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Diffusive logistic equation with non-linear boundary conditions

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ABSTRACT

We analyze the solutions of a population model with diffusion and logistic growth. In particular, we focus our study on a population living in a patch, $\Omega \subseteq \mathbb{R}^n$ with $n \geqslant 1$, that satisfies a certain non-linear boundary condition and on its survival when constant yield harvesting is introduced. We establish our existence results by the method of sub-super solutions.

1. Introduction

In this paper, we study the following reaction–diffusion model with non-linear boundary conditions:

$$u_t = d\Delta u + au - bu^2 - ch(x); \quad \Omega, \tag{1.1}$$

$$d\alpha(x,u)\frac{\partial u}{\partial \eta} + \big[1 - \alpha(x,u)\big]u = 0; \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geqslant 1$, Δ is the Laplace operator, d is the diffusion coefficient, a, b are positive parameters, $c \geqslant 0$ is the harvesting parameter, $h(x) : \overline{\Omega} \to \mathbb{R}$ is a C^1 function, $\frac{\partial u}{\partial \eta}$ is the outward normal derivative, and $\alpha(x,u) : \partial\Omega \times \mathbb{R} \to [0,1]$ is a non-decreasing C^1 function. Reaction–diffusion equations like the one in (1.1) have been used extensively in describing the spatiotemporal distributaries and abundance of organisms living in a patch, Ω. They are often represented in the form,

$$u_t = d\Delta u + u\widetilde{f}(u); \quad \Omega \tag{1.3}$$

where $u(t,x)$ denotes the population density and $\widetilde{f}(u)$ is the per capita growth rate affected by the heterogenous environment. Since Skellam's pioneering work in ecological models (see [15]), various reaction–diffusion models have been studied by such authors as Kolmogoroff in [10]. The most classic of these models is Fisher's equation with $\widetilde{f}(x,u) = (1-u)$ (see [7]), which was first used by Skellam. More recently, reaction–diffusion models have been used to describe spatiotemporal phenomena in disciplines other than ecology, such as physics, chemistry, and biology (see [3,6,12,13,16]). For this paper, we consider logistic growth with $\widetilde{f}(x,u) = (a - bu)$ and a, b positive parameters.

In addition, most ecological systems have some form of predation or harvesting of the population. For example, hunting or fishing is often used as an effective means of wildlife management. In this paper, we are interested in constant yield

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harvesting, in which harvesting is not dependent upon the population density, u, or t. The addition of a harvesting term to (1.3) leads to the following

$$u_t = d\Delta u + au - bu^2 - ch(x); \quad \Omega \tag{1.4}$$

where the parameter $c \geqslant 0$ represents the level of harvesting, $h(x) \geqslant 0$ for $x \in \Omega$, $h(x) = 0$ for $x \in \partial\Omega$, and $\|h\|_\infty = 1$. Here $ch(x)$ can be understood as the rate of the harvesting distribution. The non-linear boundary condition (1.2) has only been recently studied by such authors as [2–4], among others. Here

$$\alpha(x, u) = \frac{u}{u - d\frac{\partial u}{\partial \eta}}$$

represents the fraction of the population that remains on the boundary when reached. For the case when $\alpha(x, u) \equiv 0$, (1.2) becomes the well-known Dirichlet boundary condition. If $\alpha(x, u) \equiv 1$ then (1.2) becomes the Neumann boundary condition. The $\alpha(x, u)$'s of biological importance are those that are zero at $u = 0$ and increase to one as $u \to \infty$. With this in mind we will be interested in the study of positive steady state solutions of (1.1)–(1.2) when $d = 1$ and

$$\alpha(x, u) = \alpha(u) = \frac{u}{u + g(u)}; \quad \partial\Omega$$

where $g \in C^1([0, \infty), [\delta, \infty))$ for some $\delta > 0$, $\frac{g(u)}{u}$ is decreasing, and tends to 0 as $u \to \infty$. Hence, we study the model

$$-\Delta u = au - bu^2 - ch(x) =: f(x, u); \quad \Omega, \tag{1.5}$$

$$u\left[\frac{\partial u}{\partial \eta} + g(u)\right] = 0; \quad \partial\Omega. \tag{1.6}$$

The structure of positive solutions to the differential equation, (1.5), with Dirichlet boundary conditions ($u = 0$; $\partial\Omega$) has been studied by Oruganti, Shi, and Shivaji in [14].

