Composition type operator from Bergman space to $\mu$-Bloch type space in $\mathbb{C}^n$ ⇤

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Abstract
Let $\phi$ be a holomorphic self-map of $B$ and $\psi \in H(B)$. A composition type operator $T_{\psi,\phi}$ is defined by $T_{\psi,\phi}(f) = \psi f \circ \phi$. We can regard this operator as a generalization of a multiplication operator and a composition operator. For normal functions $\mu$, some necessary and sufficient conditions are given for which $T_{\phi,\psi}$ is a bounded or compact operator from Bergman space to $\mu$-Bloch type space $\beta_\mu$ on the unit ball of $\mathbb{C}^n$. As a Corollary, we obtain the pointwise multiplier from Bergman space to $\mu$-Bloch type space on $B$.

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1. Introduction

Let $B$ denote the unit ball of $\mathbb{C}^n$ and $dv$ denote Lebesgue measure in $B$, normalized so that $v(B) = 1$. The class of all holomorphic functions with domain $B$ will be denoted by $H(B)$. For $0 < p < \infty$, the Bergman space $A^p$ is the set of all holomorphic functions $f$ on $B$ such that

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\[ \|f\|_{A^p} = \left\{ \int_B |f(z)|^p \, dv(z) \right\}^{\frac{1}{p}} < \infty. \]

\( A^p \) is a Banach space with the norm \( \| \cdot \|_{A^p} \) as \( 1 \leq p < \infty \). If \( 0 < p < 1 \), then \( A^p \) is a Fréchet space with \( \| \cdot \|_{A^p} \).

A positive continuous function \( \mu \) on \( [0, 1) \) is called normal if there are two constants \( a, b \) (\( 0 < a < b \)) such that

1. \( \frac{\mu(r)}{(1 - r)^a} \) is decreasing for \( r \in [0, 1) \) and \( \lim_{r \to 1^-} \frac{\mu(r)}{(1 - r)^a} = 0 \);
2. \( \frac{\mu(r)}{(1 - r)^b} \) is increasing for \( r \in [0, 1) \) and \( \lim_{r \to 1^-} \frac{\mu(r)}{(1 - r)^b} = \infty \).

The normal function \( \mu \), as a weight, has been usually used to define the mixed norm spaces (see [1]). Let \( \mu \) be normal. \( f \in H(B) \) is said to belong to the \( \mu \)-Bloch type spaces \( \beta_{\mu} \) provided that

\[ \sup_{z \in B} \mu(|z|) |\nabla f(z)| < \infty, \quad \text{where } \nabla f(z) = \left( \frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_n} \right). \]

It is well known that \( \beta_{\mu} \) is a Banach space under the norm

\[ \|f\|_{\beta_{\mu}} = |f(0)| + \sup_{z \in B} \mu(|z|) |\nabla f(z)|. \]

When \( \mu(r) = 1 - r^2 \) and \( \mu(r) = (1 - r^2)^{1-\alpha} \) (\( 0 < \alpha < 1 \)), the induced spaces \( \beta_{\mu} \) are the Bloch space and Lipschitz type spaces, respectively.

Let \( \varphi \) be a holomorphic self-map of \( B \) and \( \psi \in H(B) \). The composition type operator \( T_{\psi, \varphi} \) is defined by

\[ T_{\psi, \varphi}(f) = \psi f \circ \varphi, \quad f \in H(B). \]

It is easy to see that an operator defined in this manner is linear. We can regard this operator as a generalization of a multiplication operator \( M_{\psi} \) and a composition operator \( C_{\varphi} \). In complex plane, composition operator or composition type operator has been studied for a long time by many students (for example, [2–8]). In several complex variables case, Shi and Luo [9] and Zhou and Zeng [10] got the characterization on \( \varphi \) for which the induced composition operator is bounded or compact on \( \beta_{1-r^2} \) or \( \beta_{(1-r^2)^p} \) in the unit ball. And in the polydiscs, Zhou et al. [11–13] studied the same problems. The main purpose of this paper is to discuss the conditions for which \( T_{\psi, \varphi} \) is a bounded operator or compact operator from Bergman space \( A^p \) to \( \mu \)-Bloch space \( \beta_{\mu} \) on the unit ball of \( C^n \) and our main results are the following:

**Theorem A.** Let \( 0 < p < \infty \). Let \( \mu \) be normal on \( [0, 1) \), \( \psi \in H(B) \) and \( \varphi \) be a holomorphic self-map of \( B \). Then \( T_{\psi, \varphi} \) is a bounded operator from \( A^p \) to \( \beta_{\mu} \) if and only if the following are all satisfied:

1. \( \sup_{u \in C^n \setminus \{0\}, z \in B} \frac{\mu(|u|) |\psi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}} < \infty}. \)
\[
\left(1 - |\varphi(z)|^2\right)J\varphi(z)u^2 + |\varphi(z), J\varphi(z)u|^2 \right)^{\frac{1}{2}} < \infty, \quad (1.1)
\]

(ii) \[ \sup_{z \in B} \frac{\mu(|z|)\nabla \varphi(z)}{1 - |\varphi(z)|^2} < \infty \] \quad (1.2)

where \( J\varphi(z) \) denotes a Jacobian matrix of \( \varphi(z) \) and \( J\varphi(z)u \) denotes a vector as follows:

\[
J\varphi(z) = \left( \frac{\partial \varphi_j(z)}{\partial z_k} \right)_{1 \leq j, k \leq n}, \quad J\varphi(z)u = \left( \sum_{k=1}^{n} \frac{\partial \varphi_1(z)}{\partial z_k} u_k, \ldots, \sum_{k=1}^{n} \frac{\partial \varphi_n(z)}{\partial z_k} u_k \right)^T.
\]

**Theorem B.** Let \( 0 < p < \infty \). Let \( \mu \) be normal on \([0, 1)\), \( \psi \in H(B) \) and \( \varphi \) be a holomorphic self-map of \( B \). Then \( T_{\psi, \varphi} \) is a compact operator from \( A^p \) to \( \beta_\mu \) if and only if the following are all satisfied:

(i) \( \psi \in \beta_\mu \) and \( \psi l \in \beta_\mu \) for all \( l \in \{1, 2, \ldots, n\} \). \quad (1.3)

(ii) \[ \lim_{|\varphi(z)| \to 1} \sup_{u \in C^{n-1}} \frac{\nu(|\varphi(z)|)}{(1 - |\varphi(z)|^2)^{\frac{2n+1}{p}}} \times \frac{\left(1 - |\varphi(z)|^2\right)J\varphi(z)u^2 + |\varphi(z), J\varphi(z)u|^2}{(1 - |\varphi(z)|^2)u^2 + |z, u|^2} = 0, \quad (1.4)\]

(iii) \[ \lim_{|\varphi(z)| \to 1} \frac{\mu(|\varphi(z)|)}{(1 - |\varphi(z)|^2)^{\frac{2n+1}{p}}} = 0. \quad (1.5)\]

2. The boundedness of \( T_{\psi, \varphi} \)

We will use the symbol \( c \) to denote a finite positive number which does not depend on variables \( z, w \) and may depend on some norms, not necessarily the same at each occurrence. We call \( E \) and \( F \) are comparable (denoted by \( E \approx F \) in the following) if there exist two positive constants \( c_1 \) and \( c_2 \) such that \( c_1 E \leq F \leq c_2 E \).

**Lemma 2.1.** Let \( \mu \) be normal on \([0, 1)\) and \( f \in H(B) \). Then \( f \in \beta_\mu \) if and only if

\[
\sup_{u \in C^{n-1}} \frac{\mu(|z|)}{\sqrt{(1 - |z|^2)|u|^2 + |z, u|^2}} \leq \infty.
\]

Furthermore,

\[
\|f\|_{\beta_\mu} \approx |f(0)| + \sup_{u \in C^{n-1}} \frac{\mu(|z|)|\nabla f(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |z, u|^2}}.
\]

**Proof.** By Bergman metric in [14], there are constants \( A_1 > 0 \) and \( A_2 > 0 \) such that

\[
\frac{A_1|\nabla f(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |z, u|^2}} \leq |\nabla f(z)| \leq \frac{A_2|\nabla f(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |z, u|^2}}.
\]
for all \( u \in C^n - \{0\} \). Let \( C_1 = \min\{A_1, 1\}, C_2 = \max\{A_2, 1\} \). Then

\[
C_1 \left\{ \left| f(0) \right| + \sup_{z \in B} \frac{\mu(|z|) |\nabla f(z)|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}} \right\} \leq \| f \|_{\beta^\mu}
\]

\[
\leq C_2 \left\{ \left| f(0) \right| + \sup_{z \in B} \frac{\mu(|z|) |\nabla f(z)|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}} \right\}.
\]

