Asymptotic Behavior and Oscillations of Second Order Differential Equations with Deviating Argument

Sawsan M. Aziz

Department of Mathematics, Faculty of Science for Girls, Al-Azhar University, Nasr City, Cairo, Egypt

and

A. H. Nasr

Department of Mathematics, Ain Shams University College for Women, Asma Fahmi Street, Heliopolis, Cairo, Egypt

Submitted by G. F. Webb

Received July 7, 1993

The asymptotic behavior of the solutions of the second order differential equation with deviating argument

\[(ax')' + bx' + cx = f[t, x(t), x'(t), x(\phi(t)), x'(\phi(t))]
\]

is studied. Our technique depends on an integral inequality containing a deviating argument. For special classes of these equations the explicit asymptotic behavior is obtained. From this asymptotic behavior the oscillation of the solutions is evident. © 1996 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to study the asymptotic behavior as \( t \to \infty \) of the solutions of second order differential equations with deviating argument of the form

\[
(ax')' + b(t)x'(t) + c(t)x(t) = f[t, x(t), x'(t), x(\phi(t)), x'(\phi(t))],
\]

\[t \in \mathbb{R}_+ = [0, \infty], \quad t' = d/dt,
\]

(1)
where $a, b, c,$ and $\phi$ are continuous functions defined on $\mathbb{R}_+$ and $f$ is a continuous function defined on $\mathbb{R}_+ \times \mathbb{R}^4$. We shall prove, under certain conditions, that the solution of Eq. (1) satisfying any given initial conditions behaves asymptotically (as $t \to \infty$) like a solution of the differential equation

$$(a(t)y')' + b(t)y' + c(t)y = 0, \quad t \in \mathbb{R}_+. \quad (2)$$

As a consequence, we obtain explicit asymptotic behavior of the solutions of some special types of Eq. (1) which shows the oscillatory behavior of these solutions.

We recall that the initial value problem of Eq. (1) is defined as follows (see [1, 2]).

Let $\inf_{t \in [0, \infty)} \phi(t) = \gamma$ and $\theta$ be a given differentiable function defined on $[\gamma, 0]$. The initial value problem of (1) in the interval $[0, \beta)$ is to find a function $x(t)$ such that

1. $x(0^+) = \theta(0^-), \quad x'(0^+) = \theta'(0^-)$
2. $x^{(j)}(t) = \theta^{(j)}(t), \quad t \in [\gamma, 0), \quad j = 0, 1$.
3. $(a(t)x')'$ exists on $(0, \beta)$ and Eq. (1) is satisfied.

A solution that is continuable throughout the whole interval $[\gamma, \infty)$ will be called a regular solution. Here we deal with the asymptotic behavior of such regular solutions. We shall use a technique previously used in similar situations for ordinary differential equations [3] and integrodifferential equations [4, 5]. In our case, the technique depends on a generalization of the famous Gronwall–Bellman and Bihari inequalities [6] to include the case of a deviating argument.

As in [4] and [5], we suppose that the general solution of the second order linear differential equation (2) is known and for any two linearly independent solutions $Z_1, Z_2$ of (2) we define

$$\xi(t) = |Z_1(t)| + |Z_2(t)|,$$

$$\eta(t) = |Z_1'(t)| + |Z_2'(t)|,$$

and

$$N(t) = \xi(t)/a(t)W(t),$$

where $W(t)$ denotes the Wronskian of $Z_1$ and $Z_2$;

$$W(t) = Z_1Z_2' - Z_2Z_1' \neq 0.$$
Assume the following:

1. \( f \in C(\mathbb{R}_+ \times \mathbb{R}^4, \mathbb{R}), \quad |f(t, x_1, x_2, x_3, x_4)| \)
\( \leq e(t) + r_1(t)|x_1| + r_2(t)|x_2| + r_3(t)|x_3|^p + r_4(t)|x_4|^p, \quad 0 < p \leq 1, \)

where
\( r_i \in C(\mathbb{R}_+, \mathbb{R}_+), \quad i = 1, 2, 3, 4, \quad e \in C(\mathbb{R}_+, \mathbb{R}_+). \)

2. The functions \( N(t)e(t), N(t)r_1(t)\xi(t), N(t)r_2(t)\eta(t), N(t)r_3(t)\xi(t), \) and \( N(t)r_4(t)\eta(t) \) belong to \( L_1(\mathbb{R}_+). \)

2. AN INTEGRAL INEQUALITY WITH DEVIATING ARGUMENT

We begin by proving a nonlinear integral inequality that is a mixture of Gronwall–Bellman inequality and Behari inequality (see [6]).

