Local cohomology, arrangements of subspaces and monomial ideals

Josep Àlvarez Montaner, a, *,1 Ricardo García López, b,2 and Santiago Zarzuela Armengou b,3

a Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Avinguda Diagonal 647, Barcelona 08028, Spain
b Departament d’Àlgebra i Geometria, Universitat de Barcelona, Gran Vía 585, Barcelona 08007, Spain

Received 1 December 2000; accepted 8 March 2002
Communicated by Anders Björner

0. Introduction

Let \( \mathbb{A}^n_k \) denote the affine space of dimension \( n \) over a field \( k \), let \( X \subset \mathbb{A}^n_k \) be an arrangement of linear subvarieties. Set \( R = k[x_1, \ldots, x_n] \) and let \( I \subset R \) denote an ideal which defines \( X \). In this paper we study the local cohomology modules

\[ H^i_I(R) = \text{indlim}_j \text{Ext}^i_R(R/I^j, R), \]

with special regard of the case where the ideal \( I \) is generated by monomials.

If \( k \) is the field of complex numbers (or, more generally, a field of characteristic zero), the module \( H^i_I(R) \) is known to have a module structure over the Weyl algebra \( A_n(k) \), and one can therefore consider its characteristic cycle, denoted \( \text{CC}(H^i_I(R)) \) in this paper (see e.g. [3, I.1.8.5]). On the other hand, the arrangement \( X \) defines a partially ordered set \( P(X) \) whose elements correspond to the intersections of irreducible components of \( X \) and where the order is given by inclusion.

Our first result is the determination of the characteristic cycles \( \text{CC}(H^i_I(R)) \) in terms of the cohomology of some simplicial complexes attached to the poset \( P(X) \). It follows from the formulas obtained that, in either the complex or the real case, these characteristic cycles determine the Betti numbers of the complement of the arrangement in \( \mathbb{A}^n_k \). In fact, it was proved by Goresky and MacPherson that the

*Corresponding author.
E-mail addresses: joalvarez@ma1.upc.es (J. Àlvarez Montaner), rgarcia@mat.ub.es (R. García López), zarzuela@mat.ub.es (S. Zarzuela Armengou).
1 Partially supported by the University of Nice.
2 Partially supported by the DGCYT PB98-1185 and by INTAS 97 1644.
3 Partially supported by the DGCYT PB97-0893.

0001-8708/03/$ - see front matter © 2003 Elsevier Science (USA). All rights reserved.
doi:10.1016/S0001-8708(02)00050-6
Betti numbers of the complement of an arrangement $X$ can be computed as a sum of non-negative integers, one for each non-empty intersection of irreducible components of $X$. These integers are dimensions of certain Morse groups (cf. [9, Part III, Theorems 1.3 and 3.5]). We will see that, over a field of characteristic zero, one can give a purely algebraic interpretation of them in terms of local cohomology.

More precisely, in Section 1, and following closely Björner–Ekedahl’s proof of the $\ell$-adic version of Goresky–MacPherson’s formula, we will establish the existence of a Mayer–Vietoris spectral sequence

$$E_2^{-i,j} = \text{indlim}_{P(X)}^{(i)} H^j_{I_p}(R) \Rightarrow H^{i-j}_I(R),$$

where $p$ runs over $P(X)$, $I_p$ is the (radical) ideal of definition of the irreducible variety corresponding to $p$, and $\text{indlim}_{P(X)}^{(i)}$ is the $i$th left-derived functor of the inductive limit functor in the category of inductive systems of $R$-modules indexed by $P(X)$. The main ingredient in the proof is the Matlis–Gabriel theorem on structure of injective modules. Our formula for the characteristic cycle of local cohomology follows essentially from the fact, proved in Section 1 as well, that this spectral sequence degenerates at the $E_2$-term.

In case the arrangement $X$ is defined by a monomial ideal $I$, the local cohomology modules $H^j_I(R)$ have a natural $\mathbb{Z}^n$-grading. In Section 2, we relate the multiplicities of the characteristic cycle of $H^j_I(R)$ to its graded structure (Proposition 2.1). Using results of Mustaţă, we can conclude that the multiplicities of the characteristic cycle of $H^j_I(R)$ determine and are determined by the graded Betti numbers of the Alexander dual ideal of $I$. The relation between the $\mathbb{Z}^n$-graded structure of $H^j_I(R)$ and the multiplicities of its characteristic cycle has also been established by Yanagawa in a preprint [17].

The degeneration of the Mayer–Vietoris spectral sequence provides a filtration of each local cohomology module $H^j_I(R)$, where the successive quotients are given by the $E_2$-term. In general, not all the extension problems attached to this filtration have a trivial solution (this happens for example for the arrangement given by all coordinate hyperplanes in $\mathbb{A}^n_k$, see Remark 1.4(i)). This is a major difference between the case we consider here and the cases considered by Björner and Ekedahl. Namely, in the analogous situation for the $\ell$-adic cohomology of an arrangement defined over a finite field the extensions appearing are trivial not only as extensions of $\mathbb{Q}_\ell$-vector spaces but also as Galois representations, and for the singular cohomology of a complex arrangement the extensions appearing are trivial as extensions of mixed Hodge structures (cf. [4, p. 179]). This contrasts with the fact that the proof of the degeneration of the Mayer–Vietoris spectral sequence for $\ell$-adic or singular cohomology uses deeper facts than the proof of the degeneration for local cohomology (it relies on the strictness of Deligne’s weight filtration).

In Section 3, we solve these extensions problems in case the ideal $I$ is monomial (the result stated is actually more general, in that we work in the category of $\varepsilon$-straight modules, which is a slight variation of a category introduced by Yanagawa [16], and includes as objects the local cohomology modules considered above).
turns out that these extensions can be described by a finite set of linear maps, which for local cohomology modules supported at monomial ideals can be effectively computed in combinatorial terms from certain Stanley–Reisner simplicial complexes (using the results in [13]).

