JOURNAL OF ALGEBRA **201,** 363–372 (1998)
ARTICLE NO. JA977245

# Higher Derivations and Local Cohomology Modules*

Caijun Zhou†

*Department of Mathematics, Shanghai Normal University, Shanghai, 200234, People's Republic of China*



## 1. INTRODUCTION

Throughout this paper $R$ is a commutative noetherian ring with unit. If $M$ is an $R$-module and $I$ an ideal of $R$, we denote by $H_I^i(M)$ the $i$th local cohomology module of $M$ with support in $V(I) = \{P | P \supset I, P \in \operatorname{Spec} A\}$. The local cohomology modules have been widely studied by a number of authors. Yet despite much effort, the structure of these modules is still full of mystery. In recent years, some important properties of the local cohomology module $H_I^i(R)$ were established when $R$ is a regular ring containing a field.

Let us recall some of these achievements in the following. If $R$ is a regular ring containing a field $K$ and $I$ an ideal of $R$, then the local cohomology modules of $R$ have the following properties:

(1) $H_m^j(H_I^i(R))$ is injective, where $m$ is any maximal ideal of $R$;

(2) $\operatorname{id}_R H_I^i(R) \leq \dim H_I^i(R)$;

(3) The set of the associated primes of $H_I^i(R)$ is finite;

(4) All the Bass numbers of $H_I^i(R)$ are finite.

Here $\operatorname{id}_R H_I^i(R)$ stands for the injective dimension of $H_I^i(R)$, $\dim H_I^i(R)$ stands for the dimension of the support of $H_I^i(R)$ in Spec $R$, and the $j$th Bass number of $H_I^i(R)$ with respect to a prime $P$ of $R$ is defined as $\operatorname{length}_K(\operatorname{Ext}^j_{R,p}(K, (H_I^i(R)_P)))$, where $K$ is the fraction field of $R/P$ [Ba].

The above results were established by Huneke and Sharp [Hu-Sh] when $K$ is of positive characteristic $p$ and by Lyubeznik [Ly] when $K$ is of characteristic zero. The method of Lyubeznik is completely different from

* Partially supported by the NSF of China and STF of Shanghai Higher Education.
† E-mail: Zhoucj@dns.shtu.edu.cn.



0021-8693/98 $25.00

that of Huneke and Sharp. While Huneke and Sharp use the Frobenius functor, Lyubeznik uses the theory of $D$-modules developed in [Bj].

In this paper, we will use higher derivations to study the local cohomology modules of a unramified regular ring, especially the local cohomology modules of the formal power series ring $K[[X_1, X_2, \dots, X_n]]$ over either a field or a complete $p$-ring $K$. We can give a uniform proof for results (1) and (2) cited above. The treatment of the paper should be thought as a natural extension of that of Lyubeznik [Ly]. One important thing of our method is that it gives one an insight to the study of the local cohomology modules of unramified regular local rings containing no field. In the paper, we can prove that if $R$ is a unramified regular local ring containing no field and $m$ is its unique maximal ideal, then

(i) $\mathrm{id}_R H_m^i(H_I^j(R)) \leq 1$ for all ideal $I$ and for all $i, j$;

(ii) $\mathrm{id}_R H_I^i(R) \leq \dim H_I^i(R) + 1$ for all ideal $I$ and for all $i$.

## 2. HIGHER DERIVATIONS

It is well known that, in the context of positive characteristic $p$, the concept of derivation, thus the $D$-modules, has less interest, and a more meaningful concept is that of higher derivation. The theory of higher derivation was initiated by Hasse and Schmidt [Ha-Sc]. In this section, we recall a few standard facts about higher derivations and give some special examples needed in the rest of the paper.

Let $K$ be a subring of the ring $R$. The notation $\mathrm{Der}_K(R)$ stands for the set of $K$-derivations of $R$. A higher $K$-derivation from $R$ to $R$ is a sequence $D = (D_0, D_1, D_2, \dots)$ of $K$-linear maps such that $D_0$ is the identity map of $R$ and for all $n \geq 1, x, y \in R$

$$D_n(xy) = \sum_{i+j=n} D_i(x) D_j(y).$$

These conditions are equivalent to saying that the map $E_t\colon R \to R[[t]]$ ($t$ an indeterminate) defined by

$$E_t(x) = \sum_{i=0}^{\infty} D_i(x) t^i$$

is a $K$-algebra homomorphism with $E_t(x) = x \bmod (t)$. We will use $\mathrm{HDer}_K(R)$ to denote the set of all higher $K$-derivations of $R$.