Hence, the main purpose of this paper is to initiate the extension of their results to the non-linear boundary condition in (1.6). In the literature, (1.5)–(1.6) is known as a semipositone problem since $f(x, u) < 0$ when $c > 0$, at least for some $x \in \Omega$. Determining the structure of positive solutions for such problems is known to be challenging (see [1,11,14]).

We discuss the case when $n \geqslant 1$ and Ω is any bounded domain in $\mathbb{R}^n$. Clearly, the positive solutions of

$$-\Delta u = au - bu^2 - ch(x); \quad \Omega, \tag{1.7}$$

$$u = 0; \quad \partial\Omega \tag{1.8}$$

and

$$-\Delta u = au - bu^2 - ch(x); \quad \Omega, \tag{1.9}$$

$$\frac{\partial u}{\partial \eta} = -g(u); \quad \partial\Omega \tag{1.10}$$

are positive solutions of (1.5)–(1.6). For (1.7)–(1.8) the following results are known from [14]:

Theorem 1. *(See [14].) If $a \leqslant \lambda_1$ then (1.7)–(1.8) has no positive solution for any $c \geqslant 0$, where λ_1 is the principle eigenvalue for Laplace's equation with Dirichlet boundary conditions.*

Theorem 2. *(See [14].) If $a > \lambda_1$ and $b > 0$ then there exists a $c_0 = c_0(\Omega, a, b) > 0$ such that for $0 \leqslant c < c_0$, (1.7)–(1.8) has a positive solution. Further, this solution, $u(x)$, is such that $u(x) \geqslant ch(x)$ for $x \in \overline{\Omega}$.*

In Section 2, we prove the following new non-existence and existence results for (1.9)–(1.10):

Theorem 3. *There is a constant, $\tilde{a} = \tilde{a}(\Omega, b, N) > 0$, such that if $a < \tilde{a}$ then (1.9)–(1.10) has no positive solution for any $c \geqslant 0$, where $N := \inf_{u \in [0,\infty)}\{g(u)\}$.*

Theorem 4. *There exists a constant, $\bar{a} = \bar{a}(\Omega, b) > 0$, such that if $a > \bar{a}$ then there is a $c_1 = c_1(\Omega, a, b) > 0$ such that if $0 \leqslant c < c_1$ then (1.9)–(1.10) has a strict positive solution, $u > 0$; $\overline{\Omega}$. Moreover, this solution, $u(x)$, satisfies $u(x) \geqslant ch(x)$ for $x \in \overline{\Omega}$.*

Also, the moreover statement in Theorems 2 and 4 is sometimes referred to as an "obstacle problem". To prove Theorem 4, we employ the method of sub-super solutions given in Theorem 5 (see below) which was adapted from [9]. By a subsolution (supersolution) of (1.9)–(1.10) we mean a function $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

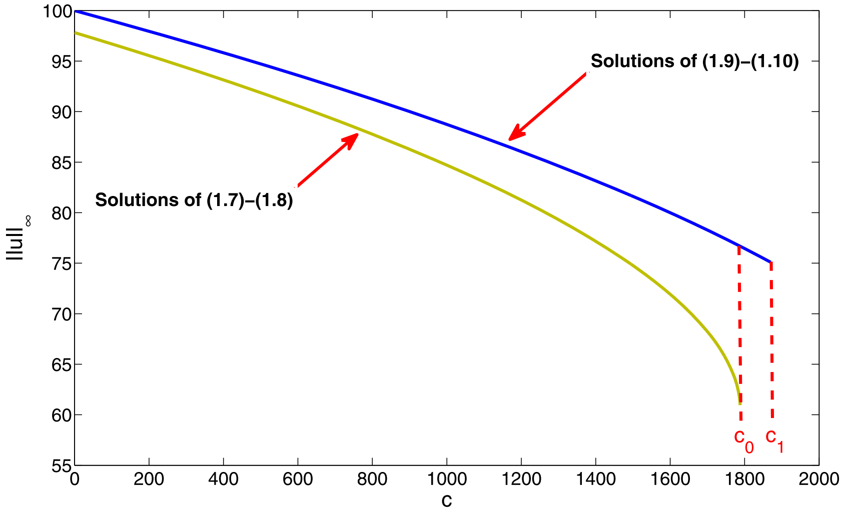

Fig. 1.1. Typical bifurcation diagram of (1.5)–(1.6).