If the base field is the field of complex numbers, the category of $\varepsilon$-straight modules considered in Section 3 is a full subcategory of the category of regular holonomic $A_n(\mathbb{C})$-modules. Then, by the Riemann–Hilbert correspondence, the category of $\varepsilon$-straight modules is equivalent to a full subcategory of the category of perverse sheaves in $\mathbb{C}^n$ (with respect to the stratification given by the coordinate hyperplanes). This category has been described in terms of linear algebra by Galligo et al. [6]. Given such a perverse sheaf, one can attach it a set of partial variation maps. In Section 4, and using the description in [6], we prove that the category of $\varepsilon$-straight modules is equivalent to the category of those perverse sheaves where all partial variations vanish (and then all partial monodromies are the identity map).

If $(P, \leq)$ is a poset, we will denote by $K(P)$ the simplicial complex which has as vertexes the elements of $P$ and where a set of vertexes $p_0, \ldots, p_r$ determines an $r$-dimensional simplex if $p_0 \prec \cdots \prec p_r$. If $K$ is a simplicial complex and $E$ is a $k$-vector space, we will denote by $\text{Simp}_* (K; E)$ the complex of simplicial chains of $K$ with coefficients in $E$.

On dealing with arrangements defined over fields of characteristic zero we will use some notions from $\mathcal{D}$-module theory, we refer to [5] or [3] for unexplained terminology. All modules over a non-commutative ring (or over a sheaf of non-commutative rings) will be assumed to be left modules. On dealing with arrangements defined over fields of positive characteristic we will refer to the notion of $F$-module introduced in [11, Definition 1.1]. If $A$ is a ring, we denote by $\text{Mod}_A$ the category of $A$-modules. For any ring $A$, if $M$ is an $A$-module endowed with a filtration $\{F_k\}_{k \geq 0}$, it will be always assumed that the filtration is exhaustive, i.e. $M = \bigcup_k F_k$, and we will agree that $F_{-1} = \{0\}$. We denote $\text{Vect}_k$ the category of $k$-vector spaces.

For arrangements of subspaces over a field of characteristic zero defined by monomial ideals, an algorithm to compute the characteristic cycles of the local cohomology modules considered in this paper was given by Alvarez Montaner [1]. In the same case, some partial results were obtained by Barkats in her thesis [2].

1. Filtrations on local cohomology modules

Let $S$ be an inductive system of $R$-modules. Roos [14] introduced a complex which has as $i$th cohomology the $i$th left-derived functor of the inductive limit functor evaluated at $S$ (and the dual notion for projective systems as well, this is actually the case treated by Roos in more detail). We recall his definition in the case of interest for us.
Let \((P, \leq)\) be a partially ordered set, let \(\mathcal{C}\) be an abelian category with enough projectives and such that the direct sum functors are exact (usually, \(\mathcal{C}\) will be a category of modules, sometimes with enhanced structure: a \(D\)-module structure, a \(F\)-module structure or a grading). We will regard \(P\) as a small category which has as objects the elements of \(P\) and, given \(p, q \in P\), there is one morphism \(p \to q\) if \(p \leq q\). A diagram over \(P\) of objects of the category \(\mathcal{C}\) is by definition a covariant functor \(F: P \to \mathcal{C}\):

Note that the image of \(F\) is an inductive system of objects of \(\mathcal{C}\) indexed by \(P\). The category which has as objects the diagrams of objects of \(\mathcal{C}\) and as functors the natural transformations is abelian and will be denoted \(\text{Diag}(P, \mathcal{C})\):

**Definition.** The Roos complex of \(F\) is the homological complex of objects of \(\mathcal{C}\) defined by

\[
\text{Roos}_k(F) := \bigoplus_{p_0 < \ldots < p_k} F_{p_0} \ldots p_k,
\]

where \(F_{p_0} \ldots p_k = F(p_0)\) and, if \(i > 0\) and we denote by \(\pi_{p_0 \ldots p_i} \ldots p_k\) the projection from \(\bigoplus_{p_0 < \ldots < p_k} F_{p_0} \ldots p_k\) onto \(F_{p_0} \ldots p_i \ldots p_k\), the differential on \(F_{p_0} \ldots p_k\) is given by

\[
F(p_0 \to p_1) + \sum_{i=1}^k (-1)^i \pi_{p_0 \ldots p_i} \ldots p_k.
\]

This construction defines a functor \(\text{Roos}_* (\cdot) : \text{Diag}(P, \mathcal{C}) \to \mathcal{C}(\mathcal{C})\), where \(\mathcal{C}(\mathcal{C})\) denotes the category of chain complexes of objects of \(\mathcal{C}\). It is easy to see that this functor is exact and commutes with direct sums.

Let \(X \subset \mathbb{A}^n_k\) be an arrangement defined by an ideal \(I \subset R\). Given \(p \in P(X)\), we will denote by \(X_p\) the linear affine variety in \(\mathbb{A}^n_k\) corresponding to \(p\) and by \(I_p \subset R\) the radical ideal which defines \(X_p\) in \(\mathbb{A}^n_k\). Note that the poset \(P(X)\) is isomorphic to the poset of ideals \(\{I_p\}_p\), ordered by reverse inclusion. We denote by \(h(p)\) the \(k\)-codimension of \(X_p\) in \(\mathbb{A}^n_k\) (that is, \(h(p)\) equals the height of the ideal \(I_p\)). The height of an ideal \(J \subset R\) will be denoted by \(h(J)\).

Let \(M\) be a \(R\)-module, \(i \geq 0\) an integer. Then one can define a diagram of \(R\)-modules \(H^i_{\lfloor x \rfloor}(R)\) on the poset \(P(X)\) by

\[
H^i_{\lfloor x \rfloor}(M) : p \mapsto H^i_{L_p}(M).
\]

This defines a functor \(H^i_{\lfloor x \rfloor}(\cdot) : \text{Mod}_R \to \text{Diag}(P(X), \text{Mod}_R)\).

