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Addendum to "Almost Split Sequences in Subcategories"*

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In Corollary 4.2 we showed that for subcategories **C** of mod Λ closed under extensions of the form $\mathbf{C} = \operatorname{Sub} M$ there are only a finite number of nonisomorphic indecomposable objects in **C** which are Ext-injective and only a finite number which are Ext-projective. In this addendum we show that these numbers are the same, and at the same time we show a close connection with the theory of tilting modules ([2, see also [1]).

We first make the following observation.

LEMMA A.1. *Let M be a Λ-module and $0 \to^{f_0} M \to^{f_1} J_0 \to J_1$ be a minimal injective resolution of M. Then there is an exact sequence* $0 \to \operatorname{Ext}^1(\ ,M) \to D(\operatorname{Tr} DM,\ ) \to (\ ,J_1)(\ ,\operatorname{Im} f) \to 0.$

Proof. We have a minimal projective resolution $D(J_1) \to D(J_0) \to D(M) \to 0$, hence an exact sequence $D(J_0)^* \to D(J_1)^* \to \operatorname{Tr} D(M) \to 0$. From this we get an exact sequence of functors $D(D(J_1)^*,\ ) \to D(D(J_0)^*,\ ) \to D(\operatorname{Tr} DM,\ ) \to 0$. Since $D(J_i)$ is projective, we have a natural isomorphism $(D(J_i)^*,\ ) \xrightarrow{\sim} D(J_i) \otimes -$, and further a natural isomorphism $D(J_i) \otimes - \xrightarrow{\sim} D(\ ,J_1)$. This gives an exact sequence $(\ ,J_1) \to (\ ,J_0) \to D(\operatorname{Tr} DM,\ ) \to 0$. We now get our desired exact sequence by considering the commutative exact diagram.

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\begin{array}{ccccccccccc}
0 \to & (\ ,M) & \to & (\ ,J_0) & \longrightarrow & (\ ,\operatorname{Im} f_1) & \to & \operatorname{Ext}^1(\ ,M) & \to 0 \\
 & \| & & \| & & \downarrow & & \downarrow & \\
0 \to & (\ ,M) & \to & (\ ,J_0) & \xrightarrow{(\ ,f_1)} & (\ ,J_1) & \to & D(\operatorname{Tr} DM,\ ) & \to 0 \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & \dfrac{(\ ,J_1)}{(\ ,\operatorname{Im} f_1)} & \xrightarrow{\sim} & C & \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & 0 & & 0 &
\end{array}
$$

PROPOSITION A.2. *Assume* $M = \bar{M}$. *Then* Sub M *is closed with respect to extensions if and only if* $\operatorname{Ext}^1(M,M) = 0$ *and* $\tau_M(J_1) \subseteq \Omega^{-1}(M)$, *where* $0 \to M \to J_0 \to J_1$ *is part of a minimal injective resolution of* M *and* $\tau_M(J_1) = \{\sum \operatorname{Im} f \mid f \in (M, J_1)\}$.

Proof. From Corollaries 5.3 and 5.7 we have that Sub M is closed with respect to extensions if and only if $(\operatorname{Tr} DM, M) = 0$ and hence if and only if $D(\operatorname{Tr} DM, M) = 0$. Now by using Lemma A.1 we have that $D(\operatorname{Tr} DM, M) = 0$ if and only if $\operatorname{Ext}^1(M,M) = 0$ and $(M, \Omega^{-1}(M)) \to^{(M,i)} (M, J_1)$ is an isomorphism, where i is the inclusion $\Omega^{-1}(M) \hookrightarrow J_1$, which is the same as saying that $\operatorname{Ext}^1(M,M) = 0$ and $\tau_M(J_1) \subseteq \Omega^{-1}(M)$.

COROLLARY A.3. *Assume* $M = \bar{M}$ *is a faithful module. Then* Sub M *is closed with respect to extensions if and only if* $\operatorname{Ext}^1(M,M) = 0$ *and* inj · dim $M \leqslant 1$.

Proof. If M is faithful, we have a monomorphism $\Lambda \to M^n$ for some n. If I is injective, $\tau_\Lambda(I) = I$ then implies that $\tau_M(I) = I$. Hence $\tau_M(J_1) = J_1 \subseteq \Omega^{-1}(M)$ if and only if inj · dim $M \leqslant 1$. Our result now follows from Proposition A.2.

COROLLARY A.4. *Assume* $M = \bar{M}$ *is a faithful module such that* Sub M *is closed with respect to extensions. Then the* Ext-*injective modules in* Sub M *all have injective dimension at most one.*

Proof. Let I be the injective Λ-module $D\Lambda^{\mathrm{op}}$ and $\mathbf{C} = \operatorname{Sub} M$. Then in the notation of Section 4, we have the exact sequence $0 \to A^I_{\mathbf{C}} \to I_{\mathbf{C}} \to I$ which induces the exact sequence of functors $0 \to (\ , A^I_{\mathbf{C}}) \mid \mathbf{C} \to (\ , I_{\mathbf{C}}) \mid \mathbf{C} \to (\ , I) \mid \mathbf{C} \to 0$. Hence the fact that M is faithful implies that $I_{\mathbf{C}} \to I \to 0$ is exact. Also we know by Theorem 4.1, that the indecomposable summands of $A^I_{\mathbf{C}}$ and $I_{\mathbf{C}}$ are precisely the indecomposable Ext-injective modules in **C**. Since $I_{\mathbf{C}}$ is in add M, we have by Corollary A.2 that inj · dim $I_{\mathbf{C}} \leqslant 1$, which gives the desired result.

COROLLARY A.5. *Assume $M = \bar{M}$ is a faithful Λ-module such that* Sub *M is closed with respect to extensions. Then the number of nonisomorphic indecomposable* Ext*-injectives and nonisomorphic indecomposable* Ext*-projectives in Sub M coincide.*

Proof. Let $A = A_1 \amalg \cdots \amalg A_k$ be a direct sum of the nonisomorphic indecomposable Ext-injectives in Sub M. Then DA satisfies the following axioms for a tilting module given by Happel and Ringel [2]:

(1) proj · dim $DA \leqslant 1$.

(2) $\mathrm{Ext}^1(DA, DA) = 0$.

(3) There exists an exact sequence $0 \to \Lambda^{\mathrm{op}} \to A' \to A'' \to 0$ with A' and A'' in add DA.

Hence by Corollary 3.2 of [2] we have that the number of nonisomorphic indecomposable summands of A is the same as the number of nonisomorphic indecomposable projective Λ-modules, which by Proposition 3.1 are exactly the Ext-projectives in Sub M when M is faithful.

We now get our main result.

THEOREM A.6. *Assume that* Sub *M is closed with respect to extensions. Then* Sub *M has the same number of nonisomorphic indecomposable* Ext*-projectives and* Ext*-injectives.*

Proof. We only make the observation that M is a faithful $\Lambda/\mathrm{ann}\, M$-module and that we have the embedding Sub $M \subseteq \mathrm{mod}(\Lambda/\mathrm{ann}\, M) \subseteq \mathrm{mod}\, \Lambda$. Hence the result follows from Corollary A.5.

REFERENCES

1. K. BONGARTZ, Tilted algebras, preprint
2. D. HAPPEL AND C. M. RINGEL, Tilted algebras, preprint.