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Composition type operator from Bergman space to μ-Bloch type space in C^n ☆

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Abstract

Let φ be a holomorphic self-map of B and $\psi \in H(B)$. A composition type operator $T_{\psi,\varphi}$ is defined by $T_{\psi,\varphi}(f) = \psi f \circ \varphi$. We can regard this operator as a generalization of a multiplication operator and a composition operator. For normal functions μ, some necessary and sufficient conditions are given for which $T_{\psi,\varphi}$ is a bounded or compact operator from Bergman space to μ-Bloch type space β_μ on the unit ball of C^n. As a Corollary, we obtain the pointwise multiplier from Bergman space to μ-Bloch type space on B.

Keywords: Bergman space; μ-Bloch type space; Composition type operator; Boundedness; Compactness

1. Introduction

Let B denote the unit ball of C^n and dv denote Lebesgue measure in B, normalized so that $v(B) = 1$. The class of all holomorphic functions with domain B will be denoted by $H(B)$. For $0 < p < \infty$, the Bergman space A^p is the set of all holomorphic functions f on B such that

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$$\|f\|_{A^p} = \left\{ \int_B \left|f(z)\right|^p \, dv(z) \right\}^{\frac{1}{p}} < \infty.$$

A^p is a Banach space with the norm $\|\cdot\|_{A^p}$ as $1 \leqslant p < \infty$. If $0 < p < 1$, then A^p is a Frechet space with $\|\cdot\|_{A^p}^p$.

A positive continuous function μ on $[0,1)$ is called normal if there are two constants a, b $(0 < a < b)$ such that

(i) $\dfrac{\mu(r)}{(1-r)^a}$ is decreasing for $r \in [0,1)$ and $\lim_{r\to 1^-} \dfrac{\mu(r)}{(1-r)^a} = 0$;

(ii) $\dfrac{\mu(r)}{(1-r)^b}$ is increasing for $r \in [0,1)$ and $\lim_{r\to 1^-} \dfrac{\mu(r)}{(1-r)^b} = \infty$.

The normal function μ, as a weight, has been usually used to define the mixed norm spaces (see [1]). Let μ be normal. $f \in H(B)$ is said to belong to the μ-Bloch type spaces β_μ provided that

$$\sup_{z\in B} \mu\big(|z|\big)\big|\nabla f(z)\big| < \infty, \quad \text{where } \nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_n}\right).$$

It is well known that β_μ is a Banach space under the norm

$$\|f\|_{\beta_\mu} = \big|f(0)\big| + \sup_{z\in B} \mu\big(|z|\big)\big|\nabla f(z)\big|.$$

When $\mu(r) = 1 - r^2$ and $\mu(r) = (1-r^2)^{1-\alpha}$ $(0 < \alpha < 1)$, the induced spaces β_μ are the Bloch space and Lipschitz type spaces, respectively.

Let φ be a holomorphic self-map of B and $\psi \in H(B)$. The composition type operator $T_{\psi,\varphi}$ is defined by

$$T_{\psi,\varphi}(f) = \psi f \circ \varphi, \quad f \in H(B).$$

It is easy to see that an operator defined in this manner is linear. We can regard this operator as a generalization of a multiplication operator M_ψ and a composition operator C_φ. In complex plane, composition operator or composition type operator has been studied for a long time by many students (for example, [2–8]). In several complex variables case, Shi and Luo [9] and Zhou and Zeng [10] got the characterization on φ for which the induced composition operator is bounded or compact on β_{1-r^2} or $\beta_{(1-r^2)^p}$ in the unit ball. And in the polydiscs, Zhou et al. [11–13] studied the same problems. The main purpose of this paper is to discuss the conditions for which $T_{\psi,\varphi}$ is a bounded operator or compact operator from Bergman space A^p to μ-Bloch space β_μ on the unit ball of C^n and our main results are the following:

Theorem A. *Let* $0 < p < \infty$. *Let* μ *be normal on* $[0,1)$, $\psi \in H(B)$ *and* φ *be a holomorphic self-map of* B. *Then* $T_{\psi,\varphi}$ *is a bounded operator from* A^p *to* β_μ *if and only if the following are all satisfied*:

