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## Algorithms for Del Pezzo surfaces of degree 5 (construction, parametrization)

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### ABSTRACT

It is well known that every Del Pezzo surface of degree 5 defined over a field  $k$  is parametrizable over  $k$ . In this paper, we give an algorithm for parametrizing, as well as algorithms for constructing examples in every isomorphism class and for deciding equivalence.

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## 0. Introduction

It is well-known that every Del Pezzo surface of degree 5 over a field  $k$  (not necessarily algebraically closed) has a proper parametrization with coefficients in  $k$  (see Swinnerton-Dyer, 1972; Sheperd-Barron, 1992). In this paper, we give a simple algorithm for constructing such a parametrization. The construction is a slight modification of Sheperd-Barrons construction (the modification is important for getting a good performance). In addition, we give a simple algorithm for construction of examples of Del Pezzo surfaces of degree 5. The algorithm takes as input a quintic squarefree univariate polynomial over  $k$ . The construction is complete in the sense that any Del Pezzo surface of degree 5 is obtained up to a projective coordinate change with matrix entries in the ground field.

The classification of quintic Del Pezzo surfaces up to projective isomorphisms defined over  $k$  is also well-known (see Skorobogatov, 2001, Lemma 3.1.7). Any such surface has 10 lines defined over the algebraic closure  $\bar{k}$ , and the incidence graph of the configuration of these lines has symmetry group

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$S_5$ . There is a Galois action on the set of lines with image in  $S_5$ . Two quintic Del Pezzo surfaces are isomorphic if the two Galois actions on the line configurations are  $S_5$ -conjugate. The Galois action may also be described as the Galois action on the roots of the quintic input polynomial in our construction algorithm. We use this explicit description to give an algorithm for deciding isomorphy of two given quintic Del Pezzo surfaces.

The parametrization algorithm was motivated by the more general problem of computing rational parametrizations over  $k$  for arbitrary rational surfaces (if possible). By Enriques–Manin reduction (see Iskovskih (1980) for the theory and (Schicho, 1998) for an algorithm), one can birationally reduce either to a conical fibration or to a Del Pezzo surface. For conic fibration and for Del Pezzo surfaces of degree 6, 8, or 9, parametrization algorithms are available (Schicho, 2000; de Graaf et al., 2006; Harrison and Schicho, 2006; de Graaf et al., 2009) and implemented in Bosma et al. (1997). For all other degrees except 5 the surfaces are either not properly parametrizable with coefficients in the ground field, or they can be reduced to degree 6, 8, or 9 (see also Polo-Blanco and Top, 2009; Swinnerton-Dyer, 1970; Iskovskih, 1980).

### 1. Theory

Throughout, we assume that  $k$  is a perfect field and  $\bar{k}$  is an algebraic closure of  $k$ . We are primarily interested in the case  $k = \mathbb{Q}$ , but all constructions represented work also in the general case. Projective algebraic varieties are defined as subsets of projective space over  $\bar{k}$ , but we assume that all varieties are defined by equations with coefficients in  $k$ , and consequently all constructions will be possible within  $k$ .

Abstractly, a Del Pezzo surface is defined as a complete nonsingular surface such that the anticanonical divisor  $-K$  is ample. The integer  $d := K^2$  is called the degree of the Del Pezzo surface. If  $d \geq 3$ , then  $-K$  is very ample and defines a natural embedding in  $\mathbb{P}^d$  as a surface of degree  $d$ . Conversely, it is known that every non-singular surface of degree  $d$  in  $\mathbb{P}^d$  is either Del Pezzo or ruled or the projection of the Veronese surface of degree 4 in  $\mathbb{P}^5$  to  $\mathbb{P}^4$ .

The general theory of Del Pezzo surfaces which is relevant to this paper may be summarized by the following well-known theorems.

**Theorem 1.** *If  $F \subseteq \mathbb{P}^d$  is a Del Pezzo surface of degree  $d$ , then  $3 \leq d \leq 9$ .*

*If  $d \neq 8$ , then  $F$  is  $\bar{k}$ -isomorphic to the blowing up of  $\mathbb{P}^2$  at  $9 - d$  points in general position, i.e., no 3 points lie on a line and no 6 points on a conic.*

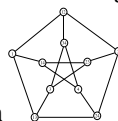
*If  $d = 8$ , then  $S$  is  $\bar{k}$ -isomorphic to either the blowup of  $\mathbb{P}^2$  at a point or to  $\mathbb{P}^1 \times \mathbb{P}^1$ .*

**Proof.** See Manin (1974, Chap. IV Theorem 24.3 and Theorem 24.4).  $\square$

**Theorem 2.** *Let  $F \subseteq \mathbb{P}^d$  be a Del Pezzo surface of degree  $d$ ,  $d \neq 8$ . Then there exists a birational parametrization  $\phi : \mathbb{P}^2 \rightarrow F$ ,  $p \mapsto (P_0(p) : \dots : P_d(p))$ , such that  $(P_0, \dots, P_d)$  are a basis for the vectorspace of all cubics vanishing at  $9 - d$  points in general position. This parametrization has coefficients in  $\bar{k}$ .*

**Proof.** See Manin (1974, Chap. IV, proof of Theorem 24.5) or Schicho (2005, Corollary 2).  $\square$

Let  $F \subseteq \mathbb{P}^5$  be a Del Pezzo surface of degree 5. By Theorem 2, the surface can be parametrized by cubics vanishing at four base points  $q_1, \dots, q_4$ . The surface  $F$  contains 4 exceptional lines  $E_1, \dots, E_4$ , which are the preimages of  $q_1, \dots, q_4$  under the inverse of the parametrization  $\phi : \mathbb{P}^2 \rightarrow F$ . For any two distinct base points  $q_i, q_j$ , the image of the line connecting  $q_i$  and  $q_j$  under  $\phi$  is also a line, which we denote by  $L_{ij}$ . These lines are all lines on  $F$ . Drawing a vertex for every line and an edge between



vertices such that the corresponding lines meet, we obtain the Petersen graph

**Theorem 3.** *The ideal  $I$  of a Del Pezzo surface  $F \subseteq \mathbb{P}^5$  of degree 5 is generated by 5 quadrics  $P_1, \dots, P_5 \in R := k[x_0, \dots, x_5]$ . The syzygy module*

$$\text{Syz}_{(P_1, \dots, P_5)} = \{(A_1, \dots, A_5) \in R^5 \mid A_1P_1 + \dots + A_5P_5 = 0\}$$

*is generated by 5 vectors  $V_1, \dots, V_5 \in R^5$  of linear forms. The second syzygy module*

$$\text{Syz}_{(V_1, \dots, V_5)} = \{(B_1, \dots, B_5) \in R^5 \mid B_1V_1 + \dots + B_5V_5 = 0\}$$