Lemma 1. Let \( u(t), f(t), g(t), \) and \( a(t) \) be nonnegative functions with \( a(t) \) monotonic increasing. If
\[ u(t) \leq a(t) + \int_a^t f(s)u(s) \, ds + \int_a^t g(s)u^p(s) \, ds, \quad 0 < p \leq 1, \]

then

(A) for \( 0 < p < 1, \)
\[ u(t) \leq e^{\int_a^t f(s) \, ds} \left[ a(t)^{1-p} + (1 - p) \int_a^t e^{-(1-p)\int_a^u f(s) \, ds} g(s) \, ds \right]^{1/(1-p)}, \]

(B) for \( p = 1, \)
\[ u(t) \leq a(t)e^{\int_a^t [f(s)+g(s)] \, ds}. \]

Proof. For the case \( 0 < p < 1. \) Let \( T \) be a fixed arbitrary positive number and \( t \in [a, T]. \) Then
\[ u(t) \leq a(T) + \int_a^t f(s)u(s) \, ds + \int_a^t g(s)u^p(s) \, ds. \]

Put \( y(t) = a(T) + \int_a^t f(s)u(s) \, ds + \int_a^t g(s)u^p(s) \, ds. \) Then
\[ u(t) \leq y(t). \quad (3) \]
Differentiating $y(t)$, we get
\[ y'(t) = f(t)u'(t) + g(t)u^p(t). \]
From (3), we obtain the Bernoulli differential inequality
\[ y'(t) \leq f(t)y(t) + g(t)y^p(t), \quad y(\alpha) = a(T). \]
Solving this inequality by the method used in solving Bernoulli equations, we get for $t \leq T$
\[ y(t) \leq e^{\int_{a}^{t} f(s) ds} \left[ a(T)^{1-p} + (1-p) \int_{a}^{t} e^{-(1-p) \int_{a}^{\xi} f(s) ds} g(s) d\xi \right]^{1/(1-p)}. \]
In this inequality, we put $t = T$. Since $T$ is arbitrary and the last inequality holds for all $t \in [\alpha, T]$, the required result follows from (3).

The case $p = 1$ is similar to the Gronwall–Bellman inequality.

**Lemma 2.** Let $u(t)$, $a(t)$, $f(t)$, $g(t)$ be nonnegative continuous functions and $a(t)$ nondecreasing. Let $\phi(t)$ be a continuously differentiable function satisfying that $\phi(t) \leq t$, $\phi'(t) > 0$, and $\phi(t)$ is eventually positive. Suppose that the inequality
\[ u(t) \leq a(t) + \int_{0}^{t} f(s)u(s) ds + \int_{0}^{t} g(s)u^p(\phi(s)) ds, \quad t \geq 0, \quad (4) \]
holds for all $t \in \mathbb{R}_+$, where $p \in (0, 1]$ is a constant. Then
(A) for $0 < p < 1$,
\[ u(t) \leq e^{\int_{0}^{t} f(s) ds} \left[ a_1^{1-p}(t) + (1-p) \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} e^{-(1-p) \int_{\phi^{-1}(0)}^{\xi} f(s) ds} g(s) d\xi \right]^{1/(1-p)} \]
(B) for $p = 1$,
\[ u(t) \leq a_1(t) \exp \left[ \int_{0}^{t} f(s) ds + \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} g(s) ds \right]. \]
where $a_1(t) = a(t) + \int_{0}^{\phi^{-1}(0)} g(\xi)u^p(\phi(\xi)) d\xi \quad (= a(t) + \text{const.})$.
Proof. From (4) we deduce that for sufficiently large $t$,
\[
u(\phi(t)) \leq a(\phi(t)) + \int_{0}^{\delta(t)} f(s)u(s) \, ds \\
+ \int_{0}^{\delta(t)} g(s)u^p(\phi(s)) \, ds, \quad \phi(t) \geq 0.
\]
(5)

