JOURNAL OF COMBINATORIAL THEORY **1**, 157-173 (1966)

The Power Group Enumeration Theorem*

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ABSTRACT

The number of orbits of a permutation group was determined in a fundamental result of Burnside. This was extended in a classical paper by Pólya to a solution of the problem of enumerating the equivalence classes of functions with a given weight from a set X into a set Y, subject to the action of the permutation group A acting on X. A generalization by de Bruijn solved the counting problem when two permutation groups are involved, A acting on X and B on Y. Thus the Pólya formula is the special case of the de Bruijn result in which B is the identity group. The Power Group Enumeration Theorem achieves the same result using only one permutation group: the power group, B^A, acting on the set Y^X of functions. de Bruijn's method was used to count self-complementary graphs by R. C. Read and finite automata by M. Harrison. These results as well as the number of self-converse directed graphs and others are easily obtained by the proper use of the Power Group Enumeration Theorem.

INTRODUCTION

Many problems in combinatorial analysis are formulated in such a way that, in order to find the number of objects having a specified property, it is necessary to count the number of equivalence classes of objects with regard to some permutation group. Often these equivalence classes consist of functions f from one set X into another set Y such that two permutation groups are involved: A acting on X and B acting on Y.

* Work supported by the U. S. Air Force Office of Scientific Research under grant AF-AFOSR-754-65.

Then two functions f and g from X into Y are regarded as equivalent with respect to this pair of groups if there exist permutations α in A and β in B such that for all x in X, $\beta f(\alpha x) = g(x)$.

N. G. de Bruijn [1] has provided a method for solving this type of problem by generalizing the classical enumeration theorem of Pólya [10].

Our main result can be interpreted as an independent derivation of de Bruijn's result as it has actually been used in solving enumeration problems; see [1, 9, 11, 12]. The implementation of the conjecture that such a derivation could be accomplished on the basis of Pólya's original theorem had to wait on the discovery of the appropriate combination of the two groups A and B into a single permutation group.

The power group of A and B is denoted by B^A and acts on Y^X, the functions from X into Y. When used in conjunction with its cycle index, the power group enables us to derive directly and naturally a formula for the number of classes of functions.

The matter can be further complicated when integral weights are assigned to the functions in such a way that each function in a given class has the same weight. Then the problem is to find the number of classes of functions which have a given weight. Both de Bruijn's formulation and our application of the power group to Pólya's method can readily be modified to handle this situation.

1. The Power Group

Let A be a permutation group of order $m = |A|$ acting on the set $X = \{x_1, \ldots, x_d\}$ of objects. Let B be another permutation group of order $n = |B|$ acting on the set $Y = \{y_1, \ldots, y_e\}$. Thus the degrees of A and B are d and e, respectively. The set of functions from X into Y is denoted by Y^X; we assume there are at least two such functions.

In [6] the permutation group $[B]^A$, called the *exponentiation*[1] of A and B, acting on Y^X is defined in the following way:

For each permutation α in A and each sequence $(\beta_1, \beta_2, \ldots, \beta_d)$ of d permutations from B there is a unique permutation $(\alpha; \beta_1, \beta_2, \ldots, \beta_d)$ in $[B]^A$ so that for every f in Y^X and every x_i in X,

$$(\alpha; \beta_1, \beta_2, \ldots, \beta_d) f(x_i) = \beta_i f(\alpha x_i). \tag{1}$$

[1] Previously denoted by B^A in [6].

It is clear that the order of $[B]^A$ is mn^d and that the degree is e^d.

Now we introduce another permutation group B^A, called the *power group* of A and B, which also acts on Y^X. For each pair of permutations α in A and β in B, there is a unique permutation $(\alpha;\beta)$ in B^A so that for every f in Y^X and x in X,

$$(\alpha;\beta)f(x) = \beta f(\alpha x). \tag{2}$$

Obviously the order of B^A is mn and the degree is e^d.

Note that, when all β_i are the same β, then the permutation

$$(\alpha;\beta_1,\beta_2,\ldots,\beta_d)$$

in $[B]^A$ may be written $(\alpha;\beta)$. Thus the power group may be regarded as the "diagonal" of the exponentiation group.

The power group B^A induces an equivalence relation on the set of functions Y^X. Any two functions f and g in Y^X are said to be "equivalent" if there is a permutation γ in B^A such that $\gamma(f) = g$. Thus f and g are equivalent whenever there are permutations α in A and β in B so that, for each x in X, $\beta f(\alpha x) = g(x)$. Note that this equation is precisely the same as in de Bruijn's equivalence of functions.