*is generated by a single vector  $W = (W_1, \dots, W_5) \in R^5$  of quadrics. The entries  $W_i, i = 1, \dots, 5$  generate the ideal  $I$ . With a suitable choice of basis  $(V_1, \dots, V_5)$  of the linear part of  $\text{Syz}_{(P_1, \dots, P_5)}$ , one can achieve that  $W_i = P_i$  for  $i = 1, \dots, 5$ . Then the matrix  $M := (V_{ij})_{i,j}$  of linear forms is skew symmetric, and the ideal is generated by the 5 first Pfaffian minors of  $M$ .*

**Proof.** See Theorem 2.2 in Brodmann and Schenzel (2006).  $\square$

**Example 1.** Consider  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (0 : 1 : 0)$ ,  $p_3 = (0 : 0 : 1)$  and  $p_4 = (1 : 1 : 1)$ . Let  $V$  be the space of cubics in  $\mathbb{Q}[t_0, t_1, t_2]$  vanishing at  $p_1, p_2, p_3$  and  $p_4$ . The following is a basis for  $V$ :

$$P_1 = t_1^2t_2 - t_1t_2t_0, \quad P_2 = t_1^2t_0 - t_1t_2t_0, \quad P_3 = t_0^2t_1 - t_1t_2t_0, \\ P_4 = t_0^2t_2 - t_1t_2t_0, \quad P_5 = t_2^2t_0 - t_1t_2t_0, \quad P_6 = t_2^2t_1 - t_1t_2t_0.$$

The map  $p \rightarrow (P_0(p) : P_1(p) : P_2(p) : P_3(p) : P_4(p) : P_5(p))$  defines a parametrization of a Del Pezzo surface  $S$  of degree 5 which is isomorphic to the blow up of  $\mathbb{P}^2$  at the points  $p_1, p_2, p_3$  and  $p_4$ . With the matrix

$$M = \begin{pmatrix} 0 & -x_0 + x_1 & -x_1 & x_1 - x_5 & x_5 \\ x_0 - x_1 & 0 & -x_2 & -x_5 & x_5 \\ x_1 & x_2 & 0 & x_2 & -x_3 \\ -x_1 + x_5 & x_5 & -x_2 & 0 & x_4 \\ -x_5 & -x_5 & x_3 & -x_4 & 0 \end{pmatrix},$$

the surface  $S$  is generated by the 5 Pfaffians of the  $4 \times 4$  diagonally symmetric submatrices, and the syzygies are generated by the columns (or rows) of  $M$ .

## 2. Construction

We are going to describe a simple algorithm that takes as input a quintic normed and squarefree univariate polynomial  $Q \in k[x]$ , called the *seed*, and produces a quintic Del Pezzo surface. Roughly speaking, the seed is used to construct 5 points in  $\mathbb{P}^2$  in general position (defined over  $\bar{k}$ ), and the surface is the image of  $\mathbb{P}^2$  under the map defined by quintics vanishing doubly at the 5 points.

**First step.** Using the seed, we construct a zero-dimensional subvariety  $B \subset \mathbb{P}^2$  defined over  $k$ , which is the set of 5 points  $\{q_1, \dots, q_5\}$  defined over  $\bar{k}$  in general position. This means, no three points are collinear. Geometrically,  $B$  is the image of the zeros of  $Q$  under the map  $\psi : \bar{k} \rightarrow \mathbb{P}^2, x \mapsto (x^2 : x : 1)$ . Algebraically, we get the ideal of  $B$  as the quotient of the ideal  $I_1$  generated by the homogenization  $Q_h(t, u)$  and  $t^2 - su$  by the ideal  $I_2$  generated by  $u$ .

**Lemma 4.** *Any set  $B \subset \mathbb{P}^2$  of 5 points defined over  $k$  in general position can be obtained by the above construction, up to a projective coordinate change of  $\mathbb{P}^2$  with matrix entries in  $k$ .*

**Proof.** For any such set  $B$ , there is a unique conic  $C_1$  that contains  $B$ . Since  $B$  is defined over  $k$ ,  $C_1$  is also defined over  $k$ . The conic  $C_1$  contains also a  $k$ -rational point: it can be constructed by choosing generically a cubic  $C_2$  through  $B$  and computing the 6-th intersection point of  $C_1$  and  $C_2$ . Any two conics in  $\mathbb{P}^2$  defined over  $k$  with  $k$ -rational points are projectively equivalent by a projective map with coefficients in  $k$ , so we may transform the conic  $C_1$  to the conic  $C$  with equation  $t^2 - su$ . We may assume that the point at infinity  $(1 : 0 : 0)$  is not in  $B$ ; this can be achieved by applying an element of the transitive group of projective automorphisms of  $\bar{k}$ . Then  $\psi^{-1}(B)$  is a subset of five points in  $\bar{k}$  which is invariant under the Galois group  $\text{Gal}(\bar{k}/k)$ , and so there is a quintic normed squarefree polynomial  $Q \in k[x]$  that has these points as zeros.  $\square$

**Second step.** We construct a rational map  $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$ ; we will show that this map is birational, and that the image  $F$  is a Del Pezzo surface of degree 5. The map is defined by the vector space of quintics vanishing with order at least 2 at the points in  $B$ . In terms of computations, we compute the square of the ideal of  $B$ , saturate it with respect to the irrelevant ideal, and take a basis of the homogeneous part of degree 5.

**Theorem 5.** *Let  $q_1, \dots, q_5$  be five points on  $\mathbb{P}^2$  in general position and let  $\phi$  be the rational map from  $\mathbb{P}^2$  into a projective space defined by the space of quintics passing through  $q_1, \dots, q_5$  with order at least 2. Then  $\phi$  maps  $\mathbb{P}^2$  into a del Pezzo surface  $F$  of degree 5 in  $\mathbb{P}^5$ .*

*The 10 lines on  $F$  are the strict transforms of the lines connecting two of the points  $q_i$  and  $q_j$ , for  $1 \leq i, j \leq 5$ .*

**Proof.** It is well known that a linear system of quintics with 5 double points has dimension 5 and has no unassigned base points (see Nagata, 1960). It follows that  $\phi$  is regular and maps into  $\mathbb{P}^5$ .

Because the self-intersection number of the linear system is 5, and this is equal to the degree of the image times the mapping degree, we also conclude that  $\phi$  is birational and the image is a surface  $F$  of degree 5.

