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Advances in Mathematics 226 (2011) 2618–2628

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The number of unit distances is almost linear for most norms

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Received 14 February 2010; accepted 21 September 2010

Available online 28 September 2010

Communicated by Gil Kalai

Abstract

We prove that there exists a norm in the plane under which no n-point set determines more than $O(n \log n \log\log n)$ unit distances. Actually, most norms have this property, in the sense that their complement is a meager set in the metric space of all norms (with the metric given by the Hausdorff distance of the unit balls).

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Keywords: Discrete geometry; Erdős unit distance problem; Baire category

1. Introduction

What is the maximum possible number $u(n)$ of unit distances determined by an n-point set in the Euclidean plane? This tantalizing question, raised by Erdős [4] in 1946, has motivated extensive research (see, e.g., Brass, Moser, and Pach [2] for a survey), but it remains wide open.

Erdős [4] proved a lower bound $u(n) = \Omega(n^{1+c/\log\log n})$ for a constant $c > 0$, attained for the $\sqrt{n} \times \sqrt{n}$ grid, and he conjectured that it has the right order of magnitude (and in particular, that $u(n) = O(n^{1+\varepsilon})$ for every fixed $\varepsilon > 0$). However, the current best upper bound is only $O(n^{4/3})$. It was first proved by Spencer, Szemerédi, and Trotter [10], based on the method of

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doi:10.1016/j.aim.2010.09.009

Szemerédi and Trotter [12], and several simpler proofs are available by now (by Clarkson et al. [3], by Aronov and Sharir [1], and the simplest one by Székely [11]).

The problem of unit distances has also been considered for norms other than the Euclidean one. For a norm[1] $\|\cdot\|$ on $\mathbb{R}^2$, let $u_{\|\cdot\|}(n)$ denote the maximum possible number of unit distances determined by n points in $(\mathbb{R}^2, \|\cdot\|)$.

If the boundary of the unit ball $B_{\|\cdot\|}$ of $\|\cdot\|$ contains a straight segment, then there are n-point sets with $\Omega(n^2)$ unit distances (indeed, if s is such a straight segment, we can create a complete bipartite pattern of unit distances by arranging the points in two straight rows parallel to s). On the other hand, if $\|\cdot\|$ is *strictly convex*, meaning that the boundary of $B_{\|\cdot\|}$ contains no straight segment, then $u_{\|\cdot\|}(n) = O(n^{4/3})$, as can be shown by a straightforward generalization of the known proofs for the Euclidean case.

Valtr [13], strengthening an earlier result of Brass, constructed a strictly convex norm $\|\cdot\|$ in the plane with $u_{\|\cdot\|} = \Omega(n^{4/3})$, thus showing that the upper bound cannot be improved in general for strictly convex norms.

A simple construction shows that $u_{\|\cdot\|}(n) = \Omega(n \log n)$ holds for every norm $\|\cdot\|$ (see [5] or, e.g., [2]). Here we will show that there exists a norm $\|.\|$ with $u_{\|\cdot\|}(n) = O(n \log n \log\log n)$, almost matching the lower bound. Actually, we show that most norms, in the sense of Baire category, have this property.

To formulate this result, we recall the relevant notions, referring, e.g., to Gruber [6, Chapter 13] for more background, original sources, and details. Let $\mathcal{B}$ be the set of all unit balls of norms in $\mathbb{R}^2$, i.e., of all closed bounded $\mathbf{0}$-symmetric convex sets containing $\mathbf{0}$ in the interior. Endowed with the Hausdorff metric[2] d_H, the set $\mathcal{B}$ forms a Baire space, meaning that each meager set[3] has a dense complement.

If P is some property that a norm on $\mathbb{R}^2$ may or may not have, we say that *most norms have property P* if the (unit balls of the) norms not having property P form a meager set in $\mathcal{B}$. A similar terminology is commonly used for convex bodies.

If most norms have property P_1 and most norms have property P_2, then most norms have both P_1 and P_2 (and similarly for countably many properties), which makes this approach a powerful tool for proving existence results. Starting with a paper of Klee [7], who proved that most norms are smooth and strictly convex, there have been many papers establishing that most norms or most convex bodies have various properties (see [6]). We add the following item to this collection.

Theorem 1.1. *There exists a constant C_0 such that most norms $\|\cdot\|$ on $\mathbb{R}^2$ satisfy*

$$u_{\|\cdot\|}(n) < C_0 n \log n \log\log n$$

[1] We recall that a (real) *norm* on a real vector space Z is a mapping that assigns a nonnegative real number $\|\mathbf{x}\|$ to each $\mathbf{x} \in Z$ so that $\|\mathbf{x}\| = 0$ implies $\mathbf{x} = \mathbf{0}$, $\|\alpha\mathbf{x}\| = |\alpha| \cdot \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$, and the triangle inequality holds: $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$. The *unit ball* of the norm $\|\cdot\|$ is the set $B_{\|\cdot\|} = \{\mathbf{x} \in Z\colon \|\mathbf{x}\| \leqslant 1\}$. The unit ball of any norm is a closed convex body B that is centrally symmetric about $\mathbf{0}$ and contains $\mathbf{0}$ in the interior. Conversely, every $B \subset Z$ with the listed properties is the unit ball of a (uniquely determined) norm.