**Lemma.** If \(E\) is an injective \(R\)-module, then the augmented Roos complex

\[
\text{Roos}_*(H^0_{\lfloor x \rfloor}(E)) \to H^0_I(E) \to 0
\]

is exact.
Proof. Since both Roos,\((\cdot)\) and \(H^0_{[\cdot]}(\cdot)\) commute with direct sums, by the Matlis–Gabriel theorem we can assume that there is a prime ideal \(p \subset R\) such that \(E = E_R(R/p)\), the injective envelope of \(R/p\) in the category of \(R\)-modules. Note also that for any ideal \(J \subset R\), \(H^0_J(E_R(R/p)) = E_R(R/p)\) if \(p \subseteq J\) and is zero otherwise. It will be enough to prove that if \(m \subset R\) is a maximal ideal, then the complex

\[
(\text{Roos}_{\cdot}(H^0_{[\cdot]}(E)))_m \to (H^0_{[\cdot]}(E))_m \to 0
\]

is exact. If \(I \subset m\), this complex is zero. Otherwise, it equals the augmented complex \(\text{Simp}_{\cdot}(K, E_{R_m}(R_m/pR_m)) \to E_{R_m}(R_m/pR_m) \to 0\), where \(K\) is the simplicial complex attached to the subposet of \(P(X)\) which has as vertexes those linear subspaces \(X_p\) such that \(I_p \subseteq m\). As \(K\) has a unique maximal element, it is contractible and then the lemma follows. \(\Box\)

Fix now an injective resolution \(0 \to R \to E^*\) of \(R\) in the category of \(R\)-modules. Each of the modules \(E^j\) \((j \geq 0)\) defines a diagram \(H^0_{[\cdot]}(E^j)\) over \(P(X)\) and one obtains a double complex

\[
\text{Roos}_{\cdot-i}(H^0_{[\cdot]}(E^j)), \quad i \leq 0, \quad j \geq 0
\]

(the change of sign on the indexing of the Roos complex is because we prefer to work with a double complex which is cohomological in both degrees). This is a second quadrant double complex with only a finite number of non-zero columns, so it gives rise to a spectral sequence that converges to \(H^*_I(R)\) (because of the lemma above). More precisely, we have

\[
E^{-i,j}_1 = \text{Roos}_{i}(H^0_{[\cdot]}(E^j)) \Rightarrow H^{i-j}_{I}(R).
\]

The differential \(d_1\) is that of the Roos complex, and since this complex computes the \(i\)th left-derived functor of the inductive limit, the \(E_2\) term will be

\[
E^{-i,j}_2 = \text{indlim}_P^{(i)} H^0_{[\cdot]}(R) \Rightarrow H^{i-j}_{I}(R)
\]

(1)

(we write \(P\) for \(P(X)\) in order to simplify the writing of our formulas). Hereafter this sequence will be called Mayer–Vietoris spectral sequence.

Remark 1.1. (i) Instead of an injective resolution of \(R\) in the category of \(R\)-modules, one could take as well an acyclic resolution with respect to the functors \(\Gamma_J(\cdot)\), \(J \subset R\) an ideal (recall that \(\Gamma_J(M) := \{m \in M \mid \exists r \geq 0\ \text{such that} \ J^rm = 0\}\)). This fact will be used in the next sections.

(ii) If the base field \(k\) is of characteristic zero, we can choose an injective resolution of \(R\) in the category of modules over the Weyl algebra \(A_n(k)\). Since \(A_n(k)\) is free as an \(R\)-module, it follows (see e.g. [3, II.2.1.2]) that this is also an injective resolution of \(R\) in the category of \(R\)-modules. If the base field \(k\) is of characteristic \(p > 0\), the ring \(R\)
has a natural $F$-module structure and its minimal injective resolution is a complex of $F$-modules and $F$-module homomorphisms (see [11, (1.2.b)]). Therefore, the spectral sequence above may be regarded as a spectral sequence in the category of $A_n(k)$-modules (respectively, of $F$-modules). The main result of this section is the following:

**Theorem 1.2.** Let $X \subset \mathbb{A}_k^n$ be an arrangement of linear varieties. Let $K(>p)$ be the simplicial complex attached to the subposet $\{ q \in P(X) \mid q > p \}$ of $P(X)$. Then:

(i) There are $R$-module isomorphisms

$$\text{indlim}_p \bigl( H_\bullet^i(R) \simeq \bigoplus_{h(p)=j} [H^j_{I_p}(R) \otimes k \check{H}_{i-1}(K(>p); k)],$$

where $\check{H}$ denotes reduced simplicial homology. We agree that the reduced homology with coefficients in $k$ of the empty simplicial complex is $k$ in degree $-1$ and zero otherwise.

(ii) The Mayer–Vietoris spectral sequence

$$E_2^{-ij} = \text{indlim}_p \bigl( H_\bullet^j(R) \Rightarrow H^i_{I_p}(R) \bigr)$$

degenerates at the $E_2$-term.

(iii) If $k$ is a field of characteristic zero, the isomorphisms in (i) are also isomorphisms of $A_n(k)$-modules and the spectral sequence in (ii) is a spectral sequence of $A_n(k)$-modules. If $k$ is a field of positive characteristic, the isomorphisms in (i) are also isomorphisms of $F$-modules and the spectral sequence in (ii) is a spectral sequence of $F$-modules.

**Proof** (cf. [4, Proposition 4.5]). (i) Given $p \in P(X)$ and a $R$-module $M$ we consider the following three diagrams:

- $F_{M, > p}$, defined by $F_{M, > p}(q) = M$ if $q > p$ and $F_{M, > p}(q) = 0$ otherwise,
- $F_{M, \geq p}$, defined by $F_{M, \geq p}(q) = M$ if $q \geq p$ and $F_{M, > p}(q) = 0$ otherwise,
- $F_{M, p}$, defined by $F_{M, p}(q) = M$ if $q = p$ and $F_{M, p}(q) = 0$ otherwise

(in all three cases $F(p \rightarrow q) = \text{id}$ if $F(p) = F(q)$ and it is zero otherwise). In the category of diagrams of $R$-modules over $P(X)$ we have an exact sequence

$$0 \rightarrow F_{M, > p} \rightarrow F_{M, \geq p} \rightarrow F_{M, p} \rightarrow 0.$$ 

Let $K(\geq p)$ be the simplicial complex attached to the subposet $\{ q \in P(X) \mid q \geq p \}$ of $P(X)$. Then one has

$$\text{Roos}_*(F_{M, \geq p}) = \text{Simp}_*(K(\geq p), M) \quad \text{and} \quad \text{Roos}_*(F_{M, > p}) = \text{Simp}_*(K(> p), M).$$
Since the complex $K(\geq p)$ is contractible (to the vertex corresponding to $p$), the long exact homology sequence obtained from the sequence of complexes

$$0 \to \text{Roos}_*(F_{M,>p}) \to \text{Roos}_*(F_{M,\geq p}) \to \text{Roos}_*(F_{M,p}) \to 0$$

gives

$$\text{indlim}_p^{(i)} F_{M,p} \cong \tilde{H}_{i-1}(K(\geq p); M),$$

where the tilde denotes reduced homology and we agree that the reduced homology of the empty simplicial complex is $M$ in degree $-1$ and zero otherwise.