To show that the image is nonsingular, we start by resolving the 5 base points. Let  $\mathbb{Y}$  be the blowup of these 5 points. Then  $\phi$  induces a regular map from  $\mathbb{Y}$  to the image surface  $F$ , associated to a divisor class  $H$ . We can write  $H = 5L - 2E$ , where  $L$  is the pullback of the class of lines and  $E = E_1 + \dots + E_5$  is the class of the exceptional divisor of the blowup consisting of 5 components. Note that  $L^2 = 1$ ,  $LE = 0$  and  $E^2 = -5$ . The only curve on  $\mathbb{Y}$  which is contracted to a point is the proper transform of the conic  $C$ , with class  $2L - E$ , because this is the only curve with intersection number 0 with  $H$ . This is a  $-1$ -curve, hence the image is nonsingular by Theorem 21.5 of Manin (1974).

Finally, the lines on  $F$  are precisely the curves  $D$  on  $\mathbb{Y}$  such that  $D \cdot H = 1$ . This is only possible if  $D$  is the proper transform of a line through two points in  $B$ , with class  $L - E_i - E_j$ ,  $1 \leq i < j \leq 5$ . These are only finitely many. Then the image is not a ruled surface, therefore it is a Del Pezzo surface.  $\square$

**Example 2.** Consider  $Q = x^5 - 1$  and let  $p_1 = (\zeta_5^2 : \zeta_5 : 1)$ ,  $p_2 = (\zeta_5^4 : \zeta_5^2 : 1)$ ,  $p_3 = (\zeta_5 : \zeta_5^3 : 1)$ ,  $p_4 = (\zeta_5^3 : \zeta_5^4 : 1)$  and  $p_5 = (1 : 1 : 1)$  where  $\zeta_5$  is a primitive 5-root of unity. Let  $V$  be the space of quintics in  $\mathbb{Q}[t_0, t_1, t_2]$  vanishing with multiplicity 2 at  $p_1, p_2, p_3, p_4$  and  $p_5$ . A base of  $V$  is the following

$$\begin{aligned} P_1 &= t_0^5 - 5t_0t_1^3t_2 + 3t_1^5 + t_2^5, & P_2 &= t_0^4t_2 - t_0^3t_1^2 - t_0t_1t_2^3 + t_1^3t_2^2, \\ P_3 &= t_0^3t_1t_2 - t_0^2t_1^3 - t_0t_2^4 + t_1^2t_2^3, & P_4 &= t_0^3t_2^2 - 2t_0^2t_1^2t_2 + t_0t_1^4, \\ P_5 &= t_0^2t_1t_2^2 - 2t_0t_1^3t_2 + t_1^5, & P_6 &= t_0^2t_2^3 - 2t_0t_1^2t_2^2 + t_1^4t_2. \end{aligned}$$

The map  $p \rightarrow (P_1(p) : P_2(p) : P_3(p) : P_4(p) : P_5(p) : P_6(p))$  defines a parametrization of a Del Pezzo surface  $S$  of degree 5. The Del Pezzo surface  $S$  is defined by the five quadrics in  $\mathbb{Q}[x_0, x_1, x_2, x_3, x_4, x_5]$ :

$$\begin{aligned} x_1x_5 - x_2x_4 + x_3^2, \\ x_1x_4 - x_2x_3 - x_5^2, \\ x_0x_5 + x_1x_3 - x_2^2 - 3x_4x_5, \\ x_0x_4 - x_1x_2 - x_3x_5 - 3x_4^2, \\ x_0x_3 - x_1^2 + x_2x_5 - 3x_3x_4. \end{aligned}$$

In Section 3.1, we will show that every quintic Del Pezzo surface is isomorphic to a surface constructed as above with some suitable seed. However, it is not true that different choices of the seed lead to non-isomorphic surfaces. For instance, if  $Q$  is a product of linear factors, then the surface  $F$  is  $k$ -isomorphic to Example 1, for all choices of the linear factors. See Section 4 for more details.

### 3. Parametrization

In this section, we describe an algorithm that takes as an input the defining equations of a quintic Del Pezzo surface  $F$  – these are 5 quadratic equations in 6 variables with coefficients in  $k$  – and produces a proper rational parametrization, i.e., a birational map from  $\mathbb{P}^2$  to  $F$  defined over  $k$ .

**First step.** We construct a point on  $F$  with coordinates in  $k$ . The first substep of this step is to construct a point on  $F$  with coordinates in  $k$  or in a quadratic extension of  $k$ .

Let  $R := k[x_0, \dots, x_5]$  be the graded coordinate ring of  $\mathbb{P}^5$ . The ideal of  $F$  is generated by 5 quadratic equations, by **Theorem 3**. By the same theorem, we also know the degree and the number of syzygies of the ideal.

**Lemma 6.** *Let  $P_1, \dots, P_5 \in R$  be generators of the ideal of  $F$ . Let  $V_1, \dots, V_5$  be generators of the syzygy module  $\text{Syz}_{(P_1, \dots, P_5)}$ . Let  $V_5 = (V_{5,1}, \dots, V_{5,5})$  be the last syzygy vector. Then the zero set  $L$  of  $(V_{1,5}, \dots, V_{5,5})$  is a linear subspace of  $\mathbb{P}^5$  of dimension 1 or 2. Moreover, the intersection of  $L$  and the zero set of  $P_5$  is contained in  $F$ .*

**Proof.** Let  $I_1 := \langle V_{1,5}, \dots, V_{5,5} \rangle_R$ ,  $I_2 := \langle P_1, \dots, P_4 \rangle_R$ , and  $I_3 := \langle P_5 \rangle_R$ . Then we have  $I_1 = I_2 : I_3$  and it follows  $I_2 \subseteq I_1$  and  $I_2 + I_3 \subseteq I_1 + I_3$ , hence the common zero set of  $I_1$  and  $I_3$  is contained in the zero set of  $I_2 + I_3$ , which is  $F$ . This shows the second assertion.

If the  $L$  were a linear space of dimension 3 or higher, then the zero set of  $I_1 + I_3$  would be a quadratic surface or even higher dimensional. But this zero set must be contained in  $F$ , so this is not possible. This shows  $\dim(L) \leq 2$ .

By a suitable choice of the generators  $V_1, \dots, V_5$ , we can achieve that the matrix  $(V_{ij})_{i,j}$  is skew symmetric. Then  $L$  is the common zero set of  $V_{5,1}, \dots, V_{5,4}$ . This shows  $\dim(L) \geq 1$ .  $\square$

The first substep of constructing a point on  $F$  with coordinates in  $k$  or in a quadratic extension is now easy to describe: the intersection of  $L$  and the zero set of  $F_5$  is defined over  $k$ , and it is either a line, or a point, or a set of two points, or a plane conic (maybe reducible). In the first case, we take any point on the line. In the second case, we take the point. In the third case, we take one of the two points. In the fourth case, we choose any line in  $L$  and intersect it with  $F_5$ , and we have one of the first three cases. The generic case is that  $L$  is a line intersecting  $F$  in two points, which are conjugate in a quadratic field extension.

