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A note on tunnel number of composite knots

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ABSTRACT

Let K be a knot in a sphere S^3. We denote by $t(K)$ the tunnel number of K. For two knots K_1 and K_2, we denote by $K_1\sharp K_2$ the connected sum of K_1 and K_2. In this paper, we will prove that if one of K_1 and K_2 has high distance while the other has distance at least 3 then $t(K_1\sharp K_2) = t(K_1) + t(K_2) + 1$.

1. Introduction

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ W = \partial_+ V$, then we say M has a Heegaard splitting, denoted $M = V \cup_S W$. In this case, S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is minimal among all the Heegaard surfaces of M, then $g(S)$ is called the genus of M, denoted by $g(M)$. If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial B = \partial D$ (resp. $\partial B \cap \partial D = \emptyset$), then $V \cup_S W$ is said to be reducible (resp. weakly reducible). Otherwise, it is said to be irreducible (resp. strongly irreducible). See [1]. The distance between two essential simple closed curves α and β on S, denoted by $d(\alpha,\beta)$, is the smallest integer $n \geqslant 0$ so that there is a sequence of essential simple closed curves $\alpha_0 = \alpha, \ldots, \alpha_n = \beta$ on S such that α_{i-1} is disjoint from α_i for $1 \leqslant i \leqslant n$. The distance of the Heegaard surface S, denoted by $d(S)$, is defined to be $\min\{d(\alpha,\beta)\}$, where α bounds a disk in V and β bounds a disk in W. See [4].

Let K be a knot in a 3-sphere S^3, we denote by $\eta(K)$ the open regular neighborhood of K in S^3, and $E(K)$ the complement of K, i.e. the manifold $S^3 - \eta(K)$. Let $V \cup_S W$ be a minimal Heegaard splitting of $E(K)$. We may assume that $\partial E(K) \subset \partial_- W$. If $d(S)$ is minimal along all the minimal Heegaard splittings of $E(K)$, then $d(S)$ is called the distance of K, denoted by $d(K)$. It is well known that there exist knots of arbitrarily of high distance, see [7]. The tunnel number of K, denoted by $t(K)$, is defined to be $g(E(K)) - 1$. For two knots K_1 and K_2, we denote by $K_1\sharp K_2$ the connected sum of K_1 and K_2. It is well known that $t(K_1\sharp K_2) \leqslant t(K_1) + t(K_2) + 1$ from a natural construction of a Heegaard splitting of $E(K_1\sharp K_2)$. Hence there is a question called the super additivity of tunnel number as follows:

Question 1. If $t(K_1\sharp K_2) = t(K_1) + t(K_2) + 1$?

In generality, the superadditivity of tunnel number does not hold. For example, Morimoto [8] gave examples to show that $t(K_1\sharp K_2) < t(K_1) + t(K_2)$, and Kobayashi [5] gave examples to show that $t(K_1\sharp K_2) - t(K_1) - t(K_2)$ can be arbitrarily large. When does the superadditivity of tunnel number hold? Morimoto [9] proved that the superadditivity of tunnel holds

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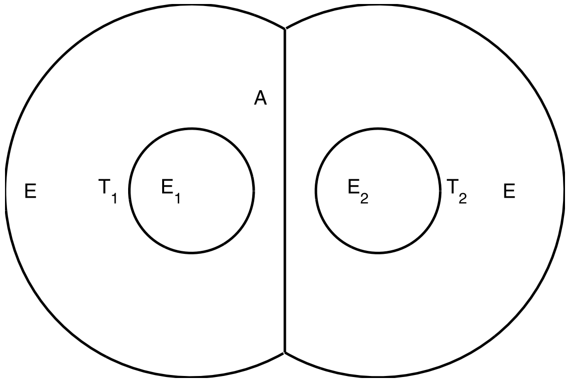

Fig. 1.

if both K_1 and K_2 are meridionally small and not m-primitive. If both K_1 and K_2 are high distance knots, the superadditivity of tunnel number does also hold, see [2] and [6]. The main result of this paper is the following:

Theorem 1. *If $d(K_1) \geqslant 2(t(K_1)+t(K_2)+2)$ while $d(K_2) \geqslant 3$, then $t(K_1 \sharp K_2) = t(K_1)+t(K_2)+1$.*

2. The proof of Theorem 1

Let F be either a properly embedded connected surface in a 3-manifold M or a connected sub-surface of ∂M. If there is an essential simple closed curve on F which bounds a disk in M or F is a 2-sphere which bounds a 3-ball in M, then we say F is compressible; otherwise, F is said to be incompressible. If F is an incompressible surface not parallel to ∂M, then F is said to be essential.

Let P be a separating connected compact surface in a 3-manifold M which cuts M into two 3-manifolds M_1 and M_2. P is said to be bicompressible if P is compressible in both M_1 and M_2. P is strongly compressible if there are compressing disks for P in M_1 and M_2 which have disjoint boundaries in P; otherwise P is weakly incompressible. By definition, if $M = V \cup_S W$ is a strongly irreducible Heegaard splitting, then S is weakly incompressible in M.

Now let P be a bicompressible closed surface in an irreducible 3-manifold M. By maximally compressing P in both sides and deleting any resulting 2-sphere components, we get a surface sum structure of M as follows:

$$M = N_1 \cup_{F_1^P} H_1^P \cup_P H_2^P \cup_{F_2^P} N_2,$$

where H_i^P is a compression body with $\partial_+ H_i^P = P$, and F_i^P is a collection (may be empty) of close surfaces of genus at least one for $i = 1, 2$. In this case, P is a Heegaard surface of the manifold $H_1^P \cup_P H_2^P$. Hence $d(P)$ is well defined as a Heegaard surface. Two weakly incompressible surfaces P and Q are said to be well-separated in M if $H_1^P \cup_P H_2^P$ is disjoint from $H_1^Q \cup_P H_2^Q$ after isotopy.

Two basic results on the distance of Heegaard splitting as follows:

Hartshorn–Scharlemann Theorem. *([3,11]) Let $M = V \cup_S W$ be a Heegaard splitting, and P be an incompressible surface in M. Then either P can be isotoped to be disjoint from S or $d(S) \leqslant 2 - \chi(P)$.*

Scharlemann–Tomova Theorem. *([13]) Let P and Q be bicompressible but weakly incompressible connected closed separating surfaces in a 3-manifold M. Then either*

(1) *P and Q are well-separated, or*
(2) *P and Q are isotopic, or*
(3) *$d(P) \leqslant 2g(Q)$.*

Let K_1 and K_2 be two knots in S^3. Then there is an essential annulus A which cuts $E(K_1 \sharp K_2)$ into $E(K_1)$ and $E(K_2)$. Furthermore, A is a meridional annulus on both $\partial E(K_1)$ and $\partial E(K_2)$. Now let $\partial E(K_i) \times I$ be a regular neighborhood of $\partial E(K_i)$ in $E(K_i)$ such that $\partial E(K_i) = \partial E(K_i) \times \{0\}$ for $i = 1, 2$. We denote by T_i the surface $\partial E(K_i) \times \{1\}$. Let $E = (\partial E(K_1) \times I) \cup_A (\partial E(K_2) \times I)$, $E_i = E(K_i) - \partial E(K_i) \times [0, 1)$ for $i = 1, 2$. Then E_i is homeomorphic to $E(K_i)$. Now $E(K_1 \sharp K_2)$ has a natural decomposition as $E(K_1 \sharp K_2) = E_1 \cup_{T_1} E \cup_{T_2} E_2$. See Fig. 1. The following lemma is a simple observation.

