q-Extensions of identities of Abel–Rothe type

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Abstract

The ordinary binomial theorem may be expressed in the statement that the polynomials $x^n$ are of binomial type. Several generalizations of the binomial theorem can be stated in this form. A particularly nice one, essentially due to Rothe, is that the polynomials $a_n(x; h, w) = x(x + h + nw)(x + 2h + nw) \cdots (x + (n - 1)h + nw)$, $a_0(x; h, w) = 1$, are of binomial type. When $h = 0$, this reduces to a symmetrized version of Abel’s generalization of the binomial theorem. A different sort of generalization was made by Schützenberger, who observed that if one adds to the statement of the binomial theorem the relation $yx = qxy$, then the ordinary binomial coefficient is replaced by the $q$-binomial coefficient. There are also commutative $q$-binomial theorems, one of which is subsumed in a $q$-Abel binomial theorem of Jackson. We go further in this direction. Our two main results are a commutative $q$-analogue of Rothe’s identity with an extra parameter, and a noncommutative symmetric $q$-Abel identity with two extra parameters. Each of these identities contains many special cases that seem to be new.

1. Introduction

We begin by listing several well-known generalizations of the binomial theorem. One such is to the falling and rising factorials: put $(x)_n, h = x(x - h)(x - 2h) \cdots (x - (n - 1)h)$, with $(x)_0, h = 1$. (Let us agree that the 0th degree member of every set of polynomials we will consider will be 1.) Then we have

$$ (x + y)_{n, h} = \sum_{k=0}^{n} \binom{n}{k} (x)_{k, h} (y)_{n-k, h}. \quad (1.1) $$

Abel generalized the binomial theorem in quite a striking way:

$$ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x - kw)^{n-k-1} (y + kw)^{n-k}. \quad (1.2) $$
Abel’s theorem also comes in a symmetrized version:

\[(x + y)(x + y + nw)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} x(x + kw)^{k-1} y(y + (n - k)w)^{n-k-1}.\]  

Then too there are so-called \(q\)-extensions of the binomial theorem. To state two of these, we introduce some standard notation:

\[k!_q = [k][k-1] \cdots [1], \quad (0!_q = 1), \quad \binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)!_q}.\]

The \(q\)-numbers \([k]\) have a simple subtraction property that we will find to be very useful, namely

\[[a] - [b] = q^b [a - b].\]  

(1.4)

An example of its use is the following calculation:

\[\binom{n+1}{k} - \binom{n}{k-1} = \frac{n!_q[n+1]}{k!_q(n-k+1)!_q} - \frac{n!_q[k]}{(k-1)!_q[k](n-k+1)!_q} \]

\[= \frac{n!_q}{k!_q(n-k)!_q(n-k+1)!_q} q^k [n-k+1] \]

from which we deduce that

\[\binom{n+1}{k} = \binom{n}{k-1} + q^k \binom{n}{k}.\]  

(1.5)

The alternate form

\[\binom{n+1}{k} = \binom{n}{k} + q^{n-k+1} \binom{n}{k-1} \]

(1.6)

follows from (1.5) on replacing \(k\) by \(n-k+1\).

Schiitzenberger [18] observed that if \(yx = qxy\), then

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\]  

(1.7)

There is an older commutative \(q\)-binomial theorem, which has been attributed to various authors. Following Jackson [12] we define the polynomial \((a + x)^{[n]} := (a + x)(a + xq) \cdots (a + xq^{n-1});\) then

\[(a + x)^{[n]} = \sum_{k=0}^{n} \binom{n}{k}_q q^{[k]} x^k a^{n-k}.\]  

(1.8)

G. Andrews has remarked to the author that (1.7) and (1.8) are equivalent. To obtain (1.8) from (1.7) we replace \(x\) by \(xy\) (this is permissible since \(y(xy) = q(xy)y\)) and \(y\) by \(ay\), where \(a\) commutes with everything, and factor all the \(y\)’s out to the right on both sides and cancel them. The reverse procedure allows us to get (1.7) from (1.8).
In view of (1.7) and (1.8), one can ask for an expansion of \((a + x)^r\) when \(xa = qax\), or for a \(q\)-binomial expansion of \((x + y)^n\) when \(x\) and \(y\) commute. We will give such expansions later.

So far we have only given what we might call one-parameter generalizations of the binomial theorem, the parameter being either \(h\), \(w\), or \(q\). We conclude this section by mentioning a two-parameter generalization. (1.1) and (1.3) are subsumed in an identity essentially due to Rothe ([17]; see also [8, 9, 19]). Let

\[
a_n(x; h,w) = x(x + h + nw)(x + 2h + nw) \cdots (x + (n - 1)h + nw).
\]

Then

\[
a_n(x + y; h,w) = \sum_{k=0}^{n} \binom{n}{k} a_k(x; h,w)a_{n-k}(y; h,w).
\]

(1.9)

Note the similarity of form of (1.1), (1.3) and (1.9) to the binomial theorem. This is sometimes expressed in the statement that, e.g., the polynomials \(a_n(x; h,w)\) are of \textit{binomial type}. Polynomials of binomial type are studied extensively in [15, 16].

