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Variational calculus on q-nonuniform lattices

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Abstract

We introduce the variational calculus on q-nonuniform lattices. In particular, we discuss the basic concepts such as the Euler–Lagrange equation and its applications to the isoperimetric, the Lagrange and the optimal control problems on q-nonuniform lattices.

1. Introduction

Following [10,11], let $x(s)$ be a real valued discrete variable ($s \in \frac{1}{2}\mathbf{Z}$) function such that

$$F\left(x(s), x\left(s-\frac{1}{2}\right)\right) = F\left(x(s), x\left(s+\frac{1}{2}\right)\right) = 0, \tag{1}$$

where

$$F(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2cy + 2f = 0. \tag{2}$$

This means that

$$x\left(s+\frac{1}{2}\right) = P(x) + \sqrt{Q(x)}, \qquad x\left(s-\frac{1}{2}\right) = P(x) - \sqrt{Q(x)}, \tag{3}$$

where $P(x)$ and $Q(x)$ are polynomials of degree maximum 1 and 2, respectively.

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Next, from (3), one derives the following most important canonical forms for $x(s)$ in order of increasing complexity:

$$x(s) = x(0), \tag{4}$$

$$x(s) = s, \tag{5}$$

$$x(s) = q^s, \tag{6}$$

$$x(s) = \frac{q^s + q^{-s}}{2}, \quad q \in \mathbf{C}. \tag{7}$$

Here for concreteness, we take $0 < |q| < 1$. The forms (4)–(7) correspond to

$$\begin{aligned}
&Q(x) = 0, \qquad P(x) = x, \\
&Q(x) = \frac{1}{4}, \qquad P(x) = x, \\
&Q(x) = \frac{(q-1)^2}{4q} x^2, \qquad P(x) = \frac{(q+1)}{2\sqrt{q}} x, \\
&Q(x) = \frac{(q-1)^2}{4q} (x^2 - 1), \qquad P(x) = \frac{(q+1)}{2\sqrt{q}} x,
\end{aligned} \tag{8}$$

respectively. As seen from (1), the set of points $\{(x(s), x(s+\frac{1}{2})), (x(s), x(s-\frac{1}{2})),\ s \in \frac{1}{2}\mathbf{Z}\}$ forms a lattice on the corresponding conic. For this reason, one refers to the functions (4)–(7) as "*continuous*" (constant), "*uniform*" (linear), "*q-uniform*" and "*q-nonuniform*" lattices, respectively.

Next, define the following divided difference derivative [10,11]:

$$\mathcal{D} f\big(x(s)\big) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}. \tag{9}$$

The point here is that if $f(x)$ is a polynomial of degree n in $x(s)$, then $\mathcal{D} f(x(s))$ is a polynomial in $x(s)$ of degree $n-1$. As far as we are aware of, (9) is the most general (divided difference) derivative having this propriety. Note that when $x(s)$ is given by (4)–(6), the corresponding divided difference derivatives give respectively:

$$Df(x) = \frac{d}{dx} f(x), \tag{10}$$

$$\Delta_{\frac{1}{2}} f(x) = \Delta f(t) = f(t+1) - f(t) = \big(e^{\frac{d}{dt}} - 1\big) f(t), \quad t = x - \frac{1}{2}, \tag{11}$$

$$D_{q^{\frac{1}{2}}} f(x) = D_q f(t) = \frac{f(qt) - f(t)}{qt - t} = \frac{q^{\frac{d}{dt}} - 1}{qt - t} f(t), \quad t = q^{-\frac{1}{2}} x. \tag{12}$$

When $x(s)$ is given by (7), the corresponding derivative is usually referred to as the Askey–Wilson first order divided difference operator [1] that one can write:

$$\mathcal{D} f(x(z)) = \frac{f(x(q^{\frac{1}{2}} z)) - f(x(q^{-\frac{1}{2}} z))}{x(q^{\frac{1}{2}} z) - x(q^{-\frac{1}{2}} z)}, \tag{13}$$

where $x(z) = \frac{z+z^{-1}}{2}$, having in mind that $z = q^s$.

To deal with the inverse of the differentiation operation that is the integration, we have to solve for f from the equation

$$\mathcal{D} f\big(x(s)\big) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})} = g\big(x(s)\big). \tag{14}$$

One gets

$$\begin{aligned} & f\left(x\left(s-\frac{1}{2}\right)\right) - f\left(x\left(N+\frac{1}{2}\right)\right) \\ &\quad = \sum_{t=s}^{t=N}\left[x\left(t-\frac{1}{2}\right) - x\left(t+\frac{1}{2}\right)\right] g\big(x(t)\big), \quad N \geqslant s. \end{aligned} \tag{15}$$

Hence the definition of the integral on lattices:

$$\begin{aligned} & \int\limits_{x(N)}^{x(s)} g\big(x(t)\big)\, d_q x(t) \\ &\quad \stackrel{\text{def}}{=} \sum_{t=s+\frac{1}{2}}^{N-\frac{1}{2}} \left[x\left(t-\frac{1}{2}\right) - x\left(t+\frac{1}{2}\right)\right] g\big(x(t)\big) \qquad (16) \\ &\quad = \sum_{x(t)=x(s+\frac{1}{2})}^{x(N-\frac{1}{2})} \left[x\left(t-\frac{1}{2}\right) - x\left(t+\frac{1}{2}\right)\right] g\big(x(t)\big). \qquad (17) \end{aligned}$$

Varying t in the opposite sense, one gets from (14)

$$\begin{aligned} & f\left(x\left(s+\frac{1}{2}\right)\right) - f\left(x\left(N-\frac{1}{2}\right)\right) \\ &\quad = \sum_{t=s}^{t=N}\left[x\left(t+\frac{1}{2}\right) - x\left(t-\frac{1}{2}\right)\right] g\big(x(t)\big), \quad N \leqslant s, \end{aligned} \tag{18}$$

and the integral

$$\begin{aligned} & \int\limits_{x(N)}^{x(s)} g\big(x(t)\big)\, d_q x(t) \qquad (19) \\ &\quad \stackrel{\text{def}}{=} \sum_{t=s-\frac{1}{2}}^{N+\frac{1}{2}} \left[x\left(t+\frac{1}{2}\right) - x\left(t-\frac{1}{2}\right)\right] g\big(x(t)\big) \qquad (20) \\ &\quad = \sum_{x(t)=x(s-\frac{1}{2})}^{x(N+\frac{1}{2})} \left[x\left(t+\frac{1}{2}\right) - x\left(t-\frac{1}{2}\right)\right] g\big(x(t)\big). \qquad (21) \end{aligned}$$

Remark that when $x(s) = (q^s + q^{-s})/2$, we have $x(-s) = x(s)$ and Eq. (18) is obtained from (15) by replacing quite simply s, t and N by $-s$, $-t$ and $-N$, respectively. It is also understood that in the formulae (15)–(17) and (18)–(21), one can take $N \to +\infty$ and $N \to -\infty$, respectively. Let finally note that the differentiation or integration on any one of the lattices in (5)–(7) generalizes the differentiation or integration on the lattices of low complexity. Hence the calculus on a given lattice generalizes the calculus on a lattice of low complexity.

