Partial regularity of a minimizer of the relaxed energy for biharmonic maps

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Abstract

In this paper, we study the relaxed energy for biharmonic maps from an \( m \)-dimensional domain into spheres for an integer \( m \geq 5 \). By an approximation method, we prove the existence of a minimizer of the relaxed energy of the Hessian energy, and that the minimizer is biharmonic and smooth outside a singular set \( \Sigma \) of finite \((m - 4)\)-dimensional Hausdorff measure. When \( m = 5 \), we prove that the singular set \( \Sigma \) is 1-rectifiable. Moreover, we also prove a rectifiability result for the concentration set of a sequence of stationary harmonic maps into manifolds.

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1. Introduction

Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^m \) for an integer \( m \geq 5 \) and let \( N \) be a compact manifold without boundary, which is embedded in \( \mathbb{R}^l \). For a map \( u \in W^{2,2}(\Omega, N) \), we define its Hessian energy by

\[
\mathbb{H}(u) = \int_{\Omega} |\Delta u|^2 \, dx.
\] (1.1)
A critical point of the Hessian energy functional in $W^{2,2}(\Omega, N)$ is called a (weakly) biharmonic map.

The partial regularity for stationary biharmonic maps has attracted much attention. Motivated by the partial regularity result for stationary harmonic maps [3], Chang, Wang and Yang in [6] introduced a study of stationary biharmonic maps and proved partial regularity of stationary biharmonic maps into spheres. Wang in [26] generalized their result for stationary biharmonic maps into a compact manifold $N$. Recently, the regularity problem for stationary biharmonic maps was revisited by Struwe in [23] from a new point of view. We recall that a stationary biharmonic map $u$ is a (weakly) biharmonic map and satisfies

$$\frac{d}{dt} \mathbb{H}(u(x + t\phi(x))) \bigg|_{t=0} = 0$$

for any $\phi \in C^1_0(\Omega, \mathbb{R}^m)$ and any $x \in \Omega$. Typical stationary biharmonic maps are minimizing biharmonic maps. The first author and Wang in [14] proved that the Hausdorff dimension of the singular set of minimizing biharmonic maps into spheres is at most $m - 5$. Recently, Scheven in [20] generalized the result for minimizing biharmonic maps into a general manifold $N$. This is an analogous result to the optimal partial regularity for minimizing harmonic maps due to Giaquinta and Giusti [8] and Schoen and Uhlenbeck [22].

On the other hand, motivated by a gap phenomenon for the Dirichlet energy discovered by Hardt and Lin [12], Bethuel, Brezis and Coron in [5] introduced a relaxed energy for the Dirichlet energy of maps in $W^{1,2}(B^3, S^2)$ and proved that a minimizer of the relaxed energy is a harmonic map. Giaquinta, Modica and Soucek in [10] proved the partial regularity of the minimizers of the relaxed energy for harmonic maps. A gap phenomenon for the Hessian energy functional similar to the one for the Dirichlet energy was observed in [14]. More precisely, there is a smooth domain $\Omega$ in $\mathbb{R}^5$ and a boundary value map $\psi : \partial \Omega \to S^4$ such that

$$\min_{u \in W^{2,2}_\psi(\Omega, S^4)} \mathbb{H}(u) < \inf_{v \in W^{2,2}_\psi(\Omega, S^4) \cap C^0(\bar{\Omega}, S^4)} \mathbb{H}(v).$$

Following the context of harmonic maps (see [4]), a family of $\lambda$-relaxed energy functionals for bi-harmonic maps was considered in [14] as follows:

$$\mathbb{H}_\lambda(u) = \mathbb{H}(u) + 16\lambda \sigma_4 L(u), \quad \forall u \in W^{2,2}_{\phi}(\Omega, S^4) \text{ and } \lambda \in [0, 1],$$

where $\sigma_4$ is the area of the unit sphere $S^4 \subset \mathbb{R}^5$ and

$$L(u) = \frac{1}{\sigma_4} \sup_{\xi : \Omega \to \mathbb{R}, \|\nabla \xi\|_{L^\infty} \leq 1} \left\{ \int_{\Omega} \nabla u \cdot \nabla \xi \, dx + \int_{\partial \Omega} D(u) \cdot \nu \xi \, dH^{n-1} \right\}$$

for the $D$-field $D(u)$. Moreover, it was proved in [14] that $\mathbb{H}_\lambda$ are sequentially lower semi-continuous and that their minimizers are partially regular biharmonic maps for $\lambda \in [0, 1)$. However, it is not known whether $\mathbb{H}_\lambda(u)$ is a relaxed energy for the Hessian functional or not. Thus, there is an open question on the existence and partial regularity of minimizers of the relaxed energy for biharmonic maps.
In order to define a relaxed energy for biharmonic maps from $\Omega \subset \mathbb{R}^m$ into the sphere $S^n$ with $m \geq 5$, we denote by $W^{2,2}_{u_0}(\Omega, S^n)$ the set of all maps $u \in W^{2,2}(\Omega, S^n)$ satisfying the boundary condition
\[ u - u_0|_{\partial \Omega} = 0, \quad \nabla(u - u_0)|_{\partial \Omega} = 0, \quad (1.2) \]
where $u_0$ is smooth on $\Omega$. Similarly, we denote by $C_{u_0}^{\infty}(\Omega, S^n)$ the space of smooth maps satisfying (1.2). Following a strategy in [11], we can define the relaxed energy $F(u)$ of biharmonic maps in an abstract way; i.e.

**Definition 1.1.** For each $u \in W^{2,2}_{u_0}(\Omega, S^n)$, we define the relaxed energy $F(u)$ by
\[ F(u) = \inf \left\{ \liminf_{k \to \infty} \mathcal{H}(u_k) \mid \{u_k\} \subset C_{u_0}^{\infty}(\Omega, S^n), \ u_k \rightharpoonup u \text{ weakly in } W^{2,2}(\Omega, S^n) \right\}. \]

It can be proved (see below Lemmas 2.1–2.2) that there is a minimizer of $F$ in $W^{2,2}_{u_0}(\Omega, S^n)$ and
\[ \min_{u \in W^{2,2}_{u_0}(\Omega, S^n)} F(u) = \inf_{u \in W^{2,2}_{u_0}(\Omega, S^n) \cap C^0(\overline{\Omega}, S^n)} \mathcal{H}(u). \quad (1.3) \]

However, without the explicit form of $F(u)$, we do not know how to prove the partial regularity of a minimizer of $F$. To overcome this difficulty, we consider a family of perturbed functionals $\mathcal{H}_\varepsilon(\varepsilon > 0)$ defined as follows:

**Definition 1.2.** For each $\varepsilon > 0$, we define the perturbed functional $\mathcal{H}_\varepsilon W^{2,2}_{u_0} : W^{1,m+1}(\Omega, S^n) \to \mathbb{R}$ by
\[ \mathcal{H}_\varepsilon(u) = \int_\Omega |\Delta u|^2 + \varepsilon |\nabla u|^{m+1} \, dx. \]

A similar approximation for the relaxed energy for harmonic maps was recently studied by Giaquinta and the two authors in [9].

The first result of this paper is:

**Theorem 1.1.** For each $\varepsilon > 0$, there exists a minimizer $u_\varepsilon$ of $\mathcal{H}_\varepsilon$ in the space $W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n)$ with $m \geq 5$. Then, for each sequence $\varepsilon \to 0$, there is a subsequence $\varepsilon_i$ such that $u_{\varepsilon_i}$ converges to a map $u$ weakly in $W^{2,2}(\Omega, S^n)$, and $u$ is a minimizer of the relaxed energy $F$ in $W^{2,2}_{u_0}(\Omega, S^n)$ and a biharmonic map. Moreover, the minimizer $u$ is smooth outside a relatively closed singular set $\Sigma$, whose $(m-4)$-Hausdorff measure is finite, where the singular set is defined by
\[ \Sigma = \bigcap_{R > 0} \left\{ x \in \Omega \mid B_R(x) \subset \Omega, \liminf_{\varepsilon_i \to 0} \int_{B_R(x)} |\Delta u_{\varepsilon_i}|^2 \, dx \geq \varepsilon_0 \right\} \]
for some constant $\varepsilon_0 > 0$. 
For the proof of Theorem 1.1, the regularity of $u$ outside the singular set simply follows from proving a suitable Morrey bound on $u$ and applying Struwe’s result [23] (The condition on ‘stationarity’ is not used once such a bound is obtained). To obtain the Morrey bound, we need to use a monotonicity formula available for $u$ through the method of [23], and then let $\varepsilon \to 0$. It is well known that one of main difficulties in the proof of partial regularity of stationary biharmonic maps is that the monotonicity formula for biharmonic maps involves boundary terms of undetermined sign. Chang, Wang and Yang [6] used a complicated iteration to deal with this difficulty. Struwe in [23] had a nice observation and gave a simple proof, on which our proof to Theorem 1.1 is based. In fact, our case is more complicated, since the stationarity of the limit map $u$ is not known, and no ‘nice’ monotonicity formula is available. Our approach is to prove a monotonicity formula for $u_\varepsilon$ and pass to the limit $\varepsilon \to 0$. We also remark that the fact that $H^{m-4}(\Sigma) < \infty$ follows exactly as in [20] with the same proof.

On the other hand, in the study of the limit of a sequence of stationary harmonic maps $v_i$, there is a nice monotonicity formula for each stationary harmonic map $v_i$. It implies that the density

$$\lim_{r \to 0} r^{2-m} \lim_{i \to \infty} \int_{B_r(x)} |\nabla v_i|^2 \, dx$$

exists for $H^{m-4}$ a.e. $x_0 \in \Omega$. By assuming that $u_{\varepsilon_i}$ converges weakly to a map $u$ in $W^{2,2}$, we can define a measure $\mu = \lim_{\varepsilon_i \to 0} |\Delta u_{\varepsilon_i}|^2 \, dx$ in the sense of Radon measures. Then it implies that the quantity

$$\Theta(x) = \lim_{r \to 0} r^{4-m} \mu(B_r(x))$$

exists for $H^{m-4}$ a.e. $x \in \Omega$.

Our proof also works for a sequence of stationary biharmonic maps into any compact manifold. Following the important paper [16] of Lin on harmonic maps, we can obtain a similar result on the concentration set of a sequence of stationary harmonic maps. Therefore, we study further properties of the boundary terms in the monotonicity formula of the minimizers $u_{\varepsilon_i}$ of $H_{\varepsilon_i}$ in Theorem 1.1 and prove that the limit

$$\lim_{r \to 0} \lim_{\varepsilon_i \to 0} \int_{B_r(x)} |\Delta u_{\varepsilon_i}|^2 + \varepsilon_i |\nabla u_{\varepsilon_i}|^{m+1} \, dx$$

exists for $H^{m-4}$ a.e. $x_0 \in \Omega$. By assuming that $u_{\varepsilon_i}$ converges weakly to a map $u$ in $W^{2,2}$, we can define a measure $\mu = \lim_{\varepsilon_i \to 0} |\Delta u_{\varepsilon_i}|^2 \, dx$ in the sense of Radon measures. Then it implies that the quantity

$$\Theta(x) = \lim_{r \to 0} r^{4-m} \mu(B_r(x))$$

exists for $H^{m-4}$ a.e. $x \in \Omega$.

Let $\tilde{u}_i$ be a sequence of stationary biharmonic maps from $\Omega \subset \mathbb{R}^m$ into a compact manifold $N \subset \mathbb{R}^l$ satisfying $\int_\Omega |\Delta \tilde{u}_i|^2 \, dx \leq C$ for a uniform constant $C > 0$. Assume that $\tilde{u}_i$ converges weakly to a map $\tilde{u}$ in $W^{2,2}$, and

$$\tilde{\mu} = \lim_{i \to \infty} |\Delta \tilde{u}_i|^2 \, dx = |\Delta \tilde{u}|^2 \, dx + \tilde{\nu}$$

in the sense of Radon measures. It was shown (see Section 3 in [20]) that $\tilde{u}_i$ converges smoothly to $\tilde{u}$ in $\Omega \setminus \tilde{\Sigma}$, where

$$\tilde{\Sigma} = \left\{ x \in \Omega \left| \liminf_{\rho \to 0} \left( \rho^{2-m} \int_{B_\rho(x)} |\nabla \tilde{u}|^2 \, dx + \rho^{4-m} \tilde{\mu}(B_\rho(x)) \right) \geq \tilde{\varepsilon}_0 \right\}$$
for a positive constant $\varepsilon_0$ and $\tilde{\mathcal{S}}$ is a relatively closed set of finite $(m - 4)$-dimension Hausdorff measure. However, there is no result on the rectifiability of the concentrated set $\tilde{\mathcal{S}}$. Luckily, we can show that the density

$$\tilde{\Theta}(x) = \lim_{r \to 0} r^{4 - m} \tilde{\mu}(B_r(x))$$

exists except for a set of $\mathcal{H}^{m-4}$ measure zero. Then, thanks to a result of Preiss [19], we have

**Theorem 1.2.** For $m \geq 5$, let $\tilde{u}_i$ be a sequence of stationary biharmonic maps from $\Omega \subset \mathbb{R}^m$ into a compact manifold $N \subset \mathbb{R}^l$ satisfying $\int_{\Omega} |\Delta\tilde{u}_i|^2 \, dx \leq C$ for a uniform constant $C > 0$. Then the measure $\tilde{\nu}$ defined in (1.4) is a $(m - 4)$-rectifiable measure and the singular set $\tilde{\mathcal{S}}$ is also $(m - 4)$-rectifiable.

We would like to point out that the proof of Theorem 1.2 strongly relies on many previous results in [20]. A similar result was also obtained by Tian [24] for the concentration set of the limit by a sequence of smooth Yang–Mills connections.