1. $-\Delta\phi \leqslant (\geqslant)\, a\phi - b\phi^2 - ch(x);\ \Omega$,
2. $\frac{\partial\phi}{\partial\eta} \leqslant (\geqslant) -g(\phi);\ \partial\Omega$.

Theorem 5. *(See [9].) If there exist a subsolution, $\phi(x)$, and supersolution, $\psi(x)$, of (1.9)–(1.10) with $\phi(x) \leqslant \psi(x)$ then (1.9)–(1.10) has a solution, $u(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\phi(x) \leqslant u(x) \leqslant \psi(x)$ on Ω.*

Since (1.9) is a semipositone problem, finding a positive subsolution is difficult. In [14] the authors make use of the Anti-Maximum Principle (see [5]) to overcome this difficulty for (1.7)–(1.8). To construct the crucial subsolution for (1.9)–(1.10) we need to further do delicate analysis to accommodate our boundary condition, (1.10).

Finally, combining Theorems 2 and 4 we obtain the following multiplicity result for (1.5)–(1.6).

Theorem 6. *If $a > \max\{\bar{a}, \lambda_1\}$ and $0 \leqslant c < \min\{c_0, c_1\}$ then (1.5)–(1.6) has at least two positive solutions, u_i; $i = 1, 2$ with $u_1(x) = 0$; $\partial\Omega$ while $u_2(x) > 0$; $\partial\Omega$. Further, the solutions satisfy $u_i(x) \geqslant ch(x)$; $i = 1, 2$ for $x \in \overline{\Omega}$.*

A typical bifurcation diagram of (1.5)–(1.6) for the case $n = 1$, $h(x) \equiv 1$, and $\Omega = (0, 1)$ is shown in Fig. 1.1 (see [8] for more details).

2. Proofs of our non-existence and multiplicity results

2.1. Proof of Theorem 3

Let $\mu_1 > 0$ and $\phi > 0$ be the first eigenvalue and corresponding positive eigenfunction of

$$\begin{cases} -\Delta\phi = \mu_1\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\eta} = -\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Define $\tilde{a} = \tilde{a}(\Omega, b, N) = \min\{\mu_1, Nb\}$. For $a < \tilde{a}$, multiplying (1.9)–(1.10) by ϕ, and integrating over Ω, we obtain

$$\int_\Omega \phi(-\Delta u)\,dx = \int_\Omega au\phi - bu^2\phi - ch(x)\phi\,dx.$$

We know

$$\begin{aligned} \int_\Omega \phi(-\Delta u)\,dx &= \int_\Omega u(-\Delta\phi)\,dx + \int_{\partial\Omega} \frac{\partial\phi}{\partial\eta}u - \frac{\partial u}{\partial\eta}\phi\,ds \\ &= \int_\Omega \mu_1\phi u\,dx + \int_{\partial\Omega} -\phi u + \phi g(u)\,ds. \end{aligned}$$

Thus we have

$$\int_{\partial\Omega} -\phi u + \phi g(u)\,ds = \int_{\Omega} (a-\mu_1)u\phi - bu^2\phi - ch(x)\phi\,dx. \tag{2.2}$$

Since $a < \mu_1$, we can see that the right-hand side of (2.2) is negative. By the maximum principle, we know that $\|u\|_\infty \leqslant \frac{a}{b} < N$ which gives

$$\int_{\partial\Omega} -\phi u + \phi g(u)\,ds = \int_{\partial\Omega} \big(g(u) - u\big)\phi\,ds > 0,$$

and by this contradiction Theorem 3 is proven.

2.2. Proof of Theorem 4

We prove this theorem by using the method of sub-super solutions. To construct the subsolution, we recall the anti-maximum principle from [5] in the following form. There exists a $\delta_1 = \delta_1(\Omega) > 0$ such that the solution z_λ of

$$\begin{cases} -\Delta z - \lambda z = -1 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ is positive in Ω and $\frac{\partial z_\lambda}{\partial\eta} < 0$ on $\partial\Omega$. Let $\alpha_\lambda = \|z_\lambda\|_\infty$, $m_\lambda = \inf\{m \mid \frac{\partial(mz_\lambda)}{\partial\eta} \leqslant -g(0) - 1\}$, and $\bar{a} = \bar{a}(\Omega, b) = \inf_{\lambda\in(\lambda_1,\lambda_1+\delta_1)} \max\{2\lambda, 2bm_\lambda\alpha_\lambda\}$. For $a > \bar{a}$, we can choose $\lambda^* \in (\lambda_1, \lambda_1 + \delta_1)$ such that $a > \max\{2\lambda^*, 2bm_{\lambda^*}\alpha_{\lambda^*}\}$. Let $K_1 = \inf\{K \mid Kz_{\lambda^*} \geqslant h(x)\}$. Define