This work is concerned in the generalization of the variational calculus. The variational calculus on the uniform lattices was proposed in [3]. In the time to follow, most of researches in the area were mainly directed to the study of the complete integrability of the discrete Euler–Lagrange equation (see, e.g., [6–9,12,17]). That is to say that as far as we are aware of, the question of the generalization of the continuous (differential) variational calculus, to the calculus of variation on lattices more general than the uniform one (treated in [3]), had never been considered until [2]. In [2], the variational calculus on the q-uniform lattices was discussed. Here, we consider the variational calculus on the q-nonuniform lattices. More precisely, we discuss q-nonuniform lattices versions of the basic concepts of variational calculus such as the Euler–Lagrange equation, the isoperimetric, Lagrange and optimal control problems. Also, some interconnections between the Euler–Lagrange equation, the Hamilton and the Hamilton–Pontriaguine systems on q-nonuniform lattices are discussed. In the following section, we first outline some basic formulae for differentiation and integration on q-nonuniform lattices, useful for the sequel.

Before closing this section, let us note that others motivation and derivation of the lattices (4)–(7) can be found in [13]. On the other side, a widely different generalization of the derivative in (12) (the so-called Jackson derivative) can be found in [5] and references therein. Also, it is to be understood that others kinds of nonuniform lattices had already been used in various discretization problems (see, e.g., [15,16]).

2. Differentiation and integration on q-nonuniform lattices

Here for clarity, we outline basic formulae of the differentiation and integration on q-nonuniform lattices. This means that setting $z = q^s$ in (7), the current variable is now $x(z) = \frac{z+z^{-1}}{2}$.

Derivative of a product

$$\begin{aligned}\mathcal{D}(fg)\big(x(z)\big) &= f\big(x\big(q^{\frac{1}{2}}z\big)\big)\mathcal{D}g\big(x(z)\big) + g\big(x\big(q^{-\frac{1}{2}}z\big)\big)\mathcal{D}f\big(x(z)\big)\\ &= g\big(x\big(q^{\frac{1}{2}}z\big)\big)\mathcal{D}f\big(x(z)\big) + f\big(x\big(q^{-\frac{1}{2}}z\big)\big)\mathcal{D}g\big(x(z)\big).\end{aligned} \tag{22}$$

Derivative of a ratio

$$\mathcal{D}(f/g)\big(x(z)\big) = \frac{g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) - f(x(q^{-\frac{1}{2}}z))\mathcal{D}g(x(z))}{g(x(q^{\frac{1}{2}}z))g(x(q^{-\frac{1}{2}}z))}. \tag{23}$$

Derivative of a composite function

$$\mathcal{D}\big(f(g)\big)\big(x(z)\big) = \frac{f(g(x(q^{\frac{1}{2}}z))) - f(g(x(q^{-\frac{1}{2}}z)))}{g(x(q^{\frac{1}{2}}z)) - g(x(q^{-\frac{1}{2}}z))} \cdot \frac{g(x(q^{\frac{1}{2}}z)) - g(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} \stackrel{\text{def}}{=} (D_g f).D_x g. \tag{24}$$

Derivative of the inverse function

Let $y = f(x)$. Then $x = f^{-1}(y)$, where f^{-1} is the inverse to f function. Applying the divided difference derivative on both sides of the preceding equation, one obtains

$$\begin{aligned} 1 &= \frac{f^{-1}(y(x(q^{\frac{1}{2}}z))) - f^{-1}(y(x(q^{-\frac{1}{2}}z)))}{x(q^{\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} \\ &= \frac{f^{-1}(y(x(q^{-\frac{1}{2}}z))) - f^{-1}(y(x(q^{-\frac{1}{2}}z)))}{y(x(q^{\frac{1}{2}}z)) - y(x(q^{-\frac{1}{2}}z))} \cdot \frac{y(x(q^{\frac{1}{2}}z)) - y(x(q^{\frac{1}{2}}z))}{x(q^{-\frac{1}{2}}z) - x(q^{-\frac{1}{2}}z)} \\ &\stackrel{\text{def}}{=} \mathcal{D}_y f^{-1}.\mathcal{D}_x y. \end{aligned}$$

Hence

$$\mathcal{D}_y f^{-1} = \frac{1}{\mathcal{D}_x f}. \tag{25}$$

"Fundamental principles" of analysis

$$\text{(i)} \quad \mathcal{D}\Bigg[\int_{x(q^N)}^{x(z)} g\big(x(z)\big)\, d_q x(z)\Bigg]$$

$$= \mathcal{D}\Bigg[\sum_{x(zq^{\frac{1}{2}})}^{x(q^{N-\frac{1}{2}})} \big[x\big(zq^{-\frac{1}{2}}\big) - x\big(zq^{\frac{1}{2}}\big)\big] g\big(x(z)\big)\Bigg]$$

$$= \frac{\big[\sum_{x(z)}^{x(q^{N-\frac{1}{2}})} - \sum_{x(qz)}^{x(q^{N-\frac{1}{2}})}\big][x(zq^{-\frac{1}{2}}) - x(zq^{\frac{1}{2}})]g(x(z))}{x(zq^{-\frac{1}{2}}) - x(zq^{\frac{1}{2}})} \tag{26}$$

$$= g\big(x(z)\big); \tag{27}$$

$$\text{(ii)} \quad \int_{x(q^N)}^{x(z)} (\mathcal{D}f)\big(x(z)\big)\, d_q x(z) = \sum_{x(zq^{\frac{1}{2}})}^{x(q^{N-\frac{1}{2}})} \big[x\big(zq^{-\frac{1}{2}}\big) - x\big(zq^{\frac{1}{2}}\big)\big](\mathcal{D}f)\big(x(z)\big)$$

$$= \sum_{x(zq^{\frac{1}{2}})}^{x(q^{N-\frac{1}{2}})} \left[f\left(x\left(zq^{-\frac{1}{2}}\right)\right) - f\left(x\left(zq^{\frac{1}{2}}\right)\right)\right] = f\left(x(z)\right) - f\left(x(q^N)\right). \tag{28}$$