2. \(q\)-Delta operators

Another two-parameter generalization of the binomial theorem was made by Jackson, who combined (1.2) with (1.8) in showing [12] that

\[
(a + x)^n = \sum_{k=0}^{n} \binom{n}{k} x(q^{k-1} - [k]w)(a + [k]w)^{n-k}.
\]

(2.1)

We aim in this paper to go further in this direction; that is, to combine \(q\)-binomial theorems with other generalized binomial theorems. Our first such combination is a common generalization of (1.1) and (1.7). We define \(q\)-analogues of the shifted factorials by

\[
(x)_{h,q} = x(x - h[1]) \cdots (x - h[n - 1]).
\]

(2.2)

If \(yx = qxy\) and \(hx = qxh\), then

\[
(x + y)_{h,h,q} = \sum_{k=0}^{n} \binom{n}{k} (x)_{h,q}(y)_{n-k,h,q}.
\]

(2.3)

Eq. (2.3) holds trivially if \(n = 0, 1\). We assume it is true for \(n\), and show it for \(n + 1\):

\[
(x + y)_{h+1,h,q} = (x + y)_{h,h,q}(x + y - [n]h)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (x)_{h,q}(y)_{n-k,h,q}(x + y - [n]h)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (x)_{h,q}(y)_{n-k,h,q}(x - q^{n-k}[k]h)
\]
where we used (1.4), \(yx = qxy, hx = qxh\) and (1.6). Thus the polynomials \((x)_{n,h,q}\) satisfy a \(q\)-analogue of the binomial type property. However, this result would be much more useful if one did not need to assume that \(hx = qxh\). Probably the most natural theory of polynomials of \(q\)-binomial type, would use the Schützenberger identity as a model, as is done especially in [13]. Other authors (see, e.g., [3]) have proposed a different definition.

Define the operator \(A_{h,q}\) by

\[
A_{h,q}f(x) = \frac{f(qx + h) - f(x)}{(q - 1)x + h}.
\]  

(2.4)

This operator was considered previously by Hahn [10], who was looking for orthogonal polynomials satisfying certain \(q\)-difference equations. It contains the classical forward and backward difference operators of the calculus of finite differences as well as the \(q\)-derivative, which is the case \(h = 0\), and which we will denote by \(D_q\).

We compute the result of applying \(A_{h,q}\) to \((x)_{n,h,q}\):

\[
A_{h,q}(x)_{n,h,q} = \frac{(qx + h)x(qx - h([2] - [1])) \cdots (qx - h([n - 1] - [1])) - (x)_{n,h,q}}{(q - 1)x + h}
\]

\[
= \frac{(qx + h)x(qx - qh[1]) \cdots (qx - qh[n - 2]) - x(x - h[1]) \cdots (x - h[n - 1])}{(q - 1)x + h}
\]

\[
= \frac{x(x - h[1]) \cdots (x - h[n - 2])}{(q - 1)x + h}((qx + h)q^{n-1} - x + h(1 + q + \cdots + q^{n-2}))
\]

\[
= \frac{(x)_{n-1,h,q}}{(q - 1)x + h} \left(\frac{q^n - 1}{q - 1} + h(1 + q + \cdots + q^{n-1})\right)
\]

\[
= \left[n\right](x)_{n-1,h,q},
\]

(2.5)

where (1.4) was used in passing from the first line to the second.
Note the special case $D_x^n x^n = [n] x^{n-1}$, which is much easier to prove. The $q$-derivative has a remarkable property that $A_{h,q}$ unfortunately lacks in general. It commutes with the $q$-shift operator $E_y$, which is defined by $E_y f(x) = f(x + y)$ where $yx = qxy$. To see this, we must show that $D_y E_y f(x) = E_y D_y f(x)$ for a general formal power series $f(x) = \sum_{n \geq 0} c_n x^n$, where the $c_n$ are some coefficients which do not depend on $x$. In fact, it is not difficult to determine them explicitly, but we will not need them in this argument, which relies on (2.3) and (2.6). We have

$$E_y D_y f(x) = E_y \sum_{n \geq 1} [n] c_n x^{n-1}$$

$$= \sum_{n \geq 1} [n] c_n (x + y)^{n-1},$$

where $yx = qxy$, and on the other hand

$$D_y E_y f(x) = D_y \sum_{n \geq 0} c_n (x + y)^n = D_y \sum_{n \geq 1} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}$$

$$= \sum_{n \geq 1} c_n \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} x^{k-1} y^{n-k}$$

$$= \sum_{n \geq 1} c_n \sum_{k=1}^n \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{k-1} y^{n-1-(k-1)}$$

$$= \sum_{n \geq 1} [n] c_n (x + y)^{n-1},$$

where again $yx = qxy$.