Notice that for any $p \in P(X)$ the module $H^i_{I_p}(R)$ vanishes unless $h(I_p) = j$, so one has an isomorphism of diagrams $H_j^i(I_p(R)) \cong \oplus_{h(I_p) = j} F_{H^i_{I_p}(R),p}$. Thus,

$$\text{indlim}_p^{(i)} H^i_{I_p}(R) \cong \oplus_{h(I_p) = j} \text{indlim}_p^{(i)} F_{H^i_{I_p}(R),p} \cong \oplus_{h(I_p) = j} \tilde{H}_{i-1}(K(\geq p); H^i_{I_p}(R)).$$

By the universal coefficient theorem, for $i > 0$

$$\tilde{H}_{i-1}(K(\geq p); H^i_{I_p}(R)) \cong H^i_{I_p}(R) \otimes_k \tilde{H}_{i-1}(K(\geq p), k).$$

Although the isomorphism given by the universal coefficient theorem is a priori only an isomorphism of $k$-vector spaces, it is easy to check that in our case it is also an isomorphism of $A_n(k)$-modules (if $\text{char}(k) = 0$) or of $F$-modules (if $\text{char}(k) > 0$). In particular, it is always an isomorphism of $R$-modules.

(ii) and (iii) Observe first that if $I \subset R$ is an ideal and we set $h = h(I)$, then all associated primes of $H^i_I(R)$ are minimal primes of $I$ (this is due to the structure of the minimal injective resolution of $R$, in fact it holds for any Gorenstein ring, see e.g. [12, Theorem 18.8]). It follows that if $p, q \subset R$ are prime ideals such that $p \not\subset q$ and we set $i = h(p)$, $j = h(q)$, then $\text{Hom}_R(H^i_{I_p}(R), H^j_{I_q}(R)) = 0$. From this last fact and (i) above, it follows that the Mayer–Vietoris sequence degenerates at the $E_2$-term. Part (iii) follows from Remark 1.1(ii) above and the observation at the end of the proof of part (i). \qed

**Corollary 1.3.** Let $X \subset \mathbb{A}^n_k$ be an arrangement of linear varieties, let $I \subset R$ be an ideal defining $X$. Then, for all $r \geq 0$ there is a filtration $\{F^r_j\}_{r \leq j \leq n}$ of $H^i_I(R)$ by $R$-submodules such that

$$F^r_j / F^r_{j-1} \cong \bigoplus_{h(p) = j} [H^i_{I_p}(R) \otimes_k \tilde{H}_{h(p) - r - 1}(K(\geq p); k)]$$

for all $j \geq r$ (we agree that $F^r_{r-1} = 0$). This filtration is functorial with respect to affine transformations. Moreover, if $\text{char}(k) = 0$ it is a filtration by holonomic $A_n(k)$-modules and if $\text{char}(k) > 0$ it is a filtration by $F$-modules. Set

$$m_{r,p} = \dim_k \tilde{H}_{h(p) - r - 1}(K(\geq p); k).$$
If char($k$) = 0, the characteristic cycle of the holonomic $A_n(k)$-module $H^r_I(R)$ is

$$CC(H^r_I(R)) = \sum m_{r,p} T^*_{X_p} \mathbb{A}^n_k,$$

where $T^*_{X_p} \mathbb{A}^n_k$ denotes the relative conormal subspace of $T^* \mathbb{A}^n_k$ attached to $X_p$.

If $k = \mathbb{R}$ is the field of real numbers, the Betti numbers of the complement of the arrangement $X$ in $\mathbb{A}^n_\mathbb{R}$ can be computed in terms of the multiplicities $\{m_{i,p}\}$ as

$$\dim \mathbb{Q} \tilde{H}_i(\mathbb{A}^n_\mathbb{R} - X; \mathbb{Q}) = \sum_p m_{i+1,p}.$$

If $k = \mathbb{C}$ is the field of complex numbers, then one has

$$\dim \mathbb{Q} \tilde{H}_i(\mathbb{A}^n_\mathbb{C} - X; \mathbb{Q}) = \sum_p m_{i+1-h(p),p}.$$

**Proof.** The filtration $\{F^r_I\}$ is the one given by the degeneration of the Mayer–Vietoris spectral sequence. It is a filtration by $A_n(k)$-modules (respectively, $F$-modules) by Theorem 1.2(iii). The formula for the characteristic cycle follows from the fact that if $h(I_p) = h$, then $CC(H^h_{I_p}(R)) = T^*_{X_p} \mathbb{A}^n_k$ and the additivity of the characteristic cycle with respect to short exact sequences. The formula for the Betti numbers of the complement $\mathbb{A}^n_\mathbb{R} - X$ follows from a theorem of Goresky–MacPherson [9, III.1.3. Theorem A], which states (slightly reformulated) that

$$\tilde{H}_i(\mathbb{A}^n_\mathbb{R} - X; \mathbb{Z}) \cong \bigoplus_p H^{h(p)-i-1}(K(\geq p), K(> p); \mathbb{Z}).$$

Regarding a complex arrangement in $\mathbb{A}^n_\mathbb{C}$ as a real arrangement in $\mathbb{A}^{2n}_\mathbb{R}$, the formula for the Betti numbers of the complement of a complex arrangement follows from the formula for real arrangements.

**Remark 1.4.** (i) Set $R = k[x, y]$, consider the ideal $I = (x \cdot y) \subset k[x, y]$ and denote $I_1 = (x)$, $I_2 = (y)$, $m = (x, y)$. For the filtration of $H^1_I(R)$ introduced above one has $F_1 \cong H^1_{I_1}(R) \oplus H^1_{I_2}(R)$, $F_2 = H^2_I(R)$, and the sequence $0 \to F_1 \to F_2 \to F_2/F_1 \to 0$ is nothing but the Mayer–Vietoris exact sequence

$$0 \to H^1_{I_1}(R) \oplus H^1_{I_2}(R) \to H^1_I(R) \to H^2_m(R) \to 0.$$

This sequence is not split, e.g. because the maximal ideal $m$ is not a minimal prime of $I$ and so cannot be an associated prime of $H^1_I(R)$. Therefore, the extension problems attached to the filtration introduced in Corollary 1.3 are non-trivial in general. This question will be studied in Section 3.