Following the terminology of Carmichael [3], we say that the permutation groups A and B are isomorphic if they are isomorphic as abstract groups, and we write $A \cong B$. We say that A and B are identical if they are not only abstractly isomorphic, but the correspondence also preserves the sets being permuted; this is written $A \equiv B$. We will require another operation on A and B. The permutation group $A + B$, called[1] the *sum* of A and B, acts on the disjoint union $X \cup Y$. For every pair of permutations α in A and β in B, there is a permutation $\alpha\beta$ in $A + B$ such that for any z in $X \cup Y$:

$$\alpha\beta(z) = \begin{cases} \alpha(z), & z \in X \\ \beta(z), & z \in Y \end{cases} \tag{3}$$

Thus the order of $A + B$ is mn and the degree is $d + e$.

THEOREM 1. *The permutation groups $A + B$ and B^A are isomorphic.*

[1] This operation is also variously known as direct product, direct sum, product, union and justaposition.

PROOF. Define the map ψ from B^A onto $A + B$ by $\psi(\alpha;\beta) = \alpha^{-1}\beta$. For two permutations $(\alpha_1;\beta_1)$ and $(\alpha_2;\beta_2)$ in B^A, we know from the definition that their product in B^A is $(\alpha_1;\beta_1)(\alpha_2;\beta_2) = (\alpha_2\alpha_1;\beta_1\beta_2)$. Therefore we have:

$$\begin{aligned}\psi(\alpha_1;\beta_1)\psi(\alpha_2;\beta_2) &= \alpha_1^{-1}\beta_1\,\alpha_2^{-1}\beta_2\\ &= \alpha_1^{-1}a_2^{-1}\,\beta_1\beta_2\\ &= (\alpha_2\alpha_1)^{-1}\,\beta_1\beta_2\\ &= \psi(\alpha_2\alpha_1;\beta_1\beta_2)\\ &= \psi((\alpha_1;\beta_1)\,(\alpha_2;\beta_2)).\end{aligned}$$

Thus ψ is a homomorphism from B^A onto $A + B$. It is also clearly $1 - 1$ and hence $B^A \cong A + B$.

Now that we have established that the power group B^A is isomorphic with the sum of A and B, it follows that $B^A \cong A^B$.

The cycle index of a permutation group is defined next; it tells the cyclic structure of its permutations. Let A be any permutation group of order m and degree d. Let α be any permutation in A, written in the usual way as the product of disjoint cycles. Let $j_k(\alpha)$ be the number of cycles of length k in the disjoint cycle decomposition of α. In addition let $a_1, \ldots, a_d$ be variables. Then by the *cycle index* $Z(A)$ of A is meant the following formal sum, which is a polynomial in the variables a_k:

$$Z(A) = \frac{1}{m} \sum_{\alpha\in A} \prod_{k=1}^{d} a_k^{j_k(\alpha)}. \tag{4}$$

Now let $h(x)$ be any polynomial. By $Z(A, h(x))$ we mean the polynomial obtained from $Z(A)$ by replacing each a_k by $h(x^k)$. For polynomials $h_1(x), \ldots, h_d(x)$ it is convenient to denote by $Z(A; h_1(x), \ldots, h_d(x))$ the polynomial obtained from $Z(A)$ on substituting $h_k(x)$ for each a_k.

Pólya has shown the importance of the cycle index of a permutation group in enumeration problems. Therefore it is natural to try to find a convenient formula for the cycle index of any new permutation group. We will find that the cycle index of the power group plays an important role in enumeration theory.

To find $Z(B^A)$ we proceed as follows. By definition of the cycle index we have:

$$Z(A) = \frac{1}{m} \sum_{\alpha \in A} \prod_{k=1}^{d} a_k^{j_k(\alpha)}, \tag{5}$$

$$Z(B) = \frac{1}{n} \sum_{\beta \in B} \prod_{k=1}^{e} b_k^{j_k(\beta)}, \tag{6}$$

$$Z(B^A) = \frac{1}{mn} \sum_{\gamma \in B^A} \prod_{k=1}^{e^d} c_k^{j_k(\gamma)}. \tag{7}$$

For each $\gamma = (\alpha; \beta)$ in B^A, the formulae for $j_k(\gamma)$ in terms of the $j_k(\alpha)$ and $j_k(\beta)$ are:

$$j_1(\alpha; \beta) = \prod_{k=1}^{d} \left(\sum_{s|k} s j_s(\beta) \right)^{j_k(\alpha)}, \tag{8}$$

where

$$\left(\sum_{s|k} s j_s(\beta) \right)^{j_k(\alpha)} = 1 \text{ whenever } j_k(\alpha) = 0;$$

and for $k > 1$ we use Möbius inversion to obtain

$$j_k(\alpha; \beta) = \frac{1}{k} \sum_{s|k} \mu\left(\frac{k}{s}\right) j_1(\alpha^s; \beta^s). \tag{9}$$

Note that, if the contribution of the permutation α to $Z(A)$ is

$$\prod_{s=1}^{d} a_s^{j_s(\alpha)},$$

then that of α^k is

$$\prod_{s=1}^{d} a_{s/(s,k)}^{(s,k) j_s(\alpha)}, \tag{10}$$

where (s, k) is the *g c d* of s and k.