Integration by parts

Equation (22) can be written as

$$f\left(x\left(q^{\frac{1}{2}}z\right)\right)\mathcal{D}g\left(x(z)\right) = \mathcal{D}(fg)\left(x(z)\right) - g\left(x\left(q^{-\frac{1}{2}}z\right)\right)\mathcal{D}f\left(x(z)\right). \tag{29}$$

Multiplying the both sides of the equation by

$$\gamma(z) = x\left(q^{-\frac{1}{2}}z\right) - x\left(q^{\frac{1}{2}}z\right) \tag{30}$$

and integrating on the $x(z)$ lattice from $x(q^N)$ to $x(z)$, one obtains

$$\int_{x(q^N)}^{x(z)} f\left(x\left(q^{\frac{1}{2}}z\right)\right)\mathcal{D}g\left(x(z)\right) d_q x(z)$$
$$= [fg]_{x(q^N)}^{x(z)} - \int_{x(q^N)}^{x(z)} g\left(x\left(q^{-\frac{1}{2}}z\right)\right)\mathcal{D}f\left(x(z)\right) d_q x(z). \tag{31}$$

Convergence of integrals

Using the relation,

$$d_q\left(x(z)\right) \stackrel{\text{def}}{=} x\left(q^{-\frac{1}{2}}z\right) - x\left(q^{\frac{1}{2}}z\right) = \frac{1}{2\sqrt{q}}(1-z^{-2})(1-q)z$$
$$= \frac{1}{2\sqrt{q}}(1-z^{-2})d_q(z),$$
$$d_q(z) \stackrel{\text{def}}{=} (1-q)z, \tag{32}$$

one makes the change of integration variables from the q-nonuniform to the uniform one

$$\int_{x(q^N)}^{x(z)} f\left(x(z)\right) d_q\left(x(z)\right) = \frac{1}{2\sqrt{q}} \int_{q^N}^{z} (1-z^{-2}) f\left(x(z)\right) d_q(z). \tag{33}$$

Hence, the existence of the integral in the lhs of (33) is conditioned by the existence of the one in the rhs. In particular, when $N \to +\infty$, we have

$$\int_{x(0)}^{x(z)} f\left(x(z)\right) d_q\left(x(z)\right) = \frac{1}{2\sqrt{q}} \int_{0}^{z} (1-z^{-2}) f\left(x(z)\right) d_q(z)$$
$$= \frac{1}{2}(1-q)z \sum_{0}^{+\infty} q^i g\left(q^{i+\frac{1}{2}}z\right), \tag{34}$$

where $g(z) = (1 - z^{-2}) f(x(z))$. But what stands in the rhs of (34) is clearly a Riemann integral sum of the function $\frac{1}{2} g(q^{\frac{1}{2}} z)$ on $[0, z]$. That is why the integrability of the function $f(x(z))$ on $[x(0), x(z)]$ can be deduced from that of $g(q^{\frac{1}{2}} z)$ on $[0, z]$. Moreover, in the case of Riemann integrability of $g(q^{\frac{1}{2}} z)$ on $[0, z]$, we have the limit

$$\int_{x(0)}^{x(z)} f\big(x(z)\big)\, d_q\big(x(z)\big) \to \frac{1}{2} \int_0^z (1 - z^{-2}) f\big(x(z)\big)\, d(z), \quad q \to 1. \tag{35}$$

Example 1 (*Derivative of a polynomial*). Let $P_n(x(z))$ be a polynomial in the variable $x(z) = \frac{z+z^{-1}}{2}$. We calculate its derivative to make sure that it is a polynomial in $x(z)$ with moreover a degree equal to $n - 1$. Using the fact that any polynomial of degree k in $x(z) = \frac{z+z^{-1}}{2}$ can be written as

$$P_k\big(x(z)\big) = \sum_{i=0}^{k} a_i \left(\frac{z + z^{-1}}{2}\right)^i = \sum_{j=0}^{k} b_j (z^j + z^{-j}), \tag{36}$$

we obtain

$$\begin{aligned}
\mathcal{D} P_n\big(x(z)\big) &= \frac{\sum_{j=0}^{n} b_j (q^{\frac{j}{2}} z^j + q^{-\frac{j}{2}} z^{-j}) - \sum_{j=0}^{n} b_j (q^{-\frac{j}{2}} z^j + q^{\frac{j}{2}} z^{-j})}{\frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{2} (z - z^{-1})} \\
&= \sum_{j=1}^{n} 2b_j \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \frac{z^j - z^{-j}}{z - z^{-1}} = \sum_{j=1}^{n} 2b_j \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \sum_{k=0}^{j-1} c_k (z^k + z^{-k}) \\
&= \sum_{j=1}^{n} 2b_j \frac{q^{\frac{j}{2}} - q^{-\frac{j}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}} \tilde{P}_{j-1}\left(\frac{z + z^{-1}}{2}\right) = \tilde{\tilde{P}}_{n-1}\left(\frac{z + z^{-1}}{2}\right).
\end{aligned} \tag{37}$$

Example 2 (*Integral of a polynomial*). We now calculate the integral of a polynomial $P_n(x(z))$ of degree n in the variable $x(z) = \frac{z+z^{-1}}{2}$ and make sure that it is a polynomial in $x(z)$ with moreover a degree equal to $n + 1$. The relation (36) will also be used. So, for a given polynomial $P_n(x(z))$, we search a function $f(x(z))$ such that

$$\mathcal{D} f\big(x(z)\big) = \frac{f(x(zq^{\frac{1}{2}})) - f(x(zq^{-\frac{1}{2}}))}{x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}})} = P_n\big(x(z)\big). \tag{38}$$

Hence

$$\begin{aligned}
f\big(x\big(zq^{-\frac{1}{2}}\big)\big) - f\big(x\big(zq^{\frac{1}{2}}\big)\big) &= \big(x\big(zq^{-\frac{1}{2}}\big) - x\big(zq^{\frac{1}{2}}\big)\big) P_n\big(x(z)\big) \\
&= \frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{2} (z - z^{-1}) P_n\left(\frac{z + z^{-1}}{2}\right) = \sum_{j=1}^{n+1} a_j (z^j - z^{-j}) \\
&= P_{n+1}^1(z) + P_{n+2}^2(z),
\end{aligned} \tag{39}$$

where

$$P_{n+1}^1(z) = \sum_{j=1}^{n+1} a_j z^j, \qquad P_{n+1}^2(z) = -\sum_{j=1}^{n+1} a_j z^{-j}. \tag{40}$$