(i) $$\sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{p+n+1}{p}}}$$

$$\times \left\{ \frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle \varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2} \right\}^{\frac{1}{2}} < \infty, \tag{1.1}$$

$$\text{(ii)} \quad \sup_{z\in B} \frac{\mu(|z|)|\nabla\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1}{p}}} < \infty, \tag{1.2}$$

where $J\varphi(z)$ denotes a Jacobian matrix of $\varphi(z)$ and $J\varphi(z)u$ denotes a vector as follows:

$$J\varphi(z) = \left(\frac{\partial \varphi_j(z)}{\partial z_k}\right)_{1\leqslant j,k\leqslant n}, \qquad J\varphi(z)u = \left(\sum_{k=1}^n \frac{\partial\varphi_1(z)}{\partial z_k}u_k, \dots, \sum_{k=1}^n \frac{\partial\varphi_n(z)}{\partial z_k}u_k\right)^T.$$

Theorem B. *Let $0 < p < \infty$. Let μ be normal on $[0,1)$, $\psi \in H(B)$ and φ be a holomorphic self-map of B. Then $T_{\psi,\varphi}$ is a compact operator from A^p to β_μ if and only if the following are all satisfied*:

$$\text{(i)} \quad \psi \in \beta_\mu \quad \textit{and} \quad \psi\varphi_l \in \beta_\mu \quad \textit{for all } l \in \{1,2,\dots,n\}, \tag{1.3}$$

$$\text{(ii)} \quad \lim_{|\varphi(z)|\to 1} \sup_{u\in C^n-\{0\}} \frac{\nu(|z|)|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+p}{p}}} \times \sqrt{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} = 0, \tag{1.4}$$

$$\text{(iii)} \quad \lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|\nabla\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1}{p}}} = 0. \tag{1.5}$$

2. The boundedness of $T_{\psi,\varphi}$

We will use the symbol c to denote a finite positive number which does not depend on variables z, w and may depend on some norms, not necessarily the same at each occurrence. We call E and F are comparable (denoted by $E \approx F$ in the following) if there exist two positive constants c_1 and c_2 such that $c_1E \leqslant F \leqslant c_2E$.

Lemma 2.1. *Let μ be normal on $[0,1)$ and $f \in H(B)$. Then $f \in \beta_\mu$ if and only if*

$$\sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|\nabla f(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} < \infty.$$

Furthermore,

$$\|f\|_{\beta_\mu} \approx |f(0)| + \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|\nabla f(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}}.$$

Proof. By Bergman metric in [14], there are constants $A_1 > 0$ and $A_2 > 0$ such that

$$\frac{A_1|\nabla f(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} \leqslant |\nabla f(z)| \leqslant \frac{A_2|\nabla f(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}}$$

for all $u \in C^n - \{0\}$. Let $C_1 = \min\{A_1, 1\}$, $C_2 = \max\{A_2, 1\}$. Then

$$C_1\left\{\left|f(0)\right| + \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|\nabla f(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}}\right\} \leqslant \|f\|_{\beta_\mu}$$
$$\leqslant C_2\left\{\left|f(0)\right| + \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|\nabla f(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}}\right\}.$$

The proof is completed. □

Lemma 2.2. *Let* $0 < p < \infty$. (1) *If* $f \in A^p$, *then*

$$\left|f(z)\right| \leqslant \frac{c\|f\|_{A^p}}{(1-|z|^2)^{\frac{n+1}{p}}}.$$

(2) *If* $f \in A^p$, *then* $f \in \beta_{(1-r^2)^{(n+1+p)/p}}$ *and* $\|f\|_{\beta_{(1-r^2)^{(n+1+p)/p}}} \leqslant c\|f\|_{A^p}$.

Proof. By Bergman operator, Proposition 1.4.10 in [15] and [16] we can obtain these results. □

Proof of Theorem A. Suppose that (1.1) and (1.2) hold. Let

$$M_1 = \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{\mu(|z|)|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{p+n+1}{p}}} \times \left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}\right\}^{\frac{1}{2}},$$
$$M_2 = \sup_{z\in B} \frac{\mu(|z|)|\nabla\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1}{p}}}.$$