We now justify Eqs. (8) and (9). Consider any permutation $\gamma = (\alpha; \beta)$ in the power group B^A. First we show how to obtain formula (8) for $j_1(\alpha; \beta)$. Let z_k be any cycle of length k in the disjoint cycle decomposition of α. Let S be the set of elements of X which are permuted by z_k. Define $m_k(\beta)$ as the number of functions in Y^S which are fixed by the permutation $(z_k; \beta)$. Then obviously

$$j_1(\alpha; \beta) = \prod_{k=1}^{d} (m_k(\beta))^{j_k(\alpha)}, \tag{11}$$

where $(m_k(\beta))^{j_k(\alpha)} = 1$ whenever $j_k(\alpha) = 0$.

Also it is clear that a function f in Y^S which is fixed by $(z_k;\beta)$ must assume all of its functional values in the set of elements permuted by a single cycle z_s of length s in the disjoint cycle decomposition of β. Suppose for such a function that $f(x) = y$ for some x in z_k and y in z_s. Then $(z_k;\beta)^s f(x) = \beta^s f(z_k{}^s x) = f(z_k{}^s x)$. But since f is fixed by $(z_k;\beta)$, it is also fixed by $(z_k;\beta)^s$. Therefore $(z_k;\beta)^s f(x) = f(x)$, and so $y = f(z_k{}^s x)$. Similarly $y = f(z_k^{is}x)$ for all i. Now $f(z_k x) = \beta^{-1}y$ and so the equations involving x and y also hold for $z_k x$ and $\beta^{-1}y$. Continuing in this manner it is easy to see that $s|k$. Since y in z_s was arbitrary, there are exactly s such functions for each cycle z_s. Thus $m_k(\beta) = \Sigma_{s|k}\, s j_s(\beta)$ and on substitution in (11) we obtain (8).

Now we observe that, for any integer k, a function f in Y^X is fixed by $(\alpha^k;\beta^k)$ if and only if f is an element of a cycle of length s in the disjoint cycle decomposition of $(\alpha;\beta)$ for some $s|k$. Adding these terms, we have

$$j_1(\alpha^k;\beta^k) = \sum_{s|k} s j_s(\alpha;\beta).$$

Solving for $j_k(\alpha;\beta)$ we may obtain a recursion formula for $j_k(\alpha;\beta)$ in terms of $j_1(\alpha^k;\beta^k)$ and $j_s(\alpha;\beta)$ with $s < k$. Applying the familiar Möbius inversion formula to the preceding equation, we immediately obtain (9). The use of (8), (9), and (10) gives $j_k(\alpha;\beta)$ in terms of the numbers $j_k(\alpha)$ and $j_k(\beta)$.

Let A be any permutation group acting on the set X. A subset of X containing exactly k elements is called a *k-subset*. Two k-subsets of X, say S_1 and S_2 are called A-equivalent if there is a permutation α in A such that $\alpha(S_1) = S_2$. By applying Pólya's theorem, the observation made in [8] is obtained:

THEOREM [8]. *The number of A-inequivalent k-subsets of X is the coefficient of x^k in $Z(A, 1 + x)$.*

This theorem may be considered to be a generalization of Burnside's theorem [2] on the number of transitivity systems of a permutation group, which is the above statement with $k = 1$.

Applying the theorem to the power group we have:

COROLLARY 1. *The number of equivalence classes of functions in Y^X determined by B^A is the coefficient of x in $Z(B^A, 1 + x)$.*

Since we know the cycle index of the power group we can obtain an explicit formula for the coefficient of x. Using the definition of the cycle index the notation from above, we have

$$Z(B^A, 1 + x) = \frac{1}{mn} \sum_{(\alpha;\beta)\in B^A} \prod_{k=1}^{e^d} (1 + x^k)^{j_k(\alpha;\beta)}. \tag{12}$$

For each permutation $(\alpha;\beta)$ in B^A, the coefficient of x in the product above comes from $(1 + x)^{j_1(\alpha;\beta)} = 1 + j_1(\alpha;\beta)x + \cdots$. Hence the contribution of each $(\alpha;\beta)$ to the coefficient of x in $Z(B^A, 1 + x)$ is simply $j_1(\alpha;\beta)$. Therefore the number of equivalence classes of functions in Y^X determined by B^A is

$$N = \frac{1}{mn} \sum_{(\alpha;\beta)\in B^A} j_1(\alpha;\beta). \tag{13}$$

This is, of course, just the application of Burnside's theorem to B^A. By formula (8) and substitution we obtain:

$$N = \frac{1}{mn} \sum_{(\alpha;\beta)\in B^A} \prod_{k=1}^{d} \left(\sum_{s|k} s j_s(\beta)\right)^{j_k(\alpha)}. \tag{14}$$

Now using the definition of the cycle index we restate this useful result in the following form.