[2] We recall that the Hausdorff distance $d_H(A, B)$ of two sets in the Euclidean plane is defined as $\min(h(A, B), h(B, A))$, where $h(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_2$, with $\|\cdot\|_2$ denoting the Euclidean distance.

[3] A set S in a metric (or topological) space X is *nowhere dense* if every nonempty open set $U \subseteq X$ contains a nonempty open set V with $V \cap S = \emptyset$. A *meager set* is a countable union of nowhere dense sets.

*for all $n \geqslant 3$ (*log *stands for logarithm in base* 2 *everywhere in this paper). In particular, there exists a smooth and strictly convex norm $\|\cdot\|$ with this property.*

Since, as was mentioned above, $u_{\|\cdot\|}(n) = \Omega(n \log n)$ for all norms, the bound in the theorem is tight up to the $O(\log\log n)$ factor. This factor comes out of a graph-theoretic result, Proposition 2.1 below, and it is not clear whether it is really needed.

The proof of the theorem has two main parts. We begin with the first, purely graph-theoretic part in Section 2. The result needed for the rest of the proof is Proposition 2.1, asserting the existence of a certain subgraph in every sufficiently dense graph with a given proper edge-coloring. Its proof relies heavily on a similar result of Přívětivý, Škovroň, and the author [8] (but the presentation below is self-contained).

Then, in Section 3 we continue with the second, geometric part of the proof of Theorem 1.1. Very roughly speaking, using the graph-theoretic result from the first part of the proof, we show that if there is a set P with many unit distances, under any norm, and if $\mathbf{u}_1, \dots, \mathbf{u}_k$ are all the mutually non-parallel unit vectors defined by pairs of points of P, then there are "many" linear dependences among the $\mathbf{u}_i$. Namely, there is an integer ℓ, such that some $\ell + 1$ vectors among the $\mathbf{u}_i$ can be expressed as linear functions of some other ℓ of the $\mathbf{u}_i$ (where the linear functions don't depend on the norm). Finally, we show that most norms don't admit such linear dependences—this is done by approximating the unit ball of the considered norm by a convex polygon, and employing a linear-algebraic perturbation argument to the lines bounding the polygon.

It would be interesting to prove a similar result for some narrower class of norms. For example, one might hope to prove that the ℓ_p norms admit only a near-linear number of unit distances for most p (in the Baire category sense or even for almost all p w.r.t. the Lebesgue measure). For that, the idea of polygonal approximations seems unusable, but perhaps more powerful tools from algebraic geometry might help.

Finally, of course, it might be possible to use some parts from the method of this paper for attacking the Euclidean case. However, since the number of unit distances for the Euclidean case can be much larger than $n \log n \log\log n$, additional ideas are certainly needed.

In this context, a very recent result of Schwartz, Solymosi, and De Zeeuw [9] is worth mentioning: using a powerful result of number theory, they proved that for every n-point $P \subset \mathbb{R}^2$ there are at most $n^{1+o(1)}$ pairs $\{\mathbf{p}, \mathbf{q}\}$, $\mathbf{p}, \mathbf{q} \in P$, such that $\mathbf{p}$ and $\mathbf{q}$ have unit Euclidean distance *and* the angle of the line $\mathbf{pq}$ with the x-axis is a rational multiple of π.

2. Connected subgraphs with few colors in edge-colored graphs

Let $G = (V, E)$ be a (simple, undirected) graph. An *edge coloring* of G is a mapping $c : E \to \mathbb{N} = \{1, 2, 3, \dots\}$. The edge coloring c is called *proper* if $c(e) \neq c(e')$ whenever the edges e and e' share a vertex.

Let G be a graph with a given edge coloring. For a subset $W \subseteq V$ of vertices we let $G[W]$ stand for the subgraph of G induced by W, with the edge coloring inherited from that of G. Further, if $I \subseteq \mathbb{N}$ is a set of colors, we write $G[I, W]$ for the subgraph induced by W on the edges with colors in I, that is,

$$G[I, W] = \big(W, \big\{\{u, v\} \in E\colon\ u, v \in W,\ c(e) \in I\big\}\big)$$

(the coloring is not explicitly mentioned in the notation).

Proposition 2.1. *Let $q > 1$ be a real parameter. Let $G = (V, E)$ be a graph on $n \geqslant 4$ vertices, with at least $Cqn \log n \log\log n$ edges (where C is a suitable absolute constant), and with a given proper edge coloring. Then there exist a subset $W \subseteq V$ of vertices, $|W| \geqslant 2$, and a subset $I \subset \mathbb{N}$ of colors such that the subgraph $G[I, W]$ is connected and the edges of $G[W]$ have at least $q|I|$ distinct colors.*

As was mentioned in the introduction, this proposition is similar to a result from [8], and the proof is also quite similar to the one in [8]. It still seems worth presenting in full, since describing the required modifications would be clumsy, and moreover, the proof below is significantly simpler than that in [8], mainly because the required result is weaker (in Proposition 2.1 we obtain a single connected subgraph, while in [8] several color-disjoint connected subgraphs on the same vertex set were needed).

At the beginning of the proof, we use a well-known observation stating that every graph of average degree δ has a subgraph whose minimum degree is at least $\delta/2$ (this follows by repeatedly deleting vertices of degree below $\delta/2$ and checking that the average degree can't decrease). So we may assume that G has minimum degree at least $(C/2)q \log n \log\log n$.