$$D := \sup_{K \geqslant \max\{1, K_1\}} \frac{(a-\lambda^*)K\alpha_{\lambda^*} + (K-1)}{b(K\alpha_{\lambda^*})^2}$$

and let $\widetilde{K} \geqslant \max\{1, K_1\}$ be such that $D = \frac{(a-\lambda^*)\widetilde{K}\alpha_{\lambda^*} + (\widetilde{K}-1)}{b(\widetilde{K}\alpha_{\lambda^*})^2}$. First, we state and prove an important claim:

Claim. *If $a > \max\{2\lambda^*, 2bm_{\lambda^*}\alpha_{\lambda^*}\}$, then*

$$\frac{m_{\lambda^*}}{\widetilde{K}} < \min\left\{D, \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}\right\}.$$

To prove the claim, we note that $\frac{m_{\lambda^*}}{\widetilde{K}} < \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}$ follows immediately from $a > 2bm_{\lambda^*}\alpha_{\lambda^*}$. Now, since $a > 2\lambda^*$, $a > 2bm_{\lambda^*}\alpha_{\lambda^*}$, and $\widetilde{K} \geqslant 1$ the following are true:

$$\begin{aligned} &\frac{a}{2} - \lambda^* > 0, \\ &a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - \frac{a}{2}\alpha_{\lambda^*} > 0, \\ &a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - m_{\lambda^*}b\alpha_{\lambda^*}^2 + 1 > 1, \\ &\big[a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - m_{\lambda^*}b\alpha_{\lambda^*}^2 + 1\big]\widetilde{K} > 1, \\ &\big(a - \lambda^*\big)\widetilde{K}\alpha_{\lambda^*} + \widetilde{K} - 1 > m_{\lambda^*}b\alpha_{\lambda^*}^2\widetilde{K}. \end{aligned}$$

Hence,

$$D = \frac{(a-\lambda^*)\widetilde{K}\alpha_{\lambda^*} + (\widetilde{K}-1)}{b(\widetilde{K}\alpha_{\lambda^*})^2} > \frac{m_{\lambda^*}}{\widetilde{K}}$$

which proves the claim.

Next, let $c_1 := \min\{D, \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}\}$. Now for $c < c_1$ there exists a d_c such that

$$\max\left\{c, \frac{m_{\lambda^*}}{\widetilde{K}}\right\} < d_c < \min\left\{D, \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}\right\}.$$

Define $\psi = \widetilde{K}d_c z_{\lambda^*}$. We know

$$-\Delta\psi = \widetilde{K}d_c(-\Delta z_{\lambda^*}) = \widetilde{K}d_c\big(\lambda^* z_{\lambda^*}\big) - \widetilde{K}d_c.$$

Thus if we prove

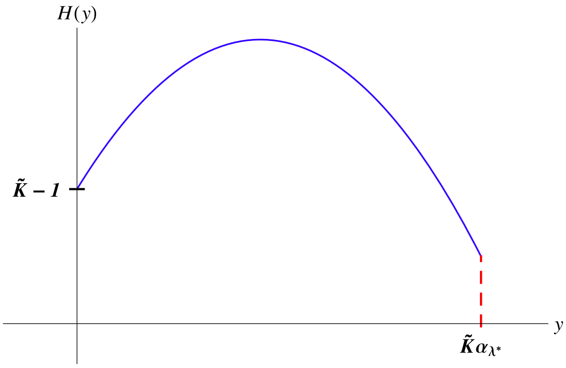

Fig. 2.1. Graph of $H(y)$.

$$\big(a-\lambda^*\big)\widetilde{K}z_{\lambda^*}-bd_c(\widetilde{K}z_{\lambda^*})^2+\widetilde{K}-1\geqslant 0, \tag{2.4}$$

then

$$\begin{aligned}-\Delta\psi&=\widetilde{K}d_c\big(\lambda^* z_{\lambda^*}\big)-\widetilde{K}d_c\\&\leqslant a(\widetilde{K}d_c z_{\lambda^*})-b(\widetilde{K}d_c z_{\lambda^*})^2-d_c\\&\leqslant a(\widetilde{K}d_c z_{\lambda^*})-b(\widetilde{K}d_c z_{\lambda^*})^2-d_c h(x)\\&=a\psi-b\psi^2-d_c h(x)\end{aligned}$$

and on $\partial\Omega$

$$\frac{\partial\psi}{\partial\eta}=\widetilde{K}d_c\frac{\partial z_{\lambda^*}}{\partial\eta}<m_{\lambda^*}\frac{\partial z_{\lambda^*}}{\partial\eta}\leqslant -g(0)-1.$$