THEOREM 2. (*Power Group Enumeration Theorem, Constant Form*). *The number of equivalence classes of functions in Y^X determined by the power group B^A is*

$$N = \frac{1}{|B|} \sum_{\beta\in B} Z(A; m_1(\beta), m_2(\beta), \ldots, m_d(\beta)) \tag{15}$$

where

$$m_k(\beta) = \sum_{s|k} s j_s(\beta).$$

Thus we have seen that counting the number of inequivalent functions from X into Y which are determined by two permutation groups A and B is a very straightforward and natural procedure, when use is made of the power group B^A. The problem is essentially one of finding the functions fixed by the permutations in B^A. It is easy to see that formula (15) is implicitly contained in de Bruijn [1, p. 173].

2. Enumeration of Weighted Functions

There are many enumeration problems in which integral weights are assigned to the functions so that each function in an equivalence class determined by the power group has the same weight. Frequently one wants to know the number of different (inequivalent) functions having a given weight. The answer can be expressed as a series.

$$N(x) = N_0 + N_1 x + N_2 x^2 + \cdots$$

in which N_i is the number of different functions of weight i.

Now consider the power group B^A acting on Y^X. Let w be a function from Y into the set $\{0, 1, 2, \ldots\}$ of non-negative integers. As usual w is called a *weight function.* For each f in Y^X, we define the weight of f, denoted $W(f)$, by

$$W(f) = \sum_{x \in X} w(f(x)). \tag{16}$$

It is important that all functions in an equivalence class determined by the power group have the same weight. A criterion for this condition to be satisfied is provided by the next theorem.

Theorem 3. *A necessary and sufficient condition for two equivalent functions to have the same weight is the existence of a partition* Y_0, $Y_1, \ldots, Y_r$ *of* Y *with* $|Y_i| = n_i$ *such that*

(a) *B is a subgroup of $S_{n_0} + \cdots + S_{n_r}$ where S_{n_i}, the symmetric group of degree n_i, acts on Y_i,*

(b) *$w(y) = i$ for each y in Y_i and $i = 0, 1, \ldots, r$.*

Proof. For the sufficiency, suppose f and g are equivalent functions. Then for some $(\alpha; \beta)$ in B^A, $(\alpha; \beta) f = g$ so that $\beta f(\alpha x) = g(x)$ for all x in X. Note that $w(\beta f(\alpha x)) = w(f(\alpha x))$. Therefore, by (16),

$$\begin{aligned} W(g) &= \sum_{x \in X} w(\beta f(\alpha x)) \\ &= \sum_{x \in X} w(f(\alpha x)) \\ &= \sum_{x \in X} w(f(x)) \\ &= W(f). \end{aligned}$$

For the necessity, let $Y_i = w^{-1}(i)$ for all $i = 0$ to r. Then condition (b) follows. Let $n_i = |Y_i|$ for each i. Let $C \equiv S_{n_0} + \cdots + S_{n_r}$ with S_{n_i} acting on Y_i. Then C is the largest permutation group acting on Y such that for any y in Y and any β in C, $w(y) = w(\beta y)$. If B is not a subgroup of C, then there is an element y in Y and a permutation β in B such that $w(y) \neq w(\beta y)$. Let f and g in Y^X be defined by $f(x) = y$ and $g(x) = \beta(y)$ for all x in X. Then obviously f and g are equivalent but $W(f) \neq W(g)$.

From now on we assume that equivalent functions have the same weight. Therefore we have B acting on Y in accordance with the conditions of Theorem 3. To obtain the generating function $N(x)$ which enumerates functions in Y^X according to weight, we now need only modify the variables $m_k(\beta)$ which appear in the statement of Theorem 2.

Let $\gamma = (\alpha; \beta)$ be any permutation in the power group B^A. Suppose z_k is any cycle of length k in the disjoint cycle decomposition of α. Again let S be the set of elements of X which are permuted by z_k. For each $i = 0, 1, 2, \ldots$, define $m_i^k(\beta)$ as the number of functions f in Y^S which are fixed by the permutation $(z_k; \beta)$ and which have

$$\sum_{x \in S} w(f(x)) = i.$$

For convenience let $m_k(\beta, x) = \sum_i m_i^k(\beta) x_j$. Note that $m_k(\beta, 1) = m_k(\beta)$ as defined in Theorem 2. Then the desired generating function $N(x)$ is given by

$$N(x) = \frac{1}{|B|} \sum_{\beta \in B} Z(A; m_1(\beta, x), m_2(\beta, x), \ldots, m_d(\beta, x)). \tag{17}$$

Once again the formula for $Z(B^A)$ provides the means for finding an explicit expression for $m_k(\beta, x)$. From condition (a) in Theorem 3 we know that each permutation β in B can be written as $\beta = \beta_0 \beta_1 \cdots \beta_r$ with β_i acting on Y_i for each i.

