \); or (3) \( \gamma((-\infty, 0), q) \subset H_2 \).
For case (3), \( \lim_{t \to -\infty} y(t, q) = -\infty \) due to (3.13). Since
\[
\frac{dz}{dt} \bigg|_{q} = -cz_0 + x_0y_0 < 0,
\]
we have \( \lim_{t \to -\infty} z(t, q) = +\infty \). It follows from (3.14) that \( \lim_{t \to -\infty} x(t, q) = 0 \). But
\[
\lim_{t \to -\infty} \frac{dz(t, q)}{dx(t, q)} = \lim_{t \to -\infty} \frac{-cz(t, q) + x(t, q)y(t, q)}{ax(t, q) - y(t, q)z(t, q)} = 0.
\]
This is a contradiction. Proposition 3.4 is thus proven. \( \square \)

Similarly, we can prove the following results.

**Proposition 3.5.** Suppose that \( b > c > 0 \) and \( a > 0 \). Then, \( \forall q \in H_6 \):

1. there exists a \( T_7 > T_6 > 0 \) such that \( \gamma([0, T_6), q) \subset H_6 \), \( \gamma(T_6, q) \in H_7 \) and \( \gamma((T_6, T_7], q) \subset H_8 \); or
2. \( \gamma([0, +\infty), q) \subset H_6 \), and \( z(t, q) \), \( y(t, q) \) and \( x(t, q) \) are all strictly decreasing on \([0, +\infty)\), with \( \lim_{t \to +\infty} x(t, q) = -\infty \) and \( \lim_{t \to +\infty} y(t, q) = \lim_{t \to +\infty} z(t, q) = 0 \).

**Proposition 3.6.** Suppose that \( b > c > 0 \) and \( a > 0 \). Then, \( \forall q \in H_6 \):

1. there exists a \( T_9 > T_8 > 0 \) such that \( \gamma((-T_8, 0), q) \subset H_6 \), \( \gamma(-T_8, q) \in H_5 \) and \( \gamma([-T_9, -T_8), q) \subset H_4 \); or
2. there exists a \( T_{10} > 0 \) such that \( \gamma((-T_{10}, 0), q) \subset H_6 \) and \( \gamma((-\infty, -T_{10}), q) \cap H_1 = \emptyset \).

Using the same techniques, we can discuss the behaviors of trajectories of system (2.1) in \( H_{11} \).

4. Existence of the heteroclinic orbit

4.1. Preliminaries

Consider a third-order autonomous system:
\[
\frac{dx}{dt} = g(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3,
\]
where the vector field \( g(x) : \mathbb{R}^3 \to \mathbb{R}^3 \) belongs to class \( C^2 \).

Assume that \( \gamma(R, q) \) is a trajectory, and \( A' \) and \( A* \) are two different critical points of system (4.1). If \( \lim_{t \to +\infty} \gamma(t, q) = A' \) and \( \lim_{t \to -\infty} \gamma(t, q) = A* \), then \( \gamma(R, q) \) is said to be a heteroclinic orbit of system (4.1). If \( \lim_{t \to +\infty} \gamma(t, q) = \lim_{t \to -\infty} \gamma(t, q) = A' \), then \( \gamma(R, q) \) is said to be a homoclinic orbit of system (4.1).

Denote by \( \Sigma \subset \mathbb{R}^3 \) a plane that cuts transversely across the recurrent system orbital flow, which occurs locally to homoclinic or heteroclinic orbits. Define a 2D map, \( P : \Sigma \to \Sigma \), called the Poincaré map. Then, \( P \) defines a 2D discrete dynamical system:
\[
x_{n+1} = P(x_n), \quad n = 0, 1, \ldots,
\]
which characterizes system (4.1). For the case of a homoclinic orbit or a heteroclinic orbit, the corresponding Poincaré map is sometimes called a Sht’nikov map (see [5–7]).
An important concept in this study is the Smale horseshoe (see [8]). This is a set of orbits analytically detected by the Shil’nikov method for discrete dynamical systems generated by the Shil’nikov map (4.2), which guarantees that the original continuous system (4.1) is chaotic in a rigorous mathematical sense. The Shil’nikov criterion for the existence of chaos is summarized in the following theorem.