### 3. Some $q$-Abel binomial theorems

In this section we prove some $q$-analogues of (1.2), including Jackson's $q$-Abel binomial theorem (2.1). Another approach to $q$-Abel identities is through $q$-Lagrange inversion (see [5, 11, 14]). We are essentially going to copy a proof of Abel's binomial theorem that is attributed in [6] to Lucas, which is in fact very similar to one of Abel's own proofs ([2]; see also [1]). We will require $q$-analogues of the Abel polynomials $\tilde{a}_k(x; w) = x(x - kw)^{k-1}$. Therefore, we define polynomials $\tilde{a}_k(x; w; q)$ by $\tilde{a}_0(x; w; q) = 1$, $\tilde{a}_1(x; w; q) = x$, and for $k \geq 2$ by

$$\tilde{a}_k(x; w; q) = x(xq - [k] w)(xq^2 - [k] w) \cdots (xq^{k-1} - [k] w).$$

Throughout this paper let $f^{(j)}(x)$ denote the $j$th $q$-derivative of $f(x)$. Then we have
Lemma 1. \( \tilde{a}_k^{(j)}(x; w; q) = k!q^j \) and, if \( 0 \leq j \leq k - 1 \), then
\[
\tilde{a}_k^{(j)}(x; w; q) = \frac{k!q^j}{(k-j)!q} \left( xq^j - w[j]\right) (xq^{j+1} - w[k]) \cdots (xq^{k-1} - w[k])
\]

Proof. Since \([0] = 0\), the case \( j = 0 \) is the definition of \( \tilde{a}_k(x; w; q) \). The case \( j = k \) would follow immediately from the case \( j = k - 1 \). Then, assuming that the lemma is true for \( j \), we show it for \( j + 1 \):
\[
\tilde{a}_k^{(j+1)}(x; w; q) = \frac{\tilde{a}_k^{(j)}(x; w; q) - \tilde{a}_k^{(j)}(xq; w; q)}{x(1-q)}
\]
\[
= \frac{k!q^j}{x(1-q)(k-j)!q} \left\{ (xq^j - [j]w)(xq^{j+1} - [k]w) \cdots (xq^{k-1} - [k]w) \right\}
\]
\[
= \frac{k!q^j}{x(1-q^k)(k-j-1)!q} \left\{ (xq^j - [j]w)(xq^{j+1} - [k]w) \right\}
\]
\[
= \frac{\tilde{a}_k^{(j+1)}(x; w; q)}{x(1-q)(k-j-1)!q} \left\{ (xq^j - [j]w)(xq^{j+1} - [k]w) \right\}.
\]
The quantity in brackets equals
\[
(x^2q^{j+1} + 1 - xw[j]q^{j+1} - xw[k]q^j + w^2[j][k]) - (x^2q^{j+k+1} - xw[j]q^k
\]
\[
- xw[k]q^{j+1} + w^2[j][k])
\]
which is
\[
xq^j\{xq^{j+1}(1 - q^{k-j}) - w[j]q(1 - q^{k-j-1}) - w[k](1 - q) \}
\]
and the coefficient of \(-xwjq^j\) is
\[
q[j](1 - q^{k-j-1}) + [k](1 - q) = 1 - q^k + q + q^2 + \cdots + q^{j-1} - q^{k-j} - q^{k-j+1}
\]
\[
= (1 + q + q^2 + \cdots + q^j)(1 - q^{k-j}).
\]
Then
\[
\tilde{a}_k^{(j+1)}(x; w; q) = \frac{k!q^j}{x(1-q^k)(k-j-1)!q} \left\{ xq^j(xq^{j+1} - [j+1]w)(1 - q^{k-j}) \right\}
\]
\[
= \frac{k!q^j}{(k-j-1)!q} \left\{ xq^j(\cdots(xq^{j+1} - [j+1]w)(1 - q^{k-j}) \right\}\prod_{l=j+2}^{k-1} (xq^l - [k]w).
\]
and so we have Lemma 1.
Corollary. For all nonnegative integers \(k\) and \(j\),
\[
\bar{a}_k^{(j)}(wq^{-j}[j]; w; q) = k!_q q^{j(1)} \delta_{kj}.
\]

Let \(f(x)\) be any function of \(x\). Subject to appropriate convergence conditions, we may write
\[
f(x) = \sum_{k \geq 0} \bar{a}_k(x; w; q) \lambda_k(w; q)
\]
for some coefficients \(\lambda_k(w; q)\) which do not depend on \(x\). To find them, take \(j\) \(q\)-derivatives with respect to \(x\):
\[
f^{(j)}(x) = \sum_{k \geq j} \bar{a}_k^{(j)}(x; w; q) \lambda_k(w; q).
\]

Set \(x = wq^{-j}[j]\) and use the above corollary:
\[
f^{(j)}(wq^{-j}[j]) = \sum_{k \geq j} k!_q q^{j(1)} \delta_{kj} \lambda_k(w; q) = j!_q q^{j(1)} \lambda_j(w; q).
\]

Hence
\[
f(x) = \sum_{k \geq 0} \frac{q^{-\binom{j}{k}}}{k!_q} \bar{a}_k(x; w; q) f^{(k)}(wq^{-k}[k]). \tag{3.1}
\]