When we try to apply the same argument to the sequence $u_{\varepsilon_i}$ in Theorem 1.1, we encounter extra difficulty; i.e. We do not know whether $u_{\varepsilon_i}$ converges strongly in $W^{2,2}$ to $u$ away from the concentration set or not. Fortunately, for $m = 5$, we use an idea of Lin [17] to prove

**Theorem 1.3.** For $m = 5$, let $u_{\varepsilon_i}$ be a minimizer of $\mathbb{H}_{\varepsilon_i}$ in Theorem 1.1 and

$$\mu = \lim_{\varepsilon_i \to 0} |\Delta u_{\varepsilon_i}|^2 \, dx = |\Delta u|^2 \, dx + \nu$$

for a measure $\nu \geq 0$. Then

(i) There is a small positive constant $\varepsilon_1 < \varepsilon_0$ such that

$$\Sigma_1 := \{x \in \Omega \mid \Theta(x) \geq \varepsilon_1\} \supset \Sigma,$$

where $\varepsilon_0$ and $\Sigma$ are defined in Theorem 1.1, and $\Sigma_1$ is a relatively closed set of finite 1-dimensional Hausdorff measure.

(ii) For $\mathcal{H}^1$-a.e. $x \in \Omega$, $\nu = \Theta(x)\mathcal{H}^1, \Sigma_1$, and for $\mathcal{H}^1$-a.e. $x \in \Sigma_1 \varepsilon_1 \leq \Theta(x) \leq C(d(x, \partial \Omega))$, where $C(d(x, \partial \Omega))$ is a constant depending on the distance from $x$ to $\partial \Omega$.

(iii) The defect measure $\nu$ is 1-rectifiable measure and hence $\Sigma_1$ is a 1-rectifiable set.

The paper is organized as follows. In Section 2, we establish a monotonicity formula and partial regularity of the weak limit of $u_{\varepsilon}$ of $\mathbb{H}_{\varepsilon}$ in Theorem 1.1. In Section 3, we prove that the quantity $\Theta(x)$ exists for $\mathcal{H}^{m-4}$ a.e. $x \in \Omega$ and give a proof of Theorem 1.2. In Section 4, we complete the proof of Theorem 1.3.

2. Perturbed variational problem and the partial regularity

Let $F(u)$ be the relaxed energy defined in Definition 1.1. The existence of a minimizer of $F(u)$ over $W^{2,2}_{\varepsilon_0}(\Omega, S^n)$ can be obtained by the direct method of calculus of variations, thanks to the lower semicontinuity of $F$. Thus
Lemma 2.1. There exists \( \tilde{u} \in W^{2,2}_{u_0}(\Omega, S^n) \) such that

\[ F(\tilde{u}) = \inf_{u \in W^{2,2}_{u_0}(\Omega, S^n)} F(u). \]

However, we do not know how to prove that the minimizer given by Lemma 2.1 is a bi-harmonic map. Instead, we start to consider a perturbed functional \( \mathbb{H}_\varepsilon \) for \( \varepsilon > 0 \). The first observation is that

Lemma 2.2.

\[ \inf_{W^{2,2}_{u_0} \cap C^0_{u_0}(\Omega, S^n)} \mathbb{H}(u) = \inf_{W^{2,2}_{u_0} \cap W^{1,1+m}(\Omega, S^n)} \mathbb{H}(u) = \inf_{C^\infty_{u_0}(\Omega, S^n)} \mathbb{H}(u). \]

Proof. By the Sobolev embedding theorem, we have

\[ C^\infty_{u_0}(\Omega, S^n) \subset W^{2,2}_{u_0} \cap W^{1,1+m}(\Omega, S^n) \subset W^{2,2}_{u_0} \cap C^0_{u_0}(\Omega, S^n). \]

It suffices to show that for each \( u \in W^{2,2}_{u_0} \cap C^0_{u_0}(\Omega, S^n) \), we can find a sequence of \( u_k \in C^\infty_{u_0}(\Omega, S^n) \) such that

\[ \lim_{k \to \infty} \|u_k - u\|_{W^{2,2}} = 0. \]

For simplicity, let us assume \( \Omega = B_1 \). Define

\[ \tilde{u} = u - u_0 \quad \text{for } x \in \overline{B_1}, \]

\[ \tilde{u} = 0 \quad \text{for } x \in B_2 \setminus \overline{B_1}. \]

Due to the boundary condition (1.2), \( \tilde{u} \) is in \( W^{2,2}(B_2, \mathbb{R}^{n+1}) \). Let \( \xi \) be a smooth function supported in \( B_1(0) \) satisfying

\[ \int_{\mathbb{R}^m} \xi \, dx = 1. \]

Set

\[ w_k(x) = \int_{\mathbb{R}^m} k^m \xi(ky)\tilde{u}(x - y) \, dy \]

and

\[ \tilde{w}_k(x) = w_k\left( \left( 1 + \frac{2}{k} \right) x \right). \]

By the definition of \( \tilde{u} \) outside \( B_1 \) and the compact support of \( \xi \), \( \tilde{w}_k \) satisfies zero Dirichlet and Neumann boundary conditions on \( \partial B_1 \). It is obvious that

\[ \lim_{k \to \infty} \|\tilde{w}_k - \tilde{u}\|_{W^{2,2}(B_1, \mathbb{R}^{n+1})} = 0. \]
We claim that \( \tilde{w}_k \) converges to \( \tilde{u} \) uniformly on \( B_1 \). In fact, \( \tilde{u}((1 + \frac{2}{k})x) \) uniformly converges to \( \tilde{u}(x) \) due to the uniform continuity of \( u \) and \( w_k(y) \) converges uniformly to \( \tilde{u}(y) \) on \( B_{3/2} \). We can now set

\[
u_k = \frac{\tilde{w}_k(x) + u_0}{|\tilde{w}_k(x) + u_0|}.
\]

It is straightforward to check that \( u_k \) satisfies the boundary conditions (1.2) and approaches \( u \) in the \( W^{2,2} \)-norm. \( \square \)

As can be seen from the above proof of Lemma 2.2, we can equivalently define \( F(u) \) to be

\[
F(u) = \inf \left\{ \lim \inf_{k \to \infty} \mathbb{H}(u_k) \mid \{u_k\} \subset W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n) \text{ and } u_k \rightharpoonup u \text{ weakly in } W^{2,2}(\Omega, S^n) \right\}.
\]

The following observation plays an important role in this paper.

**Lemma 2.3.** Let \( u_\epsilon \) be a minimizer of \( \mathbb{H}_\epsilon \) in \( W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n) \). Then

\[
\lim_{\epsilon \to 0} \int_\Omega \epsilon |\nabla u_\epsilon|^{m+1} dx = 0.
\]

**Proof.** Let \( \epsilon_i \) be any subsequence going to zero such that \( \lim_{i \to \infty} \int_\Omega \epsilon_i |\nabla u_{\epsilon_i}|^{m+1} dx \) exists. In the following, we write \( u_i = u_{\epsilon_i} \) for simplicity. Using the minimality of \( u_i \), we have

\[
\inf_{v \in W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n)} \mathbb{H}(v) \leq \lim_{i \to \infty} \inf \mathbb{H}(u_i) \leq \lim_{i \to \infty} \mathbb{H}(u_i) + \lim_{i \to \infty} \int_\Omega \epsilon_i |\nabla u_i|^{m+1} dx
\]

\[
\leq \limsup_{i \to \infty} \mathbb{H}_{\epsilon_i}(u_i)
\]

\[
\leq \inf_{v \in W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n)} \limsup_{i \to \infty} \mathbb{H}_{\epsilon_i}(v)
\]

\[
= \inf_{v \in W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n)} \mathbb{H}(v).
\]

Using Lemma 2.2, we have

\[
\lim_{i \to \infty} \int_\Omega \epsilon_i |\nabla u_{\epsilon_i}|^{m+1} dx = 0.
\]

This proves our claim. \( \square \)

We can now prove the first part of Theorem 1.1, namely,
**Proposition 2.1.** Let \( u \) be a weak limit of \( u_{\varepsilon i} \) in \( W^{2,2} \). Then \( u \) is a minimizer of \( F \). Moreover, \( u \) is a weakly biharmonic map.

**Proof.** By (2.1) and Lemma 2.3, we have

\[
F(u) \leq \liminf_{i \to \infty} H(u_{\varepsilon i}) = \inf_{v \in W^{2,2}_{u_0} \cap W^{1,m+1}(\Omega, S^n)} H(v) = \inf_{v \in C_0^\infty(\Omega, S^n)} H(v).
\]

By the definition of \( F \) again, \( u \) is a minimizer of \( F \) among all functions in \( W^{2,2}_{u_0}(\Omega, S^n) \).

It is straightforward to see that \( u_{\varepsilon} \) satisfies the Euler–Lagrange equation

\[
2\Delta^2 u_{\varepsilon} + 2(|\Delta u_{\varepsilon}|^2 + 2\nabla \cdot (\nabla u_{\varepsilon} \cdot \Delta u_{\varepsilon}) - \Delta |\nabla u_{\varepsilon}|^2) u_{\varepsilon} - \varepsilon(m + 1)[\nabla \cdot (|\nabla u_{\varepsilon}|^{m-1} \nabla u_{\varepsilon}) + |\nabla u_{\varepsilon}|^{m+1} u_{\varepsilon}] = 0
\]

in the sense of distributions. This equation can be rewritten into a ‘divergence’ form (see [25]) as follows:

\[
2\Delta (\nabla \cdot (\nabla u_{\varepsilon} \times u_{\varepsilon})) - 4\nabla \cdot (\Delta u_{\varepsilon} \times \nabla u_{\varepsilon}) - \varepsilon(m + 1)[\nabla \cdot (|\nabla u_{\varepsilon}|^{m-1} \nabla u_{\varepsilon} \times u_{\varepsilon})] = 0
\]

in the sense of distributions. Due to Lemma 2.3, we conclude that the weak limit \( u \) of \( u_{\varepsilon} \) in \( W^{2,2}_{u_0}(\Omega, S^n) \) satisfies

\[
\Delta (\nabla \cdot (\nabla u \times u)) - 2\nabla \cdot (\Delta u \times \nabla u) = 0.
\]

Hence, \( u \) is a weakly biharmonic map (see [25]). \( \square \)

The second part of Theorem 1.1 is to prove partial regularity of the limiting map \( u \) of a sequence of minimizers \( \{u_{\varepsilon i}\} \). It is well known that a monotonicity formula plays an indispensable role in the proof of partial regularity for stationary biharmonic maps. Since the minimizer \( u \) of \( F \) is not stationary, we cannot prove a monotonicity formula for \( u \) directly. Fortunately, each \( u_{\varepsilon} \) is a minimizer of \( H_{\varepsilon} \) in \( W^{2,2}_{u_0} \cap W^{1+m}(\Omega, S^n) \). Hence, we will derive a monotonicity formula for \( u_{\varepsilon} \) first and then let \( \varepsilon \) go to zero.

Angelsberg [1] gave a detailed derivation of a monotonicity formula for stationary biharmonic maps. Since the functional \( H_{\varepsilon} \) is a perturbation of the Hessian energy, most parts of the proof in [1] can be used here. For the convenience of the reader, we stick to the notation used in [1], except that we write subscripts of Greek letters to indicate partial derivatives instead of Latin letters. For example, \( u_{\varepsilon,\alpha\beta} \) means \( \frac{\partial^2}{\partial x_\alpha \partial x_\beta} u_{\varepsilon} \).

**Lemma 2.4.** Let \( u_{\varepsilon} \) be a minimizer of \( H_{\varepsilon} \) in \( W^{2,2} \cap W^{1+m}(B_{2r}, S^n) \). Then we have

\[
\int_{B_{2r}} - (\nabla \cdot \xi) (|\Delta u_{\varepsilon}|^2 + \varepsilon |\nabla u_{\varepsilon}|^{m+1}) + 4u_{\varepsilon,\gamma\gamma} u_{\varepsilon,\alpha\beta} \xi_\alpha^\beta + 2u_{\varepsilon,\gamma\gamma} u_{\varepsilon,\beta} \xi_\alpha^\beta + \varepsilon(m + 1)|\nabla u_{\varepsilon}|^{m-1} u_{\varepsilon,\alpha} u_{\varepsilon,\beta} \xi_\alpha^\beta = 0,
\]

for every test function \( \xi \in C_0^\infty(B_{2r}, \mathbb{R}^m) \).
The proof is just a direct computation (see [6] and [1]). Now we can state our monotonicity formula

**Theorem 2.1.** Let $u_\varepsilon$ be a minimizer of $H_\varepsilon$ on $B_{R_0}$ for some $R_0 > 0$. Then for all $\rho < r < R_0/2$, we have

\[
 r^{4-m} \int_{B_r} |\Delta u_\varepsilon|^2 + \varepsilon |\nabla u_\varepsilon|^{m+1} \, dx - \rho^{4-m} \int_{B_\rho} |\Delta u_\varepsilon|^2 + \varepsilon |\nabla u_\varepsilon|^{m+1} \, dx
\]

\[
= 4 \int_{B_r \setminus B_\rho} \left( \frac{(u_\varepsilon, \beta x^\alpha u_\varepsilon, \alpha \beta)^2}{|x|^{m-2}} + \frac{(m-2)(x^\alpha u_\varepsilon, \alpha)^2}{|x|^m} \right) \, dx
\]

\[
+ \varepsilon (m+1) \int_{B_r \setminus B_\rho} \frac{|\nabla u_\varepsilon|^{m-1}(x^\alpha u_\varepsilon, \alpha)^2}{|x|^{m-2}} \, dx
\]

\[
+ 2 \int_{\partial B_r \setminus \partial B_\rho} \left( -\frac{x^\alpha u_\varepsilon, \beta u_\varepsilon, \alpha \beta}{|x|^{m-3}} + 2 \frac{(x^\alpha u_\varepsilon, \alpha)^2}{|x|^{m-1}} - 2 \frac{|\nabla u_\varepsilon|^2}{|x|^{m-3}} \right) d\sigma
\]

\[
+ \varepsilon (3-m) \int_{\tau^{3-m}} \int_{B_{\tau}} |\nabla u_\varepsilon|^{m+1} \, dx \, d\tau.
\]