It is easy to see that
\[
\int_{0}^{\phi(t)} f(s)u(s) \, ds = \int_{\phi^{-1}(0)}^{t} f(\phi(s))u(\phi(s))\phi'(s) \, ds
\]
Substituting this in (5) and keeping in mind that
\[
\int_{0}^{\phi(t)} g(s)u^p(\phi(s)) \, ds \leq \int_{0}^{t} g(s)u^p(\phi(s)) \, ds
\]
\[
= \int_{\phi^{-1}(0)}^{t} g(s)u^p(\phi(s)) \, ds \\
+ \int_{0}^{\phi^{-1}(0)} g(s)u^p(\phi(s)) \, ds,
\]
we get
\[
u(\phi(t)) \leq a(\phi(t)) \\
+ \int_{\phi^{-1}(0)}^{t} f(\phi(s))u(\phi(s))\phi'(s) \, ds + \int_{\phi^{-1}(0)}^{t} g(s)u^p(\phi(s)) \, ds.
\]
Applying Lemma 1 to the function $u(\phi(t))$, we obtain
\[
u(\phi(t)) \leq \exp \left[\int_{\phi^{-1}(0)}^{t} f(\phi(s))\phi'(s) \, ds\right] \\
\times \left[a_{1}^{-p}(\phi(t)) + (1 - p)\int_{\phi^{-1}(0)}^{t} \\
\times \exp \left[-(1 - p)\int_{\phi^{-1}(0)}^{s} f(\phi(\xi))\phi'(\xi) \, d\xi\right] g(s) \, ds \right]^{1/(1 - p)},
\]
\[0 < p < 1,
\]
\[
u(\phi(t)) \leq a(\phi(t))\exp \left[\int_{\phi^{-1}(0)}^{t} f(\phi(s))\phi'(s) \, ds \\
+ \int_{\phi^{-1}(0)}^{t} g(s) \, ds \right], \quad p = 1.
\]
From this we have
\[ u(t) \leq \exp \left[ \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} f(\phi(s)) \phi'(s) \, ds \right] \]
\[ \times \left[ a_1^{1-p}(t) + (1 - p) \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} \right. \]
\[ \left. \times \exp \left[ -(1 - p) \int_{\phi^{-1}(0)}^{t} f(\phi(\xi)) \phi'(\xi) \, d\xi \right] g(s) \, ds \right]^{1/(1-p)}, \]
\[ 0 < p < 1, \]
\[ u(t) \leq a_1(t) \times \exp \left[ \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} f(\phi(s)) \phi'(s) \, ds \right] \]
\[ + \int_{\phi^{-1}(0)}^{\phi^{-1}(t)} g(s) \, ds, \quad p = 1. \]
From which we get the required result.

3. ASYMPTOTIC BEHAVIOR

**Theorem 1.** Assume the following

1. \( f \in C(\mathbb{R}_+ \times \mathbb{R}^4, \mathbb{R}), \]
   \[ |f(t, x_1, x_2, x_3, x_4)| \leq e(t) + r_1|x_1| + r_2|x_2| \]
   \[ + r_3|x_3|^p + |x_4|^p, \quad 0 < p < 1, \]
   where \( e \in C(\mathbb{R}_+, \mathbb{R}_+), r_i \in C(\mathbb{R}_+, \mathbb{R}_+), \) \( i = 1, 2, 3, 4. \)

2. The functions \( N(T)e(t), N(t)r_1(t)\xi(t), N(t)r_2(t)\eta(t), N(t)r_3(t)\xi(t), \) and \( N(t)r_4(t)\eta(t) \) belong to \( \mathcal{L}(0,\infty). \)

Then for every initial function \( \theta(t) \) defined on \( [\gamma, 0] \) there exists a solution of (1) defined on the interval \( [\gamma, 0] \cup \mathbb{R}_+ \) which can be written in the form
\[ x(t) = A(t)Z_1(t) + B(t)Z_2(t) \] (6)
satisfying the initial condition \( x(t) = \theta(t), t \in [\gamma, 0], \) and \( \lim A(t), \lim B(t) \) exist as \( t \to \infty. \)

**Proof.** Let \( x(t) \) be a solution of (1) coinciding with the initial function \( \theta(t) \) on \( [\gamma, 0]. \) Assume that this solution is written in the form
\[ x(t) = A(t)Z_1(t) + B(t)Z_2(t), \quad \text{for } t \in \mathbb{R}_+, \]
\[ x(t) = \theta(t), \quad \text{for } t \in [\gamma, 0]. \] (7)
We assume that
\[ A'(t)Z_2(t) + B'(t)Z_2(t) = 0 \]  
(8)