In case \(f(x)\) is a polynomial, there is no doubt of the convergence of the series (3.1), which was also arrived at by Jackson. His theorem (2.1) follows from (3.1) and

Lemma 2. If \(0 \leq k \leq n\), then the \(k\)th \(q\)-derivative of \((a + x)^n\) is
\[
\frac{n!_q}{(n - k)!_q} q^{(1)}(a + xq^k) \cdots (a + xq^{n-1}).
\]

Proof. The case \(k = 0\) is just the definition of \((a + x)^n\). Then assuming the lemma is true for \(k\), we show it for \(k + 1\):
\[
\{(a + x)^n\}_{a+1} = \frac{n!_q}{(n - k)!_q} q^{(1)}(a + xq^k) \cdots (a + xq^{n-1}) - (a + xq^{k+1}) \cdots (a + xq^n)
\]

\[
= \frac{n!_q}{(n - k)!_q} q^{(1)}(a + xq^{k+1}) \cdots (a + xq^{n-1}) \frac{(a + xq^n - a - xq^n)}{x(1 - q)}
\]

\[
= \frac{n!_q}{(n - k - 1)!_q} q^{(n+1)+(1)}(a + xq^{k+1}) \cdots (a + xq^{n-1}).
\]
We now have only to take \( f(x) = (a + x)^n \) in (3.1) to get Jackson’s theorem. Since \((x + a)^n \neq (a + x)^n\), we have the possibility of another theorem here. Indeed,

\[
D_q(x + a)^n = \frac{1}{x(1-q)} \left\{ (x+a)(x+aq) \cdots (x+aq^{n-1}) \right\} \\
\quad = \frac{(x+a)^{n-1}}{x(1-q)} \left\{ (x+aq^{n-2}) \cdots (x+aq) \right\} \\
\quad = [n](x + a)^{n-1}
\]

so that \((x + a)^n\) is what we might call a q-Appell polynomial. Therefore, we may easily compute its \(k\)th q-derivative by iteration, and then (3.1) gives

\[
(x + a)^n = \sum_{k=0}^{n} \binom{n}{k} q^{-\binom{k}{2}} x(qx - [k]w) \cdots (qx^{k-1} - [k]w)(wq^{-k}[k] + a)^{n-k}.
\]

(3.2)

It is now natural to ask whether there is a q-analogue of Abel’s theorem in which an honest power of \(n\) appears on the left, rather than a q-power as in (2.1) and (3.2). The difficulty in obtaining such a theorem by this method lies in the fact that one has to compute the \(k\)th q-derivative of \((x + y)^n\) with respect to \(x\), without a suitable chain rule to reduce the problem to computing q-derivatives of \(x^n\). This however is the significance of the fact that the q-derivative commutes with the q-shift operator. If we assume that \(yx = qxy\), then we can compute \(D_q(x + y)^n\) as

\[
D_q E_y x^n = E_y D_q x^n = E_y \frac{n!_q}{(n-k)!_q} x^{n-k} = \frac{n!_q}{(n-k)!_q} (x + y)^{n-k}.
\]

Since we have to evaluate this expression when \(xq^k = [k]w\), we need also to assume that \(yw = qwy\). Then we have

**Theorem 1** (Noncommutative q-analogue of Abel’s binomial theorem). If \(yx = qxy\), \(yw = qwy\), and all other pairs of variables commute, then

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} q^{-\binom{k}{2}} x(qx - [k]w) \cdots (qx^{k-1} - [k]w)(wq^{-k}[k] + y)^{n-k}.
\]

(3.3)

If we apply the remark of Andrews to this result — that is, replace \(x\) by \(xy\), \(w\) by \(wy\) and \(y\) by \(ay\), where \(a\) commutes with all the other variables, and factor all the \(y\)’s to the right and cancel them — we recover Jackson’s theorem, (2.1). If we factor the \(y\)’s out to the left instead, we get (3.2). We can get a result containing (2.1) and (3.3) by taking \(f(x) = (a + (x + y))^n\) in (3.1), where \(yx = qxy\) and all other pairs of variables commute, and computing the q-derivatives using \(E_y\) and Lemma 2. This yields
Theorem 2. If $yx = qxy$, $yw = qwy$ and all other pairs of variables commute, then

$$
(a + (x + y))^{[n]} = \sum_{k=0}^{n} \binom{n}{k} x(xq - [k]w) 
\cdots (xq^{k-1} - [k]w)(a + ([k]w + yq^k))^{[n-k]}
$$

Similarly, using $f(x) = ((x + y) + a)^{[n]}$ in (3.1), we get a result that contains (3.2) and (3.3).