Now, let us consider two functions $f_1(x(z))$ and $f_2(x(z))$ such that $f(x(z)) = f_1(x(z)) + f_2(x(z))$ and

$$f_1\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr) - f_1\bigl(x\bigl(q^{\frac{1}{2}}z\bigr)\bigr) = P_{n+1}^1(z), \tag{41}$$

$$f_2\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr) - f_2\bigl(x\bigl(q^{\frac{1}{2}}z\bigr)\bigr) = P_{n+1}^2(z). \tag{42}$$

From (41) and (42) it follows, respectively, that

$$f_1\bigl(x(z)\bigr) = \sum_{i=0}^{\infty} P_{n+1}^1\bigl(q^{i+\frac{1}{2}}z\bigr) + c_1 = \sum_{i=1}^{n+1} \frac{a_j q^{\frac{j}{2}}}{1-q^j} z^j + c_1, \tag{43}$$

$$f_2\bigl(x(z)\bigr) = -\sum_{i=0}^{\infty} P_{n+1}^2\bigl(q^{-i-\frac{1}{2}}z\bigr) + c_2 = \sum_{i=1}^{n+1} \frac{a_j q^{\frac{j}{2}}}{1-q^j} z^{-j} + c_2. \tag{44}$$

Hence

$$\begin{aligned} f\bigl(x(z)\bigr) = f_1\bigl(x(z)\bigr) + f_2\bigl(x(z)\bigr) &= c + \sum_{i=1}^{n+1} \frac{a_j q^{\frac{j}{2}}}{1-q^j}\bigl[z^j + z^{-j}\bigr] \\ &= \tilde{P}_{n+1}\left(\frac{z+z^{-1}}{2}\right). \end{aligned} \tag{45}$$

3. Euler–Lagrange equation on q-nonuniform lattices

We consider the following functional given as an integral on the q-nonuniform lattice $x(z) = \frac{z+z^{-1}}{2}$:

$$\begin{aligned} J\bigl(y\bigl(x(z)\bigr)\bigr) &= \int_a^b F\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr]\, d_q x(z) \\ &= \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr], \end{aligned} \tag{46}$$

where $\gamma(z)$ is given in (30) and $a = x(q^{\beta+\frac{1}{2}})$, $b = x(q^{\alpha-\frac{1}{2}})$ with the supposition that $\beta \geqslant \alpha$. In (46), F is a differentiable function with respect to all its arguments. The function y belongs to the variety E' of functions satisfying boundary constraints

$$y\bigl(q^{\alpha-\frac{1}{2}}\bigr) = y\bigl(q^{\beta+\frac{1}{2}}\bigr) = c, \tag{47}$$

in the linear space E of functions $f(x(z))$ defined and bounded together with $\mathcal{D}f(x(z))$, on the set

$$L=\left\{q^{\alpha-\frac{1}{2}+\frac{i}{2}},\ i=0,1,\dots,2(1+\beta-\alpha)\right\} \tag{48}$$

and equipped with the norm

$$\|f\|=\max\Bigl(\sup_{z\in L}\bigl|f\bigl(x(z)\bigr)\bigr|,\sup_{z\in L}\bigl|\mathcal{D}f\bigl(x(z)\bigr)\bigr|\Bigr). \tag{49}$$

The extremum problem consists then in finding the extremals for the functional (46) under the constraints (47). As F is a differentiable function with respect to all its arguments, we can calculate the first variation of the functional:

$$\delta J\bigl(y\bigl(x(z)\bigr),h\bigl(x(z)\bigr)\bigr)=\frac{d}{dt}J\bigl(y(x)+th(x)\bigr)\bigg|_{t=0}=\frac{d}{dt}u(t)\bigg|_{t=0},$$

$$u(t)=\int_a^b F\bigl[x(z),y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr)+th\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr),\mathcal{D}y\bigl(x(z)\bigr)+t\mathcal{D}h\bigl(x(z)\bigr)\bigr]\,d_qx(z).$$

Hence

$$\begin{aligned}&\delta J\bigl(y\bigl(x(z)\bigr),h\bigl(x(z)\bigr)\bigr)\\&\quad=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)\bigl\{F_{v_0}\bigl[x(z),y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr),\mathcal{D}y\bigl(x(z)\bigr)\bigr]h\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr)\\&\qquad+F_{v_1}\bigl[x(z),y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr),\mathcal{D}y\bigl(x(z)\bigr)\bigr]\mathcal{D}h\bigl(x(z)\bigr)\bigr\},\end{aligned} \tag{50}$$

where $F_{v_i}=\frac{\partial F}{\partial v_i}$ means the derivative of F with respect to its $(i+2)$th argument, $i=0,1$. As $y+th$ belongs also to E', it follows from (47) that

$$h\bigl(q^{\alpha-\frac{1}{2}}\bigr)=h\bigl(q^{\beta+\frac{1}{2}}\bigr)=0. \tag{51}$$

Using (31) and (51), one transforms (50) in

$$\begin{aligned}&\delta J\bigl(y\bigl(x(z)\bigr),h\bigl(x(z)\bigr)\bigr)\\&\quad=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)\bigl\{F_{v_0}\bigl[x(z),y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr),\mathcal{D}y\bigl(x(z)\bigr)\bigr]\\&\qquad-\mathcal{D}\bigl[F_{v_1}\bigl[x\bigl(zq^{-\frac{1}{2}}\bigr),y\bigl(x\bigl(q^{-1}z\bigr)\bigr),\mathcal{D}y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr)\bigr]\bigr]\bigr\}h\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr).\end{aligned} \tag{52}$$

To obtain the Euler–Lagrange equation from (52), we need the following q-nonuniform lattices version of the “fundamental lemma of variational calculus.”

Lemma 3.1. *Suppose that for a given function $f(z)$, one has*

$$\sum_{q^\alpha}^{q^\beta}\gamma(z)f(z)p(z)=0 \tag{53}$$

for any function $p(z)$ belonging to the space E, then $f(z)\equiv 0$.

Proof. For various functions $p_i(z)$, Eq. (53) gives a system of equations that one can write in matrix (may be infinite dimensional) form as $Ay = 0$, where $A_{ij} = \gamma(q^{\alpha+j})p_i(q^{\alpha+j})$ and $y_j = f(q^{\alpha+j})$, $i, j = 0, \dots, \beta - \alpha$. To obtain $f(z) \equiv 0$, it suffices to choose the $p_i(q^{\alpha+j})$ so that the matrix A be invertible, which proves the lemma. $\square$

Applying the lemma to Eq. (52), one obtains

$$F_{v_0}\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr] \\ - \mathcal{D}\bigl[F_{v_1}\bigl[x\bigl(zq^{-\frac{1}{2}}\bigr), y\bigl(x(q^{-1}z)\bigr), \mathcal{D}y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr)\bigr]\bigr] = 0, \tag{54}$$

which is the *Euler–Lagrange equation* giving the necessary condition for the extremum problem on q-nonuniform lattices. It is a second order q-difference equation, which in principle is solved uniquely under the boundary constraints (47).