Let $W \subseteq V$ be a subset of vertices of G (so far arbitrary). An *edge cut* in $G[W]$ is a partition (A, B) of W into two nonempty subsets. We define the *maximum degree* $\Delta(A, B)$ of such an edge cut as the maximum number of neighbors of a vertex from A in B or of a vertex from B in A; formally,

$$\Delta(A, B) := \max\Big\{\max_{a\in A}\big|\big\{\{a, b\} \in E\colon\ b \in B\big\}\big|, \max_{b\in B}\big|\big\{\{b, a\} \in E\colon\ a \in A\big\}\big|\Big\}.$$

The proof of Proposition 2.1 proceeds in two stages. In the first stage, we forget about the edge colors; we select the set W so that every edge cut in $G[W]$ has a sufficiently large maximum degree. In order to get the (almost tight) quantitative result in the proposition, we need to quantify the "sufficiently large maximum degree" of a cut depending on the *imbalance* of the cut, which is defined by

$$\mathrm{imb}(A, B) := \frac{|A| + |B|}{\min(|A|, |B|)}.$$

Lemma 2.2. *Let $r \geqslant 1$ be a parameter (which we will later set to $(C/2)q \log\log n$ in the application of the lemma), and let $G = (V, E)$ be a graph on $n \geqslant 2$ vertices of minimum degree at least $r \log n$. Then there exists $W \subseteq V$, $|W| \geqslant 2$, such that every edge cut (A, B) in $G[W]$ satisfies*

$$\Delta(A, B) \geqslant r \log \mathrm{imb}(A, B).$$

Proof. The proof proceeds by a recursive partitioning: As long as we can find an edge cut (A, B) of small maximum degree in the current graph, we discard the *larger* of the sets A, B.

More formally, we set $V_1 := V$. If $G[V_j]$ has already been constructed and if there is an edge cut (A_j, B_j) in $G[V_j]$ with $\Delta(A_j, B_j) < r \log \mathrm{imb}(A_j, B_j)$, we let V_{j+1} be the smaller of the sets A_j and B_j (ties broken arbitrarily) and iterate. If there is no such edge cut, we set $W := V_j$, $t := j$, and finish.

It remains to show that the resulting W is nontrivial, i.e., $|W| \geqslant 2$. This is clear for $t = 1$ (no partition step was made), so we assume $t \geqslant 2$. We show that $G[W] = G[V_t]$ has minimum degree at least 1, and thus W can't consist of a single vertex.

Initially, in G, each vertex has degree at least $r\log n$, and by passing from V_j to V_{j+1}, each vertex of V_{j+1} loses at most $\Delta(A_j,B_j) < r\log\operatorname{imb}(A_j,B_j)$ neighbors. Thus, the minimum degree in $G[V_t]$ is strictly larger than

$$r\log n - r\sum_{j=1}^{t-1}\log\operatorname{imb}(A_j,B_j) = r\log n - r\sum_{j=1}^{t-1}\log\frac{|V_j|}{|V_{j+1}|}$$
$$= r\log n - r\bigl(\log|V_1| - \log|V_t|\bigr) \geqslant 0.$$

The lemma is proved. □

Now we continue with the *second stage of the proof of Proposition* 2.1. Only here we start considering the edge colors.

According to Lemma 2.2, we now assume that $W \subseteq V$, $|W| \geqslant 2$, is such that every edge cut (A,B) in $G[W]$ has maximum degree at least $r\log\operatorname{imb}(A,B)$, with $r = (C/2)q\log\log n$. Consequently, the edges of every edge cut (A,B) have at least $r\log\operatorname{imb}(A,B)$ distinct colors (since the edge coloring is proper), and this is the only property of $G[W]$ we will use.

Let k denote the number of colors occurring on the edges of $G[W]$. We note that $k \geqslant r$ (this follows by using the condition above for an arbitrary cut). It remains to show that $G[W]$ has a connected subgraph that uses at most k/q colors.

We select the colors greedily one by one, as follows. We set $I_0 := \emptyset$, and for $j = 0,1,2,\ldots,$ we do the following: If $G[I_j,W]$ is connected, we set $I := I_j$ and finish. Otherwise, we let i_j be a color i minimizing the number of connected components of $G[I_j \cup \{i\}, W]$. Then we set $I_{j+1} := I_j \cup \{i_j\}$, and we continue with the next step. We need to show that we obtain a connected graph before exhausting more than k/q colors.

Let m_j be the number of connected components of $G[I_j,W]$. We want an upper bound on the smallest j with $m_j = 1$. First we observe that $m_{j+1} \leqslant m_j - 1$ for all j, since every edge cut contains at least one color. In the sequel, we will actually estimate the smallest j such that $m_j \leqslant 3$. Then at most two more steps suffice to get down to $m_j = 1$.

We now want to bound m_{j+1} in terms of m_j. Essentially, we will see that adding a random color to I_j is likely to connect up many components.

Let $K_1,\ldots,K_{m_j}$ be the vertex sets of the connected components of $G[I_j,W]$. The average number of vertices in a component is m_0/m_j (where $m_0 = |W|$); we call a component *small* if it has at most $2m_0/m_j$ vertices. By Markov's inequality, there are at least $m_j/2$ small components.