For \( x(t) \) to satisfy Eq. (1), we must have
\[ A'(t)Z_1(t) + B'(t)Z_1(t) = K(t), \]  
(9)

where \( K(t) = (1/a(t))f(t, x(t), x'(t), x(\phi(t)), x'(\phi(t))). \)

Solving the two equations (8) and (9), we get
\[
A(t) = A(0) + \int_0^t \frac{Z_2(s)K(s)}{W(s)} \, ds, \\
B(t) = B(0) + \int_0^t \frac{Z_1(s)K(s)}{W(s)} \, ds.
\]

Let \( Q(t) = |A(t)| + |B(t)|. \) Then, by (1),
\[
Q(t) \leq Q(0) + \int_0^t \frac{\xi(s)}{|W(s)||a(s)|} \\
\times |f[s, x(s), x'(s), x(\phi(s)), x'(\phi(s))]|ds \\
\leq Q(0) + \int_0^t N(s)e(s)ds + \int_0^t N(s)r_1(s)\xi(s)Q(s)ds \\
+ \int_0^t N(s)r_2(s)\eta(s)Q(s)ds \\
+ \int_0^t N(s)r_3(s)\xi^p(\phi(s))Q^p(\phi(s))ds \\
+ \int_0^t N(s)r_4(s)\eta^p(\phi(s))Q^p(\phi(s))ds \\
= Q(0) + \int_0^t N(s)e(s)ds + \int_0^t N(s)[r_1(s)\xi(s) \\
+ r_2(s)\eta(s)]Q(s)ds \\
+ \int_0^t N(s)[r_3(\phi(s))\xi^p(\phi(s)) \\
+ r_4(\phi(s))\eta^p(\phi(s))]Q^p(\phi(s))ds.
\]
Then from Lemma 2 we have the following:

(A) If $0 < p < 1$,

$$Q(t) \leq \exp \left( \int_0^t N(s) \left[ r_1(s) \xi(s) + r_2(s) \eta(s) \right] ds \right)$$

$$\times \left[ M_1^{1-p} + (1-p) \int_{\phi(t)}^{\phi^{-1}(t)} N(s) \right]$$

$$\times r_3(\phi(s)) \xi''(\phi(s)) + r_4(\phi(s)) \eta''(\phi(s)) \right]$$

$$\times \exp \left[ -(1-p) \int_{\phi(t)}^{x} N(z) \right]$$

$$\times \left[ r_3(z) \xi(z) + r_2(z) \eta(z) \right] dz \right]^{1/(1-p)} ds.$$

(B) If $p = 1$,

$$Q(t) \leq M_1(t) \exp \left[ \int_0^t N(s) \left[ r_1(s) \xi(s) + r_2(s) \eta(s) \right] ds \right.$$}

$$\left. + \int_{\phi(t)}^{\phi^{-1}(t)} N(s) \left[ r_3(\phi(s)) \xi(\phi(s)) + r_4(\phi(s)) \eta(\phi(s)) \right] ds \right].$$

where

$$M_1(t) = Q(0) + \int_0^t N(s) e(s) ds + \text{const.}$$

Using Condition 2, we see that $Q(t)$, and hence $A(t)$ and $B(t)$, are bounded as $t \to \infty$. From this it follows that their limits exist as $t \to \infty$.

4. APPLICATIONS

Consider the following special case of Eq. (1)

$$x''(t) + q(t)x(t) = f[t, x(t), x'(t), x(\phi(t)), x'(\phi(t))], \quad (10)$$

where $f$ satisfies the conditions of Theorem 1 and

(3) $q(t)$ tends monotonically to the positive constant $\omega^2$ as $t \to \infty$, $q'(t) \geq 0$ (or $q'(t) \leq 0$) for all $t \in \mathbb{R}_+$. 

Theorem 2. If the Conditions 1–3 are satisfied then every solution of (10) is either oscillatory with the sequence of amplitudes tending to a finite limit or tends to zero as \( t \to \infty \).