**Theorem 4.1** [7]. Suppose that two distinct critical points, denoted by $x^1_e$ and $x^2_e$, respectively, of system (4.1) are saddle foci, whose characteristic values $\delta_k$, $\sigma_k + i\omega_k$ and $\sigma_k - i\omega_k$ ($k = 1, 2$) satisfy the following Shil’nikov inequality:

$$\begin{align*}
\omega_k \neq 0, & \quad \sigma_k \delta_k < 0, \quad |\delta_k| > |\sigma_k| > 0, \quad k = 1, 2 \\
\end{align*}$$

(4.3)

under the constraint

$$\begin{align*}
\sigma_1 \sigma_2 > 0 \quad \text{or} \quad \delta_1 \delta_2 > 0. \\
(4.4)
\end{align*}$$

Suppose also that there exists a heteroclinic orbit jointing $x^1_e$ and $x^2_e$. Then,

(i) the Shil’nikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;

(ii) for any sufficiently small $C^1$-perturbation $h$ of $g$, the perturbed system

$$\frac{dx}{dt} = h(x), \quad x \in \mathbb{R}^3$$

(4.5)

has at least a finite number of Smale horseshoes in the discrete dynamics of the Shil’nikov map defined near the heteroclinic orbit;

(iii) both the original system (4.1) and the perturbed system (4.5) have horseshoe type chaos.

For convenience, a heteroclinic orbit satisfying (4.3) and (4.4) is referred to as the Shil’nikov type. Thus, the heteroclinic Shil’nikov criterion implies that if system (4.1) has one heteroclinic orbit of the Shil’nikov type, which connects two distinct saddle foci of the system, then it has both Smale horseshoes and the horseshoe type of chaos, which is rigorous in the above-defined mathematical sense.

### 4.2. Existence of the heteroclinic orbit

In this subsection, the existence of the heteroclinic orbit of system (2.1) is proved.

Recall from Section 2 that when $b > 0$, $c > 0$ and $b + c > a > 0$, system (2.1) has five critical points $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$, where $A_1$ is a saddle node, $A_2$, $A_3$, $A_4$ and $A_5$, are all saddle foci. In this subsection, we always assume that $b + c > a > 0$ and $b > c > 0$.

It first follows from the first equation of system (2.1) that

$$z = \frac{ax - \dot{x}}{y}.$$  

(4.6)

It follows easily from (2.1) that

$$\frac{d(y^2)}{dt} + 2by^2 = 2ax^2 - \frac{d(x^2)}{dt}.$$  

(4.7)
Then, solving (4.7) yields
\[
y^2(t) = e^{-2bt} \left[ \int_0^t \left( 2ax^2(s) - \frac{d(x^2(s))}{ds} \right) e^{2bs} \, ds + C_0 \right].
\]
(4.8)

Next, substituting (4.6), (4.7) and (4.8) into the third equation of system (2.1) leads to
\[
F_1(acx + (a - c)\dot{x} - \ddot{x}) - (ax - \dot{x})(ax^2 - x\dot{x} - bF_1) = xF_1^2,
\]
(4.9)

where
\[
F_1 = e^{-2bt} \left[ \int_0^t \left( 2ax^2(s) - \frac{d(x^2(s))}{ds} \right) e^{2bs} \, ds + C_0 \right],
\]
in which \(C_0\) is the integral constant.

Let
\[
x(t) = \varphi(t) = -\delta + \sum_{k=1}^{\infty} a_k e^{kat}, \quad t > 0,
\]
(4.10)

and
\[
C_0 = \frac{a\delta^2}{b} + 2 \sum_{k=2}^{\infty} \left( \frac{1}{2b + k\alpha} \sum_{i=1}^{k-1} a_{k-i}(aa_i - i\alpha a_i) \right) - 2\delta \sum_{k=1}^{\infty} \frac{2aa_k - kaak}{2b + k\alpha},
\]
(4.11)