Let i be one of the colors occurring on the edges of $G[W]$ but not belonging to I_j (so there are $k-j$ possible choices for i). We say that a component K_s *gets connected* by i if there is an edge of color i connecting a vertex of K_s to a vertex outside K_s.

By the condition on the edge cuts of $G[W]$, if K_s is a small component, then the number of colors i by which K_s gets connected is at least

$$r\log\operatorname{imb}(K_s, W\setminus K_s) \geqslant r\log\frac{m_0}{2m_0/m_j} = r\log(m_j/2).$$

Thus, the expected number of small components that get connected by a random color is at least

$$\frac{m_j}{2} \cdot \frac{r \log(m_j/2)}{k-j} \geqslant \frac{m_j}{2} \cdot \frac{r \log(m_j/2)}{k}.$$

So at least this many components get connected by the color i_{j+1}.

It is easy to check that the number of components always decreases at least by half of the number of components that get connected (an extremal case being components merged in pairs). Thus, we have

$$m_{j+1} \leqslant m_j - \frac{m_j}{4} \cdot \frac{r \log(m_j/2)}{k} \leqslant m_j \left(1 - \frac{r \log(m_j/2)}{4k}\right) \leqslant m_j e^{-r \log(m_j/2)/4k}$$

(we used $1 - x \leqslant e^{-x}$ in the last step). Assuming, as we may, that $m_j \geqslant 4$, we have $\log(m_j/2) \geqslant \frac{1}{2} \log m_j \geqslant \frac{1}{2} \ln m_j$, and so

$$\ln m_{j+1} \leqslant \ln m_j - r(\ln m_j)/8k = (1 - r/8k) \ln m_j \leqslant e^{-r/8k} \ln m_j.$$

Since $m_0 \leqslant n$, we can see that m_j drops below 4 in at most $O((k/r) \log\log n) = O(k/Cq)$ steps. We need at most two extra colors to get all the way to $m_j = 1$, so altogether the number of colors needed to build a connected graph is $O(k/Cq + 2) = O(k/Cq)$ (since $k \geqslant r$, and thus $k/Cq \geqslant \log\log n \geqslant 1$). The implicit constant in the $O(.)$ notation is independent of C, and thus we can set C so large that the number of colors is at most k/q. Proposition 2.1 is proved.

3. Unit-distance graphs

Let $\|\cdot\|$ be a norm in the plane, and let $P = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be a sequence of n distinct points in the plane. With these objects we associate a finite combinatorial object, which we will call the *decorated unit-distance graph*.

First, we define the *unit-distance graph* $G = G(\|\cdot\|, P)$ as the (undirected) graph (V, E) with vertex set $V := [n]$ (where we use the notation $[n] = \{1, 2, \ldots, n\}$) and with edges corresponding to the pairs of points of unit distance; that is, $E = \{\{a, b\}\colon \|\mathbf{p}_b - \mathbf{p}_a\| = 1\}$.

To every edge $e = \{a, b\} \in E$ we assign a vector $\mathbf{u}(e)$, in such a way that $\mathbf{u}(e) = \pm(\mathbf{p}_b - \mathbf{p}_a)$, and the sign is chosen using some globally consistent rule, so that parallel edges get the same $\mathbf{u}(e)$. For example, we may require that $\mathbf{u}(e)$ lie in the closed upper halfplane minus the negative x-axis.

Let $U := \{\mathbf{u}(e)\colon e \in E\}$ be the *unit direction set* of P, and we let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be an enumeration of all distinct elements of U, say in the lexicographic order. We call $\mathbf{u}_1, \ldots, \mathbf{u}_k$ the *unit directions* of P (under $\|\cdot\|$). Then we define a coloring $c : E \to [k]$ of the edges of the unit-distance graph, setting $c(e) = i$ if $\mathbf{u}(e) = \mathbf{u}_i$. (We note that c need *not* be a proper edge coloring, since there can be two edges with the same direction incident to a single vertex.)

Finally, we record the geometric orientation of each edge. Namely, we define a mapping $\sigma : E \to \{-1, +1\}$: For an edge $\{a, b\} \in E$ with $a < b$ we set

$$\sigma(\{a, b\}) = \begin{cases} +1 & \text{if } \mathbf{u}(e) = \mathbf{p}_b - \mathbf{p}_a, \\ -1 & \text{if } \mathbf{u}(e) = \mathbf{p}_a - \mathbf{p}_b. \end{cases}$$

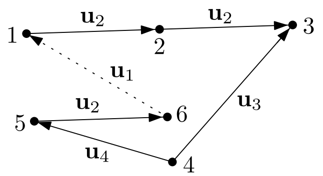

Fig. 1. Expressing $\mathbf{u}_j$ in terms of the $\mathbf{u}_i$, $i \in I$.

The *decorated unit-distance graph* of P under $\|\cdot\|$ is defined as the triple $\mathbf{G} = \mathbf{G}(\|\cdot\|, P) := (G, c, \sigma)$.