**Proof.** We follow the proof in [1]. For $\rho < \tau < r$, choose a test function $\xi(x) = \psi(t/\tau) x$, where $\psi = \psi' : \mathbb{R}^m \to [0, 1]$ is smooth with compact support on $[0, 1]$ and $\psi' \equiv 1$ on $[0, 1-t]$. Then by Lemma 2.4, we have

\[
0 = \int_{\mathbb{R}^m} \left( (4-m)|\Delta u_\varepsilon|^2 \psi - |\Delta u_\varepsilon|^2 \psi_x^\alpha x^\alpha + 4 u_\varepsilon, \alpha \alpha u_\varepsilon, \beta \gamma \psi_\beta x^\gamma \right.
\]

\[
+ 4 u_\varepsilon, \alpha \alpha u_\varepsilon, \beta \psi_\beta + 2 u_\varepsilon, \alpha \alpha u_\varepsilon, \beta \psi_x^\alpha x^\gamma 
\]

\[
+ \varepsilon \left( |\nabla u_\varepsilon|^{m+1} - |\nabla u_\varepsilon|^{m+1} \psi_x^\alpha x^\alpha + (m+1)|\nabla u_\varepsilon|^{m-1} u_\varepsilon, \alpha \psi_\alpha u_\varepsilon, \beta x^\beta \right) dx.
\]

Since $\psi_\alpha(t/\tau) = \frac{1}{\tau} \psi'(t/\tau) \frac{x^\alpha}{|x|}$, we have

\[
0 = (4-m) \int_{\mathbb{R}^m} |\Delta u_\varepsilon|^2 \psi \, dx - \frac{1}{\tau} \int_{\mathbb{R}^m} |\Delta u_\varepsilon|^2 \psi_x^\alpha x^\alpha \, dx + \frac{4}{\tau} \int_{\mathbb{R}^m} u_\varepsilon, \alpha \alpha u_\varepsilon, \beta \gamma \psi_\beta x^\gamma \, dx
\]

\[
+ \frac{2(m+1)}{\tau} \int_{\mathbb{R}^m} u_\varepsilon, \alpha \alpha u_\varepsilon, \beta \psi_x^\alpha x^\gamma \, dx + \frac{2}{\tau^2} \int_{\mathbb{R}^m} u_\varepsilon, \alpha \alpha u_\varepsilon, \beta \psi'' x^\beta \, dx
\]

\[
+ \varepsilon \left( \psi |\nabla u_\varepsilon|^{m+1} - |\nabla u_\varepsilon|^{m+1} \frac{1}{\tau} \psi_x^\alpha x^\alpha + (m+1)|\nabla u_\varepsilon|^{m-1} \frac{1}{\tau} \psi_x^\alpha u_\varepsilon, \beta x^\beta \right) dx. \quad (2.2)
\]
Set
\[ I^t(\tau) = \tau^{4-m} \int_{\mathbb{R}^m} (|\triangle u_\varepsilon|^2 + \varepsilon|\nabla u_\varepsilon|^{m+1}) \psi'\left(\frac{|x|}{\tau}\right) \, dx. \]

We have
\[
\begin{align*}
\tau^{m-3} \frac{d}{d\tau} I^t(\tau) &= (4-m) \int_{\mathbb{R}^m} (|\triangle u_\varepsilon|^2 + \varepsilon|\nabla u_\varepsilon|^{m+1}) \psi \, dx \\
&\quad - \frac{1}{\tau} \int_{\mathbb{R}^m} (|\triangle u_\varepsilon|^2 + \varepsilon|\nabla u_\varepsilon|^{m+1}) \psi' |x| \, dx \\
&\quad = (4-m) \int_{\mathbb{R}^m} |\triangle u_\varepsilon|^2 \psi \, dx - \frac{1}{\tau} \int_{\mathbb{R}^m} |\triangle u_\varepsilon|^2 \psi' |x| \, dx \\
&\quad + \varepsilon \left[ \int_{\mathbb{R}^m} |\nabla u_\varepsilon|^{m+1} \psi \, dx - \frac{1}{\tau} \int_{\mathbb{R}^m} |\nabla u_\varepsilon|^{m+1} \psi' |x| \, dx \right] \\
&\quad + \varepsilon (3-m) \int_{\mathbb{R}^m} |\nabla u_\varepsilon|^{m+1} \psi \, dx
\end{align*}
\]

Here we have used Eq. (2.2) in the last equality. Multiplying both sides by \( \tau^{3-m} \) and integrating over \( \tau \) from \( \rho \) to \( r \) yields
\[
\begin{align*}
I^t(r) - I^t(\rho) &= -4 \int_{\rho}^{r} \tau^{2-m} \int_{\mathbb{R}^m} u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta\gamma} \psi' \left( \frac{|x|}{\tau} \right) \frac{x^\beta x^\gamma}{|x|} \, dx \, d\tau \\
&\quad - 8 \int_{\rho}^{r} \tau^{2-m} \int_{\mathbb{R}^m} u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta} \psi' \left( \frac{|x|}{\tau} \right) \frac{x^\beta}{|x|} \, dx \, d\tau \\
&\quad + 2 \int_{\mathbb{R}^m} u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta} x^\beta \psi' \left( \frac{|x|}{r} \right) r^{3-m} \frac{1}{|x|} \, dx
\end{align*}
\]
\[ -2 \int_{\mathbb{R}^m} u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta\gamma} x^\beta \psi' \left( \frac{|x|}{\rho} \right) \rho^{3-m} \frac{1}{|x|} \, dx \]
\[ + \varepsilon \int_{\rho}^r \tau^{2-m} \int_{\mathbb{R}^m} - (m+1) |\nabla u_{\varepsilon}|^{m-1} \frac{\psi' (x^\alpha u_{\varepsilon,\alpha})^2}{|x|} \, dx \, d\tau \]
\[ + \varepsilon (3-m) \int_{\rho}^r \tau^{3-m} \int_{\mathbb{R}^m} |\nabla u_{\varepsilon}|^{m+1} \psi \, dx \, d\tau, \]

where the first four terms of the right-hand side of the above identity are obtained by the same calculation as in [1]. Letting \( t \) go to zero and applying Lemma 2 in the Appendix of [1], we obtain

\[ r^{4-m} \int_{B_r} |\Delta u_{\varepsilon}|^2 + \varepsilon |\nabla u_{\varepsilon}|^{m+1} \, dx - \rho^{4-m} \int_{B_\rho} |\Delta u_{\varepsilon}|^2 + \varepsilon |\nabla u_{\varepsilon}|^{m+1} \, dx \]
\[ = \int_{B_r \setminus B_\rho} \left( 4 \frac{u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta\gamma} x^\beta x^\gamma}{|x|^{m-2}} + 8 \frac{u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta\gamma} x^\beta}{|x|^{m-2}} \right) \, dx - 2 \int_{\partial B_r \setminus \partial B_\rho} \frac{u_{\varepsilon,\alpha\alpha} u_{\varepsilon,\beta\gamma} x^\beta}{|x|^{m-3}} \, d\sigma \]
\[ + \varepsilon (m+1) \int_{B_r \setminus B_\rho} \frac{|\nabla u_{\varepsilon}|^{m-1} (x^\alpha u_{\varepsilon,\alpha})^2}{|x|^{m-2}} \, dx \]
\[ + \varepsilon (3-m) \int_{\rho}^r \tau^{3-m} \int_{B_\tau} |\nabla u_{\varepsilon}|^{m+1} \, dx \, d\tau. \]

For the first line on the right-hand side of the above equation, further transformations are needed before reaching the final form appearing in the statement of the theorem as given in [1]. However, this does not concern us, since the last two terms above are in their final form.

**Remark 2.1.** If we compare Theorem 2.1 with the monotonicity formula of the biharmonic maps in [6] and [1], there are additional terms coming from the perturbed energy \( \varepsilon \int |\nabla u_{\varepsilon}|^{m+1} \, dx \). Using Lemma 2.3, we shall get rid of these addition terms by sending \( \varepsilon \) to zero.

Let \( u_{\varepsilon_i} \) be the sequence in the statement of Theorem 1.1. Due to the minimizing property of \( u_{\varepsilon_i} \),

\[ \mathbb{H}(u_{\varepsilon_i}) \leq \mathbb{H}_{\varepsilon_i} (u_{\varepsilon_i}) \leq C \]

for some constant \( C > 0 \) independent of \( i \). Set

\[ \Sigma = \bigcap_{R > 0} \left\{ x_0 \in \Omega \left| \begin{array}{l} B_R(x_0) \subset \Omega, \liminf_{i \to \infty} R^{4-m} \int_{B_R(x_0)} |\Delta u_{\varepsilon_i}|^2 \, dx \geq \varepsilon_0 \end{array} \right. \right\} \]
for a sufficiently small constant $\varepsilon_0$ to be fixed later. For the proof of $H^{m-4}(\Sigma) < +\infty$, we refer to the proof of Theorem 3.4 in [20]. For the relative closeness of $\Sigma$, an elementary proof will be given in the last section in the proof of Theorem 1.3 (see also [9] and [21]).

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** The first part is already proved. It suffices to prove the partial regularity. Let $x$ be a point in $\Omega \setminus \Sigma$. Without loss of generality, we assume that $x$ is the origin. By the definition of $\Sigma$, there exists some $R > 0$ such that $B_R \subset \Omega$ and (taking a subsequence if necessary)

$$\lim_{i \to \infty} R^{4-m} \int_{B_R} |\Delta u_{\varepsilon_i}|^2 \, dx < \varepsilon_0.$$ 

For simplicity, we will write $u_i$ for $u_{\varepsilon_i}$. It is easy to see that for each $y \in B_R/2$,

$$\lim_{i \to \infty} \frac{R}{2}^{4-m} \int_{B_{R/2}(y)} |\Delta u_i|^2 \, dx < C\varepsilon_0.$$

By scale invariance, we can assume that $R = 2$.

**We claim:** There is a constant $C > 0$ such that for almost every $y \in B_1$ and for any $r < 1/4$,

$$\left( \frac{r}{2} \right)^{4-m} \int_{B_{r/2}(y)} |\Delta u|^2 \, dx \leq C \sqrt{\varepsilon_0}. \quad (2.3)$$

Before we prove this claim, we show how Theorem 1.1 follows from this claim. Since $|u| = 1$, we have

$$-u \cdot \Delta u = |\nabla u|^2, \quad \text{which implies} \quad |\nabla u|^2 \leq |\Delta u|.$$

By (2.3) and using Hölder’s inequality, we have

$$\frac{1}{r^2} \int_{B_{r/2}(y)} |\nabla u|^2 \, dx \leq Cr^{m/2-2} \left( \int_{B_{r/2}(y)} |\nabla u|^4 \, dx \right)^{1/2} \leq Cr^{m-4} \varepsilon_0^{1/4},$$

which implies

$$\int_{B_{r/2}(y)} |\nabla^2 u|^2 + |\nabla u|^4 \, dx \leq C \int_{B_{r/2}(y)} |\Delta u|^2 + \frac{1}{r^2} |\nabla u|^2 \, dx \leq Cr^{m-4} \varepsilon_0^{1/4}.$$

By the arbitrariness of $r$ and $y$, we obtain

$$\|\nabla^2 u\|_{L^2,m-4(B_1)}^2 + \|\nabla u\|_{L^4,m-4(B_1)}^4 \leq C \varepsilon_0^{1/4} \quad (2.4)$$
for another constant $C > 0$, where $L^{p,s}(B_1)$ is the standard Morrey space with the norm

$$\|f\|_{L^{p,s}(B_1)}^p = \sup_{x_0 \in B_1, r > 0} \left( \frac{1}{r^s} \int_{B_r(x_0) \cap B_1} |f|^p \, dx \right) < +\infty.$$  

For $\varepsilon_0$ sufficiently small in (2.4), it follows from the same proof of Theorem 1.1 of [23] that $u$ is smooth in $B_{R/3}$.

Now let us prove (2.3) following an idea of Struwe in [23].

Since

$$\int_{B_1(y)} |\Delta u_i|^2 + |\nabla u_i|^4 \, dx < C\varepsilon_0 \quad \text{for } i \text{ large enough},$$

arguing as above, we obtain

$$\int_{B_{1/2}(y)} |\nabla^2 u_i|^2 + |\nabla u_i|^2 \, dx < C\sqrt{\varepsilon_0} \quad \text{for } i \text{ large enough.}$$

Then we can choose a ‘good’ radius $r_0 \in \left( \frac{1}{4}, \frac{1}{2} \right)$ such that

$$\int_{\partial B_{r_0}(y)} |\nabla^2 u_i|^2 + |\nabla u_i|^2 \, dx \leq C\sqrt{\varepsilon_0} \quad \text{for infinitely many } i \text{'s.}$$

We assume by taking a subsequence that this is true for all $i$.

For simplicity, we set $\sigma_i(y, r) = \sigma_{i,1}(y, r) + \sigma_{i,2}(y, r)$ with

$$\sigma_{i,1}(y, r) = r^{4-m} \int_{B_r(y)} |\Delta u_i|^2 + \varepsilon_i |\nabla u_i|^{m+1} \, dx$$

and

$$\sigma_{i,2}(y, r) = r^{3-m} \int_{\partial B_r(y)} \left( 2x^\alpha u_{i,\alpha} \nabla u_i + 4|\nabla u_i|^2 - 4r^{-2} |x^\alpha u_{i,\alpha}|^2 \right) d\sigma.$$  

Then we rewrite the monotonicity formula in Theorem 2.1 as follows:

$$\sigma_i(y, r) - \sigma_i(y, \rho) = \int_{B_r(y) \backslash B_{\rho}(y)} \left( \frac{|u_{i,\beta} + x^\alpha u_{i,\alpha\beta}|^2}{|x|^{m-2}} + (m-2) \frac{|x^\alpha u_{i,\alpha}|^2}{|x|^m} \right)$$

$$+ \varepsilon_i (m + 1) \int_{B_r(y) \backslash B_{\rho}(y)} \frac{|\nabla u_i|^{m-1} (x^\alpha u_{i,\alpha})^2}{|x|^{m-2}} \, dx$$

$$+ \varepsilon_i (3-m) \int_{\rho} r^{3-m} \int_{B_r(y)} |\nabla u_i|^{m+1} \, dx \, d\tau.$$  

(2.7)
Let $E_1$ be the intersection of the sets of Lebesgue points of $|\nabla u_i|^{m+1}$ for all $i$. Then the complement of $E_1$ is a set of zero Lebesgue measure. For simplicity, we also set

$$R(\varepsilon_i, \rho) := \varepsilon_i (3 - m) \int_{B_{\tau}(y)} \int_{B_\rho(y)} |\nabla u_i|^{m+1} \, dx \, d\tau.$$ 

Then, for each $y \in E_1$, the limit

$$R(\varepsilon_i, 0) := \lim_{\rho \to 0} R(\varepsilon_i, \rho)$$

exists.