where \(\delta = \sqrt{bc}, \alpha < 0\) is an undetermined constant, and \(a_k (k \geq 1)\) are also undetermined coefficients. Then,
\[
F_1 = \frac{a\delta^2}{b} - 2\delta \sum_{k=1}^{\infty} \frac{2aa_k - kaak}{2b + k\alpha} e^{kat} + 2 \sum_{k=2}^{\infty} \left( \frac{1}{2b + k\alpha} \sum_{i=1}^{k-1} a_{k-i}(aa_i - i\alpha a_i) \right) e^{kat},
\]

\[
acx + (a - c)\dot{x} - \ddot{x} = -ac\delta + \sum_{k=1}^{\infty} (aca_k + k\alpha aa_k - k\alpha ca_k - k^2\alpha^2 a_k) e^{kat}
\]

and
\[
(ax - \dot{x})(ax^2 - x\dot{x} - bF_1) = F_2 + F_3 + F_4,
\]

where
\[
F_2 = -a\delta^2 \sum_{k=1}^{\infty} \left( k\alpha aa_k - 2aa_k + \frac{2b(2aa_k - kaak)}{2b + k\alpha} \right) e^{kat},
\]
\[
F_1 = \left( i\alpha a_i - 2aa_i + \frac{2ba_i(2a - i\alpha)}{2b + i\alpha} \right) (aa_{k-i} - (k - i)\alpha a_{k-i}),
\]
\[
F_3 = -\delta \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \left( a^2a_i a_{k-i} - i\alpha aa_i a_{k-i} - \frac{2aba_i a_{k-i}(a - i\alpha)}{2b + k\alpha} - F_1 \right) e^{kat}
\]
and

\[ F_4 = \sum_{k=3}^{\infty} \left( \sum_{j=2}^{k-1} (aa_{k-j} - (k-j)\alpha a_{k-j}) \right) \times \sum_{i=1}^{j-1} \left( aa_{i}a_{j-i} - i\alpha a_{i}a_{j-i} - \frac{2ba_{i}a_{j-i}(a-i\alpha)}{2b+j\alpha} \right)e^{k\alpha t}. \]

Then, (4.9) changes to the following equation:

\[ a\delta^2 \sum_{k=1}^{\infty} \left( \frac{-a_k(2a-k\alpha)(2c+k\alpha)}{2b+k\alpha} + \frac{kaa_k}{b}(a-c-k\alpha) \right)e^{k\alpha t} = F_5 + F_6 + F_7 + F_8. \]

where

\[ F_2 = \frac{a_ia_{k-i}(2a-i\alpha)(2c(a+(k-i)\alpha) + \alpha(3i-2k)(a-(k-i)\alpha))}{2b+i\alpha}, \]

\[ F_5 = -\delta \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \left( F^3 - \frac{8bca_ia_{k-i}a_{k-j}(a-i\alpha)(2a-(k-j)\alpha)}{(2b+j\alpha)(2b+(k-j)\alpha)} + \frac{aa_{i}a_{j-i}(a-i\alpha)(2c+k\alpha)}{2b+k\alpha} + F^2 \right)e^{k\alpha t}, \]

\[ F_3 = \frac{a_i a_{j-i} a_{k-j}(a-i\alpha)((a-(k-j)\alpha)(2c+(2k-j)\alpha)-4ac)}{2b+j\alpha}, \]

\[ F_6 = \sum_{k=3}^{\infty} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \left( F^3 - \frac{8bca_ia_{j-i}a_{k-j}(a-i\alpha)(2a-(k-j)\alpha)}{(2b+j\alpha)(2b+(k-j)\alpha)} \right)e^{k\alpha t}, \]

\[ F_4 = \sum_{m=3}^{\infty} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \frac{2a_ia_{j-i}am_{j-1}(a-i\alpha)(2a-(m-j)\alpha)}{(2b+j\alpha)(2b+(m-j)\alpha)}, \]

\[ F_7 = -4\delta \sum_{k=4}^{\infty} \left( F^4 + \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \frac{a_ia_{j-i}(a-i\alpha)}{2b+j\alpha} \left( \sum_{i=1}^{j-1} \frac{a_ia_{k-j-i}(a-i\alpha)}{2b+(k-j)\alpha} \right) \right)e^{k\alpha t}, \]

and

\[ F_8 = 4 \delta \sum_{k=3}^{\infty} \sum_{m=4}^{\infty} \sum_{j=2}^{m-2} \frac{a_{k-m}}{2b+j\alpha} \left( \sum_{i=1}^{j-1} \frac{a_ia_{m-j-i}(a-i\alpha)}{2b+(m-j)\alpha} \right)e^{k\alpha t}. \]