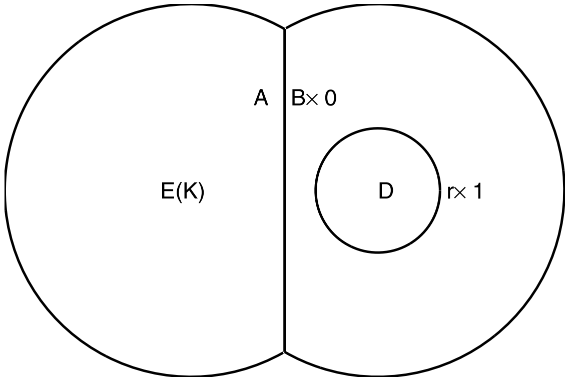

Fig. 2.

Lemma 2.1.

(1) $g(E) = 2$. *Furthermore, E has a minimal Heegaard splitting $H_1 \cup_S H_2$ such that $T_i \subset \partial_- H_i$.*
(2) *E contains no essential closed surface.*

Suppose that $t(K_1 \sharp K_2) = t(K_1) + t(K_2) + 1$. Then $g(E(K_1 \sharp K_2)) = g(E(K_1)) + g(E(K_2))$. Now by Lemma 2.1(1), from the view of amalgamation, $g(E(K_1 \sharp K_2)) = g(E_1) + g(E_2) + g(E) - g(T_1) - g(T_2)$. Hence $g(E_i \cup_{T_i} E) = g(E(K_i)) + 1$ is a necessary condition for the superadditivity of tunnel number. In generality, let K be a knot, and A be a meridian annulus on $\partial E(K)$. Let T be a torus, and B be an annulus on T. Let $N = E(K) \cup_{A = B \times \{0\}} T \times I$.

Question 2. If $g(N) = g(E(K)) + 1$?

By the argument in [9], if K is an m-primitive knot in S^3, then $g(N) \leqslant g(E(K))$. But we have the following lemma:

Lemma 2.2.

(1) $g(N) \geqslant g(E(K))$.
(2) *If $d(K) \geqslant 3$, then $g(N) = g(E(K)) + 1$.*

Proof. Recalling that $N = E(K) \cup_{A = B \times \{0\}} T \times I$.

(1) Now let r be a simple closed curve on T such that r intersects the core of the annulus B in one point. Let X be the manifold obtained by attaching a solid torus J to N along $T \times \{1\}$ so that $r \times \{1\}$ bounds a disk D in J. Then X is homeomorphic to $X = E(K)$. See Fig. 2. Let $V \cup_S W$ be a Heegaard splitting of N. Then S is also a Heegaard surface of $X = E(K)$. Hence Lemma 2.2(1) holds.

(2) Let $N = V \cup_S W$ be a minimal Heegaard splitting. If S is weakly reducible, then, by the argument in Lemma 2.2(1), S is also a weakly reducible Heegaard surface of $E(K)$. Since $d(K) \geqslant 3$, $g(S) \geqslant g(E(K)) + 1$. Hence Lemma 2.2(2) holds. Assume that $V \cup_S W$ is strongly irreducible. Now $S \cap A \neq \emptyset$. By Schultens's lemma, each component of $S \cap A$ is essential on both S and A. Furthermore, one of $S \cap E(K)$ and $S \cap (T \times I)$, say $S \cap E(K)$, is bicompressible. See [14]. Hence there are two essential disks $D \subset V$ and $B \subset W$ such that $D, B \subset E(K)$. Hence D and B are disjoint from A. Since $S \cap A \neq \emptyset$, $d(\partial D, \partial B) \leqslant 2$, $d(S) \leqslant 2$. Hence Lemma 2.2(2) holds. □

Remark. Note that if K is m-primitive, then $d(K) \leqslant 2$ and $g(N) = g(E(K))$. In addition, there are many examples such that K is m-primitive and $d(K) = 2$. Furthermore, Kobayashi and Rieck gave examples to show that K_1 and K_2 are non-primitive but $t(K_1 \sharp K_2) \leqslant t(K_1) + t(K_2)$. See [5]. Hence the condition "$d(K) \geqslant 3$" looks like natural for $g(N) = g(E(K)) + 1$.