For a fixed $i$ and any $k \in \mathbb{N}$, there is a ‘good’ radius $r_k$ with $0 < r_k < \frac{1}{k}$ such that

$$|\sigma_i(y, r_k)| \leq C r_k^{4-m} \int_{B_{r_k}(y)} |\Delta u_i|^2 + \varepsilon_i |\nabla u_i|^{m+1} \, dx$$

$$+ C r_k^{5-m} \int_{\partial B_{r_k}(y)} \left( |\nabla^2 u_i|^2 + r_k^{-2} |\nabla u_i|^2 \right) \, d\sigma$$

$$\leq C r_k^{4-m} \int_{B_{2r_k}(y)} \left( |\nabla^2 u_i|^2 + \varepsilon_i |\nabla u_i|^{m+1} + r_k^{-2} |\nabla u_i|^2 \right) \, dx.$$ 

Let $E_2$ be the intersection of the sets of Lebesgue points of $|\nabla^2 u_i|^2 + |\nabla u_i|^2$ for all $i$. The complement of $E_2$ is also of Lebesgue measure zero. If we assume $y \in E_1 \cap E_2$, we have

$$\lim_{k \to \infty} |\sigma_i(y, r_k)| = 0. \quad (2.8)$$

By (2.6), we know

$$\sigma_i(y, r_0) \leq C \sqrt{\varepsilon_0}.$$ 

In (2.7), we choose $r = r_0$ and $\rho = r_k$. Then, as $k \to \infty$, we have

$$\int_{B_0(y)} \left( \frac{|u_{i,\beta} + x^a u_{i,a\beta}|^2}{|x|^{m-2}} + (m - 2) \frac{|x^a u_{i,a}|^2}{|x|^m} \right) \leq \sigma_i(y, r_0) - \lim_{k \to \infty} \sigma_i(y, r_k) - R(\varepsilon_i, 0) \leq C \sqrt{\varepsilon_0} - R(\varepsilon_i, 0). \quad (2.9)$$

Since we know

$$\varepsilon_i \int_{B_1} |\nabla u_i|^{m+1} \, dx \to 0,$$
we may assume that there is a subsequence of \( i \) (still denoted by \( i \)) such that
\[
T(x) = (m - 3) \sum_{i=1}^{\infty} 2^i \epsilon_i |\nabla u_{\epsilon_i}|^{m+1} \in L^1(B_1).
\]

Hence,
\[
|R(\epsilon_i, 0)| = (m - 3) \epsilon_i \int_0^{r_0} \int_{B_r(y)} |\nabla u_i|^{m+1} dx \, d\tau
\leq \frac{1}{2^i} \int_0^1 \tau^{3-m} \int_{B_r(y)} T(x) \, dx \, d\tau.
\]

Let \( E_3 \) be the set of Lebesgue points of \( T \). If \( y \in E_1 \cap E_2 \cap E_3 \), then
\[
\lim_{i \to \infty} R(\epsilon_i, 0) = 0. \tag{2.10}
\]

Next, we estimate the term \( \sigma_{i,2} \). By (2.9), we have
\[
\inf_{r/2 < \rho < r} \rho^{3-m} \int_{\partial B_{\rho}(y)} \left( |u_{i,\beta} + x^\alpha u_{i,\alpha\beta}|^2 + 4 \rho^{-2} |x^\alpha u_{i,\alpha}|^2 \right) d\sigma
\leq C \int_{B_r \setminus B_{r/2}(y)} \left( \frac{|u_{i,\beta} + x^\alpha u_{i,\alpha\beta}|^2}{|x|^{m-2}} + (m - 2) \frac{|x^\alpha u_{i,\alpha}|^2}{|x|^m} \right) dx
\leq C \sqrt{\epsilon_0} - R(\epsilon_i, 0).
\]

Estimating
\[
2x^\alpha u_{i,\alpha\beta} u_{i,\beta} + 4 |\nabla u_i|^2 = 2(u_{i,\beta} + x^\alpha u_{i,\alpha\beta}) u_{i,\beta} + 2 |\nabla u_i|^2 \geq -|u_{i,\beta} + x^\alpha u_{i,\alpha\beta}|^2,
\]
we bound
\[
\sigma_{i,2}(y, \rho) \geq -\rho^{3-m} \int_{\partial B_{\rho}} \left( |u_{i,\beta} + x^\alpha u_{i,\alpha\beta}|^2 + 4 \rho^{-2} |x^\alpha u_{i,\alpha}|^2 \right) d\sigma.
\]

Therefore,
\[
\sup_{r/2 < \rho < r} \sigma_{i,2}(y, \rho) \geq -C \sqrt{\epsilon_0} + R(\epsilon_i, 0).
\]

Now from the monotonicity formula, for a suitable radius \( \rho \in (r/2, r) \),
\[
\sigma_{i,1}(y, \rho) \leq \sigma_{i}(y, \rho) - \sigma_{i,2}(y, \rho)
\leq \sigma_{i}(y, r_0) - R(\epsilon_i, 0) + C \sqrt{\epsilon_0} - R(\epsilon_i, 0).
\]
Noticing (2.10) and using the fact that \( \lim_{i \to \infty} \mathcal{R}(\varepsilon_i, \rho) = 0 \), we have

\[
\rho^{4-m} \int_{B_\rho(y)} |\nabla u|^2 \, dx \leq C \sqrt{\varepsilon_0}.
\]

Thus, we have proven our claim for \( y \in E_1 \cap E_2 \cap E_3 \). Since the Lebesgue measure of the complement of \( E_1 \cap E_2 \cap E_3 \) is zero, the claim (2.3) follows. \( \square \)

### 3. Further results on the monotonicity formula

Let \( u_{\varepsilon_i} \) be a sequence in Theorem 1.1 with a uniformly bounded total energy \( \mathbb{H}(u_{\varepsilon_i}) \leq C \). The main purpose of this section is to show that for \( \mathcal{H}^{m-4} \)-a.e. \( y \in \Omega \), the limit \( \lim_{r \to 0} r^{4-m} \liminf_{\varepsilon_i \to 0} \int_{B_r(y)} |\nabla u_{\varepsilon_i}|^2 \, dx \) exists. We will use this result to show that the defect measure is rectifiable. The arguments in this section work both for a sequence of stationary biharmonic maps and for the sequence \( u_{\varepsilon_i} \) in Theorem 1.1. Hence, we present only the proofs for the latter case.

For simplicity, we write \( u_i \) for \( u_{\varepsilon_i} \) and recall some notations used in Section 3:

\[
\sigma_{i,1}(y,r) = r^{4-m} \int_{B_r(y)} |\nabla u_i|^2 + \varepsilon_i |\nabla u_i|^{m+1} \, dx
\]

and

\[
\sigma_{i,2}(y,r) = r^{3-m} \int_{\partial B_r(y)} \left( 2x^\alpha u_{i,\alpha} u_{i,\beta} + 4|\nabla u_i|^2 - 4r^{-2} |x^\alpha u_{i,\alpha}|^2 \right) \, d\sigma.
\]

**Lemma 3.1.** Suppose that \( K \) is a compact set in \( \Omega \) and \( d \) is the distance from \( K \) to \( \partial \Omega \). For every \( y \in K \) and every \( r \in (0, d/10) \), we have

\[
\limsup_{i \to \infty} \sigma_{i,1}(y,r) = \limsup_{i \to \infty} r^{4-m} \int_{B_r(y)} |\nabla u_i|^2 \, dx \leq C(d)
\]

for some constant \( C(d) \) depending on \( d \).

**Proof.** Without loss of generality, we assume that \( y \) is the origin. Since the total energy \( \mathbb{H}(u_i) \) is bounded, we have

\[
\sup_i \sigma_{i,1} \left( \frac{3d}{4} \right) = \sup_i \left( \frac{3d}{4} \right)^{4-m} \int_{B_{3d/4}} |\nabla u_i|^2 + \varepsilon_i |\nabla u_i|^{m+1} \, dx \leq C(d).
\]

By Lemma A.1 of [20], for each \( i \), there is a good radius \( \tilde{R}_i \in [d/8, d/4] \) such that
\[ \tilde{R}_i^{5-m} \int_{\partial B_{\tilde{R}_i}} |\nabla^2 u_i|^2 d\sigma \leq Cd^{4-m} \int_{B_{d/4}} |\nabla^2 u_i|^2 dx \]

\[ \leq Cd^{4-m} \sup_i \int_{B_{d/2}} |\Delta u_i|^2 + \frac{1}{d^2} |\nabla u_i|^2 dx \leq C(d) \]

for another new constant \( C(d) \).

Then for each \( i \) and \( \tilde{R}_i \) given above,

\[ \sigma_i(\tilde{R}_i) = \sigma_{i,1}(\tilde{R}_i) + \sigma_{i,2}(\tilde{R}_i) \leq C(d) \] (3.1)

for a constant \( C(d) > 0 \).

For each \( r \) with \( 0 < r < d/10 \) and each \( i \), there exists \( \rho_{r,i} \in [r, 3r/2] \) such that

\[ |\sigma_{i,2}(\rho_{r,i})| \leq C \rho_{r,i}^{3-m} \int_{\partial B_{\rho_{r,i}}} \rho_{r,i} |\nabla u_i| |\nabla^2 u_i| + |\nabla u_i|^2 d\sigma \]

\[ \leq Cr^{2-m} \int_{B_{3r/2}} r |\nabla u_i| |\nabla^2 u_i| + |\nabla u_i|^2 dx \]

\[ \leq \eta r^{4-m} \int_{B_{3r/2}} |\nabla^2 u_i|^2 dx + C(\eta)r^{2-m} \int_{B_{3r/2}} |\nabla u_i|^2 dx \] (3.2)

for a small constant \( \eta \) which will be fixed later.

For each \( r > 0 \), set

\[ h(r) = \limsup_{i \to \infty} \sigma_{i,1}(r) + \limsup_{i \to \infty} r^{2-m} \int_{B_r} |\nabla u_i|^2 dx. \]

By an interpolation inequality of Nirenberg [18], we have

\[ \limsup_{i \to \infty} r^{2-m} \int_{B_{3r/2}} |\nabla u_i|^2 dx \leq C \limsup_{i \to \infty} \left( r^{4-m} \int_{B_{3r/2}} |\nabla u_i|^4 dx \right)^{1/2} \]

\[ \leq C \limsup_{i \to \infty} \|u_i\|_{L^\infty} \left( \limsup_{i \to \infty} r^{4-m} \int_{B_{3r/2}} |\nabla^2 u_i|^2 dx \right)^{1/2} \]

\[ + C \limsup_{i \to \infty} \|u_i\|^2_{L^\infty} \]

\[ \leq C (h(2r))^{1/2} + C. \] (3.3)

Using the monotonicity formula (2.7) for each \( u_i \), we obtain
\[
\limsup_{i \to \infty} \sigma_{i,1}(\rho_{r,i}) + \sigma_{i,2}(\rho_{r,i}) \leq \limsup_{i \to \infty} \sigma_{i,1}(\tilde{R}_i) + \sigma_{i,2}(\tilde{R}_i) = \limsup_{i \to \infty} \sigma_i(\tilde{R}_i) \leq C(d) \tag{3.4}
\]

for the above \(\rho_{r,i} \in [r, 3r/2]\) and \(\tilde{R}_i \in [d/8, d/4]\).

Hence, using (3.2)–(3.4), we obtain

\[
h(r) \leq C \limsup_{i \to \infty} \sigma_{i,1}(\rho_{r,i}) + \limsup_{i \to \infty} r^{2-m} \int_{B_r} |\nabla u_i|^2 \, dx
\]
\[
\leq C \limsup_{i \to \infty} \sigma_i(\tilde{R}_i) + C \limsup_{i \to \infty} |\sigma_{i,2}(\rho_{r,i})| + \limsup_{i \to \infty} r^{2-m} \int_{B_r} |\nabla u_i|^2 \, dx
\]
\[
\leq C(d) + C \eta \limsup_{i \to \infty} r^{4-m} \int_{B_{3r/2}} |\nabla^2 u_i|^2 \, dx + C(\eta) \limsup_{i \to \infty} r^{2-m} \int_{B_{3r/2}} |\nabla u_i|^2 \, dx
\]
\[
\leq C(d) + C \eta h(2r) + \limsup_{i \to \infty} C(\eta) r^{2-m} \int_{B_{3r/2}} |\nabla u_i|^2 \, dx.
\]

Choosing \(\eta\) sufficiently small and by (3.3), we have

\[
h(r) \leq \frac{1}{2} h(2r) + C h^{1/2}(2r) + C(d) \leq \frac{3}{4} h(2r) + C(d)
\]

for any \(r \leq \frac{d}{16}\). An iteration argument yields

\[
h(r) \leq C(d).
\]

This proves the claim. \(\Box\)

Set

\[
E = \left\{ x_0 \in \Omega \mid \limsup_{r \to 0} r^{4-m} \int_{B_r(x_0)} |\nabla u|^4 \, dx > 0 \right\}. \tag{3.5}
\]

By Corollary 3.2.3 in [27], \(\mathcal{H}^{m-4}(E) = 0\).