Comparing coefficients of \( e^{k\alpha t} \) \( (k \geq 1) \) of the same power terms, we obtain the following:

For \( k = 1 \),

\[ \frac{\alpha}{b}(a-c-\alpha) - \frac{(2a-\alpha)(2c+\alpha)}{2b+\alpha} = 0. \]

(4.13)
Let \( F(\alpha) = \frac{a}{b} (a - c - \alpha) - \frac{(2a-\alpha)(2c+\alpha)}{2b+\alpha} \). Then, \( F(0) = -\frac{2ac}{b} < 0 \). When \( b > c \), if \( -2b < \alpha < 0 \) and \( 2b + \alpha \) is small enough, then \( F(\alpha) > 0 \). So, there exist an \( \alpha \in (-2b, 0) \) such that \( F(\alpha) = 0 \). For \( k = 2 \),

\[
a_2 = \left[ F_9 + \frac{4bca_1^2(2a - \alpha)^2}{(2b + \alpha)^2} \right] \left[ a\sqrt{bc} \left( \frac{(2a - 2\alpha)(c + \alpha)}{b + \alpha} - \frac{2\alpha}{b} (a - c - 2\alpha) \right) \right],
\]

where

\[
F_9 = \frac{a_1^2(a - \alpha)(c + \alpha)}{b + \alpha} + \frac{a_1^2(2a - \alpha)(2c(a + \alpha) - \alpha(a - \alpha))}{2b + \alpha}.
\]

For \( k = 3 \),

\[
a_3 = -[F_{10} + F_{11} + F_{12}] \left[ abc \left( \frac{(2a - 3\alpha)(2c + 3\alpha)}{2b + 3\alpha} - \frac{3\alpha}{b} (a - c - 3\alpha) \right) \right],
\]

where

\[
F_{10} = -\frac{8bca_1a_2\sqrt{bc}(2a - \alpha)(2a - \alpha)}{(b + \alpha)(2b + \alpha)} - \frac{aa_1a_2\sqrt{bc}(2c + 3\alpha)(2a - 3\alpha)}{2b + 3\alpha},
\]

\[
F_{11} = -\frac{a_1a_2\sqrt{bc}(2a - \alpha)(2c(a + 2\alpha) - 3\alpha(a - 2\alpha))}{2b + \alpha} - \frac{2ca_1a_2\sqrt{bc}(a + \alpha)(a - \alpha)}{b + \alpha}
\]

and

\[
F_{12} = \frac{4bca_1^2(a - \alpha)(2a - \alpha)}{(b + \alpha)(2b + \alpha)} + \frac{4bca_1^2(2a - \alpha)^2}{(2b + \alpha)^2} + \frac{ca_1^3(a + \alpha)(a - \alpha)}{b + \alpha}.
\]

For \( k = 4 \),

\[
a_4 = -\frac{F_{13} + F_{14} + F_{15}}{abc \left( \frac{2(a - 2\alpha)(c + 2\alpha)}{b + 2\alpha} - \frac{4\alpha}{b} (a - c - 4\alpha) \right)},
\]

where

\[
F_5 = \frac{a_1a_4-i(2a - i\alpha)(2c(a + (4 - i)\alpha) + \alpha(3i - 8)(a - (4 - i)\alpha))}{2b + i\alpha},
\]

\[
F_{13} = -\delta \sum_{i=1}^{3} \left( \frac{4bca_1a_4-i(2a - i\alpha)(2a - (4 - i)\alpha)}{(b + i\alpha)(2b + (4 - i)\alpha)} + \frac{aa_1a_4-i(a - i\alpha)(c + 2\alpha)}{b + 2\alpha} + F_5 \right),
\]