*Remark* 2.1. (i) Let $D$ be a higher $K$-derivation of $R$. Then $D_1 \in \mathrm{Der}_K(R)$.

(ii) Suppose $R$ contains a field of characteristic zero, and $d \in \mathrm{Der}_K(R)$. Then $D = (1, d, d^2/2!, \dots, d^n/n!, \dots)$ is a higher $K$-derivation of $R$.

(iii) Suppose $R$ is a ring of prime characteristic $p$. Let $d \in \mathrm{Der}_K(R)$, $D \in \mathrm{HDer}_K(R)$. Then for $x \in R$ and $r$ an positive integer

$$d(x^p) = 0, \qquad D_{p^r}(x^{p^r}) = \left(D_1(x)\right)^{p^r}, \qquad \text{and}$$

$$D_n(x^{p^r}) = 0 \qquad \text{for all } 0 < n < p^r.$$

(iv) Let $S$ be a multiplicative subset of $R$. Then every $K$-derivation (resp. higher $K$-derivation) of $R$ extends uniquely to the ring $R_S$.

EXAMPLES 2.2. (a) Let $R = K[X_1, X_2, \dots, X_n]$ be the ring of polynomials in $n$ variables over a field $K$ of characteristic $p > 0$. By [Mat2, Theorem 30.2 and 30.3], one can easily see that $R$ is 0-smooth over $K$. Thus for any $d \in \mathrm{Der}_K(R)$ there exists a $D \in \mathrm{HDer}_K(R)$ such that $D_1 = d$ by [Mat2, Theorem 27.1].

(b) Let $R = K[[X_1, X_2, \dots, X_n]]$ be the ring of formal power series in $n$ variables over a field $K$. The $K$-linear derivations form a free $R$-module on the $n$ generators $d_1, d_2, \dots, d_n$, where $d_j : R \to R$ is the partial differential with respect to $X_j$, $d_i d_j = d_j d_i$, and $d_i r - r d_i = \partial r/\partial X_i$ for all $i, j$ and all $r \in R$. If ch $K = 0$, then $(1, d_i, \dots, d_i^n/n!, \dots)$ is a higher $K$-derivations on $R$. If ch $K = p > 0$, it is known that $R$ is not always 0-smooth over $K$. In fact, Tanimoto [Tan] proved that $R$ is 0-smooth over $K$ only when ch $K = p$, and $[K : K^p]$ is finite. Thus we cannot use Theorem 27.1 in [Mat2] to assert that there exists a higher $K$-derivation $D_i$ such that $(D_i)_1 = d_i$ for each $i$ $(1 \le i \le n)$. However, $R$ is the completion of the polynomial ring $S = K[X_1, X_2, \dots, X_n]$ with respect to the ideal $m = (X_1, X_2, \dots, X_n)$ of $S$, and each $d_i$ is the natural extension of the partial differential $d_i'$ with respect to $X_i$ of $S$. Let $D'$ be a higher $K$-derivation on $S$ such that $D_1' = d'$. It is easy to check that $D_k'(m^s) \subset m^{s-k}$ for all $s \ge k$. Hence $D'$ can be extended naturally as a higher $K$-derivation $D$ on $R$. In particular, we have proved that, for each $d_i$ $(1 \le i \le n)$ there exists a higher $K$-derivation $D_i \in \mathrm{HDer}_K(R)$ such that $(D_i)_1 = d_i$.

The higher derivations given in Examples 2.2(b) will play an important role in the rest of the paper. In fact, we need the following generalization of Examples 2.2(b) to treat the case of unramified regular local ring containing no field.

PROPOSITION 2.3. *Let $R = K[[X]]$ be the ring of formal power series in one variables over a ring $K$; let $d : R \to R$ be the partial differential with respect to $X$. Then there exists a higher $K$-derivation $D$ such that $D_1 = d$.*

*Proof.* Set $D_0$ to be the identity map on $R$. For each positive integer $n$, define $D_n : R \to R$ as $D_n(X^s) = \binom{s}{n}X^{s-n}$ for all $s$ (here, we use the convention $X^0 = 1$, and note that $\binom{s}{n} = 0$ for $s < n$), $D_n(a) = 0$ for $a \in K$, and $D_n$ is $K$-linear on $R$. We assert that $D = (D_0, D_1, D_2, \dots)$ is a higher $K$-derivation on $R$.