For any \(y \in \Omega\) and any \(r > 0\), we denote

\[
\sigma(y, r) := \liminf_{i \to \infty} \sigma_i(y, r) = \liminf_{i \to \infty} (\sigma_{i,1}(y, r) + \sigma_{i,2}(y, r))
\]

and

\[
\sigma_1(y, r) := \liminf_{i \to \infty} \sigma_{i,1}(y, r).
\]

**Lemma 3.2.** For \(y \notin E\), the limit \(\lim_{r \to 0} \sigma(y, r)\) exists and is nonnegative.
Proof. By (2.7) and Lemma 2.3, \( \sigma(y, r) \) is non-decreasing in \( r \), so it suffices to show that for some sequence of \( r_k \) going to zero, \( \lim_{k \to \infty} \sigma(y, 2r_k) \) exists and is nonnegative. For any sequence \( r_k \to 0 \) and each \( u_i \), there is a 'good' radius \( \rho_{k,i} \in [r_k, 2r_k] \) such that as in (3.2)

\[
\limsup_{i \to \infty} \left| \sigma_{i, 2}(y, \rho_{k,i}) \right| \\
\leq \limsup_{i \to \infty} C \rho_{k,i}^{3-m} \int_{\partial B_{\rho_{k,i}}(y)} \rho_{k,i} |\nabla u_i| |\nabla^2 u_i| + |\nabla u_i|^2 d\sigma \\
\leq \limsup_{i \to \infty} C r_k^{2-m} \int_{B_{2r_k}(y)} r_k |\nabla u_i| |\nabla^2 u_i| + |\nabla u_i|^2 dx \\
\leq C \left( \limsup_{i \to \infty} r_k^{4-m} \int_{B_{2r_k}(y)} |\nabla^2 u_i|^2 dx \right)^{1/2} \left( \limsup_{i \to \infty} r_k^{2-m} \int_{B_{2r_k}(y)} |\nabla u_i|^2 dx \right)^{1/2} \\
+ C \limsup_{i \to \infty} r_k^{2-m} \int_{B_{2r_k}(y)} |\nabla u_i|^2 dx.
\]

By Lemma 3.1, the term

\[
\limsup_{i \to \infty} r_k^{4-m} \int_{B_{2r_k}(y)} |\nabla^2 u_i|^2 dx
\]

is bounded by a constant. By our choice of \( y \) and the fact that \( u_i \) converges to \( u \) strongly in \( W^{1,2} \), we have

\[
\lim_{k \to \infty} \limsup_{i \to \infty} r_k^{2-m} \int_{B_{2r_k}(y)} |\nabla u_i|^2 dx = \lim_{k \to \infty} r_k^{2-m} \int_{B_{2r_k}(y)} |\nabla u|^2 dx = 0. \quad (3.6)
\]

Combining these yields

\[
\lim_{k \to \infty} \limsup_{i \to \infty} \left| \sigma_{i, 2}(y, \rho_{k,i}) \right| = 0
\]

for above \( \rho_{k,i} \).

Due to the monotonicity of \( \sigma_i(y, r) \) in (2.7) for each \( i \), we have

\[
\lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i(y, 2r_k) \geq \lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i(y, \rho_{k,i}) = \lim_{k \to \infty} \liminf_{i \to \infty} \sigma_{i,1}(y, \rho_{k,i}) \geq 0. \quad \square
\]

The following theorem is the main result of this section.
Theorem 3.1. For all $y \notin E$, the limit
\[
\lim_{r \to 0} \sigma_1(y, r) = \lim_{r \to 0} \liminf_{i \to \infty} r^{4-m} \int_{B_r(y)} |\nabla u_i|^2 \, dx
\]
exists and
\[
\lim_{r \to 0} \sigma_1(y, r) = \lim_{r \to 0} \sigma(y, r).
\]

Proof. It suffices to show that for any sequence $r_k$ going to zero,
\[
\lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i,1(y, r_k) = \lim_{r \to 0} \liminf_{i \to \infty} \sigma_i(y, r).
\]
Let $\theta_k$ be a sequence of positive numbers in $(0, 1/2)$ to be determined later. For each $u_i$, there exists a $\rho_{k,i} \in [r_k, r_k(1 + \theta_k)]$ such that

\[
\limsup_{i \to \infty} \left| \sigma_i,2(y, \rho_{k,i}) \right| \leq \limsup_{i \to \infty} C r_{k,i}^{3-m} \int_{B_{\rho_{k,i}}(y)} \rho_{k,i} |\nabla u_i| |\nabla^2 u_i| + |\nabla u_i|^2 \, d\sigma
\]
\[
\leq \limsup_{i \to \infty} C \theta_k^{-1} r_{k}^{2-m} \int_{B_{1/2}^2(y)} r_k |\nabla u_i| |\nabla^2 u_i| + |\nabla u_i|^2 \, dx
\]
\[
\leq C \left( \limsup_{i \to \infty} r_{k}^{4-m} \int_{B_{2r_k}(y)} |\nabla^2 u_i|^2 \, dx \right)^{1/2} \left( \theta_k^{-2} \limsup_{i \to \infty} r_{k}^{2-m} \int_{B_{2r_k}(y)} |\nabla u_i|^2 \, dx \right)^{1/2}
\]
\[
+ C \theta_k^{-1} \limsup_{i \to \infty} r_{k}^{2-m} \int_{B_{2r_k}(y)} |\nabla u_i|^2 \, dx.
\]

Since $y \notin E$, which implies that (3.6) is true, we can choose $\theta_k$ going to zero so that
\[
\lim_{k \to \infty} \limsup_{i \to \infty} |\sigma_i,2(y, \rho_{k,i})| = 0.
\]

By Lemma 3.2 and using (2.7) for each $u_i$, we see
\[
\lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i,1(y, \rho_{k,i}) \leq \lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i(y, (1 + \theta_k)r_k) = \lim_{r \to 0} \liminf_{i \to \infty} \sigma_i(y, r).
\]

For the same reason, we can find a sequence $\rho_{k,i}^{'} \in [r_k(1 - \theta_k), r_k]$ such that
\[
\lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i,1(y, \rho_{k,i}^{'}) \geq \lim_{k \to \infty} \liminf_{i \to \infty} \sigma_i(y, (1 - \theta_k)r_k) = \lim_{r \to 0} \liminf_{i \to \infty} \sigma_i(y, r).
\]

However,
Taking the limit of \( i \) going to infinity and then \( k \) going to infinity, we obtain

\[
\lim_{k \to \infty} \liminf_{i \to \infty} \sigma_{i,1}(y, \rho^*_k, i) \leq \lim_{k \to \infty} \liminf_{i \to \infty} \sigma_{i,1}(y, r_k) \leq \lim_{k \to \infty} \limsup_{i \to \infty} \sigma_{i,1}(y, r_k),
\]

This proves Theorem 3.1. \( \Box \)

Let us assume that \( u_i \) converges to a map \( u \) weakly in \( W^{2,2}(\Omega) \). Then, passing to a subsequence of \( u_i \) if necessary, we can assume \( \mu = \lim_{\varepsilon_i \to 0} |\Delta u_{\varepsilon_i}|^2 dx \) in the sense of Radon measures. By the proof of Theorem 3.1 and Lemma 2.3, we know that for \( \mathcal{H}^{m-4} \)-a.e. \( x \in \Omega \), we have

\[
\lim_{r \to 0} \liminf_{i \to \infty} \sigma_i(x, r) = \lim_{r \to 0} \liminf_{i \to \infty} \sigma_{i,1}(x, r) = \lim_{r \to 0} \limsup_{i \to \infty} \sigma_i(x, r),
\]

so the density \( \Theta(x) = \lim_{r \to 0} r^{4-n} \mu(B_r(x)) \) and \( \lim_{r \to 0} \lim_{i \to \infty} \sigma_i(x, r) \) exist.

One can see from the above proofs that the same argument works for a sequence of stationary biharmonic maps. Let \( \tilde{u}_i : \Omega \to N \) be a sequence of stationary biharmonic maps from \( \Omega \subset \mathbb{R}^m \) to compact manifold \( N \). Assume that the \( \mathbb{H}(\tilde{u}_i) \) are bounded and \( \tilde{u}_i \) converges weakly to \( \tilde{u} \). Set

\[
\tilde{\mu} = \lim_{i \to \infty} |\Delta \tilde{u}_i|^2 dx = |\Delta \tilde{u}|^2 dx + \tilde{\nu},
\]

where \( \tilde{\nu} \) is the defect measure. Now, we prove Theorem 1.2.

**Proof of Theorem 1.2.** According to Theorem 3.4 in [20], \( \tilde{\nu} \) is supported in \( \tilde{\Sigma} \) defined as the set of points \( a \in \Omega \) with

\[
\liminf_{\rho \to 0} \left( \rho^{4-n} \int_{B_{\rho}(a)} (|\Delta \tilde{u}|^2 + \rho^{-2} |\nabla \tilde{u}|^2) dx + \rho^{4-m} \tilde{\nu}(B_{\rho}(a)) \right) \geq \tilde{\varepsilon}_0,
\]

where \( \tilde{\varepsilon}_0 \) is given in Corollary 2.7 of the same paper.

The same proof as Theorem 3.1 implies that

\[
\tilde{\Theta}(x) := \lim_{r \to 0} r^{4-m} \tilde{\mu}(B_r(x)) = \lim_{r \to 0} r^{4-m} \tilde{\nu}(B_r(x))
\]

exists for \( \mathcal{H}^{m-4} \)-a.e. \( x \in \Omega \) and moreover, for \( \mathcal{H}^{m-4} \)-a.e. \( x \in \tilde{\Sigma} \)

\[
0 < \tilde{\varepsilon}_0 \leq \tilde{\Theta}(x) \leq C.
\]
Since \( \tilde{\nu} \) is absolutely continuous with respect to \( \mathcal{H}^{m-4} \cdot \tilde{\Sigma} \) [20] and \( \tilde{\nu} \equiv 0 \) outside \( \tilde{\Sigma} \), \( \tilde{\Theta}(x) \) is positive for \( \tilde{\nu} \)-a.e. \( x \in \tilde{\Omega} \). Hence, by Preiss’s result [19], \( \tilde{\nu} \) is \((m-4)\)-rectifiable, which implies that \( \tilde{\Sigma} \) is \((m-4)\)-rectifiable. \( \square \)

4. The rectifiability of the defect measure

In this section, we will present a proof of Theorem 1.3 with \( m = 5 \). Our proof is based on an idea of Lin in [17] (see also [13]). However, we are not able to prove this for dimension \( m \) greater than 5.

Let \( u_{\varepsilon_i} \) be a minimizer of \( \mathbb{H}_{\varepsilon_i} \) in \( W^{2,2}_{u_0} \cap W^{1,6}(\Omega, S^n) \). For simplicity, we write \( u_i \) for \( u_{\varepsilon_i} \). By taking a subsequence (still denoted by \( u_i \)), we may assume

\[
 u_i \rightharpoonup u \quad \text{in} \quad W^{2,2}(\Omega, S^n)
\]

and

\[
 \mu_i = |\triangle u_i|^2 \, dx \rightharpoonup \mu = |\triangle u|^2 \, dx + \nu
\]

in the sense of Radon measures. By Fatou’s lemma, \( \nu \geq 0 \).

For the proof, we need the following lemmas.

**Lemma 4.1.** Assume that \( \rho \) is a fixed positive constant and \( u \) is a smooth map from \( \overline{B}_\rho \) to \( S^n \) with

\[
 \rho^4 \| \nabla^4 u \|_{C^0(B_\rho)} + \rho^2 \| \nabla^2 u \|_{C^0(B_\rho)} \leq C,
\]

where the constant \( C \) is independent of \( \rho \). Then there exists a positive constant \( \eta_0 < 1 \), depending only on the bound \( C \), but not on \( \rho \), such that for any \( 0 < \eta \leq \eta_0 \) and for the solution \( v \) of the boundary value problem

\[
 \begin{cases}
 \triangle^2 v = 0 & \text{in} \quad B_\rho \setminus B_\rho(1-\eta); \\
 v = u & \text{on} \quad \partial B_\rho \cup \partial B_\rho(1-\eta); \\
 \frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} & \text{on} \quad \partial B_\rho \cup \partial B_\rho(1-\eta),
\end{cases}
\]

we have

\[
 \frac{1}{2} \leq |v| \leq \frac{3}{2}
\]

on \( B_\rho \setminus B_\rho(1-\eta) \).

**Proof.** By scaling, we may assume that \( \rho = 1 \). For simplicity, we set \( A_\eta := B_1 \setminus B_{(1-\eta)} \). For a fixed \( \eta > 0 \), the solution \( v \) to (4.1) is denoted by \( v_\eta \). Since \( v_\eta \) is a biharmonic function, we have

\[
 \int_{A_\eta} |\triangle v_\eta|^2 \, dx \leq \int_{A_\eta} |\triangle u|^2 \, dx \leq C \eta.
\]
Setting $w_\eta = v_\eta - u$, we have

$$\int_{A_\eta} |\Delta w_\eta|^2 \, dx \leq C_\eta$$

(4.2)

and

$$\begin{cases}
-\Delta^2 w_\eta = \Delta^2 u & \text{in } A_\eta, \\
w_\eta = 0 & \text{on } \partial A_\eta, \\
\frac{\partial w_\eta}{\partial n} = 0 & \text{on } \partial A_\eta.
\end{cases}$$

Since $|u| = 1$, it suffices for Lemma 4.1 to prove that

$$\lambda_\eta = \max_{A_\eta} |w_\eta| \to 0$$

(4.3)

as $\eta$ goes to 0.