Now we define an *abstract decorated unit-distance graph* as expected, i.e., as a triple $\mathbf{G} = (G, c, \sigma)$, where G is a graph with vertex set $[n]$ for some n, c is a mapping $E(G) \to [k]$ for some k, and σ is a mapping $E \to \{-1, +1\}$. We say that a sequence P of distinct points in $\mathbb{R}^2$ is a *realization* of an abstract decorated unit-distance graph $\mathbf{G}$ under $\|\cdot\|$ if $\mathbf{G}$ is equal to the decorated unit-distance graph of P under $\|\cdot\|$. (We require equality to keep the definitions simple; we could as well introduce a suitable notion of isomorphism, but there is no need.)

Here is the main result of this section. Roughly speaking, it tells us that if $\mathbf{G}$ is a sufficiently dense abstract decorated unit-distance graph, then for every realization, the unit directions satisfy certain fixed linear dependences—some $\ell + 1$ of the unit directions can be expressed using some other ℓ of the unit directions.

Lemma 3.1. *The following holds for a sufficiently large constant C_0. Let $\mathbf{G}$ be an abstract decorated unit-distance graph with $n \geqslant 4$ vertices, at least $f(n) := C_0 n \log n \log\log n$ edges, and k colors. Then there exists an integer $\ell \geqslant 1$, a sequence $(i(1), i(2), \dots, i(2\ell+1))$ of distinct indices in $[k]$, and linear maps $L_1, L_2, \dots, L_{\ell+1} : (\mathbb{R}^2)^\ell \to \mathbb{R}^2$ such that for every realization P of $\mathbf{G}$ (under any norm), we have*

$$\mathbf{u}_{i(\ell+j)} = L_j(\mathbf{u}_{i(1)}, \mathbf{u}_{i(2)}, \dots, \mathbf{u}_{i(\ell)}), \quad j = 1, 2, \dots, \ell+1,$$

where $\mathbf{u}_1, \dots, \mathbf{u}_k$ are the unit directions of P.

Proof. Let $\mathbf{G} = (G, c, \sigma)$. In order to apply Proposition 2.1, we may need to prune the graph so that c becomes a proper edge coloring. If $\mathbf{G}$ has any realization at all, then, for geometric reasons, no color occurs on more than two edges incident to each vertex. Hence, for each i, the subgraph made of edges of color i consists of paths and cycles, and so by deleting at most $\frac{2}{3}$ of the edges, we can turn this subgraph into a matching, and hence obtain a subgraph $\tilde{G}$ of G with at least $\frac{1}{3} f(n)$ edges for which c is a proper edge coloring. (By using more geometry, it is easily seen that it even suffices to delete only at most $\frac{1}{2}$ of the edges, rather than $\frac{2}{3}$.)

Now we are ready to apply Proposition 2.1 on the graph $\tilde{G}$ with the proper edge coloring c, and with $q = 2.001$, say. This yields a subset $W \subseteq V(\tilde{G})$ and a subset $I \subset [k]$ of colors, such that the subgraph $\tilde{G}[I, W]$ is connected, and $\tilde{G}[W]$ uses at least $2|I| + 1$ colors. Let J be a set of $|I| + 1$ colors used on the edges of W but not belonging to I.

Now we can define the objects whose existence is claimed in the lemma. We set $\ell := |I|$, let $(i(1), \dots, i(\ell))$ be an enumeration of I, and let $i(\ell+1), \dots, i(2\ell+1)$ be an enumeration of J.

Let us consider some color $j \in J$, and let $\{a, b\}$ be an edge of color j in $\tilde{G}[W]$. Then there is a path π from a to b in $\tilde{G}[W]$ whose edges have only colors in I, and for every realization P of $\mathbf{G}$, $\mathbf{u}_j$ is a signed sum of the unit directions along this path. An example is given in Fig. 1: If $a = 1$, $b = 6$, $j = 1$, the edge $\{1, 6\}$ has sign -1, the path π goes through the vertices $2, 3, 4, 5$ in this order, and its edges have colors $2, 2, 3, 4, 2$ and signs $+1, +1, -1, +1, +1$, then $\mathbf{u}_1 = -3\mathbf{u}_2 + \mathbf{u}_3 - \mathbf{u}_4$. This yields the desired linear maps $L_1, \ldots, L_{\ell+1}$, and the lemma is proved. □

4. Proof of Theorem 1.1

Let us call a norm $\|\cdot\|$ on $\mathbb{R}^2$ *bad* if $u_{\|\cdot\|}(n) \geqslant f(n) = C_0 n \log n \log\log n$ for some $n \geqslant 3$, and let $\mathcal{M} \subseteq \mathcal{B}$ be the set of all bad norms. We want to show that $\mathcal{M}$ is meager, and thus we want to cover it by countably many nowhere dense sets.

In our proof, the nowhere dense sets $\mathcal{M}_{\mathbf{G},\eta}$ are indexed by two parameters: $\mathbf{G}$, which runs through all abstract decorated unit-distance graphs with n vertices and at least $f(n)$ edges, $n = 3, 4, \ldots$, and η, which runs through all positive numbers of the form $\frac{1}{m}$, m an integer.

To define $\mathcal{M}_{\mathbf{G},\eta}$, we first define that a realization P of $\mathbf{G}$ under a norm $\|\cdot\|$ is *η-separated* if for every two unit direction vectors $\mathbf{u}_i, \mathbf{u}_j$ of this realization, the lines spanned by $\mathbf{u}_i$ and $\mathbf{u}_j$ have angle at least η.