Using the same approach as made in the derivation of formula (8) for $j_1(\alpha; \beta)$ we have:

$$\begin{aligned}
m_1(\beta, x) &= j_1(\beta_0) + j_1(\beta_1)x + j_1(\beta_2)x^2 + \cdots, \\
m_2(\beta, x) &= [j_1(\beta_0) + 2j_2(\beta_0)] + [j_1(\beta_1) + 2j_2(\beta_1)]x^2 \\
&\quad + [j_1(\beta_2) + 2j_2(\beta_2)]x^4 + \cdots, \\
m_3(\beta, x) &= [j_1(\beta_0) + 3j_3(\beta_0)] + [j_1(\beta_1) + 3j_3(\beta_1)]x^3 \\
&\quad + [j_1(\beta_2) + 3j_3(\beta_2)]x^6 + \cdots,
\end{aligned}$$

and in general

$$m_k(\beta, x) = \left[\sum_{s|k} sj_s(\beta_0)\right] + \left[\sum_{s|k} sj_s(\beta_1)\right] x^k + \left[\sum_{s|k} sj_s(\beta_2)\right] x^{2k} + \cdots.$$

Therefore

$$m_k(\beta, x) = \sum_{t=0}^{r} \left(\sum_{s|k} sj_s(\beta_t)\right) x^{kt}. \tag{18}$$

Note that the coefficient of x^{kt} depends only on k and the partition of β_t. Collecting these observations, we have the following result.

THEOREM 4. (*Power Group Enumeration Theorem, Polynomial Form*). *The polynomial which enumerates according to weight the equivalence classes of functions in Y^X determined by the power group B^A is*

$$N(x) = \frac{1}{|B|} \sum_{\beta \in B} Z(A; m_1(\beta, x), m_2(\beta, x), \ldots, m_d(\beta, x)),$$

where

$$m_k(\beta, x) = \sum_{t=0}^{r} \left(\sum_{s|k} sj_s(\beta_t)\right) x^{kt}.$$

Pólya's theorem is immediately obtained from Theorem 4 when B is identity group on Y. Then the above formula becomes

$$N(x) = Z(A; m_1(x), m_1(x^2), \ldots, m_1(x^d))$$

where $m_1(x)$ is obtained from the formula above for $m_1(\beta, x)$ by taking β as the identity permutation acting on Y. In Pólya's terminology, $m_1(x)$ is known as the "figure counting series."

The difference between this expression for $N(x)$ in Theorem 4 and that given by de Bruijn's theorem is now shown to be merely formal. It is just a matter of a change in notation and some routine algebraic manipulations.

First assume that $B \equiv B_0 + B_1 + \cdots + B_r$ with B_i acting on Y_i and $w(y) = i$ for all y in Y_i. In most applications B has this form. Let

$$Z(B_i) = \frac{1}{|B_i|} \sum_{\beta \in B_i} \prod_k b_k^{j_k(\beta)}.$$

It is very easy to see, as observed by Pólya [10], that

$$Z(B) = Z(B_0)\, Z(B_1) \cdots Z(B_r). \tag{19}$$

For any non-negative integer j and any variable b, we have

$$b^j = \left[\left(\frac{\partial}{\partial z}\right)^j \exp(bz)\right]_{z=0} . \tag{20}$$

Here it is understood that if $j = 0$ then $b^j = 1$ even when $b = 0$.

By

$$Z\left(A; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_d}\right)$$

is meant the formal differential operator obtained from $Z(A)$ on replacing each a_k in formula (4) by $\partial/\partial z_k$. Then, for any function $f(z_1, \ldots, z_d)$,

$$Z\left(A; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_d}\right) f(z_1, \ldots, z_d)$$

has the usual meaning. Using this notation and that of (20), it is obvious that

$$Z(A; m_1(\beta, x), m_2(\beta, x), \ldots, m_d(\beta, x)) = \Bigg[Z\left(A; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_d}\right) \exp\{m_1(\beta, x)z_1 + m_2(\beta, x)z_2 + \cdots + m_d(\beta, x)z_d\}\Bigg]_{z_i=0} . \tag{21}$$

Using the derivative notation, the formula for $m_k(\beta, x)$ in Theorem 4, and (19) for the cycle index of B, one can obtain with routine algebraic operations the formula for $N(x)$ given by de Bruijn's theorem:

THEOREM (DE BRUIJN). *The polynomial which enumerates according to weight the number of equivalence classes of functions from X into Y with respect to the permutation groups A and B is given by*

$$N(x) = \left[Z\left(A; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_d}\right) \prod_{t=0}^{r} Z(B_t; b_{t,1}, b_{t,2}, b_{t,3}, \ldots)\right]_{z_i=0} \tag{22}$$

where

$$b_{t,s} = \exp\left\{s \sum_{k=1}^{[d/s]} z_{sk}(x^t)^{sk}\right\}.$$

Here are the details for obtaining (22) from Theorem 4:

$$N(x) = \frac{1}{|B|} \sum_{\beta \in B} Z(A; m_1(\beta, x), m_2(\beta, x), \ldots, m_d(\beta, x))$$