(ii) The formalism of Mayer–Vietoris sequences can be applied to functors other than $H^*_I(\cdot)$ (and other than those considered in [4]). For example, one can consider
the diagrams

$$\operatorname{Ext}^i_R(R/\{\}, R) : p \mapsto \operatorname{Ext}^i_R(R/I_p, R)$$

and, similarly as for the local cohomology modules, one has a spectral sequence

$$E_2^{i,j} = \text{indlim}_p \operatorname{Ext}^i_R(R/I_p, R) \Rightarrow \operatorname{Ext}^j_R(R/I, R),$$

which degenerates at the $E_2$-term. Therefore, one can endow the module $\operatorname{Ext}^i_R(R/I, R)$ with a filtration $\{G_j^i\}_{r \leq j \leq n}$ such that

$$G_j^i/G_j^{i-1} \cong \bigoplus_{h(p) = j} [\operatorname{Ext}^i_R(R/I_p, R) \otimes_k \hat{H}_{p(h(p)) - 1}(K(>p); k)].$$

The functoriality of the construction gives that the natural morphism $\operatorname{Ext}^i_R(R/I, R) \to H^i_f(R)$ is filtered. In case $I \subseteq R$ is a monomial ideal, it would be interesting to compare this filtration with the one defined in [13, Theorem 3.3].

2. Betti numbers vs. multiplicities

The ring $R = k[x_1, \ldots, x_n]$ has a natural $\mathbb{Z}^n$-gradation given by $\deg(x_i) = e_i$, where $e_1, \ldots, e_n$ denotes the canonical basis of $\mathbb{Z}^n$. If $M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_{\alpha}$ and $N = \bigoplus_{\alpha \in \mathbb{Z}^n} N_{\alpha}$ are graded $R$-modules, a morphism $f : M \to N$ is said to be graded if $f(M_{\alpha}) \subseteq N_{\alpha}$ for all $\alpha \in \mathbb{Z}^n$. Henceforth, the term graded will always mean $\mathbb{Z}^n$-graded.

We denote by $\textbf{Mod}_R$ the category which has as objects the graded $R$-modules and as morphisms the graded morphisms. If $M, N$ are graded $R$-modules, we denote by $\text{Hom}_R(M, N)$ the group of graded morphisms from $M$ to $N$ (this group should not be confused with the internal Hom in the category $\textbf{Mod}_R$, in particular it is usually not a graded $R$-module). Its derived functors will be denoted $\text{Ext}^i_R(M, N)$, $i \geq 0$.

We recall some facts about $\textbf{Mod}_R$ which will be relevant for us (see [10] for proofs and related results). If $M$ is a graded module one can define its $*$-injective envelope $*E(M)$ (in particular, $\textbf{Mod}_R$ is a category with enough injectives). A graded version of the Matlis–Gabriel theorem holds: the indecomposable injective objects of $\textbf{Mod}_R$ are the shifted injective envelopes $*E(R/p)(\alpha)$, where $p$ is an homogeneous prime ideal of $R$ and $\alpha \in \mathbb{Z}^n$, and every graded injective module is isomorphic to a unique (up to order) direct sum of indecomposable injectives.

Let $I \subseteq R$ be a monomial ideal. Then, $R/I^j$ is a finitely generated graded $R$-module for any $j \geq 0$, and $\text{Ext}^i_R(R/I^j, N) \cong \text{Ext}^i_R(R/I^j, N)$ for any graded $R$-module $N$. It follows that injective objects of $\textbf{Mod}_R$, which usually are not injective as objects of $\textbf{Mod}_R$, are acyclic with respect to the functor $\Gamma_I$ (see [10]). It also follows that the local cohomology modules $H^i_I(R)$ are objects of $\textbf{Mod}_R$. Our aim is to relate the dimensions of its homogeneous components to the multiplicities of its characteristic cycle.
A homogeneous prime ideal of $R$ is of the form $p = (x_{i_1}, \ldots, x_{i_k})$ $1 \leq i_1 < \cdots < i_k \leq n$. We will denote by $\mathcal{P}$ the set of homogeneous prime ideals of $R$. Putting $\Omega = \{-1, 0\}^n$, there is a bijection

$$\Omega \to \mathcal{P},$$

$$\alpha \mapsto p_\alpha = \langle x_{i_k} \mid x_{i_k} = -1 \rangle.$$

For $\alpha \in \Omega$, we will set $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$.

Let $I \subset R$ be a monomial ideal, $r \geq 0$ an integer, let $X \subset \mathbb{A}_k^n$ be the arrangement defined by $I$. It will be convenient to reindex the multiplicities introduced in Section 1 as follows:

$$m_{r, \alpha} := \begin{cases} m_{r, p} & \text{if there is a } p \in P(X) \text{ with } p_\alpha = I_p, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have:

**Proposition 2.1.** In the situation and with the notations described above,

$$m_{r, \alpha} = \dim_k H^r_I(R)_\alpha \text{ for all } r \geq 0, \ \alpha \in \Omega.$$

**Proof.** Since $*$-injective modules are $I$-acyclic, by Remark 1.1(i) one can assume that the filtration $\{F_j\}_{r \leq j \leq n}$ of $H^r_I(R)$ introduced in Corollary 1.3 has been constructed from an injective resolution of $R$ in $\text{Mod}_R$. From this fact, it follows that we can assume that the $R$-modules $F_j^r$ are graded, the inclusion maps $F_{j-1}^r \hookrightarrow F_j^r$ are graded morphisms and one has isomorphisms of graded modules

$$F_j^r / F_{j-1}^r \cong \bigoplus_{|\alpha| = j} (H^r_{\text{ps}}(R))^{\oplus m_{r, \alpha}}.$$

Note that for $\alpha, \beta \in \Omega$, one has

$$(H^{|\alpha|}_{\text{ps}}(R))_\beta = \begin{cases} 0 & \text{if } \beta \neq \alpha, \\ k & \text{if } \beta = \alpha. \end{cases}$$