Next, we prove (4.3) by contradiction. If (4.3) is not true, there exists a positive number $\tilde{\delta} > 0$, a sequence of $\eta_i \to 0$, $u_i : B_1 \to S^n$ satisfying $\|\nabla^4 u_i\|_{C^0(B_1)} \leq C$, and a sequence of points $p_i \in A_{\eta_i}$ such that for $w_{\eta_i}$ given by

$$\begin{cases}
\Delta^2 w_{\eta_i} = \Delta^2 u_i & \text{in } A_{\eta_i}, \\
w_{\eta_i} = 0 & \text{on } \partial A_{\eta_i}, \\
\frac{\partial w_{\eta_i}}{\partial n} = 0 & \text{on } \partial A_{\eta_i},
\end{cases}$$

we have

$$|w_{\eta_i}(p_i)| = \lambda_{\eta_i} > \tilde{\delta}. \quad (4.4)$$

By a rotation if necessary, we may assume that $p_i = (0, 0, 0, 0, p_5^i)$ with $p_5^i < 0$. Define

$$\tilde{w}_i(\tilde{x}) = \frac{1}{\lambda_{\eta_i}} w_{\eta_i}(\eta_i \tilde{x} + p_i).$$

Let $\tilde{A}_i$ be the corresponding set defined by

$$\tilde{A}_i = \{ \tilde{x} \in \mathbb{R}^5 : \eta_i \tilde{x} + p_i \in A_{\eta_i} \}.$$  

We write $\tilde{\Delta}$ for the new Laplace operator in $\tilde{x}$:

$$\begin{cases}
\tilde{\Delta}^2 \tilde{w}_i = \frac{1}{\lambda_{\eta_i}} \eta_i^4 (\Delta^2 u_i)(\eta_i \tilde{x} + p_i) & \text{in } \tilde{A}_i, \\
\tilde{w}_i = 0 & \text{on } \partial \tilde{A}_i, \\
\frac{\partial \tilde{w}_i}{\partial n} = 0 & \text{on } \partial \tilde{A}_i.
\end{cases}$$
Moreover, by (4.2) and (4.4), we have
\[
\int_{\tilde{A}_i} |\tilde{\Delta} \tilde{w}_i|^2 d\tilde{x} \leq C.
\]
Using zero Dirichlet and Neumann boundary conditions for \( \tilde{w}_i \), we have
\[
\int_{\tilde{A}_i} |\tilde{\nabla}^2 \tilde{w}_i|^2 + |\tilde{\nabla} \tilde{w}_i|^4 d\tilde{x} \leq C,
\]
where we note
\[
\int_{\tilde{A}_i} |\tilde{\nabla} \tilde{w}_i|^4 d\tilde{x} = -\int_{\tilde{A}_i} \{\tilde{w}_i, \tilde{\nabla} \cdot (|\tilde{\nabla} \tilde{w}_i|^2 \tilde{\nabla} \tilde{w}_i)\} d\tilde{x}
\leq C \left( \int_{\tilde{A}_i} |\tilde{\nabla}^2 \tilde{w}_i|^2 d\tilde{x} \right)^{1/2} \left( \int_{\tilde{A}_i} |\tilde{\nabla} \tilde{w}_i|^4 d\tilde{x} \right)^{1/2}.
\]
Consider two hypersurfaces \( H_1 \) and \( H_2 \) defined by
\[
H_1 := \{ \tilde{x} \in \mathbb{R}^5 \mid \tilde{x}_5 = 0 \}
\]
and
\[
H_2 := \{ \tilde{x} \in \mathbb{R}^5 \mid \tilde{x}_5 = 1 \}.
\]
For each large positive \( l \), set
\[
Q_l = \left\{ \tilde{x} \in \mathbb{R}^5 \mid 0 \leq \tilde{x}_5 \leq 1, \sum_{i=1}^4 \tilde{x}_i^2 \leq l^2 \right\}.
\]
We also denote the unbounded domain between \( H_1 \) and \( H_2 \) by \( Q_\infty \). There is a sequence of diffeomorphisms
\[
\Phi_i : \mathbb{R}^5 \to \mathbb{R}^5
\]
such that when \( i \) is sufficiently large compared to \( l \),
\begin{enumerate}
\item it maps \( Q_l \) to a part of the annulus \( \tilde{A}_i \) containing the origin in the middle;
\item \( \| \Phi_i - T_a \|_{C^4(Q_l)} \to 0 \)
\end{enumerate}
as \( i \to \infty \), where \( T_a \) is a translation sending \( \tilde{x} \) to \( \tilde{x} + (0, 0, 0, 0, a) \) with \( a = \lim_{i \to \infty} \frac{-1 - \rho^5_i}{\eta_i} \).
Fix \( l \) and for \( i \) large, set
\[
\tilde{w}_i = \tilde{w}_i \circ \Phi_i : Q_l \to \mathbb{R}^5.
\]
Then \( \tilde{w}_i \) satisfies
\[
\begin{cases}
(\tilde{\Delta}^2 \tilde{w}_i \circ \Phi_i^{-1}) \circ \Phi_i = \frac{1}{\kappa_{n_i}} \eta_i^4 (\Delta^2 u_i) \circ \Phi_i & \text{in } Q_l, \\
\tilde{w}_i = 0 & \text{on } Q_l \cap (H_1 \cup H_2), \\
\frac{\partial}{\partial x_5} \tilde{w}_i = ((\Phi_i)_* \frac{\partial}{\partial x_5}) \tilde{w}_i & \text{on } Q_l \cap (H_1 \cup H_2).
\end{cases}
\]

Letting \( i \to \infty \) and then letting \( l \to \infty \), we obtain a biharmonic function \( \tilde{w}_i \) as a limit of \( \tilde{w}_i \) such that
\[
\begin{cases}
\Delta^2 \tilde{w} = 0 & \text{in } Q_\infty, \\
\tilde{w} = 0 & \text{on } H_1 \cup H_2, \\
\frac{\partial}{\partial x_5} \tilde{w} = 0 & \text{on } H_1 \cup H_2.
\end{cases}
\]

From the construction, we have
\[
\int_{Q_\infty} |\nabla^2 \tilde{w}|^2 + |\nabla \tilde{w}|^4 \, dx \leq C \quad (4.5)
\]
and \( \tilde{w} \) is bounded but non-zero, since \( \tilde{w}_i \) is a limit of \( \tilde{w}_i \circ \Phi_i \) and by (4.4)
\[
\max_{\tilde{A}_i} \tilde{w}_i = \tilde{w}_i(0) = 1. \quad (4.6)
\]

Let \( \phi(t) \) be a cut-off function in \( \mathbb{R} \) with \( \phi \equiv 1 \) on \([-R, R]\) and \( \phi \equiv 0 \) outside \([-2R, 2R]\) satisfying \( |\nabla \phi| \leq C/R \) and \( |\nabla^2 \phi| \leq C/R^2 \). Note that
\[
\nabla (w \phi^2) = \phi^2 \nabla w + 2w \phi \nabla \phi,
\]
\[
\Delta (w \phi^2) = \phi^2 \Delta w + 4 \nabla w \cdot \phi \nabla \phi + 2w \phi (\Delta \phi + |\nabla \phi|^2).
\]

Multiplying the biharmonic equations by \( w \phi(t) \) with \( t = \sqrt{x_1^2 + \cdots + x_4^2} \) yields
\[
\int_{Q_\infty} |\Delta w|^2 \phi^2 \, dx = - \int_{Q_\infty} \left[ \Delta w, 4 \nabla w \cdot \phi \nabla \phi + 2w \phi (\Delta \phi + |\nabla \phi|^2) \right] \, dx
\]
\[
\leq C \left( \int_{(B^4_{2R} \setminus B^4_R) \times [0, 1]} |\Delta w|^2 \, dx \right)^{1/2} \left( \int_{(B^4_{2R} \setminus B^4_R) \times [0, 1]} |\nabla w|^4 + |\Delta \phi|^2 + |\nabla \phi|^4 \right)^{1/2}
\]
\[
\leq C \left( \int_{(B^4_{2R} \setminus B^4_R) \times [0, 1]} |\Delta w|^2 \, dx \right)^{1/2} \to 0 \quad \text{as } R \to \infty,
\]
where \( B^4_R \) is the ball in \( \mathbb{R}^4 \) with radius \( R \). This implies that \( w \) is harmonic in \( Q_\infty \). Using zero boundary conditions, the \( w \) can be extended in \( \mathbb{R}^5 \) and bounded, so it is a constant, which is a contradiction to (4.6). □

The following lemma is an elliptic estimate involving the Sobolev space of fractional order. For the definition and properties of such functional spaces, we refer to [15]. We denote by \( \| \cdot \|_{W^{s,2}} \) the \( W^{s,2} \) Sobolev norm obtained by complex interpolation if \( s \) is not a positive integer.

**Lemma 4.2.** Let \( u \) be a biharmonic function on \( \Omega \). Assume \( u \) satisfies the boundary conditions

\[
 u|_{\partial \Omega} = f, \quad \frac{\partial^2 u}{\partial n} \bigg|_{\partial \Omega} = g.
\]

Then for any \( s > 0 \), there exists a constant \( C \) depending on the dimension and \( \Omega \) such that

\[
 \| u \|_{W^{s,2}(\Omega)} \leq C \left( \| f \|_{W^{(s-1)/2,2}(\partial \Omega)} + \| g \|_{W^{(s-3)/2,2}(\partial \Omega)} \right).
\]

**Proof.** This result might be known, but it is not easy to find a proper reference. For completeness, we give a proof here.

For \( x \in \partial \Omega \), take two open neighborhoods of \( x \), \( U, V \) such that \( x \in V \subset \overline{V} \subset U \).

Assume that \( \overline{U} \cap \Omega \) is diffeomorphic to \( B^1_1 = \{ x \in B_1 \mid x_1 \geq 0 \} \). Let \( \varphi \) be a smooth cut-off function satisfying \( \varphi(y) \equiv 0 \) for \( y \notin U \) and \( \varphi(y) \equiv 1 \) for \( y \in V \). A direct computation implies

\[
 \begin{cases}
 \Delta^2 (\varphi u) = \nabla^3 u \# \nabla \varphi + \nabla^2 u \# \nabla \varphi + \nabla u \# \nabla^3 \varphi + \nabla^4 \varphi \# u, \\
 (\varphi u)|_{\partial \Omega} = \varphi f, \\
 \frac{\partial}{\partial n} (\varphi u) \bigg|_{\partial \Omega} = \varphi g + \frac{\partial \varphi}{\partial n} f.
\end{cases}
\]  

(4.7)

Here \( \nabla^k \) means partial derivatives of order \( k \) and \( \nabla^k u \# \nabla \varphi \) means linear combinations of the product of \( \nabla^k u \) and \( \nabla \varphi \) and so on. By [2], we have

\[
 \| u \|_{W^{s,2}(V)} \leq C(\| u \|_{W^{s-1,2}(U)} + \| f \|_{W^{(s-1/2),2}(\partial \Omega \cap U)} + \| g \|_{W^{(s-3/2),2}(\partial \Omega \cap U)}) \leq C. \]  

(4.8)

Since the boundary \( \partial \Omega \) is compact, we can find a finite number of points \( x_1, \ldots, x_k \) such that \( \partial \Omega \) is covered by \( V_i \)'s. Adding (4.8) up for all \( V_i \) and using the interior estimate, we obtain

\[
 \| u \|_{W^{s,2}(\Omega)} \leq C \left( \| f \|_{W^{(s-1/2),2}(\partial \Omega)} + \| g \|_{W^{(s-3/2),2}(\partial \Omega)} + \| u \|_{W^{s-1,2}(\Omega)} \right).
\]  

(4.9)

Next, we claim that the \( \| u \|_{W^{s-1,2}(\Omega)} \) term on the right-hand side is not necessary for our case. This is proved by contradiction. If otherwise, there exist sequences \( f_i, g_i, u_i \) such that

(1) \( u_i \) is a biharmonic function on \( \Omega \) with \( u_i|_{\partial \Omega} = f_i \) and \( \frac{\partial u_i}{\partial n}|_{\partial \Omega} = g_i \);

(2) assume by scaling that \( \| f_i \|_{W^{(s-1/2),2}(\partial \Omega)} + \| g_i \|_{W^{(s-3/2),2}(\partial \Omega)} = 1 \);

(3) \( \| u_i \|_{W^{s,2}(\Omega)} \geq i \).
It follows from (3) and (4.9) that \( \lim_{i \to \infty} \| u_i \|_{W^{2,1}(\Omega)} = \infty \). Let \( \lambda_i = \| u_i \|_{W^{2,1}(\Omega)} \) and set \( \tilde{u}_i = \frac{u_i}{\lambda_i}, \tilde{f}_i = \frac{f_i}{\lambda_i} \) and \( \tilde{g}_i = \frac{g_i}{\lambda_i} \). Using (4.9) again, we have that \( \| u_i \|_{W^{2,2}(\Omega)} \) is bounded. Therefore, \( u_i \) converges weakly in \( W^{2,2}(\Omega) \) to a biharmonic function with homogeneous boundary conditions. On one hand, due to the compactness of embedding from \( W^{2,2} \) to \( W^{1,2} \), we have \( \| u \|_{W^{1,2}(\Omega)} = 1 \). On the other hand, the only biharmonic function with homogeneous boundary conditions is zero. This is a contradiction. \( \square \)

**Lemma 4.3.** Assume that \( u \) is a smooth map satisfying the assumption in Lemma 4.1. Let \( \eta_0 \) be the constant given in Lemma 4.1. For each \( \eta \in [0, \eta_0] \), there exists an \( \varepsilon_\eta > 0 \), depending only on \( \eta \), such that if \( u_i \) is a sequence of \( W^{2,2} \cap C^0(\Omega, S^n) \) and weakly converges to \( u \) in \( W^{2,2} \) satisfying

\[
\int_{\partial B_\rho} |\nabla^2 u_i|^2 + |\nabla u_i|^4 \, d\sigma < \varepsilon_\eta \quad (4.10)
\]

for some ball \( B_\rho \subset \Omega \), then we can find a new sequence \( \tilde{u}_i \in W^{2,2} \cap C^0(\Omega, S^n) \) satisfying \( \tilde{u}_i \equiv u_i \) on \( \Omega \setminus B_\rho \), \( \tilde{u}_i \equiv u \) on \( B_\rho(1-\eta) \) and

\[
\int_{B_\rho \setminus B_\rho(1-\eta)} |\Delta \tilde{u}_i|^2 \, dx \leq C \int_{B_\rho \setminus B_\rho(1-\eta)} |\Delta u|^2 \, dx \quad (4.11)
\]

for sufficiently large \( i \), where the constant \( C \) does not depend on \( \eta, \rho \) or \( u \).