To establish (2.4) we show $H(y):=(a-\lambda^*)y-bd_c y^2+(\widetilde{K}-1)\geqslant 0$ for all $y\in[0,\widetilde{K}\alpha_{\lambda^*}]$. Since $a>\lambda^*$, $\widetilde{K}\geqslant 1$, and $H''(y)\leqslant 0$ it suffices to show that $H(\widetilde{K}\alpha_{\lambda^*})=(a-\lambda^*)\widetilde{K}\alpha^*_\lambda-bd_c(\widetilde{K}\alpha_{\lambda^*})^2+(\widetilde{K}-1)\geqslant 0$. This easily follows from the fact that $d_c<D$. Fig. 2.1 illustrates $H(y)$.

To construct a subsolution $\overline{\psi}>0$; $\overline{\Omega}$, let $\overline{f}(x,u)=au-bu^2-d_c h(x)$. Then $\overline{f}$ is increasing with respect to u on $[0,\frac{a}{2b}]$ for each x. Since $\widetilde{K}d_c\alpha_{\lambda^*}<\frac{a}{2b}$, there is an $\varepsilon>0$ such that $\widetilde{K}d_c\alpha_{\lambda^*}+\varepsilon<\frac{a}{2b}$ and $g(\epsilon)\leqslant g(0)+1$.

Now define $\overline{\psi}=\psi+\varepsilon$, then $\|\overline{\psi}\|_\infty=\widetilde{K}d_c\alpha_{\lambda^*}+\varepsilon$. Also,

$$\begin{aligned}-\Delta\overline{\psi}&=-\Delta\psi\leqslant\overline{f}(x,\psi)<\overline{f}(x,\psi+\varepsilon)=\overline{f}(x,\overline{\psi})\\&=a\overline{\psi}-b\overline{\psi}^2-d_c h(x)\leqslant a\overline{\psi}-b\overline{\psi}^2-ch(x)\end{aligned}$$

and on $\partial\Omega$

$$\frac{\partial\overline{\psi}}{\partial\eta}=\frac{\partial\psi}{\partial\eta}\leqslant -g(0)-1\leqslant -g(\epsilon)=-g(\overline{\psi}).$$

Hence, $\overline{\psi}$ is a subsolution of (1.9)–(1.10) and clearly $\overline{\psi}>0$; $\overline{\Omega}$.

Now we choose a large constant $M=M(c)$ such that $aM-bM^2-ch(x)\leqslant 0$ and $M\geqslant\overline{\psi}(x)$ for $x\in\overline{\Omega}$. Then $Z:=M$ is a super solution of (1.9)–(1.10) with $Z\geqslant\overline{\psi}$. Thus there is a strict positive solution, $u(x)>0$; $\overline{\Omega}$, such that $\overline{\psi}\leqslant u\leqslant Z$ which is clearly not a solution of (1.7)–(1.8) (solutions of (1.7)–(1.8) have to satisfy Dirichlet Boundary Conditions). Also since $\widetilde{K}z^*_\lambda(x)\geqslant h(x)$ on $\overline{\Omega}$ and $c<d_c$ it is easy to see that

$$\psi=\widetilde{K}d_c z^*_\lambda\geqslant ch(x)$$

and hence, $u(x)\geqslant ch(x)$; $\overline{\Omega}$. Thus, Theorem 4 is proven.

2.3. *Proof of Theorem 6*

Let $a>\max\{\bar{a},\lambda_1\}$ and $0\leqslant c<\min\{c_0,c_1\}$. By Theorem 2, (1.5)–(1.6) has a positive solution, $u_1(x)$ with $u_1(x)=0$; $\partial\Omega$. Also, Theorem 4 implies that (1.5)–(1.6) has a second positive solution, $u_2(x)$, with $u_2(x)>0$; $\partial\Omega$. Clearly, $u_1(x)\neq u_2(x)$ and $u_i(x)\geqslant ch(x)$; $\overline{\Omega}$ for $i=1,2$.

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