\[
F_6 = \frac{a_1a_{j-i}a_{4-j}(a - i\alpha)((a - (4 - j)\alpha)(2c + (8 - 3j)\alpha) - 4ac)}{2b + j\alpha},
\]

\[
F_{14} = -\sum_{j=2}^{3} \sum_{i=1}^{j-1} \left( \frac{F_6 - 8bca_1a_{j-i}a_{4-j}(a - i\alpha)(2a - (4 - j)\alpha)}{(2b + j\alpha)(2b + (4 - j)\alpha)} \right. \left. - \frac{4bca_1a_{j-i}a_{4-j}(2a - i\alpha)(2a - (j - i)\alpha)}{(2b + i\alpha)(2b + (j - i)\alpha)} \right),
\]

\[
F_{15} = -4\delta \left( \frac{a_1^4(a - \alpha)(2a - \alpha)}{(b + \alpha)(2b + \alpha)} + \frac{a_1^4(a - \alpha)^2}{(2b + 2\alpha)^2} \right).
\]
For \( k \geq 5 \),
\[
    a_k = -\frac{\sqrt[bc]{F_{16} - F_{17} - 4\sqrt[bc]{F_{18} + F_{19}} + 4F_{20}}}{abc\left(\frac{2a-k\alpha}{2b+k\alpha} - \frac{k\alpha}{b}(a-c-k\alpha)\right)},
\]
where
\[
    F^7 = \frac{aiak_i - 2i(a-i\alpha)(2c(a + (k - i)\alpha) + \alpha(3i - 2k)(a - (k - i)\alpha))}{2b + i\alpha},
\]
\[
    F_{16} = \sum_{i=1}^{k-1} \left(\frac{4bcia_iak_i - 2i(a-i\alpha)(2a - (k - i)\alpha)}{2b + i\alpha}(2b + (k - i)\alpha) + \frac{aiak_i(a - i\alpha)(2c + k\alpha)}{2b + k\alpha} + F^7\right),
\]
\[
    F_8 = \frac{aiak_i - (a - i\alpha)((a - (k - j)\alpha)(2c + (2k - 3j)\alpha) - 4ac)}{2b + j\alpha},
\]
\[
    F_{17} = \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \left(F^8 - \frac{8bcia_ia_j - (a - i\alpha)(2a - (k - j)\alpha)}{2b + j\alpha}(2b + (k - j)\alpha) - \frac{4bcia_ia_j - (a - i\alpha)(2a - (j - i)\alpha)}{2b + i\alpha}(2b + (j - i)\alpha)\right),
\]
\[
    F_{18} = \sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \frac{aiak_i a_m - a_k - (a - i\alpha)(2a - (m - j)\alpha)}{2b + j\alpha}(2b + (m - j)\alpha),
\]
\[
    F_{19} = \sum_{j=2}^{k-2} \left(\sum_{i=1}^{j-1} aiak_i(a - i\alpha)\right) \left(\sum_{i=1}^{k-1} aiak_i(a - i\alpha)\right)
\]
and
\[
    F_{20} = \sum_{m=4}^{k-2} \sum_{j=2}^{m-2} ak_m \left(\sum_{i=1}^{j-1} aiak_i(a - i\alpha)\right) \left(\sum_{i=1}^{m-1} aiak_i(a - i\alpha)\right).
\]
So, \( \alpha \) is completely determined by \( a, b \) and \( c \), and \( a_k (k \geq 2) \) is completely determined by \( a, b, c, \alpha \) and \( a_1 \).