Now $\mathcal{M}_{\mathbf{G},\eta}$ consists of all norms $\|\cdot\|$ under which $\mathbf{G}$ has an η-separated realization.

It is easily checked that the $\mathcal{M}_{\mathbf{G},\eta}$ cover all of $\mathcal{M}$. Indeed, for every bad norm $\|\cdot\|$ we can choose n and an n-point sequence P with at least $f(n)$ unit distances. We define $\mathbf{G}$ as the decorated unit-distance graph of P under $\|\cdot\|$. It remains to observe that, trivially, every realization of $\mathbf{G}$ under some norm is η-separated for some $\eta > 0$. Thus $\|\cdot\| \in \mathcal{M}_{\mathbf{G},\eta}$.

The main part of the proof consists of showing that each $\mathcal{M}_{\mathbf{G},\eta}$ is nowhere dense. Explicitly, this is expressed in the following lemma; once we prove it, we will be done with Theorem 1.1 (the smoothness and strict convexity asserted in the theorem follows from Klee's result [7] mentioned in the introduction, namely, that most norms are smooth and strictly convex).

Lemma 4.1. *Let $\mathbf{G}$ be an abstract decorated unit-distance graph with n vertices and at least $f(n)$ edges, let $B_0 \in \mathcal{B}$ be the unit ball of some norm, and let $\eta, \varepsilon > 0$. Then there exist $B \in \mathcal{B}$ with $d_H(B, B_0) < \varepsilon$ (where d_H denotes the Hausdorff distance) and $\delta > 0$ such that no $B' \in \mathcal{B}$ with $d_H(B', B) < \delta$ belongs to $\mathcal{M}_{\mathbf{G},\eta}$.*

Proof. First we approximate B_0 by a $\mathbf{0}$-symmetric convex polygon B_1 within Hausdorff distance at most $\frac{\varepsilon}{2}$ from B_0. We make sure that all sides of B_1 are sufficiently short, so short that two lines through $\mathbf{0}$ with angle at least η never meet the same side of B_1. (If B_0 has straight segments in the boundary, we need to "bulge" B_1 slightly; see Fig. 2.)

Let $s_1, s_2, \ldots, s_{2m}$ be the sides of B_1 listed in clockwise order, say, so that s_i and s_{m+i} are opposite (i.e., $s_{m+i} = -s_i$). Let λ_i be the line spanned by s_i, and for a real parameter t, let $\lambda_i(t)$ be the line obtained by a parallel translation of λ_i by distance t, where $t > 0$ means translation away from the origin and $t < 0$ translation towards the origin. We have $\lambda_{m+i}(t) = -\lambda_i(t)$.

Let us consider an m-tuple $\mathbf{t} = (t_1, \ldots, t_m) \in T_0 := [-\delta_0, \delta_0]^m$. For $\delta_0 > 0$ sufficiently small, the lines $\lambda_1(t_1), \ldots, \lambda_m(t_m), \lambda_{m+1}(t_1), \ldots, \lambda_{2m}(t_m)$ bound a symmetric convex polygon with $2m$ sides, which we denote by $B_1(\mathbf{t})$. Moreover, for δ_0 sufficiently small, $d_H(B_1(\mathbf{t}), B_0) < \varepsilon$, and the sides of $B_1(\mathbf{t})$ are still short in the same sense as those of B_1.

Now we digress from geometry for a moment and we apply Lemma 3.1 to the abstract decorated unit-distance graph $\mathbf{G}$. This yields an integer ℓ, indices $i(1), \ldots, i(2\ell + 1)$, and linear

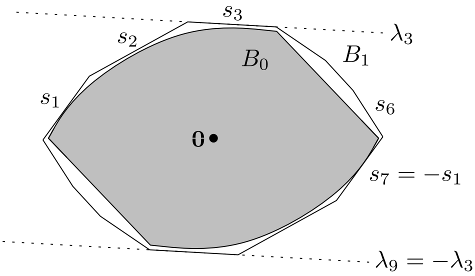

Fig. 2. Approximating the unit ball B_0 by a convex polygon.

maps $L_1, \ldots, L_{\ell+1}$ as in the lemma. In order to make the notation slightly simpler, let us pretend that $i(j) = j$ for all $j = 1, \ldots, 2\ell + 1$. Thus, for every realization of $\mathbf{G}$, the unit directions $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1}$ satisfy the linear relations $\mathbf{u}_{\ell+i} = L_i(\mathbf{u}_1, \ldots, \mathbf{u}_\ell)$, $i = 1, 2, \ldots, \ell + 1$.

Next, let us consider a particular realization of $\mathbf{G}$ under the norm induced by $B_1(\mathbf{t})$ for some $\mathbf{t} \in T_0$. Each of the unit directions $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1}$ lies on the boundary of $B_1(\mathbf{t})$, and thus on some line $\lambda_\alpha(t_\alpha)$. (Here we abuse the notation slightly, since the range of α is $[2m]$, while $\mathbf{t}$ is indexed only by $[m]$, in order to preserve the symmetry of the polygon. So we make the convention that t_{m+i} is the same as t_i.)