Using these facts and the exactness of the functors $\text{Mod}_R \to \text{Vect}_k$, $M \mapsto M_\alpha$ the desired result follows. \qed

Given a reduced monomial ideal $I \subset R$, the modules $\text{Tor}^R_i(I, k)$ are $\mathbb{Z}^n$-graded and the graded Betti numbers of $I$ are defined as

$$\beta_{i, \alpha}(I) := \dim_k \text{Tor}^R_i(I, k)_\alpha.$$
where $\alpha \in \mathbb{Z}^n$. The Alexander dual ideal of $I$ is the ideal

$$I^\vee = \left\langle \prod_{i \in S} x_i \mid S \subseteq \{1, \ldots, n\}, \prod_{i \notin S} x_i \notin I \right\rangle.$$ 

Mustață [13] showed that if $\alpha \in \Omega$, then one has

$$\beta_{i,-2}(I^\vee) = \dim_k H_{i-1}^{[\alpha]}(R)_{\alpha},$$

and all other graded Betti numbers are zero. So, we have:

**Corollary 2.2.** If $I$ is a reduced monomial ideal and $\alpha \in \Omega$, then

$$\beta_{i,-2}(I^\vee) = m_{|\alpha|-i,\alpha}.$$ 

In particular, if $\mathrm{char}(k) = 0$, the graded Betti numbers of a monomial ideal $I$ can be obtained from the ($\mathcal{O}$-module theoretic) characteristic cycles of the local cohomology of $R$ supported at its Alexander dual $I^\vee$.

Also, if $k = \mathbb{C}$ or $k = \mathbb{R}$, it follows from the corollary above and Corollary 1.3 that the (topological) Betti numbers of the complement in $\mathbb{A}^n_k$ of the arrangement defined by a monomial ideal $I$ can be obtained from the (algebraic) graded Betti numbers of $I^\vee$. This fact was already proved using a different approach in [8].

### 3. Extension problems

If $M$ is a graded $R$-module and $\alpha \in \mathbb{Z}^n$, as usual we denote by $M(\alpha)$ the graded $R$-module whose underlying $R$-module structure is the same as that of $M$ and where the grading is given by $(M(\alpha))_{\beta} = M_{\alpha + \beta}$. If $\alpha \in \mathbb{Z}^n$, we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\text{supp}_+(\alpha) = \{i \mid \alpha_i > 0\}$. We recall the following definition of Yanagawa:

**Definition** (Yanagawa [16, 2.7]). A $\mathbb{Z}^n$-graded module is said to be **straight** if the following two conditions are satisfied:

(i) $\dim_k M_\alpha < \infty$ for all $\alpha \in \mathbb{Z}^n$.

(ii) The multiplication map $M_\alpha \circ \gamma \mapsto \gamma^\beta M_{\alpha + \beta}$ is bijective for all $\alpha, \beta \in \mathbb{Z}^n$ with $\text{supp}_+(\alpha + \beta) = \text{supp}_+(\alpha)$.

The full subcategory of the category $\ast \text{Mod}_R$ which has as objects the straight modules will be denoted $\text{Str}$. Let $\varepsilon = -\sum_{i=1}^n \varepsilon_i = (-1, \ldots, -1)$. In order to avoid shiftings in local cohomology modules, we will consider instead the following (equivalent) category:
**Definition.** We will say that a graded module $M$ is $e$-straight if $M(e)$ is straight in the above sense. We denote $\mathbf{eStr}$ the full subcategory of $^*\text{Mod}_R$ which has as objects the $e$-straight modules. It follows from [13,16, Theorem 2.13] that if $I \subset R$ is a monomial ideal and $r \geq 0$ is an integer, $H^r_I(R)$ is an $e$-straight module.

**Proposition 3.1.** Let $M$ be a $e$-straight module. There is a finite increasing filtration $\{F_j\}_{0 \leq j \leq n}$ of $M$ by $e$-straight submodules such that for all $0 \leq j \leq n$ one has graded isomorphisms

$$F_j/F_{j-1} \cong \bigoplus_{z \in \Omega, |z|=j} (H^l_{p_z}(R))^\oplus m_z,$$

where $m_z = \dim_k M_z$.

**Proof.** The existence of an increasing filtration $\{G_j\}_j$ of $M$ by $e$-straight submodules such that all quotients $G_j/G_{j-1}$ are isomorphic to local cohomology modules supported at homogeneous prime ideals is an immediate transposition to $e$-straight modules of [16, 2.12] (which relies on [15, 2.5]). Inspection of Yanagawa’s proof shows that, in order to prove the existence of a filtration $\{F_j\}_j$ satisfying the condition of the proposition, it is enough to show that if $p_a, p_b$ are homogeneous prime ideals with $|a| = |\beta| = l$, then $^*\text{Ext}^1_R(H^l_{p_a}(R), H^l_{p_b}(R)) = 0$. The minimal $^*$-injective resolution of $H^l_{p_b}(R)$ is

$$0 \to H^l_{p_b}(R) \to ^*\text{Ext}_R(R/p_{\beta}(a,c)) \to \bigoplus_{l|\beta|=0} ^*\text{Ext}_R(R/p_{\beta-\epsilon_{\beta}}(a,c)) \to \cdots,$$

(see e.g. [16, 3.12]). Thus the vanishing of the above $^*$Ext module follows from the following:

**Claim.** For all homogeneous prime ideals $q \supset p_{\beta}$ with $h(q) - h(p_{\beta}) = 1$, one has $^*\text{Hom}_R(H^l_{p_q}(R), ^*\text{Ext}_R(R/q)(-\epsilon)) = 0$.

**Proof.** If $q$ is such an homogeneous prime ideal, we will assume that $q \supset p_{a}$ (otherwise, the statement follows easily from the fact that $q$ is the only associated prime of $^*\text{Ext}_R(R/q)$). Set $\alpha := c - a \in \Omega$. It suffices to prove the vanishing of $^*\text{Hom}_R(H^l_{p_a}(R), ^*\text{Ext}_R(R/q))$. By the equivalence of categories proved in [16, 2.8], there is a bijection between this group and $^*\text{Hom}_R(R/p_{a}(\alpha,c), R/q)$. A graded morphism $\varphi : R/p_{a}(\alpha,c) \to R/q$ is determined by $\varphi(1) \in (R/q)_{-\alpha}$. But $(R/q)_{-\alpha} = 0$, so we are done.