The proof is completed. \( \square \)

**Lemma 2.2.** Let \( 0 < p < \infty \). (1) If \( f \in A^p \), then

\[
|f(z)| \leq \frac{c \| f \|_{A^p}}{(1 - |z|^2)^{\frac{n+1}{p}}}.
\]

(2) If \( f \in A^p \), then \( f \in \beta_{(1-2(n+1)p)/p} \) and \( \| f \|_{\beta_{(1-2(n+1)p)/p}} \leq c \| f \|_{A^p} \).

**Proof.** By Bergman operator, Proposition 1.4.10 in [15] and [16] we can obtain these results. \( \square \)

**Proof of Theorem A.** Suppose that (1.1) and (1.2) hold. Let

\[
M_1 = \sup_{z \in B} \frac{\mu(|z|)|\psi(z)|}{(1 - |\psi(z)|^2)^{\frac{n+1}{p}}} \times \left\{ \frac{(1 - |\psi(z)|^2)|J\psi(z)u|^2 + |\psi(z), J\psi(z)u|^2}{(1 - |z|^2)|u|^2 + |(z, u)|^2} \right\}^{\frac{1}{2}},
\]

\[
M_2 = \sup_{z \in B} \frac{\mu(|z|)|\nabla \psi(z)|}{(1 - |\psi(z)|^2)^{\frac{n+1}{p}}}.
\]

Then \( M_1 < \infty, M_2 < \infty \). For any \( f \in A^p \), by Lemmas 2.1 and 2.2 we have

\[
\mu(|z|)|\nabla [T_{\psi, \varphi}(f)](z)| \leq \mu(|z|) \left( |\nabla \psi(z)| |f(\psi(z))| + |\psi(z)| |\nabla C_{\varphi}(f)(\psi(z))| \right)
\]

\[
\leq \frac{c \mu(|z|)|\nabla \psi(z)| \| f \|_{A^p}}{(1 - |\psi(z)|^2)^{\frac{n+1}{p}}} + \sup_{z \in C^n - \{0\}} \frac{c \mu(|z|)|\psi(z)|}{(1 - |\psi(z)|^2)^{\frac{n+1}{p}}} \times \left\{ \frac{(1 - |\psi(z)|^2)|J\psi(z)u|^2 + |\psi(z), J\psi(z)u|^2}{(1 - |z|^2)|u|^2 + |(z, u)|^2} \right\}^{\frac{1}{2}}
\]

\[
\times \left( \frac{(1 - |\psi(z)|^2)^{\frac{n+1}{p}}|\nabla f(\psi(z)) J\psi(z)u|}{\sqrt{(1 - |\psi(z)|^2)|J\psi(z)u|^2 + |\psi(z), J\psi(z)u|^2}} \right) \leq c M_2 \| f \|_{A^p} + c M_1 \| f \|_{\beta_{(1-2(n+1)p)/p}} \leq c \| f \|_{A^p}.
\]

This means that \( T_{\psi, \varphi} \) is a bounded operator from \( A^p \) to \( \beta^\mu \).
Conversely, suppose that $T_{\psi, \psi}$ is a bounded operator from $A^p$ to $\beta_u$. Then we can easily obtain $\psi \in \beta_u$ and $\psi \psi \in \beta_u$ by taking $f(z) = 1$ and $f(z) = z_i$ ($i = 1, \ldots, n$) in $A^p$, respectively.

For any given $w \in B$ and $u \in C^n - \{0\}$. If $|\psi(w)| \leq \sqrt{3}$, it follows from $\psi \in \beta_u$ and $\psi \psi \in \beta_u$

$$
\frac{\mu(|w|)|\psi(w)|}{(1 - |\psi(w)|^2)^{\frac{n+1}{2}}} \left\{ \frac{(1 - |\psi(w)|^2)|J \psi(w)u|^2 + |\langle \psi(w), J \psi(w)u \rangle|^2}{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2} \right\}^{\frac{1}{2}} \\
\leq c \mu(|w|) \left\{ \frac{|J \psi(w)u|}{\sqrt{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2}} \right\}^{\frac{1}{2}} \\
= c \left\{ \sum_{i=1}^{n} \mu(|w|) \right\} \left\{ \frac{|\nabla \psi(w)u| + |\nabla \psi(w)u||\psi(w)|}{\sqrt{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2}} \right\}^{\frac{1}{2}} \\
\leq c \left\{ \sum_{i=1}^{n} \mu(|w|) \right\} \left\{ \frac{|\nabla \psi(w)u| + |\nabla \psi(w)u||\psi(w)|}{\sqrt{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2}} \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{n} \mu(|w|) \right\}.$$ 

This shows that (1.1) and (1.2) hold.