Let $\alpha(i) \in [2m]$ be the index such that $\mathbf{u}_i$ lies on $\lambda_{\alpha(i)}(t_{\alpha(i)})$ (if $\mathbf{u}_i$ is a vertex of the polygon and thus lies on two of the lines, we pick one arbitrarily). Since the sides of $B_1(\mathbf{t})$ are short, we have $\alpha(i) \neq \alpha(i')$ whenever $i \neq i'$, and also $\alpha(i) + m \neq \alpha(i')$ (where $\alpha(i) + m$ is to be understood modulo $2m$).

Let us call a mapping $\alpha : [2\ell + 1] \to [2m]$ an *admissible assignment of lines* if it satisfies the condition in the previous sentence. Let us define a *box* $T \subseteq T_0$ as a product of closed intervals with a nonempty interior; each box can be written as an m-dimensional "interval" $[\mathbf{t}_{\min}, \mathbf{t}_{\max}]$. Our next goal is establishing the following claim.

Claim 4.2. *There exists a box $\tilde{T} \subseteq T_0$ such that for every admissible assignment of lines α and for every $\mathbf{t} \in \tilde{T}$ there are no vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1} \in \mathbb{R}^2$ such that each $\mathbf{u}_i$ lies on the appropriate line, i.e., $\mathbf{u}_i \in \lambda_{\alpha(i)}(t_{\alpha(i)})$, and the $\mathbf{u}_i$ satisfy the linear relations $\mathbf{u}_{\ell+i} = L_i(\mathbf{u}_1, \ldots, \mathbf{u}_\ell)$, $i = 1, 2, \ldots, \ell + 1$.*

Proof. We will kill all admissible assignments α one by one inductively, progressively shrinking the current box. The following statement allows us to make an inductive step: *Let $T \subseteq T_0$ be a box, and let α be an admissible assignment of lines. Then there exists a box $T' \subseteq T$ such that for every $\mathbf{t} \in T'$ there are no vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1} \in \mathbb{R}^2$ with $\mathbf{u}_i \in \lambda_{\alpha(i)}(t_{\alpha(i)})$ for all i and with $\mathbf{u}_{\ell+i} = L_i(\mathbf{u}_1, \ldots, \mathbf{u}_\ell)$, $i = 1, 2, \ldots, \ell + 1$.*

To prove this, let us consider a vector $\mathbf{x} \in \mathbb{R}^{2\ell}$, which we think of as a concatenation of $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$, and let us think of its components x_i as unknowns.

For each $i = 1, 2, \ldots, \ell$, the condition $\mathbf{u}_i \in \lambda_{\alpha(i)}(t_{\alpha(i)})$ translates to a single linear equation for $\mathbf{x}$, of the form $\mathbf{a}_i^T \mathbf{x} = b_i$, where the coefficient vector $\mathbf{a}_i$ on the left-hand side doesn't depend on $\mathbf{t}$, while $b_i = b_i(t_{\alpha(i)})$ is a *nonconstant* linear function of $t_{\alpha(i)}$.

Similarly, for $i = 1, 2, \ldots, \ell + 1$, the condition $\mathbf{u}_{\ell+i} \in \lambda_{\alpha(\ell+i)}(t_{\alpha(\ell+i)})$ together with $\mathbf{u}_{\ell+i} = L_i(\mathbf{u}_1, \ldots, \mathbf{u}_\ell)$ translate to a similar linear equation $\mathbf{a}_{\ell+i}^T \mathbf{x} = b_{\ell+i}$, again with $\mathbf{a}_{\ell+i}$ independent of $\mathbf{t}$ and with $b_{\ell+i} = b_{\ell+i}(t_{\alpha(\ell+i)})$ a nonconstant linear function of $t_{\alpha(\ell+i)}$.

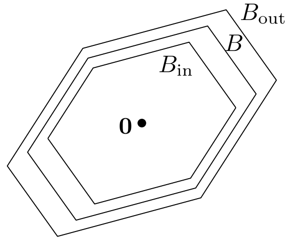

Fig. 3. The polygons $B_{\rm in}$, $B_{\rm out}$, and B.

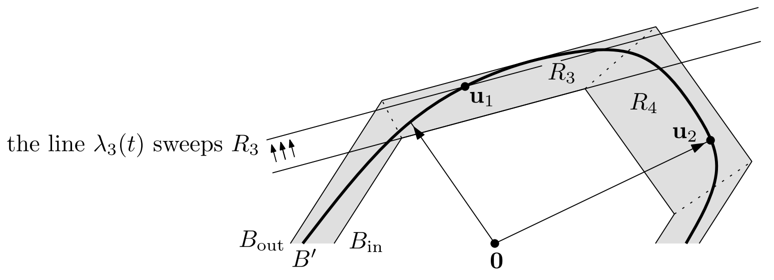

Fig. 4. Dividing the region $B_{\rm out} \setminus B_{\rm in}$ into trapezoids.

Since the $\alpha(i)$ are all distinct, altogether we get that if the appropriate $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1}$ exist, then $\mathbf{x}$ satisfies the system $A\mathbf{x} = \mathbf{b}$ of $2\ell + 1$ linear equations with 2ℓ unknowns, where A is a fixed matrix and the right-hand side $\mathbf{b} = \mathbf{b}(\mathbf{t})$ is a *surjective* linear function $\mathbb{R}^m \to \mathbb{R}^{2\ell+1}$.