$$= \frac{1}{|B|} \Big[\sum_{\beta \in B} Z\left(A; \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right)$$

$$\exp\{m_1(\beta, x)z_1 + \cdots + m_d(\beta, x)z_d\} \Big]_{z_i=0}$$

$$= \Big[Z\left(A; \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}\right) \frac{1}{|B|} \sum_{\beta \in B}$$

$$\exp\{m_1(\beta, x)z_1 + \cdots + m_d(\beta, x)z_d\} \Big]_{z_i=0}.$$

But

$$\frac{1}{|B|} \sum_{\beta \in B} \exp\{m_1(\beta, x)z_1 + \cdots + m_d(\beta, x)z_d\}$$

$$= \frac{1}{|B|} \sum_{\beta \in B} \Big[\exp\left\{ \sum_{k=1}^{d} \left(\sum_{s|k} s j_s(\beta_0) \right) z_k \right\} \exp\left\{ \sum_{k=1}^{d} \left(\sum_{s|k} s j_s(\beta_1) \right) x^k z_k \right\}$$

$$\exp\left\{ \sum_{k=1}^{d} \left(\sum_{s|k} s j_s(\beta_2) \right) x^{2k} z_k \right\} \cdots \Big]$$

$$= \frac{1}{|B|} \sum_{\beta \in B} [\exp\{j_1(\beta_0) \textstyle\sum z_k + 2j_2(\beta_0) \sum z_{2k} + 3j_3(\beta_0) \sum z_{3k} + \cdots\}$$

$$\exp\{j_1(\beta_1) \textstyle\sum x^k z_k + 2j_2(\beta_1) \sum x^{2k} z_{2k} + 3j_3(\beta_1) \sum x^{3k} z_{3k} + \cdots\}$$

$$\exp\{j_1(\beta_2) \textstyle\sum (x^2)^k z_k + 2j_2(\beta_2) \sum (x^2)^{2k} z_{2k} + 3j_3(\beta_2) \sum (x^2)^{3k} z_{3k} + \cdots\} \cdots]$$

$$= \frac{1}{|B|} \sum_{\beta \in B} [(\exp \textstyle\sum z_k)^{j_1(\beta_0)} (\exp 2 \sum z_{2k})^{j_2(\beta_0)} (\exp 3 \sum z_{3k})^{j_3(\beta_0)} \cdots$$

$$(\exp \textstyle\sum x^k z_k)^{j_1(\beta_1)} (\exp 2 \sum x^{2k} z_{2k})^{j_2(\beta_1)} (\exp 3 \sum x^{3k} z_{3k})^{j_3(\beta_1)} \cdots$$

$$(\exp \textstyle\sum (x^2)^k z_k)^{j_1(\beta_2)} (\exp 2 \sum (x^2)^{2k} z_{2k})^{j_2(\beta_2)}$$

$$(\exp 3 \textstyle\sum (x^2)^{3k} z_{3k})^{j_3(\beta_2)} \cdots]$$

$$= \prod_{t=0}^{r} Z(B_t; \exp \textstyle\sum (x^t)^k z_k, \exp 2 \sum (x^t)^{2k} z_{2k}, \exp 3 \sum (x^t)^{3k} z_{3k}, \cdots).$$

Hence it is convenient to introduce the otherwise contrived-looking notation

$$b_{t,s} = \exp\left\{ s \sum_{k=1}^{[d/s]} (x^t)^{ks} z_{ks} \right\}$$

and one then obtains the formula (22) of de Bruijn.

Formula (22) and Theorem 4 are thus seen to yield precisely the same results.

3. Graphical Applications

Two significant applications of de Bruijn's theorem to graphical enumeration problems were made by R. C. Read. In [11] Read has shown how to compute the number of self-complementary graphs with p points. In [12] Read gives the generating function for the enumeration of graphs with p points whose lines are colored with n colors such that the colors are interchangeable.

Before discussing these problems in detail, we give some of the usual notation and definitions associated with the enumeration of graphs. As above, the symmetric group acting on the set $\{1, \ldots, p\}$ of p objects is denoted by S_p. The "pair group" $S_p^{(2)}$ of the symmetric group is the permutation group which acts on all subsets of $\{1, \ldots, p\}$ containing exactly two objects, as induced by the elements of S_p. A formula for $Z(S_p^{(2)})$ is given in [4]. The permutation group consisting of a single permutation, the identity element, which acts on k objects, is denoted by E_k.

Self-Complementary Graphs

The complement of a graph G is denoted by $\bar{G}$. A graph G is self-complementary if $G \cong \bar{G}$, i.e., G and $\bar{G}$ are isomorphic. Read showed how to compute the number of self-complementary graphs with p points by first applying de Bruijn's theorem to count the number of graphs with p points up to complementation. In counting graphs up to complementation, two graphs G_1 and G_2 are regarded as equivalent whenever $G_1 \cong G_2$ or $G_1 \cong \bar{G}_2$.