**Proof.** For the proof, we need to define \( \tilde{u}_i \) in \( B_\rho \setminus B_\rho(1-\eta) \) and verify (4.11). The construction consists of the following four steps.

**Step one.** Let \( v \) be the solution of the boundary value problem

\[
\begin{aligned}
\Delta^2 v &= 0 \quad \text{in } B_\rho \setminus B_\rho(1-\eta); \\
v &= u \quad \text{on } \partial B_\rho \cup \partial B_\rho(1-\eta); \\
\frac{\partial v}{\partial n} &= \frac{\partial u}{\partial n} \quad \text{on } \partial B_\rho \cup \partial B_\rho(1-\eta).
\end{aligned}
\]

Define \( v_i \) as the biharmonic extension of \( u \) and \( \tilde{u}_i \) as follows:

\[
\begin{aligned}
\Delta^2 v_i &= 0 \quad \text{in } B_\rho \setminus B_\rho(1-\eta); \\
v &= u \quad \text{on } \partial B_\rho(1-\eta); \\
v &= u_i \quad \text{on } \partial B_\rho; \\
\frac{\partial v}{\partial n} &= \frac{\partial u}{\partial n} \quad \text{on } \partial B_\rho(1-\eta); \\
\frac{\partial v}{\partial n} &= \frac{\partial u_i}{\partial n} \quad \text{on } \partial B_\rho.
\end{aligned}
\]

We need to prove some estimates of \( v_i \). Recall that both \( u_i \) and \( u \) are bounded in \( W^{2,2} \) (in fact \( u \) is smooth.) The restriction of a \( W^{2,2} \) function to a hypersurface belongs to \( W^{1,5,2} \) and

\[
\frac{\partial u_i}{\partial n} \in W^{0,5,2}(\partial B_\rho).
\]
By Lemma 4.2, we have

\[
\|v_i\|_{W^{2,2}(B\rho \setminus B\rho(1-\eta))} \leq C \left( \|u_i\|_{W^{1.5,2}(\partial B\rho)} + \left\| \frac{\partial u_i}{\partial n} \right\|_{W^{0.5,2}(\partial B\rho)} \right) + C(u) \\
\leq C \|u_i\|_{W^{2,2}(B\rho \setminus B\rho(1-\eta))} + C(u) \\
\leq C(u). \quad (4.13)
\]

Moreover, we can obtain a better estimate if we take (4.10) into account. Due to (4.10), we have

\[
\|u_i\|_{W^{2,2}(\partial B\rho)} \leq C(\varepsilon \eta)
\]

and

\[
\|\partial_n u_i\|_{W^{1,2}(\partial B\rho)} \leq C(\varepsilon \eta).
\]

This combined with the fact that \(u\) is smooth implies that (by Lemma 4.2 again)

\[
\|v_i\|_{W^{2.5,2}(B\rho \setminus B\rho(1-\eta))} \leq C(n, \varepsilon \eta, u). \quad (4.14)
\]

It implies that \(v_i\) converges to \(v\) in \(W^{2,2}(B\rho \setminus B\rho(1-\eta))\).

**Step two.** Without loss of generality, we can assume that \(\rho = 1\) by rescaling. We will show there exists a small positive \(\lambda < \eta\) such that

\[
\frac{1}{2} \leq |v_i| \leq 2
\]

on

\(B_1 \setminus B_{1-\lambda}\).

According to Green’s formula for the biharmonic equation [7],

\[
v_i(x) = \int_{\partial B_{1-\eta} \cup \partial B_1} K_0(x, y)v_i(y)\,d\sigma_y + \int_{\partial B_{1-\eta} \cup \partial B_1} K_1(x, y)\partial_n v_i(y)\,d\sigma_y. \quad (4.15)
\]

Following [13], set \(\xi_0 = \frac{x}{|x|}\) and \(s = 1 - |x|\). Since we will only consider the estimate near \(\partial B_1\), we may require \(x \in B_1 \setminus B_{1-\eta/2}\). Therefore, \(s\) is the distance from \(x\) to \(\partial(B_1 \setminus B(1-\eta))\). It follows from [7] that for \(y \in \partial B_1 \cup \partial B_{1-\eta}\) and \(x \in B_1 \setminus B_{1-\eta/2}\)

\[
|K_0(x, y)| \leq C \frac{s^2}{d^6(x, y)} \quad (4.16)
\]

and

\[
|K_1(x, y)| \leq C \frac{s^2}{d^5(x, y)}, \quad (4.17)
\]
where \( C \) is a constant depending on \( \eta \). For some \( k > 1 \) with \( ks \leq \frac{\eta}{4} \), we write

\[
v_{ks, \xi_0} = \frac{1}{|\partial B_1 \cap B_{ks}(\xi_0)|} \int_{\partial B_1 \cup B_{ks}(\xi_0)} v_i \, d\sigma.
\]

Using the Poincaré inequality, we see

\[
\frac{1}{|\partial B_1 \cap B_{ks}(\xi_0)|} \int_{\partial B_1 \cap B_{ks}(\xi_0)} |v_i - v_{ks, \xi_0}| \, d\sigma \leq C \| \nabla u_i \|_{L^4(\partial B_1 \cap B_{ks}(\xi_0))} \leq C \varepsilon^{1/4}. \tag{4.18}
\]

Hence \( |1 - |v_{ks, \xi_0}|| \leq C \varepsilon^{1/4} \). Since the constant function \( v_{ks, \xi_0} \) is a biharmonic function with constant Dirichlet boundary value and zero Neumann boundary value, we have

\[
v_i(x) - v_{ks, \xi_0} = \int_{\partial B_1 \setminus \partial B} K_0(x, y)(v_i(y) - v_{ks, \xi_0}) \, d\sigma_y
+ \int_{\partial B_1 \setminus \partial B} K_1(x, y) \partial_n v_i(y) \, d\sigma_y. \tag{4.19}
\]

To estimate the first integral, we divide the integral domain into two parts,

\[ \Omega_1 = \partial B_1 \cap B_{ks}(\xi_0) \quad \text{and} \quad \Omega_2 = (\partial B_1 \setminus B_{ks}(\xi_0)) \cup \partial B_1 \setminus \partial B. \]

For the second part, we estimate

\[
\int_{\Omega_2 \setminus B_{\eta/2}(\xi_0)} |K_0(x, y)(v_i - v_{ks, \xi_0})| \, d\sigma_y \leq \left( \int_{\Omega_2 \cap B_{\eta/2}(\xi_0)} + \int_{\Omega_2 \setminus B_{\eta/2}(\xi_0)} \right) |K_0(x, y)(v_i - v_{ks, \xi_0})| \, d\sigma_y.
\]

For \( y \in \Omega_2 \setminus B_{\eta/2}(\xi_0) \), we note

\[
|K_0(x, y)| \leq Cs^2.
\]

Hence,

\[
\int_{\Omega_2 \setminus B_{\eta/2}(\xi_0)} |K_0(x, y)(v_i - v_{ks, \xi_0})| \, d\sigma_y \leq Cs^2. \tag{4.20}
\]

Using (4.16), we have

\[
\int_{\Omega_2 \cap B_{\eta/2}(\xi_0)} |K_0(x, y)(v_i - v_{ks, \xi_0})| \, d\sigma_y \leq \int_{ks}^{\eta/2} \frac{Cs^2}{t^6} \, dt \leq \frac{C}{k^2} - Cs^2. \tag{4.21}
\]
Here $t$ is the distance between $\xi_0$ and $y$ on the sphere $\partial B_1$ and we estimate $d(x, y)$ from below by $C t$. We add (4.20) and (4.21) to get

$$\int_{\Omega_2} |K_0(x, y)(v_i - v_{ks, \xi_0})| d\sigma_y \leq C s^2 + \frac{C}{k^2} \leq \frac{C}{k^2}. \tag{4.22}$$

To estimate the integral over $\Omega_1$, we notice that for $y \in \partial B_1 \cup \partial B_{1-\eta}$ and $x \in B_1 \setminus B_{1-\eta/2}$, we have

$$d(x, y) \geq s,$$

which implies

$$|K_0(x, y)| \leq \frac{C}{s^4},$$

where the constant $C$ depends on $\eta$. Hence, by (4.18), we have

$$\left| \int_{\partial B_1 \cap B_{ks}(\xi_0)} K_0(x, y)(u_i - v_{ks, \xi_0}) \right| \leq \frac{C}{s^4} \int_{\partial B_1 \cap B_{ks}(\xi_0)} |u_i - v_{ks, \xi_0}| d\sigma_y \leq C k^4 \varepsilon_\eta^{1/4}, \tag{4.23}$$

where we used the fact that the volume of $\partial B_1 \cap B_{ks}(\xi_0)$ is comparable with $(ks)^4$.

The second integral in (4.19) is estimated similarly.

$$\left| \int_{\Omega_2} K_1(x, y) \partial_n v_i \sigma_y \right| \leq \left( \int_{\Omega_2 \cap B_{\eta/2}(\xi_0)} + \int_{\Omega_2 \setminus B_{\eta/2}(\xi_0)} \right) |K_1(x, y) \partial_n v_i| d\sigma_y$$

$$\leq \left( \int_{\Omega_2 \cap B_{\eta/2}(\xi_0)} |K_1|^{4/3} \right)^{3/4} \left( \int_{\Omega_2 \setminus B_{\eta/2}(\xi_0)} |\partial_n v_i|^{4/3} \right)^{1/4}$$

$$+ C s^2 \int_{\Omega_2 \setminus B_{\eta/2}(\xi_0)} |\partial_n v_i| d\sigma_y$$

$$\leq C \left( s^{8/3} \int_{ks}^{\eta/2} \frac{1}{t^{20/3}} t^3 dt \right)^{3/4} + C s^2$$

$$\leq C \left( \frac{1}{k^{8/3}} - C s^{8/3} \right)^{3/4} + C s^2$$

$$\leq \frac{C}{k^2} + C s^2 \leq \frac{C}{k^2}, \tag{4.24}$$

where we used (4.17), (4.10) and the Hölder inequality. On the other hand,
\[
\left| \int_{\Omega_1} K_1(x, y) \partial_n u_i \, d\sigma \right| \leq \frac{C}{s^3} \int_{\Omega_1} |\partial_n u_i| \, d\sigma_y \\
\leq C k^3 \left( \int_{\Omega_1} |\partial_n u_i|^4 \, d\sigma_y \right)^{\frac{1}{4}} \leq C k^3 \varepsilon_{\eta}^{1/4}.
\]

Combining (4.22), (4.23), (4.24) with (4.25), we have
\[
|v_i(x) - v_{ks, \xi_0}| \leq \frac{C}{k^2} + C \varepsilon_{\eta}^{1/4} (k^4 + k^3).
\]

Since the above constant \( C \) depends only on \( \eta \), we can choose \( k \) large so that \( \frac{C}{k^2} < \frac{1}{10} \) and then choose \( \varepsilon_{\eta} \) sufficiently small so that
\[
C \varepsilon_{\eta}^{1/4} (k^4 + k^3) + |1 - |v_{ks, \xi_0}|| < \frac{1}{10}.
\]

Hence, if we set \( \lambda = \min\{\frac{1}{4k}, \eta/2\} \), we have
\[
\frac{3}{4} \leq |v_i| \leq 2
\]
for any point \( x \in B_1 \setminus B_{1-\lambda} \).

**Step three.** We will establish an estimate of \( v_i \) on 
\[
B_{\rho(1-\lambda/2)} \setminus B_{\rho(1-\eta)}.
\]

Due to the interior estimate for biharmonic functions and (4.13) in Step two,
\[
\|v_i\|_{C^i(B_{\rho(1-\lambda/2)} \setminus B_{\rho(1-\eta/2)})} \leq C(l, u).
\]

Given this, the elliptic boundary value problem on \( B_{\rho(1-\eta/2)} \setminus B_{\rho(1-\eta)} \) implies
\[
\|v_i\|_{C^i(B_{\rho(1-\eta/2)} \setminus B_{\rho(1-\eta)})} \leq C(l, u),
\]

since both boundary values are bounded in appropriate norms. Combining the above two estimates, we see \( v_i \) is uniformly bounded in the \( C^2 \)-norm on \( B_{\rho(1-\lambda/2)} \setminus B_{\rho(1-\eta)} \).