Then $M_1 < \infty$, $M_2 < \infty$. For any $f \in A^p$, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
\mu(|z|)\left|\nabla\left[T_{\psi,\varphi}(f)\right](z)\right| &\leqslant \mu(|z|)\Big(\left|\nabla\psi(z)\right|\left|f(\varphi(z))\right| + \left|\psi(z)\right|\left|\nabla\left[C_\varphi(f)\right](z)\right|\Big)\\
&\leqslant \frac{c\mu(|z|)|\nabla\psi(z)|\|f\|_{A^p}}{(1-|\varphi(z)|^2)^{\frac{n+1}{p}}} + \sup_{\substack{u\in C^n-\{0\}\\ z\in B}} \frac{c\mu(|z|)|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+p}{p}}}\\
&\quad\times\left\{\left(\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}\right)^{\frac{1}{2}}\right.\\
&\quad\left.\times\frac{(1-|\varphi(z)|^2)^{\frac{n+1+p}{p}}|(\nabla f)(\varphi(z))J\varphi(z)u|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}}\right\}\\
&\leqslant cM_2\|f\|_{A^p} + cM_1\|f\|_{\beta_{(1-r^2)^{(n+1+p)/p}}} \leqslant c\|f\|_{A^p}.
\end{aligned}$$

This means that $T_{\psi,\varphi}$ is a bounded operator from A^p to β_μ.

Conversely, suppose that $T_{\psi,\varphi}$ is a bounded operator from A^p to β_μ. Then we can easily obtain $\psi \in \beta_\mu$ and $\psi\varphi_l \in \beta_\mu$ by taking $f(z) = 1$ and $f(z) = z_l$ $(l = 1, \dots, n)$ in A^p, respectively.

For any given $w \in B$ and $u \in C^n - \{0\}$. If $|\varphi(w)| \leqslant \sqrt{2/3}$, it follows from $\psi \in \beta_\mu$ and $\psi\varphi_l \in \beta_\mu$ that

$$\begin{aligned}
&\frac{\mu(|w|)|\psi(w)|}{(1-|\varphi(w)|^2)^{\frac{n+1+p}{p}}}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2 + |\langle\varphi(w), J\varphi(w)u\rangle|^2}{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}\right\}^{\frac{1}{2}}\\
&\quad\leqslant c\mu\big(|w|\big)\big|\psi(w)\big|\frac{|J\varphi(w)u|}{\sqrt{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}}\\
&\quad\leqslant c\sum_{l=1}^{n}\mu\big(|w|\big)\frac{|\nabla\varphi_l(w)u||\psi(w)|}{\sqrt{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}}\\
&\quad\leqslant c\sum_{l=1}^{n}\mu\big(|w|\big)\frac{|\nabla(\psi\varphi_l)(w)u| + |\nabla\psi(w)u||\varphi_l(w)|}{\sqrt{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}}\\
&\quad\leqslant c\sum_{l=1}^{n}\big(\|\psi\varphi_l\|_{\beta_\mu} + \|\psi\|_{\beta_\mu}\big).
\end{aligned}$$

This shows that (1.1) and (1.2) hold.

In the following, we always assume that $|\varphi(w)| > \sqrt{2/3}$. First we suppose that $\varphi(w) = r_w e_1$, where $r_w = |\varphi(w)|$, e_1 is a vector $(1, 0, \dots, 0)$.

(i) $\sqrt{(1-r_w^2)(|\xi_2|^2 + \cdots + |\xi_n|^2)} \leqslant |\xi_1|$, where $(\xi_1, \dots, \xi_n)^T = J\varphi(w)u$. Let

$$f_w(z) = \frac{z_1 - r_w}{1 - r_w z_1}\left\{\frac{1 - r_w^2}{(1 - r_w z_1)^2}\right\}^{\frac{n+1}{p}}.$$

Then, by Proposition 1.4.10 in [15] we have $\|f_w\|_{A^p} \leqslant c$. It is clear that

$$\begin{aligned}
c\|T_{\psi,\varphi}\| &\geqslant \|T_{\psi,\varphi}\|\|f_w\|_{A^p} \geqslant \big\|T_{\psi,\varphi}(f_w)\big\|_{\beta_\mu}\\
&\geqslant \mu\big(|w|\big)\big|\nabla\psi(w)f_w\big(\varphi(w)\big) + \psi(w)\nabla\big[C_\varphi(f_w)\big](w)\big|\\
&\geqslant \frac{c\mu(|w|)|\psi(w)||(\nabla f_w)(\varphi(w))J\varphi(w)u|}{\sqrt{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}} - \mu\big(|w|\big)\big|\nabla\psi(w)\big|\big|f_w\big(\varphi(w)\big)\big|\\
&= \frac{c\mu(|w|)|\psi(w)||\xi_1|}{(1-r_w^2)^{\frac{n+1+p}{p}}\sqrt{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}}.
\end{aligned}\tag{2.1}$$

