Note

A conjecture concerning Ramsey’s theorem

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1. Introduction

One version of Ramsey’s theorem [2] states that for any positive integer c, if the edges of a complete graph G on a countably infinite number of vertices are partitioned into c classes (c-colored) there results a complete countably infinite subgraph H of G all of whose edges are in the same class (all edges of H have the same color).

What happens if all c colors must be used in the coloring? To investigate this, we define an exact c-coloring of a set S to be a surjection from S to \( T \) with \( \text{card}(T) = c \) and we consider the following proposition \( P(c, m) \) for positive integers \( c, m \).

\[ P(c, m) : \text{If the edges of a countable infinite complete graph } G \text{ are exactly } c\text{-colored, then there exists a countable infinite complete subgraph } H \text{ of } G \text{ whose edges are exactly } m\text{-colored.} \]

The purpose of this note is to inquire as to which pairs \( c, m \) of positive integers make \( P(c, m) \) a true statement.

2. Sufficient conditions

If \( m = 1 \), then \( P(c, m) \) is just Ramsey’s theorem with the hypothesis strengthened from \( c \)-coloring to exact \( c \)-coloring, so \( P(c, m) \) is true a fortiori.
If \( m = 2 \), then it turns out that \( P(c, m) \) is true with \( c \geq 2 \). Suppose that \( G \) is exactly \( c \)-colored with \( c \geq 2 \). Ramsey's theorem guarantees a one-colored infinite complete subgraph \( F \); say its edges have color 1. Take \( F \) to be maximal with respect to the property that all its edges have color 1. \( G \) must contain points not in \( F \) since \( c \geq 2 \). Choose a point \( p \) not in \( F \) and consider the joins from \( p \) to \( F \). Some color other than color 1 must occur among these joins since \( F \) is maximal. If some color other than color 1, call it color 2, occurs infinitely often, then simply take \( H \) to be the complete graph on \( p \) together with those vertices in \( F \) to which \( p \) is attached by edges colored 2.

On the other hand, if no color other than color 1 occurs infinitely often among the joins from \( p \) to \( F \), then simply take \( H \) to be the complete graph on \( p \) together with those vertices in \( F \) to which \( p \) is attached by color 1, together with one additional vertex in \( F \).

If \( c = m \), then \( P(c, m) \) is trivially true as we can take \( H = G \).

We consolidate the above information.

**Theorem.** \( P(c, m) \) is true if one of the following conditions holds:

1. \( m = 1 \),
2. \( m = 2, c \geq 2 \),
3. \( c = m \geq 3 \).

It may seem surprising that these conditions appear to be both necessary and sufficient, but I conjecture that this is so.

**Conjecture.** \( P(c, m) \) is true if and only if one of the following holds:

1. \( m = 1 \),
2. \( m = 2, c \geq 2 \),
3. \( c = m \geq 3 \).

### 3. Necessary conditions

It is trivially true that \( P(c, m) \) is false for \( m > c \). Therefore, in light of the theorem, the values of \( c, m \) which remain to be examined are those for which \( c > m \geq 3 \), and all of the cases below refer to this 'critical region'. The constructions which are displayed as counterexamples of \( P(c, m) \) have appeared in [1].

We let \( N_n = \{1, \ldots, n\} \) and say a set \( S \) of ordered pairs of positive integers has density \( d \) when

\[
\lim_{m, n \to \infty} \frac{\text{card}(S \cap (N_m \times N_n))}{mn} = d.
\]

**Case 1:** \( P(c, m) \) is false if \( c \) is even and \( m \) is odd.

(Note that this covers 1/4 of the ordered pairs \((c, m)\) by density.) Let \( c = 2x \) and let \( U \) be a set of vertices designated \( 1, \ldots, x \). The vertices of \( G \) are the elements of
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U together with a countable infinity of other points. Color with 0 all the edges on
\( G - U \). The edges from \( i \in U \) to \( G - U \) get color \( i \). The edge from \( i \) to \( j \) \((i < j, i,j \in U)\) gets
color \( i' \). Altogether this takes \( 2x = c \) colors. Any subgraph \( H \) of \( G \) contains only
vertices of \( G - U \), in which case it is exactly 1-colored, or it contains \( y \) vertices of \( U \), in
which case it is exactly 2-colored.

Case 2: \( P(c, m) \) is false if \( c \) is odd, \( m \) even, and \( m > (c + 3)/2 \).

(Density 1/8.) Let \( c = 2x + 1 \). Color each edge of the complete bipartite graph
\( K_{2,x} \) with a different color. Color all other edges of \( G \) with the one remaining color.
Any subgraph \( H \) which uses more than \( (c + 3)/2 \) colors must contain both points in the
component of \( K_{2,x} \) which contains two points. Therefore, \( H \) is exactly \( m \)-colored,
where \( m = 2y + 1 \), an odd number.

Case 3: \( P(c, m) \) is false if \( c = C(x, 2) + C(y, 2) + 1 \) and \( m \neq C(a, 2) + C(b, 2) + 1 \), where
\( C(x, y) \) is a binomial coefficient.

(Density 0, since the condition on \( c \) is equivalent to \( 8c - 6 = (2x - 1)^2 + (2y - 1)^2 \), and
almost no positive integers are the sum of two squares.) Color each edge of \( K_{x,y} \), with a different color. Color all other edges of \( G \) with yet a different color. This uses
\( c = C(x, 2) + C(y, 2) + 1 \) colors, and it is clear that \( H \) must use \( m = C(a, 2) + C(b, 2) + 1 \)
colors (with \( 0 \leq a \leq x, 0 \leq b \leq y \)).

Case 4: \( P(c, m) \) is false if \( c = 1 + xy \) and \( m \neq 1 + ab \) with \( 1 \leq a \leq x, 1 \leq b \leq y \).

Standard number theory arguments show that for almost all \( c, c - 1 = xy \) with
\( x, y \geq M \), where \( M \) is a fixed positive integer. We define

\[
N(x, y) = \text{card}\{z: 1 \leq z \leq xy \text{ and } z \neq ab \text{ with } 1 \leq a \leq x \text{ and } 1 \leq b \leq y\}.
\]

Again, one uses elementary number theory to show \( \lim_{x, y \to \infty} N(x, y)/xy = 1 \). Thus, for
a density 1 of ordered pairs \( (c, m) \), \( c - 1 = xy \) and \( m - 1 \neq ab \) with \( a \leq x \) and \( b \leq y \).

Now color each edge of \( K_{x,y} \) with a different color, and color all other edges of
\( G \) with yet a different color. This uses \( c = 1 + xy \) colors, and it is clear that \( H \) must use
\( m = 1 + ab \) colors with \( 1 \leq a \leq x, 1 \leq b \leq y \). This construction includes the construction
in Case 2 as a special case.

Altogether the four cases cover a density 1 of ordered pairs \( (c, m) \). (Indeed, Case 4
covers a density 1 by itself.) However, there are pairs, such as \( (120, 16) \), which are not
covered by any case.

In proving the Conjecture in its entirety it may be necessary to finitize the graphs
under consideration. It is easy to show that the following assertion about finite graphs
is equivalent to the 'only if' part of the conjecture.

**Finite version of the conjecture.** Let \( c, m \) be positive integers with \( c > m \geq 3 \). There
exists a positive integer \( n \) and an exact \( c \)-coloring of the set of edges and vertices of
a complete graph \( K_n \) using the set of colors \( N_c \), such that no complete subgraph has its
vertices and edges colored using the colors of a set \( T \) with \( \text{card} (\{2, \ldots, c\} \cap T) = m \).
References