**Remark 7.** A closer analysis shows that in the case where  $L$  is a line, the intersection is either a point or two points, and if  $L$  is a plane, then the projection  $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$  with center  $L$  restricts to a birational map  $F \dashrightarrow \mathbb{P}^2$ , which gives a shortcut to the parametrization problem. We will omit the proof because it is not necessary for the correctness proof of the algorithm.

The second substep is necessary if the constructed point  $q \in F$  has coordinates in a quadratic extension  $K$ . Then there is also a conjugate point  $\bar{q} \in F$ .

**Lemma 8.** *Let  $q \in F$  be a point with coordinates in a quadratic field extension  $K$  of  $k$  (but not in  $k$ ) and  $\bar{q}$  its conjugate. Assume that the line  $q\bar{q}$  is not contained in  $F$ . Then there exist at most 5 tangent lines  $T \in T_q F$  such that either  $T$  and its conjugate  $\bar{T} \in T_{\bar{q}} F$  are coplanar, or the 3-plane  $N$  generated by  $T$  and  $\bar{T}$  intersects  $F$  in a set of positive dimension.*

*In all remaining cases, either  $T$  is contained in  $F$ , and  $N \cap F = T \cup \bar{T} \cup \{p\}$  where  $p$  is a point in  $F$  defined over  $k$ , or  $N$  intersects  $F$  in precisely 3 points, namely  $q$  and  $\bar{q}$  (both with multiplicity 2), and a third point  $p$  which is defined over  $k$ .*

**Proof.** We will show that  $T_q F \cap T_{\bar{q}} F$  is either a point or the empty set. Let us first prove that the intersection  $T_q F \cap T_{\bar{q}} F$  does not contain the line  $q\bar{q}$ . If  $q\bar{q} \subseteq T_q F \cap T_{\bar{q}} F$ , then  $q\bar{q}$  is tangent to  $F$  at the points  $q$  and  $\bar{q}$ . We claim that, in such a case, the line  $q\bar{q}$  would be in  $F$  which would be in contradiction with the hypothesis. Indeed, if  $q\bar{q}$  is not in  $F$  one can take  $x$  and  $y$  two generic points in  $F$ , then the 3-plane generated by  $q, \bar{q}, x$  and  $y$  intersects  $F$  in  $q$  and  $\bar{q}$  with multiplicity 2 and in  $x$  and  $y$  with multiplicity 1 which contradicts the degree of  $F$  being 5. Therefore, the line  $q\bar{q}$  is not contained in  $T_q F \cap T_{\bar{q}} F$ .

Suppose now that  $T_q F \cap T_{\bar{q}} F = m \neq q\bar{q}$  where  $m$  is a line in  $\mathbb{P}^5$  defined over  $k$ . Then neither  $q$  nor  $\bar{q}$  is in  $m$  (since if  $q \in m$ , so is  $\bar{q}$  and therefore  $m = q\bar{q}$ ). Now we define  $\pi : \mathbb{P}^5 \rightarrow \mathbb{P}^3$  to be the projection from the points  $q$  and  $\bar{q}$ . Then  $\pi|_F$  is the blow up of  $F$  at the points  $q$  and  $\bar{q}$  and the Zariski closure of  $\pi(F)$  is a cubic surface  $S$  in  $\mathbb{P}^3$  (note that  $S$  is a cubic surface if and only if  $q\bar{q}$  is not contained in  $F$ ). If  $T_q F \cap T_{\bar{q}} F = m$  and  $q$  and  $\bar{q}$  are not in the line  $m$ , then both tangent planes  $T_q(F)$  and  $T_{\bar{q}}(F)$  map to the same line  $\ell$  in  $S$ . This line would then be the exceptional divisor of two different points (or tangent directions) in  $\mathbb{P}^2$ , among the six that are blown up, which is not possible.

Now we are in the situation that  $T_qF \cap T_{\bar{q}}F$  is either a point or the empty set. In this situation the tangent planes  $T_q(F)$  and  $T_{\bar{q}}(F)$  map to (conjugate) lines  $\ell$  and  $\bar{\ell}$  in  $S$ . If  $T_q(F)$  and  $T_{\bar{q}}(F)$  intersect at a point, then the lines  $\ell$  and  $\bar{\ell}$  intersect at a point  $s \in S$ . In such a case there is another line in  $S$  intersecting both  $\ell$  and  $\bar{\ell}$ . If  $T_q(F)$  and  $T_{\bar{q}}(F)$  do not intersect, then  $\ell$  and  $\bar{\ell}$  are disjoint and therefore there exist at most 5 lines in  $S$  intersecting both  $\ell$  and  $\bar{\ell}$  (see Polo-Blanco and Top, 2009, Lemma 1.2).

Via the projection  $\pi$  the 3-plane  $N$  goes to a line  $n$  in  $\mathbb{P}^3$  defined over  $k$  that intersects both  $\ell$  and  $\bar{\ell}$  at two conjugate points  $t = \pi(T)$  and  $\bar{t} = \pi(\bar{T})$  respectively. When the line  $n$  is a line in  $S$ , we are in the previous cases described above. Otherwise  $n \cap S = \{t, \bar{t}, x\}$  where  $x$  is a point in  $S$  defined over  $k$ . In this case  $N \cap F = T \cup \bar{T} \cup \{p\}$  if  $T \subseteq F$  or  $N \cap F = \{q, \bar{q}, p\}$  otherwise, with  $p$  being a point defined over  $k$  and  $\pi(p) = x$ .  $\square$

The lemma gives a construction for a  $k$ -rational point on  $F$  from a point  $q \in F$  defined over a quadratic extension  $K$ : if the line  $q\bar{q}$  accidentally is in  $F$ , then projection from this line is a birational map to a nonsingular quadric surface in  $\mathbb{P}^3$ , and there are known constructions for points on a nonsingular quadric in case such points exist. If not (this is the general case), let  $r \neq q$  be a point with coordinates in  $K$  in  $T_qF$ , but not in  $(T_qF \cap T_{\bar{q}}F)$ . Let  $\bar{r}$  be its conjugate. Then intersect the 3-plane  $N$  generated by  $q, \bar{q}, r, \bar{r}$  with  $F$ . If we are not unlucky, then the intersection contains a single point defined over  $k$ . We can be unlucky at most 5 times.