It follows from (2.1) and Lemma 2.3 that

$$\begin{aligned}
&\frac{\mu(|w|)|\psi(w)|}{(1-|\varphi(w)|^2)^{\frac{n+1+p}{p}}}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2 + |\langle\varphi(w), J\varphi(w)u\rangle|^2}{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}\right\}^{\frac{1}{2}}\\
&\quad= \frac{\mu(|w|)|\psi(w)|}{(1-r_w^2)^{\frac{n+1+p}{p}}}\left\{\frac{(1-r_w^2)(|\xi_2|^2 + \cdots + |\xi_n|^2) + |\xi_1|^2}{(1-|w|^2)|u|^2 + |\langle w,u\rangle|^2}\right\}^{\frac{1}{2}}
\end{aligned}$$

$$\leqslant \frac{\mu(|w|)|\psi(w)|}{(1-r_w^2)^{\frac{n+1+p}{p}}} \frac{\sqrt{2}|\xi_1|}{\sqrt{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}} \leqslant c\|T_{\psi,\varphi}\|.$$

(ii) $\sqrt{(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)} > |\xi_1|$. For $j=2,\ldots,n$, let $\theta_j = \arg\xi_j$ and $a_j = e^{-i\theta_j}$ as $\xi_j \neq 0$ or $a_j = 0$ as $\xi_j = 0$. Take

$$f_w(z) = (a_2z_2+\cdots+a_nz_n)\left\{\frac{1-r_w^2}{(1-r_wz_1)^2}\right\}^{\frac{n+1}{p}} \frac{1}{1-r_wz_1},$$

then

$$\|f_w\|_{A^p} \leqslant \left\{(1-r_w^2)^{n+1}\int_B \frac{(|z_2|+\cdots+|z_n|)^p}{|1-r_wz_1|^{2n+2+p}}\,dv(z)\right\}^{\frac{1}{p}}$$
$$\leqslant \left\{(1-r_w^2)^{n+1}\int_B \frac{[(n-1)(1-|z_1|^2)]^{\frac{p}{2}}}{|1-r_wz_1|^{2n+2+p}}\,dv(z)\right\}^{\frac{1}{p}} \leqslant \frac{c}{\sqrt{1-r_w^2}}.$$

Similarly to the proof of (2.1) we can get

$$\frac{\mu(|w|)|\psi(w)|}{(1-r_w^2)^{\frac{n+1+p}{p}}} \frac{|\xi_2|+\cdots+|\xi_n|}{\sqrt{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}} \leqslant \frac{c\|T_{\psi,\varphi}\|}{\sqrt{1-r_w^2}}.$$

Thus

$$\frac{\mu(|w|)|\psi(w)|}{(1-|\varphi(w)|^2)^{\frac{n+1+p}{p}}}\left\{\frac{(1-|\varphi(w)|^2)|J\varphi(w)u|^2+|\langle\varphi(w),J\varphi(w)u\rangle|^2}{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}\right\}^{\frac{1}{2}}$$
$$= \frac{\mu(|w|)|\psi(w)|}{(1-r_w^2)^{\frac{n+1+p}{p}}}\left\{\frac{(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)+|\xi_1|^2}{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}\right\}^{\frac{1}{2}}$$
$$\leqslant \frac{\mu(|w|)|\psi(w)|}{(1-r_w^2)^{\frac{n+1+p}{p}}} \frac{\sqrt{2(1-r_w^2)(|\xi_2|^2+\cdots+|\xi_n|^2)}}{\sqrt{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}}$$
$$\leqslant \sqrt{2}\frac{\mu(|w|)|\psi(w)|}{(1-r_w^2)^{\frac{n+1+p}{p}}} \frac{\sqrt{1-r_w^2}(|\xi_2|+\cdots+|\xi_n|)}{\sqrt{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}} \leqslant c\|T_{\psi,\varphi}\|.$$

This shows that (1.1) holds.

In a general situation, if $\varphi(w) \neq |\varphi(w)|e_1$, we use the unitary transformation U_w to make $\varphi(w) = \rho_w e_1 U_w$, where $\rho_w = |\varphi(w)| > \sqrt{2/3}$. We can prove (1.1) by taking $g_w = f_w \circ U_w^{-1}$.