The equality $m_z = \dim_k M_z$ is proved as in the proof of Proposition 2.1. □

**Remark.** Even if $M$ is a local cohomology module supported at a monomial ideal, in general the submodules $F_j$ introduced in the proposition above are not. This is one of the reasons to consider the category of $e$-straight modules.
Henceforth we will assume that a \(\varepsilon\)-straight module \(M\) together with an increasing filtration \(\{F_j\}_j\) of \(M\) as in Proposition 3.1 have been fixed. For each \(0 \leq j \leq n\), one has an exact sequence

\[
(s_j): 0 \to F_{j-1} \to F_j \to F_j/F_{j-1} \to 0,
\]

which defines an element of \(\text{Ext}^1_R(F_j/F_{j-1}, F_{j-1})\). In this section, our aim is to show that this element is determined, in a sense that will be made precise below, by the \(k\)-linear maps \(\cdot x_i : M_x \to M_{x+\varepsilon_i}\), where \(|x| = j\) and \(i\) is such that \(x_i = -1\). In particular, the sequence \((s_j)\) splits if and only if \(\chi_i M_x = 0\) for \(x, i\) in this range. We will prove first the following lemma:

**Lemma.** The natural maps

\[
\text{Ext}^1_R(F_j/F_{j-1}, F_{j-1}) \to \text{Ext}^1_R(F_j/F_{j-1}, F_{j-1}/F_{j-2})
\]

are injective for all \(j \geq 2\).

**Proof.** From the short exact sequence \((s_{j-1})\), applying \(\text{Hom}(F_j/F_{j-1}, \cdot)\) we obtain the exact sequence

\[
\text{Ext}^1_R(F_j/F_{j-1}, F_{j-2}) \to \text{Ext}^1_R(F_j/F_{j-1}, F_{j-1}) \to \text{Ext}^1_R(F_j/F_{j-1}, F_{j-1}/F_{j-2}),
\]

and we have to prove that \(\text{Ext}^1_R(F_j/F_{j-1}, F_{j-2}) = 0\). Applying again \(\text{Hom}(F_j/F_{j-1}, \cdot)\) to the exact sequences \((s_l)\) for \(l \leq j - 2\), and descending induction, the assertion reduces to the statement \(\text{Ext}^1_R(F_j/F_{j-1}, F_l/F_{j-1}) = 0\) for \(l \leq j - 2\). By Proposition 3.1, it suffices to prove that if \(p, q \in R\) are homogeneous prime ideals of heights \(h(p) = j\) and \(h(q) = l\) and \(l \leq j - 2\), then \(\text{Ext}^1_R(H^l_p(R), H^j_q(R)) = 0\). We will have \(q = p_\beta\), \(l = |\beta|\), for some \(\beta \in \Omega\). As observed in the proof of Proposition 3.1, the minimal \(\varepsilon\)-injective resolution of \(H^{|\beta|}_p(R)\) is

\[
0 \to H^{|\beta|}_p(R) \to \text{E}_R(R/p_\beta)(-\varepsilon) \to \bigoplus_{i|\beta|=0} \text{E}_R(R/p_{\beta-\varepsilon_i})(-\varepsilon) \to \cdots.
\]

Thus, again as in the proof of Proposition 3.1, it suffices to prove that

\[
\text{Hom}_R(H^l_p(R), \text{E}_R(R/p_{\beta-\varepsilon_i})(\varepsilon)) = 0.
\]

This follows as in the proof of Theorem 1.2(ii), because \(p_{\beta-\varepsilon_i}\) is the only associated prime of \(\text{E}_R(R/p_{\beta-\varepsilon_i})\). \(\square\)

**Remark 3.2.** If \(I \subset R\) is an ideal defining an arbitrary arrangement of linear varieties in \(\mathbb{A}^n_k\), one can prove an analogous lemma for the filtration of the local cohomology module \(H^l_I(R)\) introduced in Section 1.
The extension class of \((s_j)\) maps, via the morphism in the above lemma, to the extension class of the sequence
\[(s'_j): 0 \rightarrow F_{j-1}/F_{j-2} \rightarrow F_j/F_{j-2} \rightarrow F_j/F_{j-1} \rightarrow 0.\]

Let
\[0 \rightarrow F_{j-1}/F_{j-2} \rightarrow \ast E^0 \rightarrow \ast E^1 \rightarrow \cdots\]
be the minimal \(*\)-injective resolution of \(F_{j-1}/F_{j-2}\). Given a graded morphism \(F_j/F_{j-1} \rightarrow \text{Im} \, d^0\), one obtains an extension of \(F_{j-1}/F_{j-2}\) by \(F_j/F_{j-1}\) taking the following pull-back:

\[
\begin{array}{ccc}
0 & \rightarrow & F_{j-1}/F_{j-2} \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Im} \, d^0 \\
& & \varphi \\
\end{array}
\]

and all extensions of \(F_{j-1}/F_{j-2}\) by \(F_j/F_{j-1}\) are obtained in this way. Take \(\zeta \in \Omega\) with \(|\zeta| = j\). Applying the functor \(H^*_p(\cdot)\) to this diagram, we obtain a commutative square

\[
\begin{array}{ccc}
H^0_p(\text{Im} \, d^0) & \xrightarrow{\sim} & H^1_p(F_{j-1}/F_{j-2}) \\
\varphi^\alpha \downarrow & & \downarrow \sim \\
H^0_p(F_j/F_{j-1}) & \xrightarrow{\delta^\alpha} & H^1_p(F_{j-1}/F_{j-2})
\end{array}
\]

where

(i) The upper horizontal arrow is an isomorphism because
\[\ast E^0 \simeq \bigoplus_{|\rho| = j-1} \ast(E_R(R/p_\rho)) \oplus m_\rho(-\breve{\epsilon}),\]
and then \(H^i_p(\ast E^0) = 0\) for all \(i \geq 0\).

(ii) The morphism \(\delta^\alpha\) is the connecting homomorphism of the given extension.