Proof. Consider the equation

\[ y''(t) + q(t)y(t) = 0. \]  

(11)

It is clear that this equation is oscillatory [7] and every solution has the following property. If \( \{t_n\} \) is the sequence of the zeros of the derivative \( y'(t) \) the sequence of the amplitudes \( \{|y(t_n)|\} \) has a positive limit. In fact, consider the Liapunov function

\[ W(t) = \frac{1}{q(t)} y'^2(t) + y^2(t) > 0. \]

Upon differentiation and using Eq. (10), we see that

\[ W'(t) = - \frac{q'(t)}{q^2(t)} y'^2(t) \leq 0. \]

Hence \( W(t) \) is monotonic decreasing and bounded below. It follows that every solution of (11) and its derivative are bounded functions. Moreover, \( \lim W(t) = \alpha^2 \) exists as \( t \to \infty \). Consequently,

\[ y'^2(t_n) \to \alpha^2, \quad \text{i.e., } |y(t_n)| \to |\alpha| \text{ as } n \to \infty. \]

It remains to show that \( \alpha^2 \neq 0 \). For this, consider the function

\[ V(t) = y'^2(t) + q(t)y^2(t) > 0. \]

Upon differentiating, using Eq. (11), we get

\[ V'(t) = q'(t)y'^2(t) \geq 0, \]

i.e., \( V(t) \) is monotonic increasing to some positive constant \( \beta^2 \). From the equality \( W(t) = (1/q(t))V(t) \), it follows that

\[ \alpha^2 = \frac{1}{\omega^2} \beta^2 > 0. \]

Let \( Z_1(t), Z_2(t) \) be two linearly independent solutions of (11), then, according to Theorem 1, every solution of (10) can be written in the form

\[ x(t) = [a_1Z_1(t) + a_2Z_2(t)] + o(1), \quad \text{as } t \to \infty, \]  

(12)
where \( a_1, a_2 \) are constants.

Since Eq. (11) is linear, we can write (12) in the form

\[
x(t) = y(t) + o(1) \quad \text{as } t \to \infty,
\]

where \( y(t) \) is a solution of (11). From the first part of the theorem, we see that the sequence of the amplitudes of \( y(t) \) tends to a positive constant \( a^2 \) (depending on \( y \)). The relation (13) gives the required result. In fact, the first possibility, i.e., the oscillation of the solution \( y(t) \) occurs when \((a_1, a_2) \neq (0,0)\) and the second, i.e., the tendency to zero occurs when \((a_1, a_2) = (0,0)\). The second case of the theorem when \( q'(t) \leq 0 \) can be similarly treated.

Now consider the following special case of Eq. (1)

\[
x''(t) + \omega^2 x(t) = f[t, x(t), x'(t), x(\phi(t)), x'(\phi(t))],
\]

where \( f \) satisfies the same conditions and \( \omega^2 > 0 \) is a constant.

**Theorem 3.** Every solution of Eq. (14) either is oscillatory and can be written in the form

\[
x(t) = A(t)\sin(\omega t + \delta(t)),
\]

where \( \lim A(t), \lim \delta(t) \) exist as \( t \to \infty \), or \( x(t) \) tends to zero as \( t \to \infty \).

**Proof.** Consider the equation \( y''(t) + \omega^2 y(t) = 0 \), which has the two linearly independent solutions \( Z_1 = \sin \omega t, \ Z_2 = \cos \omega t \), for which \( W(s) = 1/\omega \), and \( N(s) = 1, \xi(s), \eta(s) \) are bounded. Consequently, Theorem 1 implies

\[
x(t) = A_1(t)\sin \omega t + A_2(t)\cos \omega t,
\]

where \( \lim A_1(t), \lim A_2(t) \) exist as \( t \to \infty \). If one of these limits, say \( \lim A_2(t) \), is not equal to zero then (15) can be written in the form

\[
x(t) = A(t)\sin(\omega t + \delta(t)),
\]

where \( A(t) = [A_1^2(t) + A_2^2(t)]^{1/2}, \ \delta(t) = \arctan A_1(t)/A_2(t) \). Otherwise, \( x(t) \) has the form (15) and \( \lim x(t) = 0 \) as \( t \to \infty \).

**Corollary.** From Theorem 2, it follows that for every oscillatory solution of (14) the limit of the sequence of the zeros \( \{t_n\} \) of the solution has the asymptotic behavior

\[
t_n = \frac{\pi}{\omega} n + o(1).
\]
This generalizes a result obtained in [1] for the linear equation
\[ x^{\prime\prime}(t) + \omega^2 x(t) + r_2(t) x(t) + r_3(t) x(\phi(t)) = 0. \]

The technique here is different and, comparatively, simple.

REFERENCES