**Remark 9.** In Sheperd-Barron (1992), Sheperd-Barron suggests to choose the line  $T$  (or the line  $qr$  in the above construction) parametrically, and compute the moving intersection point in terms of this parameter. The parameter can be chosen as an element in  $K$ , or equivalently two elements in  $k$ . We implemented this method also in Magma; a comparison with random examples with 3–4 decimal digits in the coefficients of the generators of the ideal of  $F$  does not show a significant difference in the runtimes. In numbers, the runtimes varied between 14 and 52 CPU seconds for our method, and between 32 and 88 CPU seconds for Sheperd-Barron’s method (we used a P8600 dual core processor with 2.4 GHz). However, Sheperd-Barron’s method leads to a parametrization of algebraic degree 10, which is twice as large as the degree of the parametrization computed by the method below.

In Hassett (2009, Exercise 3.1.4), Hassett gives another method for constructing a  $k$ -rational point: the zero set of three generic quadrics in the ideal of  $F$  decomposes into  $F$  and a cubic rational scroll. The scroll has a unique  $-1$ -curve, which intersects  $F$  in a single  $k$ -rational point. However, generic choices are not free, they increase the coefficients, and it seems not so easy to analyze the non-generic cases for this method.

Still another method explained in Swinnerton-Dyer (1972) is due to Enriques: by generic projection, one obtains an image  $F'$  of  $F$  in  $\mathbb{P}^3$  with a rational quintic double curve. It can be parametrized over  $k$ ; compute two points on it. The line through the two points generically intersects  $F'$  in a single smooth point which lifts back to a  $k$ -rational point on  $F$ . This method is computationally very expensive, as we observed by testing it with a few simple examples.

**Second step.** Given a point  $p \in F$  defined over  $k$ , we consider the projection map  $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$  with projection center equal to the tangent plane  $T_pF$ . The restriction of  $\pi$  to  $F$  will be birational.

**Lemma 10.** *With the notation as above, the restriction  $\pi|_F : F \dashrightarrow \mathbb{P}^2$  is birational.*

*If  $p$  does not lie on one of the 10 lines, then the parametrization has algebraic degree 5. If it lies on at least one line, then the algebraic degree is smaller than 5.*

**Proof.** Two generic hyperplanes through  $T_pF$  intersect  $F$  in 5 points, including the intersection at  $p$ . The intersection multiplicity at  $p$  is equal to 4, because hyperplane sections through  $T_pF$  have a double point at  $p$ . Hence there is exactly one moving intersection, and so the map  $\pi|_F$  is birational.

The algebraic degree of the inverse map is equal to the number of intersections of the image of a generic line in the parameter plane and a generic hyperplane. In this case, the image of a generic line is also a hyperplane, hence the algebraic degree is 5. When  $p$  lies on a line, then the line is a fixed component reduces the algebraic degree by 1. If  $p$  lies on 2 lines, then the mapping degree drops by 2.  $\square$

**Remark 11.** In case  $F$  is a minimal Del Pezzo surface, i.e., it does not have Galois orbits of pairwise disjoint lines, then the smallest possible algebraic degree of a parametrization is 5 (see Schicho, 2006). So in this case the given construction has smallest possible degree.

### 3.1. Completeness of the construction

To prove that any Del Pezzo surface  $F$  of degree 5 is  $k$ -isomorphic to a surface constructed by the method in Section 2, it suffices to show that  $F$  has a parametrization defined by quintics passing with multiplicity 2 through 5 points  $q_1, \dots, q_5$  in general position (recall that this only means that no three of these points are collinear). We will do that in this short section; another proof of the completion is given in Section 4 through the classification.

It is clear that  $F$  does contain a point  $p$  with coordinates in  $k$  and is not contained in one of the 10 lines. Indeed, we have already seen that  $F$  has a parametrization, hence the set of all points defined over  $k$  is Zariski-dense and can therefore not be contained in the union of the 10 lines.

**Theorem 12.** *Assume that  $p \in F$  is defined over  $k$  and not contained in one of the 10 lines on  $F$ . Then the inverse of the birational projection  $F \rightarrow \mathbb{P}^2$  from the tangent line  $T_p F$  is a parametrization defined by quintics passing with multiplicity 2 through 5 points in general position.*

**Proof.** We have already proven that the algebraic degree of the parametrization is 5, hence the parametrization is defined by quintics. Let  $q_1, \dots, q_n$  be the base points of the parametrization (including infinitely near), and let  $m_1, \dots, m_n$  be their respective multiplicities. The number of moving intersections of two quintics in the linear system  $\Gamma$  defining the parametrization is equal to 5, and the genus of a generic element is equal to 1, because it is birational to a generic hyperplane section of  $F$ . This gives the numeric conditions

$$25 - m_1^2 - \dots - m_n^2 = 5,$$

$$6 - \frac{m_1(m_1 - 1)}{2} - \dots - \frac{m_n(m_n - 1)}{2} = 1.$$

This gives only two cases.

Case 1:  $n = 5, m_1 = \dots = m_5 = 2$ . In this case, any line through 3 of the base points would have at least 6 intersections, counted with multiplicity, with any quintic in  $\Gamma$ , hence it would be a fixed component, which is not possible; it follows that no three base points are collinear. Moreover, no base points is infinitely near to another, because this would give rise to the blow down of a  $-2$ -curve which would give a singular point on the surface  $F$ , in contrast to the assumption that  $F$  is nonsingular. Hence  $q_1, \dots, q_5$  are double points of  $\Gamma$ , and they are in generic position.

Case 2:  $n = 6, m_1 = 3, m_2 = m_3 = 2, m_4 = m_5 = m_6 = 1$ . Then the parametrization map  $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$  is the product of the blowup at the 6 base points and some birational regular map blowing down the fundamental curves. Because  $F$  is  $\bar{k}$ -isomorphic to the blowup of  $\mathbb{P}^2$  at 4 points, the number of fundamental curves is 2. The fundamental curves are the curves which have no moving intersection points with the curves in  $\Gamma$ , and these are the two lines  $L(q_1, q_2)$  and  $L(q_1, q_3)$ . Let  $p_1$  be the image of the fundamental curve  $L(q_1, q_2)$ . When we compose the parametrization with the projection from the tangent plane at  $p_1$ , then we get the rational map defined by all quintics in  $\Gamma$  having  $L(q_1, q_2)$  as a double component. Canceling the common factor, this is the system of cubics with double point in  $q_2$  and passing through  $q_1, q_4, q_5, q_6$ . On the other hand, this should be the identity map, and this is not the case. So this case does not happen.  $\square$

## 4. Deciding isomorphy

In this section, we give an algorithm for deciding whether two given anticanonically embedded Del Pezzo surfaces  $F_1$  and  $F_2$  of degree 5 are isomorphic over  $k$ . The section makes use of Galois cohomology and  $k$ -twists: an approach which gives an alternative (non-constructive) proof of the parametrizability of degree 5 Del Pezzo surfaces (see Skorobogatov, 2001). Through cohomology, we see that the isomorphism class over  $k$  is determined by the action of Galois on the graph of exceptional lines and we explain how to choose the seed in the construction in order to obtain any prescribed isomorphism class. Serre (1994), Skorobogatov (2001) and Polo-Blanco (2007) are basic references here.