In the following, we always assume that $|\psi(w)| > \sqrt{3}$. First we suppose that $\psi(w) = r_w e_1$, where $r_w = |\psi(w)|$, $e_1$ is a vector $(1, 0, \ldots, 0)$.

(i) $\sqrt{(1 - r_w^2)(|\xi_1|^2 + \cdots + |\xi_n|^2)} \leq |\xi_1|$, where $(\xi_1, \ldots, \xi_n)^T = J \psi(w)u$. Let

$$
\psi(w) = r_w \left\{ \frac{1}{1 - r_w z_1} \left( 1 - \frac{r_w}{1 - r_w z_1} \right)^{\frac{n+1}{2}} \right\}. 
$$

Then, by Proposition 1.4.10 in [15] we have $\|f_w\|_{A^p} \leq c$. It is clear that

$$
\begin{align*}
&c \left\| T_{\psi, \psi} \right\| \|f_w\|_{A^p} \geq \left\| T_{\psi, \psi} (f_w) \right\|_{\beta_u} \\
&\geq \mu(|w|) \left\| \nabla \psi(w) f_w (\psi(w)) \right\| + \psi(w) \nabla \left\{ C_{\psi} (f_w) \right\} (w) \\
&\geq c \mu(|w|) \left\| \nabla \psi(w) f_w (\psi(w)) \right\| \left\| \nabla \psi(w) \right\| - \mu(|w|) \left\| \nabla \psi(w) \right\| \left\| f_w (\psi(w)) \right\| \\
&= \frac{c \mu(|w|) \left\| \nabla \psi(w) \right\| \left\| f_w (\psi(w)) \right\|}{(1 - r_w^2)^{\frac{n+1}{2}} \sqrt{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2}}. \\
&\left(2.1\right)
\end{align*}
$$

It follows from (2.1) and Lemma 2.3 that

$$
\begin{align*}
&\frac{\mu(|w|)|\psi(w)|}{(1 - |\psi(w)|^2)^{\frac{n+1}{2}}} \left\{ \frac{(1 - |\psi(w)|^2)|J \psi(w)u|^2 + |\langle \psi(w), J \psi(w)u \rangle|^2}{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2} \right\}^{\frac{1}{2}} \\
&= \mu(|w|)|\psi(w)| \left\{ \frac{(1 - r_w^2)(|\xi_1|^2 + \cdots + |\xi_n|^2) + |\xi_1|^2}{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2} \right\}^{\frac{1}{2}}
\end{align*}
$$
Similarly to the proof of (2.1) we can get
\[
\mu(|w|)|\psi(w)| \leq c\|T_{\psi,\phi}\|.
\]
\[
\frac{\mu(|w|)|\psi(w)|}{(1 - r_w^2)^{\frac{\mu+1}{p}}} \leq c\|T_{\psi,\phi}\|.
\]

Thus
\[
\mu(|w|)|\psi(w)| \leq \frac{(1 - |\psi(w)|^2)|J\psi(w)u|^2 + |\langle \psi(w), J\phi(w)u \rangle|^2}{(1 - |w|^2)|u|^2 + |\langle w, u \rangle|^2} \leq c\|T_{\psi,\phi}\|.
\]

This shows that (1.1) holds.

In a general situation, if \( \psi(w) \neq |\psi(w)| e_1 \), we use the unitary transformation \( U_w \) to make \( \psi(w) = \rho_w e_1 U_w \), where \( \rho_w = |\psi(w)| > \sqrt{2}/3 \). We can prove (1.1) by taking \( g_w = f_w \circ U_w^{-1} \).