In order to prove (1.2), we take

$$h_w(z) = \left\{\frac{1-|\varphi(w)|^2}{(1-\langle z,\varphi(w)\rangle)^2}\right\}^{\frac{n+1}{p}},$$

then $\|h_w\|_{A^p} \leqslant c$. Thus,

$$
\begin{aligned}
c\|T_{\psi,\varphi}\| &\geqslant \|T_{\psi,\varphi}\|\|h_w\|_{A^p} \geqslant \big\|T_{\psi,\varphi}(h_w)\big\|_{\beta_\mu} \\
&\geqslant \mu\big(|w|\big)\big|\nabla\psi(w)h_w\big(\varphi(w)\big)+\psi(w)\nabla(C_\varphi h_w)(w)\big| \\
&\geqslant \mu\big(|w|\big)\big|\nabla\psi(w)\big|\big|h_w\big(\varphi(w)\big)\big| - \frac{c\mu(|w|)|\psi(w)||(\nabla h_w)(\varphi(w))J\varphi(w)u|}{\sqrt{(1-|w|^2)|u|^2+|\langle w,u\rangle|^2}} \\
&= \frac{\mu(|w|)|\nabla\psi(w)|}{(1-|\varphi(w)|^2)^{\frac{n+1}{p}}} - \frac{c\mu(|w|)|\psi(w)||\langle\varphi(w),J\varphi(w)u\rangle|}{(1-|\varphi(w)|^2)^{\frac{n+1+p}{p}}}.
\end{aligned}
\tag{2.2}
$$

This shows that (1.2) holds by (2.2) and (1.1). The proof is completed. □

Corollary 2.3. *Let $0<p<\infty$, $n=1$. Let μ be normal on $[0,1)$, $\psi\in H(D)$ and φ be a holomorphic self-map on disc D. Then $T_{\psi,\varphi}$ is a bounded operator from A^p to β_μ if and only if*

$$
\sup_{z\in D}\frac{\mu(|z|)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+2}{p}}}<\infty \quad \textit{and} \quad \sup_{z\in D}\frac{\mu(|z|)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}}<\infty.
$$

Corollary 2.4. *Let $0<p<\infty$. Let μ be normal on $[0,1)$ and φ be a holomorphic self-map of B. Then C_φ is a bounded operator from A^p to β_μ if and only if*

$$
\sup_{\substack{u\in C^n-\{0\}\\ z\in B}}\frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n+1+p}{p}}} \times\left\{\frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2+|\langle\varphi(z),J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2+|\langle z,u\rangle|^2}\right\}^{\frac{1}{2}}<\infty.
$$

Let X, Y be two spaces of holomorphic functions on the unit ball B. We call ψ a pointwise multiplier from X to Y if $\psi f\in Y$ for all $f\in X$. The collection of all pointwise multipliers from X to Y is denoted $M(X,Y)$.

Corollary 2.5. *Let $0<p<\infty$. Let μ be normal on $[0,1)$ and $\psi\in H(B)$. Then $\psi\in M(A^p,\beta_\mu)$ if and only if*

$$
\sup_{z\in B}\frac{\mu(|z|)|\psi(z)|}{(1-|z|^2)^{\frac{n+1+p}{p}}}<\infty \quad \textit{and} \quad \sup_{z\in B}\frac{\mu(|z|)|\nabla\psi(z)|}{(1-|z|^2)^{\frac{n+1}{p}}}<\infty.
$$

3. The compactness of $T_{\psi,\varphi}$

Lemma 3.1. *Let $0<p<\infty$. Let μ be normal on $[0,1)$, $\psi\in H(B)$ and φ be a holomorphic self-map of B. Then $T_{\psi,\varphi}$ is a compact operator from A^p to β_μ if and only if for any bounded sequence $\{f_j\}$ in A^p which converges to 0 uniformly on compact subset of B, we have $\|T_{\psi,\varphi}f_j\|_{\beta_\mu}\to 0$ as $j\to\infty$.*

Proof. The result can be proved by using Montel theorem and Lemma 2.2; the details are omitted here. □

Proof of Theorem C. Suppose that (1.4) and (1.5) hold. Then for any $\varepsilon > 0$, there exists $0 < \delta < 1$ such that

$$\sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{p+n+1}{p}}} \left\{ \frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2} \right\}^{\frac{1}{2}} < \varepsilon \tag{3.1}$$

and

$$\frac{\mu(|z|)|\nabla\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1}{p}}} < \varepsilon \tag{3.2}$$

as $|\varphi(z)|^2 > 1-\delta$.