In fact, for any two elements $x, y \in R$, we can write

$$x = a_0 + a_1X + \cdots + a_nX^n + \cdots, \qquad y = b_0 + b_1X + \cdots + b_nX^n + \cdots,$$

where $a_i, b_j \in K$ for all $i, j$. So $xy = c_0 + c_1X + \cdots + c_nX^n + \cdots$, where $c_n = \sum_{i+j=n} a_ib_j$. Thus for each positive integer $n$ we have

$$D_n(xy) = \sum_{k=0}^{\infty} c_kD_n(X^k) = \sum_{k=0}^{\infty} c_k\binom{k}{n}X^{k-n}.$$

On the other hand, observe that

$$\begin{aligned}\sum_{i+j=n} D_i(x)D_j(y) &= \sum_{i+j=n}\left(\sum_{r=0}^{\infty} a_r\binom{r}{i}X^{r-i}\right)\left(\sum_{s=0}^{\infty} b_s\binom{s}{j}X^{s-j}\right)\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_rb_s\sum_{i+j=n}\binom{r}{i}\binom{s}{j}X^{r+s-n}\\ &= \sum_{k=0}^{\infty} c_k\binom{k}{n}X^{k-n}\end{aligned}$$

because $\sum_{i+j=n}\binom{r}{i}\binom{s}{j} = \binom{r+s}{n}$. Hence we have proved

$$D_n(xy) = \sum_{i+j=n} D_i(x)D_j(y)$$

and thus $D$ is the desired higher $K$-derivation.

The following result is an immediate consequence of Proposition 2.3 because of $R = K[X_1, X_2, \dots, X_n]] = K[[X_1, X_2, \dots, X_{n-1}]][[X_n]]$.

COROLLARY 2.4. *Let $R = K[[X_1, X_2, \dots, X_n]]$ be a ring of formal power series in $n$ variables over a ring $K$. Let $d_i : R \to R$ be the partial differential with respect to $X_i$ $(1 \le i \le n)$. Then there exists a higher derivation $D_i$ such that $(D_i)_1 = d_i$ and $(D_i)_k$ is trivial on the subring $K[[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]]$ for $k \ge 1$.*

## 3. $\overline{D}$-MODULES

In this section we introduce the notion of $\overline{D}$-modules which is a natural extension of the usual $D$-modules studied in [Bj].

Let $K$ be a subring of a ring $R$. We define $\overline{D}(R, K)$ as the subring of $\mathrm{Hom}_K(R, R)$ generated by the set $\{D_n | D \in \mathrm{HDer}_K(R),\ n \geq 1\}$ and the multiplications by elements of $R$. By a $\overline{D}(R, K)$-module we always mean a left $\overline{D}(R, K)$-module. The injective ring homomorphism $R \to \overline{D}(R, K)$ that sends $r$ to the map $R \to R$ which is the multiplication by $r$ gives $\overline{D}(R, K)$ a structure of an $R$-algebra. Every $\overline{D}(R, K)$-module $M$ is automatically an $R$-module via this map. Note that the usual differential ring $D(R, K)$, which is a subring of $\mathrm{Hom}_K(R, R)$ generated by $K$-derivations and the multiplications by elements of $R$, is a subring of $\overline{D}(R, K)$.

EXAMPLES 3.1. (i) The natural action of $\overline{D}(R, K)$ on $R$ makes $R$ a $\overline{D}(R, K)$-module.

(ii) If $M$ is a $\overline{D}(R, K)$-module and $S \subset R$ is a multiplicative subset of $R$, then $M_S$ carries a natural structure of $\overline{D}(R, K)$-module. Namely for $r \in R$ we set $r(x/s) = rx/s$, and for D $\in \mathrm{HDer}_K(R)$ and $n \geq 1$ we set

$$D_n(x/s) = \sum_{i+j=n} D_i(1/s) D_j(x),$$

where $D_i(1/s)$ is uniquely determined by Remark 1.1(iv). In particular, $\overline{D}(R, K)_S$ has a natural structure of $\overline{D}(R, K)$-module. This implies that $\overline{D}(R, K)_S$ has a natural ring structure and $M_S$ has a natural structure of $\overline{D}(R, K)_S$-module.