In order to prove (1.2), we take
\[
\|h_w\|_{Ap} \leq c. \quad \text{Thus,}
\]
\[
\phi(w) = \frac{1 - |\psi(w)|^2}{(1 - \langle z, \phi(w) \rangle)^2}.
\]

Then \( \|h_w\|_{Ap} \leq c \). Thus,
Proof. The result can be proved by using Montel theorem and Lemma 2.2; the details are omitted here. □
Proof of Theorem C. Suppose that (1.4) and (1.5) hold. Then for any $\epsilon > 0$, there exists $0 < \delta < 1$ such that

$$\sup_{u \in C^n(0)} \frac{\mu(|z|)|\psi(z)|}{(1 - |z|^2)^{\frac{n+1}{4}}} \left\{ \frac{(1 - |z|^2)|J\psi(z)u|^2 + \left|\psi(z), J\psi(z)u\right|^2}{(1 - |z|^2)|u|^2 + |(z, u)|^2} \right\} \leq \epsilon$$

(3.1)

and

$$\frac{\mu(|z|)|\nabla \psi(z)|}{(1 - |z|^2)^{\frac{n+1}{4}}} \leq \epsilon$$

(3.2)

as $|\psi(z)|^2 > 1 - \delta$.

Let $\{f_j\}$ be any a sequence $\{f_j\}$ which converges to 0 uniformly on compact subset of $B$ satisfying $\|f_j\|_{A^p} \leq 1$. Then $\{f_j\}$ and $\{\nabla f_j\}$ converges to 0 uniformly on $\{w: \|w\|^2 \leq 1 - \delta\}$.

If $|\psi(z)|^2 > 1 - \delta$, then, from (3.1), (1.3), Lemma 2.1 and Lemma 2.2, we have

$$\sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla (T\psi f_j)(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}} \leq \sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla f_j(\psi(z))J\psi(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}}$$

$$\leq \sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla f_j(\psi(z))J\psi(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}} + c\mu(|z|)|\nabla \psi(z)||f_j(\psi(z))|$$

$$\leq \sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla f_j(\psi(z))J\psi(z)u|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}} + c\mu(|z|)|\nabla \psi(z)||f_j(\psi(z))|$$

$$\leq c\epsilon \|f_j\|_{L^p(1 - |z|^2, 1 + |z|, \mu)} + c\epsilon \|f_j\|_{A^p} \leq c\epsilon \|f_j\|_{A^p} \leq c\epsilon \rightarrow c\epsilon \quad (j \rightarrow \infty).$$

If $|\psi(z)|^2 \leq 1 - \delta$, by (1.3) and Lemma 2.1,

$$\sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla f_j(\psi(z))u|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}}$$

$$\leq \sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla \psi(z)u||f_j(\psi(z))|}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}}$$

$$+ \sup_{u \in C^n(0)} \frac{\mu(|z|)|\nabla f_j(\psi(z))|(|\nabla \psi_1(z)u| + \cdots + |\nabla \psi_n(z)u|)}{\sqrt{(1 - |z|^2)|u|^2 + |(z, u)|^2}}$$

$$\leq c\|\psi\|_{L^p} ||f_j(\psi(z))|| + c \sum_{i=1}^n (||\psi\psi_i\|_{L^p} + ||\psi\|_{L^p}) ||\nabla f_j(\psi(z))||$$

$$\rightarrow 0 \quad (j \rightarrow \infty).$$

(3.4)

By (3.3) and (3.4) we have $\|T\psi f_j\|_{L^p} \rightarrow 0 \quad (j \rightarrow \infty)$. This means that $T\psi f$ is a compact operator from $A^p$ to $\beta_{\mu}$ by Lemma 3.1.
Conversely, for any \( l \in \{1, 2, \ldots, n\} \), by taking \( f(z) = 1 \) or \( f(z) = z_l \in \bigcup^n_{l=1} \), we get \( \psi \in \beta_{\mu} \) and \( \psi \psi = T_{\psi, \phi}(f) \in \beta_{\mu} \). That is, (1.3) holds.

Assume that (1.4) fails. Then there exist sequence \( \{z'_j\} \subset B \) satisfying \( r_j = |\psi(z'_j)| \to 1 \) as \( j \to \infty \), \( \{u^j\} \subset C^n - \{0\} \) and constant \( \varepsilon_0 > 0 \) such that

\[
\left( \frac{1}{1 - r_j^2} \left( \frac{1}{1 - |z'_j|^2} |J\psi(z'_j)u^j| + |\psi(z'_j)\psi(u^j)|^2 \right)^{\frac{1}{2}} \right) \geq \varepsilon_0. \tag{3.5}
\]

To construct the sequence of functions \( \{f_j\} \), we first assume that \( \psi(z'_j) = r_je_1 \) (\( j = 1, 2, \ldots \)).