Let $\{f_j\}$ be any a sequence $\{f_j\}$ which converges to 0 uniformly on compact subset of B satisfying $\|f_j\|_{A^p} \leqslant 1$. Then $\{f_j\}$ and $\{|\nabla f_j|\}$ converges to 0 uniformly on $\{w\colon |w|^2 \leqslant 1-\delta\}$.

If $|\varphi(z)|^2 > 1-\delta$, then, from (3.1), (1.3), Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\nabla(T_{\psi,\varphi} f_j)(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} \\
&\leqslant \sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\psi(z)||\nabla f_j(\varphi(z))J\varphi(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} + c\mu\big(|z|\big)\big|\nabla\psi(z)\big|\big|f_j\big(\varphi(z)\big)\big| \\
&\leqslant \sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\psi(z)|}{(1-|\varphi(z)|^2)^{\frac{p+n+1}{p}}} \left\{ \frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2} \right\}^{\frac{1}{2}} \\
&\quad \times \frac{(1-|\varphi(z)|^2)^{\frac{p+n+1}{p}} |\nabla f_j(\varphi(z))J\varphi(z)u|}{\sqrt{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle\varphi(z), J\varphi(z)u\rangle|^2}} + \frac{c\mu(|z|)|\nabla\psi(z)|\|f_j\|_{A^p}}{(1-|\varphi(z)|^2)^{\frac{n+1}{p}}} \\
&\leqslant c\varepsilon\|f_j\|_{\beta_{(1-r^2)^{(n+1+p)/p}}} + c\varepsilon\|f_j\|_{A^p} \leqslant c\varepsilon\|f_j\|_{A^p} \leqslant c\varepsilon \to c\varepsilon \quad (j\to\infty).
\end{aligned} \tag{3.3}$$

If $|\varphi(z)|^2 \leqslant 1-\delta$, by (1.3) and Lemma 2.1,

$$\begin{aligned}
&\sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\nabla(T_{\psi,\varphi} f_j)(z)u|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} \\
&\leqslant \sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\nabla\psi(z)u||f_j(\varphi(z))|}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} \\
&\quad + \sup_{u\in C^n-\{0\}} \frac{\mu(|z|)|\psi(z)||\nabla f_j(\varphi(z))|(|\nabla\varphi_1(z)u| + \cdots + |\nabla\varphi_n(z)u|)}{\sqrt{(1-|z|^2)|u|^2 + |\langle z,u\rangle|^2}} \\
&\leqslant c\|\psi\|_{\beta_\mu}\big|f_j\big(\varphi(z)\big)\big| + c\sum_{l=1}^{n}\big(\|\psi\varphi_l\|_{\beta_\mu} + \|\psi\|_{\beta_\mu}\big)\big|\nabla f_j\big(\varphi(z)\big)\big| \\
&\to 0 \quad (j\to\infty).
\end{aligned} \tag{3.4}$$

By (3.3) and (3.4) we have $\|T_{\psi,\varphi}(f_j)\|_{\beta_\mu} \to 0$ $(j\to\infty)$. This means that $T_{\psi,\varphi}$ is a compact operator from A^p to β_μ by Lemma 3.1.

Conversely, for any $l \in \{1, 2, \dots, n\}$, by taking $f(z) = 1$ or $f(z) = z_l \in A^p$, we get $\psi \in \beta_\mu$ and $\psi\varphi_l = T_{\psi,\varphi}(f) \in \beta_\mu$. That is, (1.3) holds.

Assume that (1.4) fails. Then there exist sequence $\{z^j\} \subset B$ satisfying $r_j = |\varphi(z^j)| \to 1$ as $j \to \infty$, $\{u^j\} \subset C^n - \{0\}$ and constant $\varepsilon_0 > 0$ such that

$$\frac{\mu(|z^j|)|\psi(z^j)|}{(1-r_j^2)^{\frac{n+1+p}{p}}}\left(\frac{(1-|\varphi(z^j)|^2)|J\varphi(z^j)u^j|^2 + |\langle\varphi(z^j), J\varphi(z^j)u^j\rangle|^2}{(1-|z^j|^2)|u^j|^2 + |\langle z^j, u^j\rangle|^2}\right)^{\frac{1}{2}} \geqslant \varepsilon_0. \quad (3.5)$$

To construct the sequence of functions $\{f_j\}$, we first assume that $\varphi(z^j) = r_j e_1$ $(j = 1, 2, \dots)$.