Theorem 3. If $yx = qxy$, $yw = qwy$, and all other pairs of variables commute, then

$$
((x + y) + a)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} q^{-1} x(xq - [k]w) \cdots (xq^{k-1} - [k]w)(wq^{-k}[k] + y + a)^{[n-k]}
$$

4. $q$-Rothe and ‘symmetric’ $q$-Abel identities

To some extent, one can use $A_{h,q}$ in place of $D_q$ to obtain $q$-analogues of the Rothe identity (1.9). We cannot get noncommutative $q$-Rothe identities directly since $A_{h,q}$ does not commute with $E_y$. Once we have commutative $q$-Rothe identities, it is possible to convert them to noncommutative forms by using Andrews’ remark in the reverse direction. However, the noncommutative identities seem most interesting when they have an extra (noncommuting) parameter beyond what we can get in the commutative case, as in Theorems 2 and 3. This will only happen when we can find an operator that commutes with $E_y$, and one can prove [13] that the only such operators are invertible formal power series in the $q$-derivative.

The polynomials we will consider are defined by

$$
a_n(x; b, h, w, q) = (x + b)(xq + b + h + [n]w)(xq^2 + b + [2]h + [n]w) 
\cdots (xq^{n-1} + b + [n - 1]h + [n]w),
$$

where, as usual, $a_0(x; b, h, w, q) = 1$. We are aiming for a $q$-Rothe identity, but as a byproduct we will get both commutative and noncommutative $q$-analogues of the symmetric Abel identity (1.3). As we shall see, however, in the $q$-case these ‘symmetric’ identities are not as symmetric as one would wish. The following lemma generalizes Lemmas 1 and 2.

Lemma 3. $A_{h,q}^{n}a_{n}(x; b, h, w, q) = n!_q q^{n^2}$ and, if $0 \leq k \leq n - 1$, then

$$
A_{h,q}^{k}a_{n}(x; b, h, w, q) = \frac{n!_q}{(n-k)!_q} q^{k^2} (xq^k + b + [k]h + [k]w) 
\times (xq^{k+1} + b + [k + 1]h + [n]w) \cdots (xq^{n-1} + b + [n-1]h + [n]w).
$$
The proof is by induction on \( k \). The case \( k = 0 \) holds by definition, and the case \( k = n \) would follow easily from the case \( k = n - 1 \). It is straightforward to check that 
\[
[n] + q[k][n - k - 1] = [k + 1][n - k],
\]
a fact we will use to complete the proof.

We assume the lemma is true for \( k \), and show that it holds for \( k + 1 \):

\[
A_{h,q}^{k+1} a_n(x; b, h, w, q) = \frac{n!_q}{(n-k)!_q} \frac{q^{(i)}}{(q-1)x + h} \times \left\{ (xq^{k+1} + b + [k + 1]h + [k]w) \prod_{j=k+2}^{n-1} (xq^j + b + [j]h + [n]w) \\
- (xq^k + b + [k]h + [k]w) \prod_{j=k+1}^{n-1} (xq^j + b + [j]h + [n]w) \right\}
\]

\[
= \frac{n!_q}{(n-k)!_q} \frac{q^{(i)}}{(q-1)x + h} \left( \prod_{j=k+2}^{n-1} (xq^j + b + [j]h + [n]w) \right) \times \left\{ (xq^{k+1} + b + [k + 1]h + [k]w)(xq^n + b + [n]h + [n]w) \\
- (xq^k + b + [k]h + [k]w)(xq^{k+1} + b + [k + 1]h + [n]w) \right\}.
\]

The quantity in brackets is

\[
[n]w(xq^n(q - 1) + ([k + 1] - [k])h) + (xq^{k+1} + b + [k + 1]h)(xq^n + [n]h - xq^{k+1} - [k + 1]h) \\
+ [k]w(xq^n + [n]h - xq^{k+1} - [k + 1]h) = [n]w q^k((q - 1)x + h) + (xq^{k+1} + b + [k + 1]h)(xq^n + [n]h - xq^{k+1} - [k + 1]h) \\
+ [k]w(xq^{k+1} + b + [k + 1]h)(xq[n - k](q - 1) + q^k h[n - k]) \\
+ [k]w(xq^{k+1}[n - k](q - 1) + q^{k+1}h[n - k - 1]) \\
= ((q - 1)x + h)q^{k}((xq^{k+1} + b + [k + 1]h)[n - k] \\
+ ([n] + q[k][n - k - 1])w) \\
= ((q - 1)x + h)q^{k}[n - k](xq^{k+1} + b + [k + 1]h + [k + 1]w)
\]

and Lemma 3 follows. An immediate corollary is that

\[
A_{h,q}^{k} a_n(x; b, h, w, q) |_{x=-q^{-k}(b + [k](h + w))} = n!_q q^{(i)}(h).
\]

Hence we may formally determine the coefficients in the expansion \( f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x; h, w, q) \) by operating with \( A_{h,q} \) \( k \) times and setting \( x = -q^{-k}(b + [k](h + w)) \). The result is

\[
f(x) = \sum_{k \geq 0} q^{-\delta(k)} \frac{a_k(x; b, h, w, q)}{k!_q} (A_{h,q}^k f(x)) |_{x=-q^{-k}(b + [k](h + w))}
\]
at least for polynomial \( f(x) \), and subject to convergence conditions for more general \( f(x) \).