**Step four.** An immediate consequence of Step three is that \( v_i \) converges uniformly to \( v \) on \( B_{\rho(1-\lambda/2)} \setminus B_{\rho(1-\eta)} \). According to Lemma 4.1, \( \frac{1}{3} \leq |v| \leq \frac{2}{3} \), so we have \( \frac{1}{4} \leq |v_i| \leq 2 \) on \( B_{\rho(1-\lambda/2)} \setminus B_{\rho(1-\eta)} \) if \( i \) is sufficiently large. Combining this with the result of Step two, we know
\[
\frac{1}{4} \leq |v_i| \leq 2
\]
when \( i \) sufficiently large on \( B_{\rho} \setminus B_{\rho(1-\eta)} \). Therefore, we may define
\[
\bar{u}_i = \frac{v_i}{|v_i|}
\]
on \( B_{\rho} \setminus B_{\rho(1-\eta)} \).
Since $\tilde{u}_i = \frac{u_i}{|v_i|}$ and $|v_i| \geq \frac{1}{4}$, we have

$$|\nabla \tilde{u}_i| \leq C |\nabla v_i|,$$

and

$$|\Delta \tilde{u}_i|^2 \leq C |\Delta v_i|^2 + |\nabla v_i|^4. \tag{4.26}$$

By Lemma 4.1, $|v|$ is bounded by a constant. Hence, if we set $w = u - v$, so is $|w|$. By definition, $w$ satisfies homogeneous Dirichlet and Neumann boundary condition. Integration by parts gives

$$\int_{B^\rho \setminus B(1-\eta)^\rho} |\nabla w|^4 \, dx = - \int_{B^\rho \setminus B(1-\eta)^\rho} \langle w, \nabla (|\nabla w|^2 \nabla w) \rangle \, dx$$

$$\leq C \int_{B^\rho \setminus B(1-\eta)^\rho} |w||\nabla w|^2 |\nabla^2 w| \, dx$$

$$\leq \frac{1}{2} \int_{B^\rho \setminus B(1-\eta)^\rho} |\nabla w|^4 \, dx + C \int_{B^\rho \setminus B(1-\eta)^\rho} |\Delta w|^2 \, dx, \tag{4.27}$$

where we used $\int_{B^\rho \setminus B(1-\eta)^\rho} |\nabla^2 w|^2 = \int_{B^\rho \setminus B(1-\eta)^\rho} |\Delta w|^2 \, dx$. Therefore, we have

$$\int_{B^\rho \setminus B(1-\eta)^\rho} |\nabla v|^4 \, dx \leq C_1 \int_{B^\rho \setminus B(1-\eta)^\rho} (|\Delta v|^2 + |\nabla v|^4) \, dx.$$

By a similar argument to (4.27) and noticing that $|v_i| \leq 2$, we can prove

$$\int_{B^\rho \setminus B(1-\eta)^\rho} |\nabla (v_i - v)|^4 \, dx \leq C \int_{B^\rho \setminus B(1-\eta)^\rho} |\nabla^2 (v_i - v)|^2 \, dx$$

$$+ \int_{\partial B^\rho} \left\langle u_i - u, |\nabla (u - u_i)|^2 \frac{\partial (u_i - u)}{\partial n} \right\rangle \, dx.$$

Since $v_i$ converges to $v$ strongly in $W^{2,2}$ and $u_i$ is uniformly bounded in $W^{2,2}(\partial B^\rho)$, then $v_i$ also converges strongly in $W^{1,4}(B^\rho \setminus B(1-\eta))$ to $v$. Then for sufficiently large $i$, we have

$$\mathbb{H}(\tilde{u}_i; B^\rho \setminus B(1-\eta)^\rho) \leq C \int_{B^\rho \setminus B(1-\eta)^\rho} (|\Delta v_i|^2 + |\nabla v_i|^4)$$

$$\leq 2C \int_{B^\rho \setminus B(1-\eta)^\rho} (|\Delta v|^2 + |\nabla v|^4)$$
\[ \leq 2C(1 + C_1) \int_{B_\rho \setminus B_{(1-\eta)\rho}} (|\Delta u|^2 + |\nabla u|^4) \]
\[ \leq 4C(1 + C_1) \mathbb{H}(u; B_\rho \setminus B_{(1-\eta)\rho}). \]

This proves our claim. □

**Lemma 4.4.** Let \( \Theta(x) \) be the density in Theorem 1.3. Then there exists a positive constant \( \varepsilon_1 \) such that for each \( x_0 \in \Omega \setminus \Sigma_1 \), i.e. for \( x_0 \) with \( \Theta(x_0) < \varepsilon_1 \), \( \Theta(x_0) = 0 \).

**Proof.** Let \( \eta_0 \) be the constant given by Lemma 4.1 and \( \varepsilon_{\eta_0} \) be the constant corresponding to \( \eta_0 \) in Lemma 4.3. We may assume that \( \varepsilon_1 < \varepsilon_0 \). Hence \( u \) is smooth in \( \Omega \setminus \Sigma_1 \), where \( \Sigma_1 \) is defined in Theorem 1.3. By Theorem 3.1 and the definition of the set \( E \) in (3.5), \( \Theta \) is defined for all \( x \in \Omega \setminus \Sigma_1 \). If \( x_0 \not\in \Sigma_1 \), then there is an \( R_0 > 0 \) such that \( u \) is smooth in \( B_{R_0}(x_0) \). Moreover, we can assume by taking \( R_0 \) small that
\[ R_0^4 \| \nabla^4 u \|_{C^0(B_{R_0}(x_0))} + R_0^2 \| \nabla^2 u \|_{C^0(B_{R_0}(x_0))} < C(\varepsilon_0). \]

Then \( u \) satisfies the condition in Lemma 4.1 for each \( \rho < R_0 \) so that the ball \( B_\rho(x_0) \subset B_{R_0}(x_0) \).

Since \( \Theta(x_0) < \varepsilon_1 \), we can find a sequence \( r_j \to 0 \) such that
\[ \liminf_{i \to \infty} r_j^{-1} \int_{B_{r_j}(x_0)} |\Delta u|^2 dx < \varepsilon_1. \]

Then, a similar proof in Section 2 yields
\[ \liminf_{i \to \infty} r_j^{-1} \int_{B_{r_j/2}(x_0)} |\nabla^2 u|^2 + |\nabla u|^4 dx < C\sqrt{\varepsilon_1}. \]

By a choice of a ‘good’ radius \( \rho_j \in (r_j/4, r_j/2) \), we have
\[ \int_{\partial B_{\rho_j}(x_0)} |\nabla^2 u|^2 + |\nabla u|^4 d\sigma \leq C\sqrt{\varepsilon_1} < \varepsilon_{\eta_0} \quad (4.28) \]

for infinitely many \( i \)’s choosing \( \varepsilon_1 = \left( \frac{\varepsilon_{\eta_0}}{2C} \right)^2 \). Assume by taking a subsequence that (4.28) is true for all \( i \).

By Lemma 4.3, we construct a new sequence \( \{\tilde{u}_i\} \) which agrees with \( u_i \) outside \( B_{\rho_j}(x_0) \). As proved in Lemma 2.3, \( u_i \) is a minimizing sequence for \( \mathbb{H}(u) \) in \( W^{2,2}_{\mu_0}(\Omega, S^n) \cap C^0(\Omega, S^n) \). Then we have
\[ \lim_{i \to \infty} \mathbb{H}(u_i; \Omega) \leq \liminf_{i \to \infty} \mathbb{H}(\tilde{u}_i; \Omega). \]

Combining this with Lemma 4.3 yields
\[ \liminf_{i \to \infty} \mathbb{H}(u_i; B_{\rho_j}(x_0)) \leq \liminf_{i \to \infty} \mathbb{H}(\tilde{u}_i, B_{\rho_j}(x_0)) \leq C\mathbb{H}(u, B_{\rho_j}(x_0)). \quad (4.29) \]
This means that
\[
\Theta(x_0) = \lim_{\rho_j \to 0} \rho_j^{-1} \mu(B_{\rho_j}(x_0)) \leq C \lim_{\rho_j \to 0} \rho_j^{-1} \mathbb{H}(u, B_{\rho_j}(x_0)) = 0.
\]
This proves our claim. \(\Box\)

We are now ready to prove Theorem 1.3:

**Proof of Theorem 1.3.** The proof about finite 1-dimensional Hausdorff measure of the singular set \(\Sigma_1\) is standard (e.g. [20]).

To show \(\Sigma_1\) is relatively closed, let \(x_j\) be a sequence in \(\Sigma_1\) such that \(x_j \to x \in \Omega\). It suffices to find a contradiction if \(x \notin \Sigma_1\). In that case, \(\nabla u\) is bounded in \(B_{r_1}(x)\) for some \(r_1 > 0\). By the proof of Lemma 3.2, there is \(\delta = \delta(\varepsilon_1) > 0\) such that for each \(r \in (0, \delta]\) and \(y \in B_{r_1/2}(x)\), there is a \(\rho_{y,i,r} \in [r/2, r]\) such that we have
\[
\limsup_{i \to \infty} \sigma_{i,2}(y, \rho_{y,i,r}) \leq C r \leq C \delta = \frac{\varepsilon_1}{10}.
\] (4.30)

where \(C\) is a constant independent of \(r\) and we choose \(\delta = \frac{\varepsilon_1}{10C} < r_1\). By Lemma 4.4, we know \(\Theta(x) = 0\), which implies there exists a small constant \(\delta' = \delta'(x) < \delta\) such that
\[
\sigma_1(x, \delta') \leq \frac{\varepsilon_1}{20}.
\]

For any \(y \in B_{\delta'/2}(x)\), we know
\[
\liminf_{i \to \infty} \sigma_{i,1}(y, \rho_{y,i,\delta'/2}) \leq 4\sigma_1(x, \delta') \leq \frac{\varepsilon_1}{5}
\]
for the \(\rho_{y,i,\delta'/2} \in [\delta'/2, \delta']\). By (2.7) and Lemma 2.3, for each \(0 < r < \delta'/4\), we have
\[
\liminf_{i \to \infty} \sigma_i(y, \rho_{y,i,r}) \leq \liminf_{i \to \infty} \sigma_i(y, \rho_{y,i,\delta'/2}) \leq \frac{3\varepsilon_1}{10}.
\]

By (4.30) again, we obtain
\[
\sigma_1(y, r) = \liminf_{i \to \infty} \sigma_{i,1}(y, r) \leq 2 \liminf_{i \to \infty} \sigma_{i,1}(y, \rho_{y,i,r}) \leq 2 \liminf_{i \to \infty} \sigma_i(y, \rho_{y,i,r}) + \frac{\varepsilon_1}{5} \leq \frac{4\varepsilon_1}{5}.
\]

But, this is a contradiction since when \(j\) is large, \(x_j\) is in \(B_{\delta'/2}(x)\) and \(\Theta(x) > \varepsilon_1\).

Let \(x\) be a point in \(\Omega \setminus \Sigma_1\). For any \(\eta > 0\), there exists an \(\varepsilon_\eta > 0\) given by Lemma 4.3. Since \(\Theta(x) = 0\), there is a small constant \(\rho_x\) such that for each \(\rho < \rho_x\),
\[
\frac{1}{\rho} \lim_{i \to \infty} \int_{B_{\rho}(x)} |\nabla^2 u_i|^2 + |\nabla u_i|^4 \, dx \leq \frac{C}{\rho} \mu(B_{2\rho}(x)) + \frac{C}{\rho^3} \lim_{i \to \infty} \int_{B_{2\rho}(x)} |\nabla u_i|^2 \, dx \leq \varepsilon_\eta/4.
\]
This implies that
\[
\lim_{i \to \infty} \int_{\partial B_{\rho'}^e(x)} |\nabla^2 u_i|^2 + |\nabla u_i|^4 \, dx \leq \varepsilon \eta
\]
for some $\rho' \in (\rho/2, \rho)$.

By Lemma 4.3, we construct a new sequence $\tilde{u}_i$ such that
\[
\lim_{i \to \infty} \int_{B_{\rho'} \setminus B_{\rho'(1-\eta)}(x)} |\nabla \tilde{u}_i|^2 \, dx \leq C \int_{B_{\rho'} \setminus B_{\rho'(1-\eta)}(x)} |\nabla u|^2 \leq C \eta \rho'^5,
\]
where the constant $C$ does not depend on $\rho$ or $\eta$. (It depends on $C^2$ norm of $u$ near $x$, which is bounded since $u$ is smooth at $x$.) Hence, we have
\[
\mathbb{H}(u, B_{\rho'}(x)) \leq \mu(B_{\rho'}(x)) \leq \lim_{i \to \infty} \mathbb{H}(\tilde{u}_i, B_{\rho'}(x)) \leq \mathbb{H}(u, B_{\rho'}(x)) + C \eta |B_{\rho'}(x)|.
\]
It follows that
\[
v(B_{\rho'}(x)) \leq C \eta |B_{\rho'}(x)|,
\]
which implies that $v$ is absolutely continuous with respect to the Lebesgue measure. By the Radon–Nikodym theorem, $v = g(x) \, dx$ in $\Omega \setminus \Sigma_1$. Since $\eta$ is arbitrary, $v \equiv 0$ outside $\Sigma_1$.

By Lemma 3.1, $\mu$ is absolutely continuous with respect to $\mathcal{H}^1 \upharpoonright \Sigma_1$. Then, the Radon–Nikodym theorem tells us that there is a density function $\Theta'$ such that
\[
\mu|\Sigma_1 = \Theta'(x) \mathcal{H}^1 \upharpoonright \Sigma_1.
\]
By Corollary 3.2.3 in [27],
\[
v|\Sigma_1 = \Theta'(x) \mathcal{H}^1 \upharpoonright \Sigma_1
\]
for $\mathcal{H}^1$-a.e. $x \in \Sigma_1$. Since $v \equiv 0$ outside $\Sigma_1$, we have
\[
\Theta'(x) = \lim_{r \to 0} r^{-1} v(B_r(x) \cap \Sigma_1) = \lim_{r \to 0} r^{-1} v(B_r(x)) = \Theta(x)
\]
and
\[
0 < \varepsilon_1 \leq \Theta(x) \leq C
\]
for $\mathcal{H}^1$-a.e. $x \in \Sigma_1$. These imply that for $\mathcal{H}^1$-a.e. $x \in \Omega$, $v = \Theta(x) \mathcal{H}^1 \upharpoonright \Sigma_1$. Thus, Part (ii) of Theorem 1.3 are proved.

Define a measure $\nu_1$ on $\mathbb{R}^3$ by
\[
\nu_1(S) = v(S \cap \Omega)
\]
for any set $S \subset \mathbb{R}^5$. Then, for $\nu_1$-almost every $x \in \mathbb{R}^5$

$$0 < \lim_{r \to 0} r^{-1} \nu_1(B_r(x)) < \infty.$$ 

According to Preiss’s result in [19], $\nu_1$ is 1-rectifiable and so is $\nu$. Therefore, $\Sigma_1$ is also 1-rectifiable.

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