In general, the ring $D(R, K)$ does not have too many good properties, so neither does the ring $\overline{D}(R, K)$. It is well known that if $K$ is a field of characteristic zero and $R$ is a ring of formal power series in a finite number of variables over $K$, Then $D(R, K)$ is left and right noetherian [Bj, 3.1.6].

In the rest of the section, we assume that $R = K[[X_1, X_2, \ldots, X_n]]$ is the formal power series ring with $n$ variables over a ring $K$. Let $d_i$ $(1 \leq i \leq n)$ be the partial differential with respect to $X_i$. We say a subring $S$ of $\overline{D}(R, K)$ is admissible if $R \subset S$, $d_i \in S$ for $1 \leq i \leq n$, and for each $i$ there exists a higher $K$-derivation $D_i$ such that $(D_i)_1 = d_i$ and $(D_i)_n \in S$ for all $n \geq 1$. Clearly, the ring $\overline{D}(R, K)$ is always admissible by Corollary 2.4. If $K$ is a field of characteristic zero, then $D(R, K)$ is an admissible subring of $\overline{D}(R, K)$ by Remark 2.1.

Now, we turn to prove an interesting proposition which is a natural extension of Proposition 2.3 in [Ly].

PROPOSITION 3.2. *Let $K$ be a field, let $R = K[[X_1, X_2, \dots, X_n]]$ be a ring of formal power series in $n$ variables over $K$, and let $m$ be the maximal ideal of $R$. Let $E_R(K)$ be the injective hull of the residue field $K$ of $R$ in the category of $R$-modules and $c$ be a nonzero element of $E_R(K)$ with $mc = 0$. Suppose $S$ is an admissible subring of $\overline{D}(R, K)$. Then $E_R(K)$ has a natural structure of an $S$-module and as an $R$-module, $Sc \simeq E_R(K)$.*

*Proof.* Since $R$ is regular, $E_R(K) = H_m^n(R)$. Since $m$ is generated by $X_1, X_2, \dots, X_n$ it follows from the property of local cohomology that

$$E_R(K) \simeq R_{X_1X_2\cdots X_n}/N,$$

where $N = \sum_{i=1}^n R_{X_1\cdots X_{i-1}X_{i+1}\cdots X_n}$. So, by Examples 3.1(ii), $E_R(K)$ has a natural structure of $\overline{D}(R, K)$-module, and thus has a natural structure of $S$-module. To prove $Sc \simeq E_R(K)$, it suffices to prove $S(1/X_1X_2 \cdots X_n) = R_{X_1X_2\cdots X_n}$ for the $S$-module $R_{X_1X_2\cdots X_n}$. We use induction on $i$ to assert that if

$$1/(X_1X_2 \cdots X_n)^i \in S(1/X_1X_2 \cdots X_n)$$

then

$$1/(X_1X_2 \cdots X_2)^{i+1} \in S(1/X_1X_2 \cdots X_n)$$

and the conclusion follows immediately from this fact.

Suppose $1/(X_1X_2 \cdots X_n)^i \in S(1/X_1X_2 \cdots X_n)$. If ch $K = 0$ or ch $K = p$ and $p \nmid i$, then

$$d_1d_2 \cdots d_n\left(1/(X_1X_2 \cdots X_n)^i\right) = (-i)^n\left(1/(X_1X_2 \cdots X_n)^{i+1}\right).$$

Note that $(-i)^n \notin m$. So

$$1/(X_1X_2 \cdots X_n)^{i+1} \in S(1/X_1X_2 \cdots X_n).$$

If $p|i$, we can choose an integer $j$ such that $i = p^rj$ and $p \nmid j$. Since $S$ is an admissible subring of $\overline{D}(R, K)$, there exist higher $K$-derivations $D_1$, $D_2, \dots, D_n$ such that $(D_k)_1 = d_k$ for $1 \le k \le n$. Observe that

$$\begin{aligned}(D_1)_{p^r}(D_2)_{p^r} \cdots (D_n)_{p^r}&\left(1/(X_1X_2 \cdots X_n)^j\right)^{p^r} \\ &= (-j)^{np^r}\left(1/(X_1X_2 \cdots X_n)^{p^r(j+1)}\right).\end{aligned}$$

Hence

$$1/(X_1X_2 \cdots X_n)^{i+1} \in S(1/X_1X_2 \cdots X_n)$$

because $j \notin m$ and $p \ge 2$. That completes the proof.