Since we have more equations than unknowns, the system $A\mathbf{x} = \mathbf{b}$ has a solution only for $\mathbf{b}$ contained in a proper linear subspace of $\mathbb{R}^{2\ell+1}$. Hence, by the surjectivity of $\mathbf{b}(\mathbf{t})$, the set of all $\mathbf{t} \in \mathbb{R}^m$ for which $A\mathbf{x} = \mathbf{b}(\mathbf{t})$ is unsolvable is a dense open subset of $\mathbb{R}^m$. From this the existence of the desired box T' follows, and the Claim 4.2 is proved. □

Finishing the proof of Lemma 4.1. Let us consider the box $\tilde{T} = [\mathbf{t}_{\rm min}, \mathbf{t}_{\rm max}]$ as in Claim 4.2. We set $\mathbf{t}_{\rm mid} := (\mathbf{t}_{\rm min} + \mathbf{t}_{\rm max})/2$, and we consider the polygons $B_{\rm in} := B_1(\mathbf{t}_{\rm min})$, $B_{\rm out} := B_1(\mathbf{t}_{\rm max})$, and $B := B_1(\mathbf{t}_{\rm mid})$; see Fig. 3. We claim that B has the properties required in the lemma, i.e., no $B' \in \mathcal{B}$ sufficiently close to B belongs to $\mathcal{M}_{\mathbf{G},\eta}$.

To see this, we note that every B' sufficiently close to B satisfies $B_{\rm in} \subseteq B' \subseteq B_{\rm out}$. For contradiction, we assume that there is an η-separated realization of $\mathbf{G}$ under B'. Then the unit directions $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1}$ lie on the boundary of B'.

The region $B_{\rm out} \setminus B_{\rm in}$ is naturally divided into $2m$ trapezoids $R_1, \ldots, R_{2m}$ belonging to the sides, as in Fig. 4. Each of $\mathbf{u}_i$, $i = 1, 2, \ldots, 2\ell + 1$, lies in one of these trapezoids, let us call it $R_{\alpha(i)}$ (border disputes resolved arbitrarily). Since the considered realization is η-separated, no two of the $\mathbf{u}_i$ share the same trapezoid, and also no two of these trapezoids are opposite to one another. So α defines an admissible assignment of sides.

Let us consider the trapezoid $R_{\alpha(i)}$. As the line $\lambda_{\alpha(i)}(t)$ moves from the inner position (with $t = (\mathbf{t}_{\rm min})_{\alpha(i)}$) to the outer position (with $t = (\mathbf{t}_{\rm max})_{\alpha(i)}$), it sweeps the whole of $R_{\alpha(i)}$, and hence for some t it contains $\mathbf{u}_i$; let us denote this value of t by $\bar{t}_{\alpha(i)}$.

This defines $2\ell + 1$ of the components of a vector $\bar{\mathbf{t}} \in \mathbb{R}^m$. Let us set the remaining components to the corresponding components of $\mathbf{t}_{\rm mid}$, say. Then $\bar{\mathbf{t}}$ lies in the box $\tilde{T}$, and hence, by Claim 4.2, $\mathbf{u}_1, \ldots, \mathbf{u}_{2\ell+1}$ cannot lie on the corresponding lines. The resulting contradiction proves the lemma, and this also finishes the proof of Theorem 1.1. □

Acknowledgments

I would like to thank an anonymous referee for careful reading and numerous valuable comments, and Jozsef Solymosi for giving me a preprint of [9] and pointing out the reference [5].

References

[1] B. Aronov, M. Sharir, Cutting circles into pseudo-segments and improved bounds for incidences, Discrete Comput. Geom. 28 (4) (2002) 475–490.
[2] P. Brass, W. Moser, J. Pach, Research Problems in Discrete Geometry, Springer, New York, 2005.
[3] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, Discrete Comput. Geom. 5 (1990) 99–160.
[4] P. Erdős, On a set of distances of *n* points, Amer. Math. Monthly 53 (1946) 248–250.
[5] P. Erdős, G.B. Purdy, Some extremal problems in geometry, IV, in: Proc. 7th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Louisiana State Univ., Baton Rouge, La., 1976, Congress. Numer. XVII, Utilitas Math., Winnipeg, Man., 1976, pp. 307–322.
[6] Peter M. Gruber, Convex and Discrete Geometry, Grundlehren Math. Wiss., vol. 336, Springer, Berlin, 2007.
[7] V. Klee, Some new results on smoothness and rotundity in normed linear spaces, Math. Ann. 139 (1959) 51–63.
[8] J. Matoušek, A. Přívětivý, P. Škovroň, How many points can be reconstructed from *k* projections? SIAM J. Discrete Math. 22 (2008) 1605–1623.
[9] R. Schwartz, J. Solymosi, F. de Zeeuw, Rational distances with rational angles, preprint, arXiv:1008.3671.
[10] J. Spencer, E. Szemerédi, W.T. Trotter, Unit distances in the Euclidean plane, in: B. Bollobás (Ed.), Graph Theory and Combinatorics, Academic Press, New York, NY, 1984, pp. 293–303.
[11] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, Combin. Probab. Comput. 6 (1997) 353–358.
[12] E. Szemerédi, W. Trotter Jr., A combinatorial distinction between Euclidean and projective planes, European J. Combin. 4 (1983) 385–394.
[13] P. Valtr, Strictly convex norms allowing many unit distances and related touching questions, manuscript, Charles University, Prague, 2005.