Remark 1. If the function under the sign of integration F is given by $F = F(x, y_1, \dots, y_n, \mathcal{D}y_1, \dots, \mathcal{D}y_n)$, so the extremum necessary condition is given by n equations similar to (54), one equation for each variable, the other variables being supposed fixed.

Remark 2. If the function under the sign of integration F is given by $F = F(x, y, \mathcal{D}y, \dots, \mathcal{D}^n y)$, so the change of variables $y_1 = y$, $y_2 = \mathcal{D}y$, $\dots$, $y_n = \mathcal{D}^{n-1}y$ leads to the case of Remark 1, with additional constraints: $\mathcal{D}y_1 = y_2$, $\mathcal{D}y_2 = y_3$, $\dots$, $\mathcal{D}y_{n-1} = y_n$. This gives a particular case of the "Lagrange problem" which will be discussed in the next section.

4. Applications

4.1. The isoperimetric problem on q-nonuniform lattices

The problem Consider the integration functional on q-nonuniform lattices

$$J_0\bigl(y\bigl(x(z)\bigr)\bigr) = \int_a^b F_0\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr]\, d_q x(z) \\ = \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F_0\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr] \tag{55}$$

defined in E'. Let next be given a set of other functionals

$$J_i\bigl(y\bigl(x(z)\bigr)\bigr) = \int_a^b F_i\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr]\, d_q x(z) \\ = \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F_i\bigl[x(z), y\bigl(x\bigl(q^{-\frac{1}{2}}z\bigr)\bigr), \mathcal{D}y\bigl(x(z)\bigr)\bigr], \tag{56}$$

defined also in E' and consider the equations

$$J_i\big(y(x)\big) = c_i, \quad i = 1, \dots, m. \tag{57}$$

The isoperimetric problem consists in finding extremals of the functional $J_0(y)$, among all the functions belonging in E' and satisfying (57).

The solution The settled isoperimetric problem can be solved in a more general setting by the following theorem (see, e.g., [4]).

Theorem 4.1. *Suppose that is given a set of functionals $J_i(y)$, $i = 0, 1, \dots, m$, defined on a variety E' of a linear normed space E and admitting on E' the first variation $\delta J_i(y, h)$ with $\delta J_i(y, h)$, $i = 1, \dots, m$, linearly independent functionals. Suppose next that y_0 is an extremal of $J_0(y)$ under the constraints $J_i(y) = c_i$, $i = 1, \dots, m$, and $\delta J_i(y_0, h) \not\equiv 0$, $i = 1, \dots, m$. In that case, y_0 is an ordinary extremal for the functional $J^*(y) = J_0(y) + \sum_{i=1}^m \lambda_i J_i(y)$, where the λ_i are some constants.*

Applied to our functionals (55), (56) and constraints (57), the theorem implies that if the functions under the signs of integration F_0 and F_i are differentiable with respect to all its arguments (this is sufficient for the functionals to have the first variation), and the variational derivatives (i.e. the function in the lhs of the corresponding Euler–Lagrange equation) of the functionals (56) are linearly independent (this is sufficient for the first variations to be so), then the extremals of $J_0(y)$ under the constraints (57) are included in the union of the set of solutions of the equations $\delta J_i(y, h) = 0$, $i = 1, \dots, m$, and that of $\delta J^*(y, h) = 0$, where $J^*(y) = J_0(y) + \sum_{i=1}^m \lambda_i J_i(y)$.

Example. Suppose it required to find the extremum of the integration functional on q-nonuniform lattices

$$\begin{aligned} J_0\big(y\big(x(z)\big)\big) &= \int_a^b \left[\frac{1}{2}\big(\mathcal{D}y\big(x(z)\big)\big)^2 - a\big(q + q^2 + 2q^{\frac{3}{2}}\big)y\big(x\big(q^{-\frac{1}{2}}z\big)\big)\right] d_q x(z) \\ &= \sum_{z=q^\alpha}^{q^\beta} \gamma(z)\left[\frac{1}{2}\big(\mathcal{D}y\big(x(z)\big)\big)^2 - a\big(q + q^2 + 2q^{\frac{3}{2}}\big)y\big(x\big(q^{-\frac{1}{2}}z\big)\big)\right] \end{aligned} \tag{58}$$

under the constraints

$$\begin{aligned} J_1\big(y\big(x(z)\big)\big) &= 4aq^{\frac{3}{2}} \int_a^b \big[x^2\big(q^{-\frac{3}{4}}z\big)y\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big]\, d_q x(z) \\ &= 4aq^{\frac{3}{2}} \sum_{z=q^\alpha}^{q^\beta} \gamma(z)\big[x^2\big(q^{-\frac{3}{4}}z\big)y\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big] = c_1. \end{aligned} \tag{59}$$

According to Theorem 4.1, this is equivalent to the problem of finding the ordinary extremum for the functional

$$J^*\big(y\big(x(z)\big)\big)=\int_a^b F^*\big[x(z),y\big(x\big(q^{-\frac12}z\big)\big),\mathcal{D}y\big(x(z)\big)\big]\,d_qx(z)$$
$$=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)F^*\big[x(z),y\big(x\big(q^{-\frac12}z\big)\big),\mathcal{D}y\big(x(z)\big)\big], \tag{60}$$

where

$$F^*=\frac12\big(\mathcal{D}y\big(x(z)\big)\big)^2+\big[4\lambda aq^{\frac32}x^2\big(q^{-\frac34}z\big)-a\big(q+q^2+2q^{\frac32}\big)\big]y\big(x\big(q^{-\frac12}z\big)\big). \tag{61}$$

The Euler–Lagrange equation for this problem is

$$\mathcal{D}\big[\mathcal{D}y\big(q^{-\frac12}z\big)\big]=4\lambda aq^{\frac32}x^2\big(q^{-\frac34}z\big)-a\big(q+q^2+2q^{\frac32}\big). \tag{62}$$

Its solution reads

$$y\big(x(z)\big)=\frac{4aq^4}{(q^3+q^2+q+1)(q^2+q+1)}x^4((z))$$
$$-\frac{qa(q^5+q^4+4q^3+4q^2+q+1)}{(q^2+1)(q^2+q+1)}x^2((z))$$
$$+\frac{q(q^4+q^3+3q^2+q+1)}{2(q^3+q^2+q+1)}+p_1\big(x(z)\big),\quad \lambda=1, \tag{63}$$

where $p_1(x(z))$ is any first degree polynomial in $x(z)$.