Let the pair group $S_p^{(2)}$ act on $X^{(2)}$, the collection of all 2-subsets of $X = \{1, \ldots, p)$. Let S_2, the symmetric group of degree 2, act on $Y = \{0, 1\}$. It is easy to show that the number of graphs with p points up to complementation is the same the number of equivalence classes of functions in $Y^{X^{(2)}}$ determined by the power group $S_2^{S_p^{(2)}}$. Briefly, a function f in $Y^{X^{(2)}}$ represents a graph G whose points are $\{1, \ldots, p\}$ and in which point i as adjacent with point j whenever $f(\{i, j\}) = 1$. Thus the elements 0, 1 of Y are used to indicate the absence or presence of a line. If the permutation $(\alpha; (01))$ of $S_2^{S_p^{(2)}}$ sends f to g, then g is a function which represents the complement of G. Applying Corollary 1 we see that the number of graphs with p points up to complementation is the coefficient of x in $Z(S_2^{S_p^{(2)}}, 1 + x)$.

Now we apply Theorem 2, the Power Group Enumeration formula, with $A \equiv S_p^{(2)}$ and $B \equiv S_2$. The two permutations of S_2 are (0) (1) and (01). Therefore for $\beta = (0)\,(1)$, we have $j_1(\beta) = 2$, $j_s(\beta) = 0$ for $s > 1$, and so for this β, $m_k(\beta) = 2$ for all $k = 1$ to $\binom{p}{2}$. For $\beta = (01)$, we have $j_2(\beta) = 1$, $j_s(\beta) = 0$ for $s \neq 2$, and so for this β, $m_k(\beta) = 2$ when $2 \mid k$ and $m_k(\beta) = 0$ when $2 \nmid k$. Thus the number of graphs up to complementation is

$$\tfrac{1}{2}\,\{Z(S_p^{(2)}; 2, 2, \ldots) + Z(S_p^{(2)}; 0, 2, 0, 2, \ldots)\}. \tag{23}$$

Read observed that the number of graphs on p points up to complementation could be obtained by taking $A \equiv S_p^{(2)}$ and $B \equiv S_2$ and by applying that special case of de Bruijn's theorem which gives the formula (15) of Theorem 2; de Bruijn had already applied this special case of his theorem with A arbitrary and $B \equiv S_2$ to obtain similarly

$$\tfrac{1}{2}\,\{Z(A; 2, 2, 2, 2, \ldots) + Z(A; 0, 2, 0, 2, \ldots)\}.$$

Now to count the number of self-complementary graphs with p points, Read's procedure has four steps.

STEP 1. Count the number c_p of graphs with p points up to complementation.

STEP 2. Observe that the number of graphs with p points counted twice if self-complementary and once if not is $2c_p$.

STEP 3. From Pólya's result as given in [4] we know that the total number of graphs on p points is $Z(S_p^{(2)};\ 2, 2, \ldots)$.

STEP. 4. Substracting the result of Step 3 from that of Step 2, we find that the number of self-complementary graphs is $Z(S_p^{(2)}; 0, 2, 0, 2, \ldots)$.

Explicit formulas for the number of self-complementary graphs and digraphs with a given number p of points appear in Read [11], who thus solved one of the problems in [7].

Graphs with n-Colored Lines

Theorem 4, the polynomial form of the Power Group Enumeration Theorem, provides a simple approach to the problem of deriving the counting polynomial, $C_p{}^n(x)$, which enumerates graphs with p points whose lines are colored with n colors such that the colors are interchangeable. This result was first given by Read [12] using de Bruijn's formula, (22).

Let E_1 be the permutation group of order one and degree one acting on the set $\{0\}$. As before let $A \equiv S_p^{(2)}$ act on $X^{(2)}$. Let $B \equiv E_1 + S_n$ act on $Y = \{0\} \cup \{1, \ldots, n\}$. Define the weight function $w\colon Y \to \{0, 1\}$ by $w(y) = 0$ if $y = 0$ and $w(y) = 1$ if $y \neq 0$. Then each function f in $Y^{X^{(2)}}$ represents a graph with exactly $f^{-1}(i)$ lines of color i for $i = 1, \ldots n$. Further, the weight $W(f)$ of the function f is the number of lines in the graph represented by f. It follows that $C_p{}^n(x)$ is just the polynomial, $N(x)$, given by Theorem 4 applied to the power group $(E_1 + S_n)^{S_p^{(2)}}$.

To illustrate, we show the details for $n = 3$: In accordance with notation above, we have $B_0 \equiv E_1$ acting on $Y_1 = \{0\}$ and $B_1 \equiv S_3$ acting on $Y_2 = \{1, 2, 3\}$. For each β in $E_1 + S_3$ we must compute $m_k(\beta, x)$ as given by formula (18). Recall that for $t = 0, 1$ the coefficient of x^{tk} in $m_k(\beta, x)$ is $\sum_{s|k} sj_s(\beta_t)$. There are three cases.