We can use this result in two different ways. First take \( f(x) = a_n(x; a + b, h, w, q) \) and compute

\[
\Delta_{h, q}^k a_n(x; a + b, h, w, q) = \frac{n!_q}{(n-k)!_q} q^\binom{k}{2} (xq^k + a + b + [k]h + [k]w) \\
\times (xq^{k+1} + a + b + [k+1]h + [n]w) \cdots \\
\times (xq^{n-1} + a + b + [n-1]h + [n]w)
\]

when \( 0 \leq k \leq n - 1 \). When we evaluate this at \( xq^k = -b - [k](h + w) \), the terms involving \( h \) behave very nicely, while the terms involving \( w \) and \( b \) are not so nice. The resulting expression is

\[
\frac{n!_q}{(n-k)!_q} q^\binom{k}{2} \prod_{i=1}^{n-k-1} (a + b(1 - q^i) + [i]h + ([n] - [k]q^i)w).
\]

Eq. (4.1) now gives us

**Theorem 4 (Commutative q-Rothe identity).** If all variables commute, then

\[
a_n(x; a + b, h, w, q) = \sum_{k=0}^{n} \binom{n}{k} a_k(x; b, h, w, q) \left( a + b(1 - q^i) + [i]h + ([n] - [k]q^i)w \right)
\]

We may also take \( f(x) = a_n(x + y; a + b, 0, w, q) := a_n(x + y; a + b, w, q) \) in (4.1), where \( yx = qxy \) so that we can compute \( D^k_\Delta a_n(x + y; a + b, w, q) \) as \( E_y D^k_\Delta a_n(x; a + b, w, q) \) as before. Thus we get

\[
D^k_\Delta a_n(x + y; a + b, w, q) = \frac{n!_q}{(n-k)!_q} q^\binom{k}{2} (xq^k + yq^k + a + b + [k]w) \\
\times (xq^{k+1} + yq^{k+1} + a + b + [n]w) \cdots (xq^{n-1} + yq^{n-1} + a + b + [n]w)
\]

when \( 0 \leq k \leq n - 1 \) and \( yx = qxy \). We evaluate this when \( xq^k = -b - [k]w \), which forces us to assume that \( yw = qwy \) and \( yb = qby \). There results

\[
\frac{n!_q}{(n-k)!_q} q^\binom{k}{2} (yq^k + a) \prod_{i=1}^{n-k-1} (yq^{k+i} + a + b(1 - q^i) + ([n] - [k]q^i)w).
\]

From (4.1) we now get
Theorem 5 (Noncommutative symmetric $q$-Abel identity). If $yx = qxy$, $yw = qwy$, $yb = qby$ and all other pairs of variables commute, then

$$a_k(x + y; a + b, w, q) = \sum_{k=0}^{n} \binom{n}{k} a_k(x; b, w, q) \left( (yq^k + a) \prod_{i=1}^{n-k-1} (yq^{k+i} + a + b(1 - q^i) + ([n] - [k]q^i)w) \right).$$

Theorem 4 has many special cases of interest. For example, when $b = 0$ we have

$$(x + a)(xq + a + h + [n]w) \cdots (xq^{n-1} + a + [n - 1]h + [n]w)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x(xq + a + h + [k]w) \cdots (xq^{k-1} + [k]w) a(a + h - ([n] - [k]q)w)$$

$$\times \cdots (a + [n - k - 1]h + ([n] + [k]q^{n-k-1})w). \tag{4.2}$$

If we instead set $x = 0$ then

$$(a + b)(a + b + h + [n]w) \cdots (a + b + [n - 1]h + [n]w)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b(b + h + [k]w) \cdots (b + [k - 1]h + [k]w) a(a + b(1 - q) + h)$$

$$+ ([n] - [k]q)w) \cdots (a + b(1 - q^{n-k-1}) + [n - k - 1]h$$

$$+ ([n] - [k]q^{n-k-1})w). \tag{4.3}$$

Eqs. (4.2) and (4.3) are themselves $q$-analogues of Rothe's identity. Setting $a = 0$ in Theorem 4 reduces to a triviality, and the case $h = 0$ will be discussed below. When $w = 0$ we are getting a commutative $q$-analogue of (1.1), with an extra parameter:

$$(x + a + b)(xq + a + b + h) \cdots (xq^{n-1} + a + b + [n - 1]h)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (x + b)(xq + b + h) \cdots (xq^{k-1} + b + [k - 1]h)$$

$$\times a(a + b(1 - q) + h) \cdots (a + b(1 - q^{n-k-1}) + [n - k - 1]h) \tag{4.4}$$