If \( (1 - r_j^2)(w_1^j|^2 + \cdots + w_n^j|^2) < |w_1^j|^2 \), where \((w_1^j, \ldots, w_n^j)^T = J\psi(z'_j)u^j\), take

\[
f_j(z) = \frac{1}{1 - r_j z_1} \left( \frac{1 - r_j^2}{1 - r_j z_1^2} \right)^{\frac{n+1}{2}}.
\]

We can prove easily \( \|f_j\|_{A^F} \leq c \).

Let \( E \) be any a compact subset of \( B \). Then there exists 0 < \( r \) < 1 such that \( E \subset \{z: |z| \leq r\} \). Thus

\[
\max_{z \in E} |f_j(z)| \leq \left\{ \frac{1 - r_j^2}{1 - r^2} \right\}^{\frac{n+1}{2}} \to 0 \quad \text{as} \quad j \to \infty.
\]

That is, \( \{f_j\} \) converges to 0 uniformly on compact subset of \( B \). But from (3.5) we have

\[
\|T_{\psi, \phi}(f_j)\|_{\beta_{\mu}} \geq c \mu(|z|)|\psi(z)|(|\nabla f_j|(r_je_1)J\psi(z)u^j|) \frac{1}{\sqrt{(1 - |z|^2)|u^j|} + |\psi(z)|} - \mu(|z|)|\nabla \psi(z)| |f_j(r_je_1)|
\]

\[
= c \mu(|z|)|\psi(z)| \left( \frac{1}{1 - |z|^2}|u^j| + |\psi(z)| \right)^{\frac{1}{2}} \frac{1}{\sqrt{(1 - |z|^2)|u^j|} + |\psi(z)|}
\]

\[
\times \frac{|u^j|}{\sqrt{(1 - r_j^2)(w_1^j|^2 + \cdots + w_n^j|^2) + |u^j|^2}} \geq \frac{c\varepsilon_0}{\sqrt{2}}.
\]

This means that

\[
\lim_{j \to \infty} \|T_{\psi, \phi}(f_j)\|_{\beta_{\mu}} \neq 0.
\]

This contradicts the compactness of \( T_{\psi, \phi} \) by Lemma 3.1.

If \( (1 - r_j^2)(w_1^j|^2 + \cdots + w_n^j|^2) \geq |w_1^j|^2 \), then we let \( \theta_k^j = \arg w_k^j \) (\( k = 2, \ldots, n \)) and take

\[
f_j(z) = (e^{-i\theta_2^j}z_2 + \cdots + e^{-i\theta_n^j}z_n) \left( \frac{1 - r_j^2}{1 - r_j z_1^2} \right)^{\frac{n+1}{2}} \frac{1}{\sqrt{1 - r_j z_1^2}}.
\]

We have \( \|f_j\|_{A^F} \leq c \) and \( \{f_j\} \) converges to 0 uniformly on compact subset of \( B \). But
Lemma 2.1 we have

\[ \|T_{\psi,\phi}(f_j)\|_{\mathcal{B}_\rho} \geq \frac{c\mu(|\xi|)|\psi(\xi)|(\nabla f_j)(r_j,e_1) J\psi(\xi)u + f_j(\psi(\xi))\nabla \psi(\xi)}{\sqrt{(1 - |\xi|^2)|u|^2 + |(\xi,u)|^2}} \]

\[ = \frac{c\mu(|\xi|)|\psi(\xi)|}{(1 - r_j^2)^1/\rho} \times \left\{ \frac{(1 - |\psi(\xi)|^2) |J\psi(\xi)u|^2 + |(\psi(\xi), J\psi(\xi)u)|^2}{(1 - |\xi|^2)|u|^2 + |(\xi,u)|^2} \right\}^{1/2} \times \frac{|w_j^2 + \cdots + w_n^2|^{1/2}}{\sqrt{1 - r_j^2}} > c\epsilon_0. \]  

(3.6)

This contradicts the compactness of \( T_{\psi,\phi} \) by Lemma 3.1.