If $\sqrt{(1-r_j^2)(|w_2^j|^2 + \cdots + |w_n^j|^2)} < |w_1^j|$, where $(w_1^j, \dots, w_n^j)^T = J\varphi(z^j)u^j$, take

$$f_j(z) = \frac{z_1 - r_j}{1 - r_j z_1}\left\{\frac{1-r_j^2}{(1-r_j z_1)^2}\right\}^{\frac{n+1}{p}}.$$

We can prove easily $\|f_j\|_{A^p} \leqslant c$.

Let E be any a compact subset of B. Then there exists $0 < r < 1$ such that $E \subseteq \{z: |z| \leqslant r\}$. Thus

$$\max_{z\in E}\left|f_j(z)\right| \leqslant \left\{\frac{1-r_j^2}{(1-r)^2}\right\}^{\frac{n+1}{p}} \to 0 \quad \text{as } j \to \infty.$$

That is, $\{f_j\}$ converges to 0 uniformly on compact subset of B. But from (3.5) we have

$$\begin{aligned}
&\left\|T_{\psi,\varphi}(f_j)\right\|_{\beta_\mu} \\
&\quad\geqslant \frac{c\mu(|z^j|)|\psi(z^j)||(\nabla f_j)(r_j e_1)J\varphi(z^j)u^j|}{\sqrt{(1-|z^j|^2)|u^j|^2 + |\langle z^j, u^j\rangle|^2}} - \mu\left(|z^j|\right)\left|\nabla\psi(z^j)\right|\left|f_j(r_j e_1)\right| \\
&\quad= \frac{c\mu(|z^j|)|\psi(z^j)|}{(1-r_j^2)^{\frac{n+1+p}{p}}}\left\{\frac{(1-|\varphi(z^j)|^2)|J\varphi(z^j)u^j|^2 + |\langle\varphi(z^j), J\varphi(z^j)u^j\rangle|^2}{(1-|z^j|^2)|u^j|^2 + |\langle z^j, u^j\rangle|^2}\right\}^{\frac{1}{2}} \\
&\qquad\times \frac{|w_1^j|}{\sqrt{(1-r_j^2)(|w_2^j|^2 + \cdots + |w_n^j|^2) + |w_1^j|^2}} > \frac{c\varepsilon_0}{\sqrt{2}}.
\end{aligned}$$

This means that

$$\lim_{j\to\infty}\left\|T_{\psi,\varphi}(f_j)\right\|_{\beta_\mu} \neq 0.$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma 3.1.

If $\sqrt{(1-r_j^2)(|w_2^j|^2 + \cdots + |w_n^j|^2)} \geqslant |w_1^j|$, then we let $\theta_k^j = \arg w_k^j$ $(k = 2, \dots, n)$ and take

$$f_j(z) = \left(e^{-i\theta_2^j} z_2 + \cdots + e^{-i\theta_n^j} z_n\right)\left\{\frac{1-r_j^2}{(1-r_j z_1)^2}\right\}^{\frac{n+1}{p}} \frac{\sqrt{1-r_j^2}}{1-r_j z_1}.$$

We have $\|f_j\|_{A^p} \leqslant c$ and $\{f_j\}$ converges to 0 uniformly on compact subset of B. But

$$\begin{aligned}
\|T_{\psi,\varphi}(f_j)\|_{\beta_\mu} &\geqslant \frac{c\mu(|z^j|)|\psi(z^j)(\nabla f_j)(r_je_1)J\varphi(z^j)u^j + f_j(\varphi(z^j))\nabla\psi(z^j)|}{\sqrt{(1-|z^j|^2)|u^j|^2 + |\langle z^j, u^j\rangle|^2}} \\
&= \frac{c\mu(|z^j|)|\psi(z^j)|}{(1-r_j^2)^{\frac{n+1+p}{p}}} \\
&\quad \times \left\{\frac{(1-|\varphi(z^j)|^2)|J\varphi(z^j)u^j|^2 + |\langle\varphi(z^j), J\varphi(z^j)u^j\rangle|^2}{(1-|z^j|^2)|u^j|^2 + |\langle z^j, u^j\rangle|^2}\right\}^{\frac{1}{2}} \\
&\quad \times \frac{(|w_2^j| + \cdots + |w_n^j|)\sqrt{1-r_j^2}}{\sqrt{(1-r_j^2)(|w_2^j|^2 + \cdots + |w_n^j|^2) + |w_1^j|^2}} > \frac{c\varepsilon_0}{\sqrt{2}}.
\end{aligned} \tag{3.6}$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma 3.1.