We give a noncommutative version of (4.2) to illustrate how such an argument would go. Multiply (4.2) on the right by $y^n$, and assume that $yx = qxy$, $yw = qwy$, $yh = qhy$ and $y$ commutes with $a$. Use the commutation relations to move the $y$'s to
the left on both sides, so as to have one \( y \) on the right of each factor. Then rename \( xy \)
as \( x \), \( hy \) as \( h \), \( wy \) as \( w \) and \( ay \) as \( y \). We get that if \( yx = qxy \), \( yh = qhy \), \( yw = qwy \) and all other parts of variables commute, then
\[
(x + y)((x + y)q + h + [n]w) \cdots ((x + y)q^n - 1 + [n - 1]h + [n]w) = \sum_{k=0}^{n} \binom{n}{k} x(xq + h + [k]w) \cdots (xq^{k-1} + [k - 1]h + [k]w)yq^k \\
\times (yq^{k+1} + h + ([n] - [k]q)w) \\
\times \cdots (yq^{n-1} + [n - k - 1]h + ([n] - [k]q^{n-k-1})w). \tag{4.5}
\]
It is not hard to get from this back to (4.2) by using Andrews’ remark. An interesting special case is
\[
(x + y)((x + y)q + h) \cdots ((x + y)q^n - 1 + [n]w) = \sum_{k=0}^{n} \binom{n}{k} x(xq + h + [k]w) \cdots (xq^{k-1} + [k - 1]h + [k]w)yq^k (yq^{k+1} + h) \cdots (yq^{n-1} + [n - k - 1]h), \tag{4.6}
\]
where \( yx = qxy \) and \( yh = qhy \). This is another q-analogue of (1.1). In a similar way, (4.3) can be converted to
\[
(y + b)(yq + b + h + [n]w) \cdots (yq^n - 1 + b + [n - 1]h + [n]w) = \sum_{k=0}^{n} \binom{n}{k} b(b + h + [k]w) \cdots (b + [k - 1]h + [k]w)yq^k \\
\times (yq^{k+1} + b(1 - q) + h + ([n] - [k]q)w) \cdots (yq^{n-1} + b(1 - q^{n-k-1}) \\
+ [n - k - 1]h + ([n] - [k]q^{n-k-1})w), \tag{4.7}
\]
where \( yb = qby \), \( yh = qhy \), \( yw = qwy \) and all other variables commute.

We now turn to special cases of Theorem 5. The best commutative q-analogue of the symmetric Abel identity that we are able to get is the case \( y = 0 \), which is also the case \( h = 0 \) of Theorem 4. This is
\[
(x + a + b)(xq + a + b + [n]w) \cdots (x + a + b + [n]w) = \sum_{k=0}^{n} \binom{n}{k} (x + b)(xq + b + [k]w) \cdots (xq^{k-1} + b + [k]w)a(a + b(1 - q) \\
+ h + ([n] - [k]q)w) \cdots (a + b(1 - q^{n-k-1}) \\
+ [n - k - 1]h + ([n] - [k]q^{n-k-1})w), \tag{4.8}
\]
Eq. (4.8) contains two simpler symmetric $q$-Abel identities, the cases $x = 0$ and $b = 0$. Theorem 5 also contains simpler noncommutative symmetric $q$-Abel identities, as well as the following curious $q$-analogue of the binomial theorem:

$$
(x + y + a + b)((x + y)q + a + b) \cdots ((x + y)q^{n-1} + a + b)
$$

$$
= \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] (x + b)(xq + b) \cdots (xq^{k-1} + b)(yq^k + a)
\times (yq^{k+1} + a + b(1 - q)) \cdots (yq^{n-1} + a + b(1 - q^{n-k-1})),
$$

(4.9)

where $yx = qxy$ and $yb = qby$ and all other variables commute. Let us look at the case $b = 0$ of (4.9): if $yx = qxy$, then

$$
(x + y + a)((x + y)q + a) \cdots ((x + y)q^{n-1} + a)
$$

$$
= \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{(\alpha)k} x^{(\alpha)k}(yq^k + a)(yq^{k+1} + a) \cdots (yq^{n-1} + a).
$$

(4.10)

This is easy to prove directly by expanding both sides using (1.7) and (1.8); one sees that each side is

$$
\sum_{i+j+k=n} \left[ \begin{array}{c} n \\ i, j, k \end{array} \right] q^{(\alpha)k} x^i y^j a^k,
$$

where $yx = qxy$ and $[i, j, k]$ is a $q$-trinomial coefficient, defined in the obvious way by analogy with the $q$-binomial coefficient. The combinatorial significance of (4.10) can be seen if we rewrite it in the form

$$
(x + y + a)(xq + (yq + a)) \cdots (xq^{n-1} + (yq^{n-1} + a))
$$

$$
= \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x(xq)(xq^2) \cdots (xq^{k-1})(yq^k + a)(yq^{k+1} + a) \cdots (yq^{n-1} + a),
$$

where $yx = qxy$. The left side is evidently $\sum z_1z_2, \ldots, z_n$, where each $z_i = xq^{i-1}$ or $(yq^{i-1} + a)$ and the sum is over all the possible products. If we rewrite these products to have all of the factors containing $x$ to the left of all those containing $y$, noting that $(yq^i + a)(xq^{i+1}) = q(xq^i)(yq^{i+1} + a)$, and we recall that $[z]$ is the generating function for words in a two-letter alphabet where $q$ marks an inversion [4] then we see that we get the right side of (4.10). At least from this point of view, therefore, the combinatorial meaning of (4.10) is nearly the same as that of Schützenberger’s identity (1.7). It would be very interesting to have combinatorial interpretations of some of the more complicated identities that we have given.