4.2. The Lagrange problem on q-nonuniform lattices

The problem Let now be given a q-nonuniform lattices integration functional

$$J_0\big(\bar y(x)\big)=\int_a^b F_0\big[x(z),\bar y\big(x\big(q^{-\frac12}z\big)\big),\mathcal{D}\bar y\big(x(z)\big)\big]\,d_qx(z)$$
$$=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)F_0\big[x(z),\bar y\big(x\big(q^{-\frac12}z\big)\big),\mathcal{D}\bar y\big(x(z)\big)\big] \tag{64}$$

defined in E'^n. Here $\bar y(x)=(y_1(x),\dots,y_n(x))$. Let moreover be given a set of difference equations on q-nonuniform lattices

$$\phi_i\big[x(z),\bar y\big(x\big(q^{-\frac12}z\big)\big),\mathcal{D}\bar y\big(x(z)\big)\big]=0,\quad i=1,\dots,m<n. \tag{65}$$

The Lagrange problem consists in finding extremals of the functional (64) under the constraints (65).

The solution The Lagrange problem can be reduced to the isoperimetric one by transforming (65) in type (57) constraints. For that, we multiply the both sides of (65) by arbitrary functions $\lambda_i(x)$, and then take the integral on the q-nonuniform lattice from a to b. We obtain new constraints

$$J_i\big(\bar{y}(x)\big) = \int_a^b \lambda_i(x)\phi_i\big[x(z), \bar{y}\big(x\big(q^{-\frac{1}{2}}z\big)\big), \mathcal{D}\bar{y}\big(x(z)\big)\big]\,d_q x(z) = 0, \quad i = 1, \ldots, m. \tag{66}$$

Under the conditions of Theorem 4.1, the solutions $(y_1(x), \ldots, y_n(x))$ of the isoperimetric problem (64), (66) satisfy the Euler–Lagrange equation for the functional

$$\hat{J}(\bar{y}) = J_0(\bar{y}) + \sum_{i=1}^{m} \hat{\lambda}_i(x) J_i(\bar{y}), \quad \hat{\lambda}_i(x) = \tilde{\lambda}_i \lambda_i(x),\ i = 1, \ldots, m, \tag{67}$$

for some constants $\tilde{\lambda}_i$. But since clearly from (65) follows (66), the solutions of the Lagrange problem (64), (65) satisfy as well the Euler–Lagrange equation for the same functional (67).

Example. Suppose now that it is required to find the extremum of the functional

$$\begin{aligned} J_0(x, y, u) &= \frac{1}{2}\int_a^b \big[u^2\big(t(z)\big) - x^2\big(t\big(q^{-\frac{1}{2}}z\big)\big)\big]\,d_q x(z) \\ &= \frac{1}{2}\sum_{z=q^\alpha}^{q^\beta} \gamma(z)\big[u^2\big(t(z)\big) - x^2\big(t\big(q^{-\frac{1}{2}}z\big)\big)\big], \end{aligned} \tag{68}$$

under the constraints

$$\mathcal{D}x\big(t(z)\big) = y\big(t\big(q^{-\frac{1}{2}}z\big)\big), \qquad \mathcal{D}y\big(t(z)\big) = u\big(t(z)\big). \tag{69}$$

This is a Lagrange type problem hence it is equivalent to the problem of finding an ordinary extremum for the functional

$$\begin{aligned} &J^*(x, y, u, \lambda_1, \lambda_2) \\ &\quad = \int_a^b F^*\big[t(z), x\big(t\big(q^{-\frac{1}{2}}z\big)\big), y\big(t\big(q^{-\frac{1}{2}}z\big)\big), u\big(t(z)\big), \mathcal{D}x\big(t(z)\big), \mathcal{D}y\big(t(z)\big)\big]\,d_q x(z) \\ &\quad = \sum_{z=q^\alpha}^{q^\beta} \gamma(z) F^*\big[t(z), x\big(t\big(q^{-\frac{1}{2}}z\big)\big), y\big(t\big(q^{-\frac{1}{2}}z\big)\big), u\big(t(z)\big), \mathcal{D}x\big(t(z)\big), \mathcal{D}y\big(t(z)\big)\big], \end{aligned} \tag{70}$$

where

$$
\begin{aligned}
F^* = {} & \frac{1}{2}\big(u^2\big(t(z)\big) - x^2\big(t\big(q^{-\frac{1}{2}}z\big)\big)\big) \\
& + \lambda_1(t)\big(\mathcal{D}x\big(t(z)\big) - y\big(t\big(q^{-\frac{1}{2}}z\big)\big)\big) + \lambda_2(t)\big(\mathcal{D}y\big(t(z)\big) - u\big(t(z)\big)\big).
\end{aligned}
\tag{71}
$$

The Euler–Lagrange equation for this problem reads

$$
x\big(t(z)\big) = \mathcal{D}^+\mathcal{D}^-\mathcal{D}^-\mathcal{D}^+x\big(t(z)\big),
\tag{72}
$$

where

$$
\begin{aligned}
\mathcal{D}^+h\big(t(z)\big) &\stackrel{\text{def}}{=} \frac{h(t(z)) - h(t(qz))}{t(z) - t(qz)}, \\
\mathcal{D}^-h\big(t(z)\big) &\stackrel{\text{def}}{=} \frac{h(t(z/q)) - h(t(z))}{t(z/q) - t(z)}.
\end{aligned}
\tag{73}
$$