Our main result in the paper is the following which extends Theorem 2.4(a) in [Ly] to positive characteristic $p$.

THEOREM 3.3. *Let* $R = K[[X_1, X_2, \dots, X_n]]$ *be a ring of formal power series in n variables over a field K, let S be an admissible subring of* $\overline{D}(R, K)$*, and let M be a S-module. If* $\dim_R M = 0$*, then M is an injective R-module.*

*Proof.* Let $\{e_i\}_{i \in I}$ be a $K$-basis of the scole of $M$ (recall the scole of $M$ is the submodule annihilated by the maximal ideal of $R$). It is clear that $N = \sum_{i \in I} S(e_i)$ is a submodule of $M$. By Proposition 3.2, $N$ is injective $R$-module, so $M = N \oplus N'$, where $N'$ is an $R$-module supported only at the maximal ideal of $R$. Since the natural map $N \to M$ induces an isomorphism on the scoles, $N' = 0$, so $M = N$. This proves the theorem.

## 4. LOCAL COHOMOLOGY MODULES OF REGULAR RINGS CONTAINING A FIELD

In this section, we study the injective dimension of the $i$th local cohomology module $H_I^i(R)$ for a regular ring $R$ containing a field and an ideal $I$ of $R$ by means of Theorem 3.3 in the last section. This method is essentially as the same as that of Lyubeznik [Ly].

First of all, we recall a useful description of local cohomology modules. Let $R$ be a noetherian ring and $I = (f_1, f_2, \dots, f_s)$ be an ideal of $R$. Let $M$ be an $R$-module. We get a complex from the Čech complex

$$0 \to M \to \oplus M_{f_i} \to M_{f_i f_j} \to \oplus M_{f_i f_j f_k} \to \cdots$$

whose $i$th cohomology module is $H_I^i(M)$. Here the map $M_{f_{i_1} \cdots f_{i_j}} \to M_{f_{k_1} \cdots f_{k_{j+1}}}$ induced by the corresponding differential is the natural localization (up to sign) if $\{i_1, \dots, i_j\}$ is a subset of $\{k_1, \dots, k_{j+1}\}$ and is 0 otherwise. Thus if $R = K[[X_1, X_2, \dots, X_n]]$ is the formal power series ring in $n$ variables over a ring $K$ and $S$ is an admissible subring of $\overline{D}(R, K)$, then $H_I^i(M)$ is also a $S$-module for a $S$-module $M$ and for all ideals $I$ of $R$. In particular, $H_I^j(H_J^i(R))$ is a $S$-module for all $i, j$. The following result is an immediate consequence of Theorem 3.3.

PROPOSITION 4.1. *Let* $R = K[[X_1, X_2, \dots, X_n]]$ *be a ring of formal power series in n variables over a field K, let m be the maximal ideal of R, and I be an ideal of R. Then* $H_m^i(H_I^j(R))$ *is injective for all* $i, j$.

Now, we are ready for the main result of this section.

THEOREM 4.2 (cf. [Ly], Corollary 3.6(a)(b)]. *Let $R$ be any regular $K$-algebra over a field $K$ and let $I$ be an ideal of $R$. Then*

(a) $H^j_m(H^i_I(R))$ *is injective for every maximal ideal $m$ of $R$.*

(b) $\mathrm{id}_R(H^i_I(R)) \le \dim_R(H^i_I(R))$.

*Proof.* (a) Since $H^j_m(H^i_I(R)) \simeq H^j_{mR_m}(H^i_{IR_m}(R_m))$, we can assume that $R$ is local and $m$ is the maximal ideal of $R$. Let $\hat{R}$ be the completion of $R$ with respect to $m$. Then $H^j_m(H^i_I(R)) \otimes \hat{R} \simeq H^j_{m\hat{R}}(H^i_{I\hat{R}}(\hat{R}))$. By Cohen's structure theorem $\hat{R}$ is a ring of formal power series in several variables over a coefficient field $K$ or $\hat{R}$. By Proposition 4.1, $H^j_{m\hat{R}}(H^i_{I\hat{R}}(\hat{R}))$ is an injective $\hat{R}$-module of dimension 0. Because $E_{\hat{R}}(\hat{R}/m\hat{R}) = E_R(R/m)$, thus $H^j_m(H^i_I(R))$ is injective. This proves (a).