If there exists $\varphi(z^j)$ such that $\varphi(z^j) \neq |\varphi(z^j)|e_1$, then there is the unitary transformation U_j such that $\varphi(z^j) = \rho_j e_1 U_j$, $j \in \{1, 2, \ldots, n\}$. Now $g_j = f_j \circ U_j^{-1}$ is the desired function sequence, and the details are omitted there.

Next we prove (1.5) holds. Assume (1.5) fails, then there exist sequence $\{z^j\} \subset B$ satisfying $|\varphi(z^j)| \to 1$ as $j \to \infty$ and constant $\varepsilon_0 > 0$ such that

$$\frac{\mu(|z^j|)|\nabla\psi(z^j)|}{(1-|\varphi(z^j)|^2)^{\frac{n+1}{p}}} \geqslant \varepsilon_0. \tag{3.7}$$

We take again

$$f_j(z) = \left\{\frac{1-|\varphi(z^j)|^2}{(1-\langle z, \varphi(z^j)\rangle)^2}\right\}^{\frac{n+1}{p}}.$$

Then $\{f_j\}$ is bounded in A^p and converges to 0 uniformly on compact subset of B. From Lemma 2.1 we have

$$\begin{aligned}
\|T_{\psi,\varphi}(f_j)\|_{\beta_\mu} &\geqslant \mu(|z^j|)\left|\nabla[T_{\psi,\varphi}(f_j)](z^j)\right| \geqslant \mu(|z^j|)\left|\nabla\psi(z^j)\right|\left|f_j(\varphi(z^j))\right| \\
&\quad - \sup_{u \in C^n - \{0\}} \frac{c\mu(|z^j|)|\psi(z^j)||\nabla f_j(\varphi(z^j))J\varphi(z^j)u|}{\sqrt{(1-|z^j|^2)|u|^2 + |\langle z^j, u\rangle|^2}} \\
&= \frac{\mu(|z^j|)|\nabla\psi(z^j)|}{(1-|\varphi(z^j)|^2)^{\frac{n+1}{p}}} \\
&\quad - \sup_{u \in C^n - \{0\}} \frac{c(n+1)\mu(|z^j|)|\psi(z^j)||\langle\varphi(z^j), J\varphi(z^j)u\rangle|}{p(1-|\varphi(z^j)|^2)^{\frac{n+1+p}{p}}\sqrt{(1-|z^j|^2)|u|^2 + |\langle z^j, u\rangle|^2}}.
\end{aligned} \tag{3.8}$$

By (3.7), (3.8) and (1.4) we get

$$\lim_{j\to\infty} \|T_{\psi,\varphi} f_j\|_{\beta_\mu} \neq 0.$$

This contradicts the compactness of $T_{\psi,\varphi}$ by Lemma 3.1. This shows that (1.5) holds. The proof is completed. □

Corollary 3.2. *Let $0 < p < \infty$, $n = 1$. Let μ be normal on $[0, 1)$, $\psi \in H(D)$ and φ be a holomorphic self-map on disc D. Then $T_{\psi,\varphi}$ is a compact operator from A^p to β_μ if and only if the following are all satisfied*:

(i) $\varphi \in \beta_\mu$,

(ii) $$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+2}{p}}} = 0,$$

(iii) $$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2}{p}}} = 0.$$

Corollary 3.3. *Let $0 < p < \infty$. Let μ be normal on $[0, 1)$ and φ be a holomorphic self-map of B. Then C_φ is a compact operator from A^p to β_μ if and only if*

$$\lim_{|\varphi(z)|\to 1} \sup_{u\in C^n-\{0\}} \frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n+1+p}{p}}} \times \left\{ \frac{(1-|\varphi(z)|^2)|J\varphi(z)u|^2 + |\langle \varphi(z), J\varphi(z)u\rangle|^2}{(1-|z|^2)|u|^2 + |\langle z, u\rangle|^2} \right\}^{\frac{1}{2}} = 0$$

and $\varphi_l \in \beta_\mu$ for all $l \in \{1, 2, \ldots, n\}$.

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