CASE 1. $\beta = (0)\ (1)\ (2)\ (3)$.

We have $\beta_0 = (0)$ and $\beta_1 = (1)\,(2)\,(3)$. So $j_1(\beta_0) = 1$ and $j_1(\beta_1) = 3$. Therefore $m_k(\beta, x) = 1 + 3x^k$ for all k.

CASE 2. $\beta = (0)\,(12)\,(3)$.

Since $\beta_0 = (0)$ and $\beta_1 = (12)\,(3)$, we have $j_1(\beta_0) = 1$, $j_1(\beta_1) = 1$ and $j_2(\beta_1) = 1$. Therefore $\sum_{s|k} sj_s(\beta_1)$ is $j_1(\beta_1)$ or $(j_1(\beta_1) + 2j_2(\beta_1))$ according as k is odd or even. Hence $m_k(\beta, x)$ is $1 + x^k$ or $1 + 3x^k$ according as k is odd or even.

CASE 3. $\beta = (0)\ (123)$.

Since $\beta_0 = (0)$ and $\beta_1 = (123)$, we have $j_1(\beta_0) = 1$ and $j_3(\beta_1) = 1$. Therefore $\sum_{s|k} sj_s(\beta_1)$ is 0 or $3j_3(\beta_1)$ according as $3 \nmid k$ or $3|k$. Hence $m_k(\beta, x)$ is 1 or $1 + 3x^k$ according as $3 \nmid k$ or $3 \mid k$.

If $\beta = (0)\,(13)\,(2)$ or $\beta = (0)\,(23)\,(1)$, then, of course, $m_k(\beta, x)$ is given by Case 2. If $\beta = (0)\,(132)$, then $m_k(\beta, x)$ is as in Case 3. From Theorem 4 we now have

$$\begin{aligned} C_p{}^3(x) = \tfrac{1}{6}\{&Z(S_p^{(\ )};\ 1+3x,\ 1+3x^2, \ldots) \\ &+ 3Z(S_p^{(2)};\ 1+x,\ 1+3x^2,\ 1+x^3, \ldots) \\ &+ 2Z(S_p^{(2)};\ 1,1,\ 1+3x^3, \ldots)\}. \end{aligned}$$

For $p = 3$:

$$C_3{}^3(x) = 1 + x + 2x^2 + 3x^3.$$

Read derived a formula for $C_p{}^n(x)$ by applying de Bruijn's theorem with $A \equiv S_p^{(2)}$, $B_0 \equiv E_1$, and $B_1 \equiv S_n$. We illustrate his procedure for $n = 3$. First we have:

$$\begin{aligned} &Z(E_1; b_{0,1})Z(S_3; b_{1,1}, b_{1,2}, b_{1,3}) \\ &\quad = \tfrac{1}{6}\, b_{0,1}[b_{1,1}^3 + 3b_{1,1}b_{1,2} + 2b_{1,3}] \\ &\quad = \tfrac{1}{6}\, [b_{0,1}b_{1,1}^3 + 3b_{0,1}b_{1,1}b_{1,2} + 2b_{0,1}b_{1,3}]. \end{aligned}$$

Now

$$\begin{aligned} b_{0,1} &= \exp\Big\{\sum_k z_k\Big\} \\ b_{1,1} &= \exp\Big\{\sum_k z_k x^k\Big\} \\ b_{1,2} &= \exp\Big\{2\sum_k z_{2k} x^{2k}\Big\} \\ b_{1,3} &= \exp\Big\{3\sum_k z_{3k} x^{3k}\Big\}. \end{aligned}$$

Therefore

$$\begin{aligned} &Z(E_1; b_{0,1})Z(S_3; b_{1,1}, b_{1,2}, b_{1,3}) \\ &\quad = \tfrac{1}{6}\, [\exp\{\textstyle\sum z_k + 3\sum z_k x^k\} \\ &\qquad + 3\exp\{[\textstyle\sum z_k + \sum z_k x^k + 2\sum z_{2k} x^{2k}\} \\ &\qquad + 2\exp\{\textstyle\sum z_k + 3\sum z_{3k} x^{3k}\}] \\ &\quad = \tfrac{1}{6}\, [\exp\{(1+3x)z_1 + (1+3x^2)z_2 + (1+3x^3)z_3 + \cdots\} \\ &\qquad + 3\exp\{(1+x)z_1 + (1+3x^2)z_2 + (1+x^3)z_3 + \cdots\} \\ &\qquad + 2\exp\{z_1 + z_2 + (1+3x^3)\,z_3 + \cdots\}. \end{aligned}$$

Now using (21) we can obtain the same result as before for $C_p{}^3(x)$.

In conclusion, we note that with the aid of the Power Group Enumeration Theorem one can always determine the number of equivalence classes of functions (with or without weights) from a set X into a set Y acted on by permutation groups A and B, respectively, provided one only knows the cycle indexes of A and B.

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