Proof of Theorem 1. By the assumptions, $d(K_1) \geqslant 2(t(K_1) + t(K_2) + 2)$ while $d(K_2) \geqslant 3$. Hence $d(K_1) \geqslant 2(g(E(K_1)) + g(E(K_2)))$. Let $E(K_1 \sharp K_2) = V \cup_S W$ be a minimal Heegaard splitting. Then $g(S) \leqslant g(E(K_1)) + g(E(K_2))$.

Case 1. S is strongly irreducible.

Since $H_1 \cup_{S_1} H_2$ is a Heegaard splitting of E_1 with $d(S_1) \geqslant 2(g(E(K_1)) + g(E(K_2)))$. By Scharlemann–Tomova Theorem, either S and S_1 are well-separated, or S and S_1 are isotopic, or $d(S_1) \leqslant 2g(S)$. Since $V \cup_S W$ is a Heegaard splitting of M, S and S_1 are not well-separated and not isotopic. Hence $g(S) = g(E(K_1)) + g(E(K_2))$.

Case 2. $V \cup_S W$ is weakly reducible.

$$V \cup_S W = (V_1 \cup_{P_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{P_n} W_n),$$

where each $V_i \cup_{P_i} W_i$ is strongly irreducible, each F_i is essential. See [12]. Since $V \cup_S W$ is minimal, each $V_i \cup_{P_i} W_i$ is non-trivial. Hence $g(P_i) \leqslant g(S) - 1 \leqslant g(E(K_1)) + g(E(K_2)) - 1$, $g(F_i) < g(P_i)$. By Hartshorn–Scharlemann Theorem, each F_i can be isotoped to be disjoint from E_1. Since a compression body contains no essential closed surfaces, one component of $\bigcup_{i=1}^{n} F_i$ is isotopic to T_1. For details, see [10]. Hence $V \cup_S W$ is an amalgamation of a Heegaard splitting of E_1 and a Heegaard splitting of $E \cup_{T_2} E_2$ along T_1. By Lemma 2.2(2), $g(E(K_1 \sharp K_2)) = g(E(K_1)) + g(E(K_2))$. Hence $t(K_1 \sharp K_2) = t(K_1) + t(K_2) + 1$. $\square$

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References

[1] A. Casson, C.McA. Gordon, Reducing Heegaard splittings, Topology Appl. 27 (1987) 275–283.
[2] K. Du, J. Ma, R. Qiu, M. Zhang, Heegaard genera of annular 3-manifolds, J. Knot Theory Ramifications 20 (2011) 547–566.
[3] K. Hartshorn, Heegaard splittings of Haken manifolds have bounded distance, Pacific J. Math. 204 (2002) 61–75.
[4] J. Hempel, 3-manifolds as viewed from the curve complex, Topology 40 (2001) 631–657.
[5] T. Kobayashi, A construction of arbitrarily high degeneration of tunnel numbers of knots under connected sum, JKTR 3 (1994) 179–186.
[6] F. Li, G. Yang, F. Lei, Heegaard genera of high distance are additive under connected sum, Topology Appl. 157 (2010) 1188–1194.
[7] Y. Minsky, Y. Moriah, S. Schleimer, High distance knots, AGT 7 (2007) 1471–1483.
[8] K. Morimoto, There are knots whose tunnel numbers go down under connected sum, Proc. Amer. Math. Soc. 123 (1995) 3527–3532.
[9] K. Morimoto, On the super additivity of tunnel number of knots, Math. Ann. 317 (2000) 489–508.
[10] R. Qiu, S. Wang, M. Zhang, The Heegaard genera of surface sums, Topology Appl. 157 (2010) 1593–1601.
[11] M. Scharlemann, Proximity in the curve complex: boundary reduction and bicompressible surfaces, Pacific J. Math. 228 (2006) 325–348.
[12] M. Scharlemann, A. Thompson, Thin position for 3-manifolds, AMS Contemporary Math. 164 (1994) 231–248.
[13] M. Scharlemann, M. Tomova, Alternate Heegaard genus bounds distance, Geom. Topol. 10 (2006) 593–617.
[14] J. Schultens, Additivity of tunnel number for small knots, Comment. Math. Helv. 75 (2000) 353–367.