(b) $\mathrm{id}_R(H^i_I(R)) \le \dim_R(H^i_I(R))$ if and only if $\mathrm{id}_{R_P}(H^i_{IR_P}(R_P)) \le \dim_{R_P} H^i_{IR_P}(R_P))$ for all primes $P$ of $R$. Thus we can assume $R$ is local and $m$ is the maximal ideal of $R$. By induction on $\dim_R(H^i_I(R))$ we can assume that for all non-maximal primes $P$,

$$\mathrm{id}_{R_P} H^i_{IR_P}(R_P) \le \dim_{R_P} H^i_{R_P}(R_P) < \dim H^i_I(R).$$

Hence, if $J^*$ is a minimal injective resolution of $H^i_I(R)$, all $J^k$ with $k \ge \dim_R H^i_I(R)$ are supported at $m$, i.e., $H^0_m(J^k) = J^k$ for $k \ge \dim_R H^i_I(R)$. By (a) and [Ly, Lemma 1.4], we conclude that all the differentials $J^k \to J^{k+1}$ are zero for $k \ge \dim_R H^i_I(R)$. Since $J^*$ is minimal, $J^k = 0$ for $k > \dim_R H^i_I(R)$. This proves (b).

Now, we make a remark before the end of the section.

*Remark* 4.3. (i) Readers who have read the paper [Ly] may find that we do not consider the functor $\mathscr{T}$ defined in [Ly, Sect. 1]. For this functor, one can give a proof (similar to that of Lyubeznik) of the result which states that the injective dimension of $\mathscr{T}(R)$ is no more than the dimension of $\mathscr{T}(R)$ for a regular ring $R$ containing a field of positive characteristic.

(ii) It seems very interesting to study admissible subrings of $\overline{D}(R, K)$ for the formal power series ring $R = K[[X_1, X_2, \ldots, X_n]]$ over a field $K$ of positive characteristic. A natural question is whether there exists an admissible subring such that it is both left and right noetherian.

## 5. LOCAL COHOMOLOGY OF UNRAMIFIED REGULAR LOCAL RINGS CONTAINING NO FIELD

In this section, we study the injective dimensions of local cohomology modules of unramified regular local rings containing no field.

THEOREM 5.1. *Let $R$ be an unramified regular local ring containing no field and let $m$ be its maximal ideal. Then*

(i) $\mathrm{id}_R H^j_m(H^i_I(R)) \le 1$ *for every ideal $I$ of $R$ and for all $i, j$.*

(ii) $\mathrm{id}_R H^i_I(R) \le \dim H^i_I(R) + 1$ *for every ideal $I$ of $R$ and for all $i$.*

*Proof.* (i) Similar to the proof of Theorem 1.2(a), we may assume $R = K[[X_1, X_2, \dots, X_n]]$ is a ring of formal power series in $n$ variables over a complete $p$-ring $K$. Note that $\overline{D}(R, K)$ is always admissible, and for every element $d \in \overline{D}(R, K)$, $d$ is $K$-linear. Thus we can naturally derive from $\overline{D}(R, K)$ an admissible subring $S$ of $\overline{D}(R/pR, K/pK)$. In fact we can choose $S$ to be the subring of $\overline{D}(R/pR, KpK)$ generated by elements of $R/pR$ and $D_n$ for all $D \in \mathrm{HDer}(R, K)$ and all $n$. Since $H^i_I(R)$ is a $\overline{D}(R, K)$-module, it is clear that $H^j_m(H^i_I(R))$ is a $\overline{D}(R, K)$-module.

Consider the map $H^j_m(H^i_I(R)) \xrightarrow{p} H^j_m(H^i_I(R))$. It is easy to see that both the kernel $N$ and the cokernel $N'$ of this map are $S$-modules of dimension (as $R/pR$-modules) zero. Since $S$ is admissible, we conclude that $N, N'$ are injective $R/pR$-modules by Theorem 3.3. Thus

$$\mathrm{Ext}^k_R(R/m, N) = 0 \qquad \textit{and} \qquad \mathrm{Ext}^k_R(R/m, N') = 0$$

for all $k \ge 2$. Set $M = H^j_m(H^i_I(R))$. Consider the short exact sequences

$$0 \to N \to M \to pM \to 0$$
$$0 \to pM \to M \to N' \to 0.$$

We have the following exact sequences

$$0 \to \mathrm{Ext}^2_R(R/m, M) \to \mathrm{Ext}^2_R(R/m, pM) \to 0$$
$$\cdots \to \mathrm{Ext}^2_R(R/m, pM) \to \mathrm{Ext}^2_R(R/m, M) \to 0.$$

Hence $\mathrm{Ext}^2_R(R/m, M) \xrightarrow{p} \mathrm{Ext}^2_R(R/m, M)$ is a surjective map. Since this map is trivial, we assert that $\mathrm{Ext}^2_R(R/m, M) = 0$. Therefore $\mathrm{id}_R M \le 1$ because $\dim_R M = 0$.