Searching the solution under the form

$$
x\big(t(z)\big) = \sum_{j=0}^{\infty}(a_j z^j + a_{-j}z^{-j}),
\tag{74}
$$

one finds the following recurrence relations for the coefficients:

$$
a_j = \int_a^{+}\int_a^{-}\int_a^{-}\int_a^{+}(a_j),
\tag{75}
$$

where the applications $\int_a^{\pm}$:

$$
\int_a^{\pm}:\ (a_{\pm(j-1)}, a_{\pm j}, a_{\pm(j+1)}) \to a^{\pm}_{\pm j}
\tag{76}
$$

are given by

$$
\begin{aligned}
& a^{\pm}_j = \frac{\gamma_0 q^{\frac{j}{2}}}{1-q^j}\big(a_{j-1}q^{\mp\frac{j-1}{2}} - a_{j+1}q^{\mp\frac{j+1}{2}}\big), \quad \gamma_0 = \frac{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}{2}, \\
& a^{\pm}_{-j} = \frac{\gamma_0 q^{\frac{j}{2}}}{1-q^j}\big(a_{-j+1}q^{\pm\frac{j-1}{2}} - a_{-j-1}q^{\pm\frac{j+1}{2}}\big), \quad j = 2, \ldots, \\
& a^{\pm}_1 = \gamma_0(2a_0 - a_2 q^{\mp}), \quad a^{\pm}_{-1} = \gamma_0(2a_0 - a_{-2}q^{\pm}), \quad a^{\pm}_0 = cte,
\end{aligned}
\tag{77}
$$

such that the applications

$$
\int^{\pm}:\ \sum_{j=0}^{\infty}(a_j z^j + a_{-j}z^{-j}) \to \sum_{j=0}^{\infty}\big(a^{\pm}_j z^j + a^{\pm}_{-j}z^{-j}\big)
\tag{78}
$$

are the inverses of $\mathcal{D}^{\pm}$. Additional constraints to (74) are obtained by the fact that the applications $\int^+$ and $\int^-$ are defined on series $\sum_{j=0}^{\infty}(b_j z^j + b_{-j}z^{-j})$ for which $b_1 = qb_{-1}$ and $b_{-1} = qb_1$, respectively.

4.3. The optimal control problem on q-nonuniform lattices

The problem Consider now the integration functional on q-nonuniform lattices

$$J\left(\bar{y}(x),\bar{u}(x)\right)=\int_a^b f^0\left[x(z),\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right]d_qx(z)$$
$$=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)f^0\left[x(z),\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right],\tag{79}$$

where $\bar{y}(x)=(y_1(x),\dots,y_n(x))$ and $\bar{u}(x)=(u_1(x),\dots,u_n(x))$. The functional is defined on E'^n union the set of admissible (that is which values belong to a fixed set U in $\mathbf{R}^n$) functions $\bar{u}(x)$, where E'^n is the subset of E^n which elements satisfy the boundary constraints

$$\bar{y}\left(q^{\alpha-\frac{1}{2}}\right)=\bar{y}\left(q^{\beta+\frac{1}{2}}\right)=C\dots.\tag{80}$$

Consider then the difference equations on q-nonuniform lattices

$$\mathcal{D}y_i\left(x(z)\right)=f_i\left[x(z),\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right],\quad i=1,\dots,n.\tag{81}$$

The optimal control problem consists in finding among all admissible vector functions $\bar{u}(x)$, that for which the corresponding solution of (80), (81) is an extremal of the functional (79). The functions $\bar{y}(x)$ and $\bar{u}(x)$ are said to constitute an *optimal process* and are called *optimal trajectory* and *optimal control*, respectively.

The solution To solve the optimal control problem, we consider it as an $n+m$ dimensional Lagrange problem: Find $n+m$ functions $(y_1(x),\dots,y_n(x))$ and $(u_1(x),\dots,u_n(x))$ that are extremals for (79) under the conditions (80) and

$$\phi_i\left(x,\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right)=0,\tag{82}$$

where

$$\phi_i\left(x,\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right)$$
$$=\mathcal{D}y_i\left(x(z)\right)-f_i\left[x(z),\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right],\quad i=1,\dots,n.\tag{83}$$

According to the discussions done in the preceding subsection, the solutions of such an extremum problem satisfy necessarily the Euler–Lagrange system of the functional

$$J^*\left(\bar{y}(x),\bar{u}(x)\right)=\int_a^b F^*\left[x(z),\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right]d_qx(z)$$
$$=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)F^*\left[x(z),\bar{y}\left(x\left(q^{-\frac{1}{2}}z\right)\right),\bar{u}\left(x(z)\right)\right],\tag{84}$$

where

$$\begin{aligned}
&F^*\big[x(z), \bar{y}\big(x\big(q^{-\frac{1}{2}}z\big)\big), \bar{u}\big(x(z)\big)\big] \\
&\quad = f^0\big[x(z), \bar{y}\big(x\big(q^{-\frac{1}{2}}z\big)\big), \bar{u}\big(x(z)\big)\big] \\
&\qquad + \sum_{i=1}^{n} \psi_i(x)\big[\mathcal{D}y_i(x) - f_i\big[x(z), \bar{y}\big(x\big(q^{-\frac{1}{2}}z\big)\big), \bar{u}\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big]\big].
\end{aligned} \tag{85}$$

The corresponding Euler–Lagrange system is then

$$f^0_{v_j} - \sum_{i=1}^{n} \psi_i(x) f_{i_{v_j}} - \mathcal{D}\big[\psi_i\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big] = 0, \quad j = 1, \dots, n, \tag{86}$$

$$f^0_{w_j} - \sum_{i=1}^{n} \psi_i(x) f_{i_{w_j}} = 0, \quad j = 1, \dots, m. \tag{87}$$

Here f_i^0 and f_i have as arguments $x(z), \bar{y}(x(q^{-\frac{1}{2}}z)), \bar{u}(x(q^{-\frac{1}{2}}z))$ and g_{v_j} and g_{w_j} mean the partial derivatives of g with respect to its $(j+1)$th and $(n+j+1)$th arguments, respectively. Setting

$$\begin{aligned}
H = {} & -f^0\big[x(z), \bar{y}\big(x\big(q^{-\frac{1}{2}}z\big)\big), \bar{u}\big(x(z)\big)\big] \\
& + \sum_{i=1}^{n} \psi_i(x)\big[f_i\big[x(z), \bar{y}\big(x\big(q^{-\frac{1}{2}}z\big)\big), \bar{u}\big(x(z)\big)\big]\big],
\end{aligned} \tag{88}$$

so, (81), (86) and (87) give, respectively,

$$\mathcal{D}y_i(x) = H_{\psi_i}, \tag{89}$$

$$\mathcal{D}\big[\psi_i\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big] = -H_{y_i}, \quad j = 1, \dots, n, \tag{90}$$

and

$$H_{u_i} = 0, \quad j = 1, \dots, m. \tag{91}$$

Thus, the necessary condition for the optimal control problem is given by (91), provided is solved the system (89)–(90). Due to similarities with the continuous case [14], one can refer to H and (89)–(90) as *q-nonuniform lattices Hamilton–Pontriaguine function* and *system*, respectively.