If there exists \( \varphi(z') \) such that \( \varphi(z') \neq |\psi(z')|e_1 \), then there is the unitary transformation \( U_j \) such that \( \varphi(z') = \rho_j e_1 U_j, j \in \{1, 2, \ldots, n\} \). Now \( g_j = f_j U_j^{-1} \) is the desired function sequence, and the details are omitted there.

Next we prove (1.5) holds. Assume (1.5) fails, then there exist sequence \( \{z_j\} \subset B \) satisfying \( |\varphi(z_j)| \to 1 \) as \( j \to \infty \) and constant \( \epsilon_0 > 0 \) such that

\[ \frac{\mu(|z_j|)|\nabla \psi(z_j)|}{(1 - |\varphi(z_j)|^2)^{1/\rho}} \geq \epsilon_0. \]  

(3.7)

We take again

\[ f_j(z) = \left\{ \begin{array}{ll}
1 - |\varphi(z_j)|^2 \\
(1 - (z, \varphi(z_j))^2)
\end{array} \right\}^{\frac{\rho+1}{\rho}}. \]

Then \( \{f_j\} \) is bounded in \( A^\rho \) and converges to 0 uniformly on compact subset of \( B \). From Lemma 2.1 we have

\[ \|T_{\psi,\phi}(f_j)\|_{\mathcal{B}_\rho} \geq \mu(|z_j|)|\nabla T_{\psi,\phi}(f_j)(z_j)| \geq \mu(|z_j|)|\nabla \psi(z_j)||f_j(\psi(z_j))| \]

\[ \geq \sup_{\xi \in C^\rho([-0])} \frac{c\mu(|\xi|)|\psi(\xi)||\nabla f_j(\psi(\xi)) J\psi(\xi)u|}{\sqrt{(1 - |\xi|^2)|u|^2 + |(\xi,u)|^2}} \]

\[ = \frac{\mu(|z_j|)|\nabla \psi(z_j)|}{(1 - |\varphi(z_j)|^2)^{1/\rho}} \geq \sup_{\xi \in C^\rho([-0])} \frac{c(n + 1)\mu(|z_j|)|\psi(\xi)|||\psi(\xi), J\psi(\xi)u||}{p(1 - |\varphi(z_j)|^2)^{n+1+p/\rho} \sqrt{(1 - |\xi|^2)|u|^2 + |(\xi,u)|^2}}. \]  

(3.8)

By (3.7), (3.8) and (1.4) we get

\[ \lim_{j \to \infty} \|T_{\psi,\phi} f_j\|_{\mathcal{B}_\rho} \neq 0. \]

This contradicts the compactness of \( T_{\psi,\phi} \) by Lemma 3.1. This shows that (1.5) holds. The proof is completed. \( \Box \)
Corollary 3.2. Let $0 < p < \infty$, $n = 1$. Let $\mu$ be normal on $[0, 1)$, $\psi \in H(D)$ and $\varphi$ be a holomorphic self-map on disc $D$. Then $T_{\psi, \varphi}$ is a compact operator from $A^p$ to $\beta_{\mu}$ if and only if the following are all satisfied:

(i) $\varphi \in \beta_{\mu}$,

(ii) $\lim_{|\psi(z)| \to 1} \frac{\mu(|z|)|\psi(z)|}{1 - |\varphi(z)|^2} \frac{1}{p} = 0$,

(iii) $\lim_{|\psi(z)| \to 1} \frac{\mu(|z|)}{(1 - |\varphi(z)|^2)^{\frac{1}{p}}}$ = 0.

Corollary 3.3. Let $0 < p < \infty$. Let $\mu$ be normal on $[0, 1)$ and $\varphi$ be a holomorphic self-map of $B$. Then $C_{\varphi}$ is a compact operator from $A^p$ to $\beta_{\mu}$ if and only if

$$\lim_{|\psi(z)| \to 1} \sup_{u \in C^n \setminus \{0\}} \frac{\mu(|z|)}{(1 - |\varphi(z)|^2)^{\frac{1}{p}}} \left(1 - |\varphi(z)|^2\right)^{n+1/p} \left\{ \frac{1}{1 - |z|^2} |u|^2 + |\langle \psi(z), J\varphi(z)u \rangle|^2 \right\}^{1/2} = 0$$

and $\varphi_l \in \beta_{\mu}$ for all $l \in \{1, 2, \ldots, n\}$.

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References