Note that (4.9) has symmetry: if \( x(t) \) is a solution, then \(-x(-t)\) is also a solution. So,
\[
    x(t) = \delta - \sum_{k=1}^{\infty} a_k e^{-k\alpha t}, \quad t < 0,
\]
is also a solution of (4.9). From the continuity of the solution, we have
\[
    \sum_{k=1}^{\infty} a_k = \delta,
\]
which will determine the value of \( a_1 \). Thus, if \( a, b \) and \( c \) satisfy some conditions (for example, \( a = 0.5, b = 10 \) and \( c = 4 \)), system (2.1) has a heteroclinic orbit, which connects the critical points \( A_3 \) and \( A_2 \), and takes the following form:
\[
    \varphi(t) = \begin{cases} 
    -\delta + \sum_{k=1}^{\infty} a_k e^{k\alpha t}, & \text{for } t > 0, \\
    0, & \text{for } t = 0, \\
    \delta - \sum_{k=1}^{\infty} a_k e^{-k\alpha t}, & \text{for } t < 0.
    \end{cases}
\]
In the last part of this subsection, the uniform convergence of the series expansion (4.10) of the heteroclinic orbit is proven. For simplicity, only the case where system (2.1) has the special parameter set that generate the existence of a two double-scroll attractor is discussed. For some other parameter sets, if the heteroclinic orbit exists, the proof is similar.

When \(a = 0.5\), \(b = 10\) and \(c = 4\), \(\delta = 2\sqrt{T_0}\), the values of \(\alpha\) and \(a_1\) can be determined by (4.13)–(4.17) and (4.19). So, \(a_k (k \geq 2)\) can also be determined, with \(\sum_{k=1}^{\infty} a_k = 2\sqrt{T_0}\). Thus, \(|a_k|\) is bounded; that is, there exists a \(g > 0\) such that

\[
|a_k| \leq g, \quad k = 1, 2, \ldots.
\]

Consequently, \(\sum_{k=1}^{\infty} |a_k e^{kat}| \leq g \sum_{k=1}^{\infty} e^{kat}\) is convergent on \((0, +\infty)\). So, \(-\delta + \sum_{k=1}^{\infty} a_k e^{-kat}\) is also convergent on \((0, +\infty)\). Similarly, it can be proved that \(\delta - \sum_{k=1}^{\infty} a_k e^{-kat}\) is also convergent on \((-\infty, 0)\). Hence, there exists an orbit \(\gamma(R, q_1) \subset H_{10}\) of system (2.1) such that \(\lim_{t \to +\infty} \gamma(t, q_1) = A_3\) and \(\lim_{t \to -\infty} \gamma(t, q_1) = A_2\). Using the same technique, we can prove that there exist three orbits, \(\gamma(R, q_2) \subset H_{10}, \gamma(R, q_3) \subset H_{11}\) and \(\gamma(R, q_4) \subset H_{11}\), of system (2.1), such that \(\lim_{t \to +\infty} \gamma(t, q_2) = A_2, \lim_{t \to -\infty} \gamma(t, q_2) = A_3, \lim_{t \to +\infty} \gamma(t, q_3) = A_4, \lim_{t \to -\infty} \gamma(t, q_3) = A_5, \lim_{t \to +\infty} \gamma(t, q_4) = A_5, \lim_{t \to -\infty} \gamma(t, q_4) = A_4\) and \(\gamma(R, q_1), A_2, \gamma(R, q_2)\) and \(A_3\) constitute a cycle, \(\gamma(R, q_3), A_4, \gamma(R, q_4)\) and \(A_5\) also constitute another cycle.

5. Conclusions

When the parameters of system (2.1) satisfy some conditions, the existence of heteroclinic orbits has been proven by using the undetermined coefficient method. It has been shown that with the typical parameter values of \(a = 0.5\), \(b = 10\) and \(c = 4\), system (2.1) has four heteroclinic orbits of the Shil’nikov type: \(\gamma(R, q_1) \subset H_{10}, \gamma(R, q_2) \subset H_{10}, \gamma(R, q_3) \subset H_{11}\) and \(\gamma(R, q_4) \subset H_{11}\), such that \(\gamma(R, q_1), A_2, \gamma(R, q_2)\) and \(A_3\) constitute a cycle, while \(\gamma(R, q_3), A_4, \gamma(R, q_4)\) and \(A_5\) also constitute another cycle, as discussed in detail above.

As can be seen, system (1.1) has a simple algebraic form but possesses very rich and complex dynamics. This system deserves further studied with more subtle analysis in the future.

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References