We write  $G_k$  for  $G(\bar{k}/k)$ . Let  $F$  denote a degree 5 Del Pezzo over  $k$ . The key point for degree 5 is that all such  $F$  are still isomorphic over  $\bar{k}$  (because any two sets of four points in the projective plane

with no three collinear are conjugate under  $PGL_3$ , so are Galois twists of each other, but that the automorphism groups are finite, so that the  $H^1$  Galois cohomology group defining the set of twists is relatively easy to describe. From this, it is easy to just write down an  $F$  for each cohomology class, thus giving representatives for all of the  $k$ -isomorphism classes.

The Appendix to Section 3.1 of Skorobogatov (2001) gives a classification of these twists. However, the description there is rather abstract: they are given as quotients of the set of stable points of a Grassmannian by a twisted torus. We present the twisting theory in a more elementary fashion here, leading directly to the concrete description of the isomorphism classes in terms of seeds with given splitting behavior over  $k$ .

#### 4.1. Classification of $k$ -isomorphism classes

Let  $F_0$  denote the standard “split” surface, the projective plane  $\mathbb{P}_k^2$  blown up at the four  $k$ -rational points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(0 : 0 : 1)$ ,  $(1 : 1 : 1)$  (see Example 1).

Let  $E$  denote the incidence graph for the ten exceptional lines on  $F$  which are all defined over  $\bar{k}$ . From the root system description of  $E$  in Manin (1974), we can identify  $Aut(E)$  with the symmetric group on 5 elements  $S_5$  and the 10 exceptional lines with the set of pairs  $\{\{i, j\} : 1 \leq i < j \leq 5\}$ , so that the action of  $Aut(E)$  corresponds to the natural action of  $S_5$  on the pairs.

Let  $Aut(F)$  denote the group of algebraic automorphisms of  $F$  over  $\bar{k}$ .  $Aut(F)$  naturally acts on the exceptional lines preserving incidence relations, which leads to a homomorphism  $\psi : Aut(F) \rightarrow Aut(E)$ .

**Lemma 13.**  $\psi$  is an isomorphism.

**Proof.** This is Lemma 3.1.7 of Skorobogatov (2001). We give a more elementary proof here that does not use the moduli space description of  $F$  but instead reduces to the degree 6 Del Pezzo case.

We identify  $F$  with a blow-up of the plane at 4 points. If  $f \in Aut(F)$  fixes each exceptional line, then it comes from an automorphism of the plane that fixes each of the 4 points. Such an automorphism is trivial if no 3 points lie on a line. Thus  $\psi$  is injective.

To show surjectivity, it suffices to prove that  $Aut(F)$  is transitive on lines and that its stabilizer of any particular line is  $D_6$ . If we blow down any exceptional line  $L$ , we get a non-degenerate degree 6 Del Pezzo  $F_1$  whose exceptional lines are the images of exceptional lines of  $F$  that do not intersect  $L$ . Any automorphism of  $F_1$  that fixes the image point  $p$  of  $L$  will lift to an automorphism of  $F$  (that fixes  $L$ ). Considering the configuration  $E$ , it is then easy to see that it suffices to prove that the automorphisms of  $F_1$  that preserve a point  $p$  not on an exceptional line induce the full group of graph automorphisms of its exceptional lines. This is true because the automorphism group of  $F_1$  is a split extension of  $D_6 \cong C_2 \times D_3$  by a 1-dimensional torus  $T$  where  $T$  acts transitively on the complement of the exceptional lines and is precisely the subgroup fixing all six of these. If  $F_1$  is isomorphic to the plane blown up at three points  $P_i$ , the  $T.D_3$  part comes from automorphisms of the plane preserving  $\{P_i\}$  and the extra  $C_2$  comes from the plane Cremona transform based at the  $P_i$ .  $\square$

We will therefore identify  $Aut(F)$  with  $S_5$  through its action on the lines. Since the action of  $Aut(F)$  on these is equivariant with respect to the  $G_k$  action, if all of the exceptional lines are defined over  $k$  then all elements of  $Aut(F)$  are defined over  $k$  also. In particular, this holds for  $F_0$  and  $Aut(F_0)$  can be identified with  $S_5$  with trivial  $G_k$  action.

**Lemma 14.** We have the following bijective correspondence

$$\begin{aligned} &\{k\text{-isomorphism classes of non-degenerate degree 5 Del Pezzos}\} \\ &\updownarrow \\ &H^1(G_k, S_5) = \{\text{homomorphisms } G_k \rightarrow S_5 \text{ up to } S_5\text{-conjugacy}\} \end{aligned}$$

under which, a  $k$ -isomorphism class  $[F]$  corresponds to the homomorphism giving the action of  $G_k$  on its graph of exceptional lines  $E$ .

In particular, the  $k$ -isomorphism class of a 5 Del Pezzo is determined by the action of  $G_k$  on its exceptional lines.



**Proof.** The bijection (Theorem 3.1.3 of Skorobogatov (2001)) comes from standard twisting theory, identifying the set  $k$ -twists of a quasi-projective variety  $X$  with the elements of  $H^1(G_k, \text{Aut}_{\bar{k}}(X))$  (see Ch. 3, §1, Serre, 1994).

We get the lower equality as  $\text{Aut}(F_0)$  is equal to  $S_5$  with trivial  $G_k$ -action.

That the cocycle class  $[u_F]$  of a twist  $F$  of  $F_0$  corresponds to the homomorphism of  $G_k$  giving the action on its exceptional graph  $E$  is an easy consequence of the definition of  $u_F$  ( $u_F(\sigma) = f^{-1} \circ \sigma(f)$  where  $f : F_0 \rightarrow F$  is an isomorphism over  $\bar{k}$ ). Note that replacing  $u_F$  by an  $S_5$ -conjugate homomorphism just corresponds to relabeling the elements of  $E$ .  $\square$

We now show how to construct a particular Del Pezzo  $F$  that has a given  $G_k$  action  $f : G_k \rightarrow S_5$  on its set of exceptional lines. The fixed field  $L$  of the kernel of the  $G_k$  action is the splitting field of the 10 lines, which coincides with the splitting field of the seed  $Q$  in case the surface has been constructed as in Section 2. The possible isomorphism types for subgroups of  $S_5$  are those occurring in Table 1. Straightforward computation shows the following.