Example (*Linear quadratic problem on q-nonuniform lattices*). The problem now is that of finding a control function $u(x)$ such that the corresponding solution to the boundary value problem

$$\begin{aligned}
&\mathcal{D}y = -ay\big(x\big(zq^{-\frac{1}{2}}z\big)\big) + u\big(x(z)\big), \quad a > 0, \\
&y\big(q^{\alpha-\frac{1}{2}}\big) = y\big(q^{\beta+\frac{1}{2}}\big)
\end{aligned} \tag{92}$$

is an extremal for the functional (*quadratic cost functional on q-nonuniform lattices*)

$$J(y,u)=\frac{1}{2}\int_a^b\big[u^2\big(x(z)\big)+y^2\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big]\,d_q x(z)$$
$$=\frac{1}{2}\sum_{z=q^\alpha}^{q^\beta}\gamma(z)\big[u^2\big(x(z)\big)+y^2\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big]. \tag{93}$$

The problem is of optimal control type. The Hamilton–Pontriaguine function and system are, respectively,

$$H=-\frac{1}{2}\big[y^2\big(x\big(q^{-\frac{1}{2}}z\big)\big)+u^2\big(x(z)\big)\big]+\psi\big(x(z)\big)\big[-ay\big(x\big(zq^{-\frac{1}{2}}z\big)\big)+u\big(x(z)\big)\big] \tag{94}$$

and

$$\begin{aligned}&\mathcal{D}y=-ay\big(x\big(zq^{-\frac{1}{2}}z\big)\big)+u\big(x(z)\big),\\&\mathcal{D}\big[\psi\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big]=a\psi\big(x(z)\big)+y\big(x\big(zq^{-\frac{1}{2}}\big)\big),\\&\psi=u.\end{aligned} \tag{95}$$

The equation for $y(x(z))$ then becomes

$$\mathcal{D}\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big)+ay\big(x\big(q^{-1}z\big)\big)\big]=a\mathcal{D}y+(a^2+1)y\big(x\big(q^{-\frac{1}{2}}z\big)\big). \tag{96}$$

Searching the solution $y(x(z))$ under a series of the form (74), so the recurrence relations satisfied by the coefficients are given by

$$y=\int^{-}y-a\int^{+}y+(a^2+1)\int^{-}\int^{+}y, \tag{97}$$

where the applications $\int^{\pm}$ are defined in (77)–(78).

Remark. It is to be noted that for $q\to 1$ ($\alpha\to 0$, $\beta\to+\infty$), the preceding problems and examples tend to the corresponding problems and examples in the continuous analysis on the interval $[0,1]$ (which can naturally be transformed in any other finite interval by the well known linear change of variables).

4.4. Interconnection between the variational calculus, the optimal control and the Hamilton system on q-nonuniform lattices

Consider now the case of pure variational calculus on q-nonuniform lattices that is the control function and the control system are not present explicitly: Find extremals of the functional

$$J\big(y\big(x(z)\big)\big)=\int_a^b F\big[x(z),y\big(x\big(q^{-\frac{1}{2}}z\big)\big),\mathcal{D}y\big(x(z)\big)\big]\,d_q x(z)$$
$$=\sum_{z=q^\alpha}^{q^\beta}\gamma(z)F\big[x(z),y\big(x\big(q^{-\frac{1}{2}}z\big)\big),\mathcal{D}y\big(x(z)\big)\big] \tag{98}$$

defined in E'. Note also that the variable x is not present explicitly. Our objective is to show the following proposition.

Proposition 4.1. *On q-nonuniform lattices, are equivalent: the Euler–Lagrange equation, the Hamilton and the Hamilton–Pontriaguine systems.*

Proof. We show this in three steps.

(a) We first show how to obtain the Hamilton system from the Euler–Lagrange equation. For the functional in (98), the Euler–Lagrange equation reads

$$\begin{aligned} &F_{v_0}\big[x(z), y\big(x\big(q^{-\frac{1}{2}}z\big)\big), \mathcal{D}y\big(x(z)\big)\big] \\ &\quad - \mathcal{D}\big[F_{v_1}\big[x\big(zq^{-\frac{1}{2}}\big), y\big(x\big(q^{-1}z\big)\big), \mathcal{D}y\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big]\big] = 0. \end{aligned} \tag{99}$$

Letting

$$\psi(x) = F_{v_1}\big[x(z), y\big(x\big(q^{-\frac{1}{2}}z\big)\big), \mathcal{D}y\big(x(z)\big)\big] \tag{100}$$

and

$$H = -F + \psi(x)\mathcal{D}y, \tag{101}$$

we get from (99)–(101) the Hamilton system

$$\begin{aligned} &\mathcal{D}y\big(x(z)\big) = H_\psi\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), \psi\big(x(z)\big), \mathcal{D}y\big(x(z)\big)\big], \\ &\mathcal{D}\big[\psi\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big] = -H_y\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), \psi\big(x(z)\big), \mathcal{D}y\big(x(z)\big)\big]. \end{aligned} \tag{102}$$

(b) To get the Hamilton–Pontriaguine system from the Hamilton system (102), it suffices to suppose $u(x(z)) = \mathcal{D}y(x(z))$ to be the control equation for the given initial non-controlled extremum problem. In that case, (102) gives

$$\begin{aligned} &\mathcal{D}y\big(x(z)\big) = H_\psi\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), \psi\big(x(z)\big), u\big(x(z)\big)\big], \\ &\mathcal{D}\big[\psi\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big] = -H_y\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), \psi\big(x(z)\big), u\big(x(z)\big)\big], \end{aligned} \tag{103}$$

with

$$H = -F\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), u(x)\big] + \psi\big(x(z)\big)u\big(x(z)\big), \tag{104}$$

the Hamilton–Pontriaguine function, and from (100) we get the third equation in (91):

$$H_u = 0. \tag{105}$$

(c) We finally show how to obtain the Euler–Lagrange equation (99) from the Hamilton–Pontriaguine system (103)–(105). From (104) and (105), we have

$$\psi\big(x(z)\big) = F_{v_1}\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), u\big(x(z)\big)\big] = F_{v_1}\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), D_q y\big(x(z)\big)\big], \tag{106}$$

while from (103) we get

$$\begin{aligned} \mathcal{D}\big[\psi\big(x\big(q^{-\frac{1}{2}}z\big)\big)\big] &= F_{v_0}\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), u\big(x(z)\big)\big] \\ &= F_{v_0}\big[y\big(x\big(q^{-\frac{1}{2}}z\big)\big), D_q y\big(x(z)\big)\big]. \end{aligned} \tag{107}$$

Finally, (106) and (107) give the Euler–Lagrange equation (99), which proves the proposition. □

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