(ii) By induction on $\dim H^i_I(R)$ we can assume that for every non-maximal prime ideal $P$ of $R$, $\mathrm{id}_{R_P} H^i_{I_p}(R_p) < \mathrm{id}_R H^i_I(R)$. Let $J^*$ be a minimal injective resolution of $H^i_I(R)$. Thus all $J^k$ with $k \ge \dim H^i_I(R) + 1$ are supported at $m$, i.e., $H^0_m(J^k) = J^k$ for $k \ge \dim H^i_I(R) + 1$. Set $E^k = H^0_m(J^k)$ for all $i$. Thus the local cohomology $H^j_m(H^i_I(R))$ can be computed as the homology of the complex

$$0 \to E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots.$$

Set $Z^j = \operatorname{Ker} \partial^j, B^j = \operatorname{Im} \partial^j$. Now we use induction to conclude that $\operatorname{id}_R(Z^j) \leq 1$ and $B^j$ is injective for all $j$. Suppose the conclusion holds for those $k < j$. Consider the exact sequences

$$0 \to B^{j-1} \to Z^j \to H_m^j\big(H_I^i(R)\big) \to 0.$$

By (i), $\operatorname{id}_R H_m^j(H_I^i(R)) \leq 1$, and by induction hypothesis, $B^{j-1}$ is injective, so $\operatorname{id}_R Z^j \leq 1$. Moreover, we have the exact sequence

$$0 \to Z^j \to E^j \to B^j \to 0$$

and thus $B^j$ is injective. In particular we have proved that $\operatorname{id}_R Z^d \leq 1$, for $d = \dim H_I^i(R)$. Hence $E^{d+2} = 0$ because

$$Z^d \to E^d \to E^{d+1} \to E^{d+2} \to \cdots$$

is a minimal injective resolution of $Z^d$. This proves (ii).

## REFERENCES


[Ba] H. Bass, Ubiquity of Gorenstein rings, *Math. Z.* **82** (1963), 8–28.

[Bj] J. E. Bjork, "Rings of Differential Operators," Amsterdam, North-Holland, 1979.

[G] A. Grothendieck, "Local Cohomology," Lecture Notes in Math., Vol. 41, Springer-Verlag, Berlin/Heidelberg/New York, 1966.

[Ha] R. Hartshorne, Cohomological dimension of algebraic varieties, *Ann. of Math.* **88** (1968), 403–450.

[Ha-Sc] H. Hasse and F. K. Schmidt, Noch eine Begründung der Theorie der höheren Differential-quotienten in einem algebraischen Funktionenköroper einer Unbestimmten., *J. Reime Angew. Math.* **177** (1937), 215–237.

[Hu-Sh] C. Huneke and R. Sharp, Bass numbers of local cohomology modules, *Trans. Amer. Math. Soc.* **339** (1993), 765–779.

[Ly] G. Lyubeznik, Finiteness properties of local cohomology modules, *Invet. Math.* **113** (1994), 41–55.

[Mat1] H. Matsumura, Integrable derivations, *Nagoya Math. J.* **87** (1982), 227–245.

[Mat2] H. Matsumura, "Commutative Ring Theory," Cambridge Univ. Press, New York, 1986.

[O] A. Ogus, Local cohomology dimension of algebraic varieties, *Ann. of Math.* **98** (1973), 1–34.

[Pe-Sz] C. Peskine and L. Szpiro, Dimension projective finie et cohomology locale, *Publ. Math. Inst. Hautes Études Sci.* **42** (1973), 323–395.

[Tan] H. Tanimoto, Smoothess of Noetherian rings, *Nagoya Math. J.* **95** (1984), 163–179.

[Zh] C. J. Zhou, Functors of the category of abelian sheaves on regular affine schemes, *Acta Math. Sinica*, **12** (1996), 413–414.