1. Each isomorphism type of subgroup different from  $C_2$  or  $C_2 \times C_2$  occurs uniquely up to conjugacy in  $S_5$  (i.e.,  $A \cong B \Rightarrow A$  is conjugate to  $B$ ). The isomorphism type  $C_2$  occurs for two conjugacy classes:  $\langle (12) \rangle$  and  $\langle (12)(34) \rangle$ . The isomorphism type of  $C_2 \times C_2$  occurs also twice:  $\langle (12), (34) \rangle$  and  $\langle (12)(34), (13)(24) \rangle$ .
2. For  $A \leq S_5$ ,  $\text{Aut}(A)$  is induced by  $N_{S_5}(A)/C_{S_5}(A)$ , except if  $A \cong D_4$  or  $A \cong D_6$  when  $N_{S_5}(A)/C_{S_5}(A)$  induces the group of inner automorphisms  $\text{Inn}(A)$  with  $[\text{Aut}(A) : \text{Inn}(A)] = 2$ .

This implies that, up to  $S_5$ -conjugacy,  $f$  is completely determined by  $L$  unless  $G(L/k)$  is isomorphic to  $C_2, C_2 \times C_2, D_4$  or  $D_6$ , when there are two  $S_5$ -conjugacy classes of  $f \leftrightarrow L$ .

Explicitly, for  $G(L/k) = D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1 \ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ , the two classes are

$$[\sigma \mapsto (1234), \tau \mapsto (13)] \quad \text{and} \quad [\sigma \mapsto (1234), \tau \mapsto (12)(34)]$$

and for  $G(L/k) = D_6 = \langle \sigma, \tau \mid \sigma^6 = \tau^2 = 1 \ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ , the two classes are

$$[\sigma \mapsto (123)(45), \tau \mapsto (12)] \quad \text{and} \quad [\sigma \mapsto (123)(45), \tau \mapsto (12)(45)].$$

**Lemma 15.** Let  $Q \in k[x]$  be a quintic normed and squarefree polynomial, with roots  $P_1, \dots, P_5 \in \bar{k}$ . Let  $F$  be the Del Pezzo surface constructed as in Section 2 with seed  $Q$ . Then the homomorphism  $f : G_k \rightarrow S_5$  corresponding to the  $k$ -isomorphism class of  $F$  under the correspondence of Lemma 14 is the permutation representation of  $G_k$  on  $\{P_1, \dots, P_5\}$ .

Conversely, for any Galois extension  $L$  such that  $G(L/k)$  is isomorphic to a subgroup of  $S_5$ , and an  $S_5$ -conjugacy class of  $f : G(L/k) \hookrightarrow S_5$ ,  $f$  comes from an appropriate quintic normed and squarefree polynomial.

**Proof.** The lines on  $F$  are in bijective correspondence with the 2-element subsets of  $S := \{P_1, \dots, P_5\}$ ; since we identified the action of  $\text{Aut}(F)$  on  $E$  with the natural action of  $S_5$  on pairs, the first statement follows.

For a given  $L$ , we need to find  $Q$  such that the permutation Galois action on its roots is of the form  $G_k \twoheadrightarrow G(L/k) \hookrightarrow S_5$ . If  $\#f(G(L/k))$  is divisible by 5, then we can take  $K$  as the fixed field of  $f^{-1}(S_4) \leq G(L/k)$  and take  $S$  as the  $G(L/k)$  orbit of a primitive element (over  $k$ ) of  $K$ . An  $S$  for the other cases can be constructed similarly. In the non- $C_2, D_4, D_6$  cases, we are done. In these three cases, we also need to show that we can take an  $S$  as above that leads to either of the 2 conjugacy classes of embedding  $G(L/k) \hookrightarrow S_5$ .

In case  $G(L/k) = C_2$ , pick  $P_1, P_2$  conjugate over  $L$  and  $P_3, P_4, P_5 \in k$  for  $\langle (12) \rangle$ ; pick  $P_1, P_2$  and  $P_3, P_4$  conjugate over  $L$  and  $P_5 \in k$  for  $\langle (12)(34) \rangle$ .

In case  $G(L/k) = C_2 \times C_2$ , then we have that  $L = k(\alpha)$  where  $\alpha$  is a primitive element of  $L$  and  $L = L_1 \otimes_k L_2$  where  $L_1$  and  $L_2$  are two distinct quadratic extensions of  $k$  contained in  $L$ . If  $P_1, P_2, P_3$  and  $P_4$  are the roots of the minimal polynomials of  $\alpha$  and  $P_5$  is in  $k$  we have the embedding of  $G(L/k)$  in  $S_5$  as the Klein group  $V_4 = \langle (12)(34), (13)(24) \rangle$ . If  $P_1$  and  $P_2$  are conjugate elements in  $L_1, P_3$  and  $P_4$  are conjugate elements in  $L_2$  and  $P_5$  is in  $k$ , then the embedding of  $G(L/k)$  on  $S_5$  is the group  $\langle (12), (34) \rangle$ .

**Table 1**  
Seeds for constructing example Del Pezzo surfaces over  $\mathbb{Q}$  of prescribed isomorphism type and number of line orbits.

Isomorphism type of Galois Group, number of orbits	Polynomial defined by the five points in $\overline{\mathbb{Q}}$ that determine the Del Pezzo surface of degree 5	Parametric degree
$S_5, 1$	$x^5 - 2x^4 - 3x^3 + 6x^2 - 1$	5
$A_5, 1$	$x^5 - 11x^3 - 5x^2 + 18x + 9$	5
$S_4, 2$	$(x^4 - 4x^2 - x + 1)x$	3
$H_{20}, 1$	$x^5 - 9x^3 - 4x^2 + 17x + 12$	5
$A_4, 2$	$(x^4 - x^3 - 7x^2 + 2x + 9)x$	3
$D_6, 3$	$(x^3 - 2)(x^2 - 5)$	4
$D_6, 3$	$(x^3 + 2)(x^2 + x + 1)$	4
$D_5, 2$	$x^5 - x^4 - 5x^3 + 4x^2 + 3x - 1$	5
$D_4, 3$	$(x^4 - 4x^2 + 5)x$	3
$D_4, 3$	$(x^4 - 8x^2 - 4)x$	3
$S_3, 4$	$(x^3 - x^2 - 3x + 1)(x + 1)x$	3
$C_6, 3$	$(x^3 - x^2 - 2x + 1)(x^2 + 1)$	4
$C_5, 2$	$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$	5
$C_4, 3$	$(x^4 - x^3 - 4x^2 + 4x + 1)x$	3
$C_2 \times C_2, 4$	$(x^4 - 2x^2 + 9)x$	3
$C_2 \times C_2, 5$	$(x^2 + 1)(x^2 - 2)x$	3
$C_3, 4$	$(x^3 - x^2 - 2x + 1)(x + 1)x$	3
$C_2, 6$	$(x^2 + 1)(x^2 + 4)x$	3
$C_2, 7$	$(x^2 + 1)(x + 1)(x - 1)x$	3
$1, 10$	$(x + 2)(x - 2)(x + 1)(x - 1)x$	3