There are four nontrivial special cases of Theorem 5 with three parameters equal to zero. Two of these are (1.7) and (1.8), but the other two, also $q$-analogues of the
binomial theorem, may not have been observed before. These are companion identities in the sense of Andrews' remark. The noncommutative one is

\[(b + y)(b + yq) \cdots (b + yq^{n-1})\]

\[= \sum_{k=0}^{n} \binom{n}{k} b^k y q^k (y q^{k+1} + b(1 - q)) \cdots (y q^{n-1} + b(1 - q^{n-k-1})).\]  (4.11)

where \(yb = qby\). Its commutative counterpart is

\[(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k a(a + b(1 - q)) \cdots (a + b(1 - q^{n-k-1})).\] (4.12)

It is not difficult to prove (4.12) directly by induction. It can also be obtained from the fact ([7]) that if \((z; q)_n = (1 - z)(1 - zq) \cdots (1 - zq^{n-1})\), with \((z; q)_0 = 1\), then

\[(rs; q)_n = \sum_{k=0}^{n} \binom{n}{k} (r; q)_k s^k (s; q)_{n-k}.\]

If \(r = 0\), this is just

\[1 = \sum_{k=0}^{n} \binom{n}{k} s^k (s; q)_{n-k}.\]

Replacing \(s\) by \(b/(a + b)\) and multiplying by \((a + b)^n\) gives (4.12).

5. Some other expansions

Recalling the notation of Jackson that we have used earlier, expansions in terms of the polynomials \((a + x)^{[n]}\) are covered by the foregoing theory, but expansions in terms of \((x + a)^{[n]}\), qua function of \(x\), are not. We noted in Section 3 that the \(q\)-derivative with respect to \(x\) of \((x + w)^{[n]}\) is \([n](x + w)^{[n-1]}\). Iterating this and evaluating at \(x = -w\) we see that

\[D_q^k(x + w)^{[n]}|_{x=-w} = k! q^\delta_{nk}.\]

From this it follows as before that, if we formally expand a function of \(x\) in terms of these polynomials, this expansion takes the form

\[f(x) = \sum_{k\geq 0} \frac{(x + w)^{[k]}}{k! q^k} f^{(k)}(-w).\] (5.1)

We can also write down a noncommutative version of this expansion by applying the \(q\)-shift operator \(E_y\). Thus if \(yx = qxy\), \(yw = qwy\) and all other pairs of variables commute, we have

\[f(x + y) = \sum_{k\geq 0} \frac{(x + w)^{[k]}}{k! q^k} f^{(k)}(y - w).\] (5.2)

The case \(w = 0\) is a noncommutative \(q\)-analogue of Taylor's theorem.
The expansions of \((x + y + a)^{[n]}\) and \((a + (x + y))^{[n]}\) in terms of the polynomials \((x + w)^{[k]}\) are readily obtained from (5.2) and the above remark about the \(q\)-derivatives of these polynomials. We have

**Theorem 6.** If \(yx = qxy, yw = qwy, \) and all other pairs of variables commute, then

\[
\begin{align*}
(i) \quad (x + y + a)^{[n]} &= \sum_{k=0}^{n} \binom{n}{k} (x + w)^{[k]}((y - w) + a)^{[n-k]}, \\
(ii) \quad (a + (x + y))^{[n]} &= \sum_{k=0}^{n} \binom{n}{k} q^{[n]}(x + w)^{[k]}(a + (y - w)q^{k})^{[n-k]}.
\end{align*}
\]

Let us look at some special cases of this theorem. If we take \(w = 0\) in (i), or in Theorem 3, we get an identity similar to (4.10):

\[
(x + y + a)(x + y + aq) \cdots (x + y + aq^{n-1}) = \sum_{k=0}^{n} \binom{n}{k} x^k(y + a)(y + aq) \cdots (y + aq^{n-k-1}),
\]

where \(yx = qxy\). One can give the same sort of direct proofs of this as we did for (4.10), which can itself be obtained by taking \(w = 0\) in (ii) or in Theorem 2. In both (i) and (ii), the case \(a = 0\) is

**Corollary.** If \(yx = qxy, yw = qwy\) and all other pairs of variables commute, then

\[
(x + y)^{n} = \sum_{k=0}^{n} \binom{n}{k} (x + w)^{[k]}(y - w)^{n-k}.
\]

We leave to the reader the cases \(x = 0\) and \(y = 0\) of Theorem 6, which are of some interest, and conclude with another expansion. In view of (2.5), it is easy to see that, formally,

\[
f(x) = \sum_{k \geq 0} \binom{x}{k,h,q} A_{h,q}^{k} f(x)|_{x=0}.
\]

Using this and Lemma 3, we have

\[
a_n(x; b, h, w, q) = \sum_{k \geq 0} \binom{n}{k} q^{[n]}(x)_{k,h,q}(b + [k]h + [k]w) \\
\times (b + [k + 1]h + [n]w) \cdots (b + [n - 1]h + [n]w).
\]

When \(h = w = 0\), this reduces to (1.8).

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References