Assume  $G(L/k) = D_4 = \langle \sigma, \tau | \sigma^4 = \tau^2 = 1 \quad \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ . Let  $K_1 = L^{(\tau)}, K = L^{(\sigma\tau)}$ .  $[K_1 : k] = 4$  and  $L$  is the Galois closure of  $K_1$  in  $\bar{k}$ . Similarly for  $K$ . We pick  $P_5 \in k$  and  $P_1, \dots, P_4$  the conjugates of a primitive element of  $K/k$  for one conjugacy class of  $f$  and  $P_1, \dots, P_4$  the conjugates of a primitive element of  $K_1/k$  for the other.

The case  $G(L/k) = D_6$  is similar.  $\square$

In particular, the surfaces constructed cover every  $k$ -isomorphism class and are plainly parametrizable over  $k$  (they are all constructed by blowing up a Galois-stable set of 5 points on a plane conic and then blowing down the strict transform of the conic). Thus, we can deduce that all non-degenerate degree 5 Del Pezzos over  $k$  are parametrizable over  $k$  from this classification without the explicit construction of  $k$ -rational points in Section 3. The lemma also gives an alternative proof of the completeness result of Section 4.

Table 1 below gives examples of seeds for every isomorphism type. In the table the isomorphism type of the Galois group acting on the Del Pezzo surface and the number of orbits of the induced action on the 10 lines appear together with a polynomial  $f(x)$  of degree 5 that defines the surface. The groups  $C_n, D_n, S_n$  and  $A_n$  denote the cyclic, the dihedral, the symmetric and the alternating groups respectively and  $H_{20}$  denotes the unique subgroup up conjugation of order 20 of  $S_5$ . In case there are two conjugacy classes of embeddings of the group in  $S_5$ , a polynomial for each case is given (with the same splitting field  $L$ ). The Del Pezzo surface can be obtained by mapping  $\mathbb{P}^2$  into  $\mathbb{P}^5$  by the space of quintics passing with multiplicity two through the points  $(P_1^2 : P_1 : 1), (P_2^2 : P_2 : 1), (P_3^2 : P_3 : 1), (P_4^2 : P_4 : 1), (P_5^2 : P_5 : 1)$ , where  $P_1, \dots, P_5$  are the roots of  $f(x)$ . In the last column of the table we include the parametric degree of the Del Pezzo surface. This degree is 3 if and only if  $F$  is the blowup of  $\mathbb{P}^2$  at a Galois-invariant quadruple of points, if and only if there exists a Galois orbit of 4 pairwise disjoint lines, if and only if  $f$  has a linear factor. The degree is 4 if and only if  $F$  is the blowup of a nonsingular quadric at a Galois-invariant triple of points but not of  $\mathbb{P}^2$ , if and only if there exists a Galois orbit of 3 pairwise disjoint lines but no Galois orbit of 4 pairwise disjoint lines, if and only if  $f$  has a quadratic but no linear factor. It is 5 if and only if  $F$  is  $k$ -minimal, if and only if  $f$  is irreducible.

#### 4.2. Testing for $k$ -isomorphy

Given two quintic Del Pezzo surfaces  $F_1, F_2$  by two sets of quadric generators of their ideals, we decide whether they are  $k$ -isomorphic.

**First step.** We reduce to deciding whether the two Galois actions on the roots of two given quintic squarefree polynomials  $Q_1, Q_2$  (the seeds) are  $S_5$ -conjugate. The polynomials can be constructed by first calculating a parametrization by quintics as in [Theorem 12](#), and then construct the polynomial as in [Lemma 4](#).

**Second step.** Given  $(Q_1, Q_2)$ , we first decide whether the splitting fields coincide. Theoretically, this can be done by factoring  $Q_2$  over the splitting field of  $Q_1$  and factoring  $Q_1$  over the splitting field of  $Q_2$  (but this is, of course, not the fastest method). If the splitting fields do not coincide, then the actions have different kernels and are not conjugate. Otherwise, let  $L$  be the common splitting field of the  $Q_i$ . If the Galois group  $G(L/k)$  is not isomorphic to  $C_2, C_2 \times C_2, D_4$ , or  $D_6$ , then the actions are conjugate by the preceding section. Otherwise we get the following.

- Case (i) Assume that  $G(L/k) = C_2$ . Then the actions are conjugate if and only if the number of irreducible factors of  $Q_1$  and  $Q_2$  coincide—they have either 4 factors of degree 2, 1, 1, 1, or 3 factors of degree 2, 2, 1.
- Case (ii) Assume that  $G(L/k) = C_2 \times C_2$ . Then the actions are conjugate if and only if the number of irreducible factors of  $Q_1$  and  $Q_2$  coincide—they have either 3 factors of degree 2, 2, 1, or 2 factors of degree 4, 1.
- Case (iii) Assume that  $G(L/k) = D_4$ . Then both  $Q_1$  and  $Q_2$  have two irreducible factors, of degree 1 and 4. Both quartic factors define non-Galois degree 4 extensions of  $k$  which contain a unique quadratic subextension over  $k$ . The actions are conjugate if and only if the two quadratic extensions coincide. In fact, the Galois theory shows that for two non-Galois, non-conjugate degree 4 subfields of  $L, L_1$  and  $L_2$ , the quadratic subfield of  $L_1$  is generated by the square-root of the discriminant of a defining polynomial for  $L_2$  and vice-versa. So we find that the actions are conjugate if and only if the quotient of the discriminants of the quartic factors is a square in  $k$ .
- Case (iv) Assume that  $G(L/k) \cong D_6$ . Then both  $Q_1$  and  $Q_2$  have two irreducible factors, of degree 2 and 3. The actions are conjugate if the two splitting fields coincide.

**Remark 16.** If  $k$  is non-perfect, everything above works with  $k^{sep}$  replacing  $\bar{k}$  as the exceptional lines for  $F$  are all defined over  $k^{sep}$ . This follows from the fact that all smooth rational surfaces are separably split ([Coombes, 1988](#)).

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