Since \(F_j/F_{j-1} = \bigoplus_{|\zeta| = j} H^0_p(F_j/F_{j-1})\), the morphism \(\varphi\) is determined by the morphisms \(\varphi^\alpha\) for \(|\zeta| = j\), and these are in turn determined by the connecting homomorphisms \(\delta^\alpha\) via the commutative square above. Observe also that the module \(H^1_p(F_j/F_{j-1})\) is isomorphic to a direct sum of local cohomology modules of the form \(H^i_p(R)\). We will show next that, because of this fact, it suffices to consider the restriction of \(\delta^\alpha\) to the \(k\)-vector space of homogeneous elements of multidegree \(\zeta\).
Lemma (Linearization). Let \( k_1, k_2 \geq 0 \) be integers, let \( M_1 = H^l_{p_k}(R)^{\oplus k_1}, M_2 = H^l_{p_k}(R)^{\oplus k_2} \). The restriction map

\[ \ast \text{Hom}_R(M_1, M_2) \to \text{Hom}_{\text{Vect}}((M_1)_a, (M_2)_a) \]

is a bijection. Moreover, these bijections are compatible with composition of graded maps.

Proof. Taking the components of a graded map \( M_1 \to M_2 \), it will be enough to prove that all graded endomorphisms of \( N := H^l_{p_k}(R) \) are multiplications by constants of the base field \( k \). Let \( \varphi : N \to N \) be a graded endomorphism. Note that \( \varphi \) can be regarded also as an endomorphism of the graded module \( N(\varepsilon) \). By Yanagawa [16], \( N(\varepsilon) \) is a straight module and \( \varphi \) is determined by its restriction to the \( \mathbb{N}^d \)-graded part of \( N(\varepsilon) \), which is \( R/p_a(\varepsilon^c) \) (\( \varepsilon^c = \varepsilon - \alpha \)). A graded \( R \)-module map \( R/p_a(\varepsilon^c) \to R/p_a(\varepsilon^c) \) is determined by the image of \( 1 \in (R/p_a(\varepsilon^c))_{-\varepsilon^c} \). Since

\[ (R/p_a(\varepsilon^c))_{-\varepsilon^c} = (R/p_a)_0 = k, \]

\( \varphi \) must be the multiplication by some constant, as was to be proved. \( \square \)

Thus, any extension of \( F_{j-1}/F_{j-2} \) by \( F_j/F_{j-1} \) is determined by the \( k \)-linear maps

\[ \delta^y_x : H^0_{p_k}(F_j/F_{j-1})_y \to H^1_{p_k}(F_{j-1}/F_{j-2})_y. \]

One can easily check that \( H^0_{p_k}(F_j/F_{j-1})_y \simeq M_y \), and using a Čech complex one obtains that

\[ H^1_{p_k}(F_{j-1}/F_{j-2})_y = \text{Ker} \left[ \bigoplus_{x_j=-1} \left( (F_{j-1}/F_{j-2})_{x_j} \right)_y \to \bigoplus_{x_{j-1}=-1} \left( (F_{j-1}/F_{j-2})_{x_{j-1}x_j} \right)_y \right], \]

where the last arrow is an isomorphism given by multiplication by \( x_j \) on \( (F_{j-1}/F_{j-2})_{x_j} \). The connecting homomorphism obtained applying \( H^1_{p_k}(\cdot, -) \) to the exact sequence \( (s'_j) \) can be computed using Čech complexes as well. It turns out that, via the isomorphisms above, the corresponding map \( \delta^y_x \) is in this case precisely the map

\[ M_y \to \bigoplus_{x_j=-1} M_{y+x_j}, \]

\[ m \mapsto \bigoplus (x_j \cdot m). \]

In conclusion,
Proposition. The extension class \( (s_j) \) is uniquely determined by the \( k \)-linear maps \( \cdot x_i : M_\mathbf{a} \to M_{\mathbf{a} + \mathbf{e}_i} \) where \( |\mathbf{a}| = j \) and \( \mathbf{a}_i = -1 \).

Remarks. (i) Mustaţă [13] has proved that for local cohomology modules supported at monomial ideals, the linear maps \( \cdot x_i : H^j_I(R)_\mathbf{a} \to H^j_I(R)_{\mathbf{a} + \mathbf{e}_i} \) can be explicitly computed in terms of the simplicial cohomology of certain Stanley–Reisner complexes attached to \( I \).

(ii) The very definition of \( \varepsilon \)-straight modules shows that they are determined as graded modules by the vector spaces \( M_\mathbf{a} \) and the multiplication maps \( \cdot x_i : M_\mathbf{a} \to M_{\mathbf{a} + \mathbf{e}_i} \) with \( \mathbf{a}_i = -1 \). However, this fact alone is not very enlightening if one wishes to know how the extension problems arising from the Mayer–Vietoris sequence are related to these data. Regarding \( \varepsilon \)-straight modules as representations of a boolean lattice one can obtain an alternative proof of the results in this section. We have chosen a more algebraic approach because it seems us that it might be better suited to extend our results to local cohomology modules supported at more general types of arrangements (cf. Remark 3.2).

4. Local cohomology and perverse sheaves

In this section we will use the following notations:

- \( R = \mathbb{C}[x_1, \ldots, x_n] \) (by a slight abuse of notation, we will denote by \( R \) as well the sheaf of regular algebraic functions in \( \mathbb{C}^n \)).
- \( \mathcal{O} \) denotes the sheaf of holomorphic functions in \( \mathbb{C}^n \).
- \( \mathcal{D} \) denotes the sheaf of differential operators in \( \mathbb{C}^n \) with holomorphic coefficients.
- \( T \) denotes the union of the coordinate hyperplanes in \( \mathbb{C}^n \), endowed with the stratification given by the intersections of its irreducible components.
- \( X_\mathbf{a} \) denotes the linear subvariety of \( \mathbb{C}^n \) defined by the ideal \( p_\mathbf{a} \subset R \), \( \mathbf{a} \in \Omega \).

We denote \( \text{Perv}^T(\mathbb{C}^n) \) the category of complexes of sheaves of finitely dimensional vector spaces on \( \mathbb{C}^n \) which are perverse relatively to the given stratification of \( T \) [6, I.1]. We denote \( \mathcal{D}^r_{\text{hr}} \) the full abelian subcategory of the category of regular holonomic modules \( \mathcal{M} \) in \( \mathbb{C}^n \) such that their solution complex \( \mathbb{R}\text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O}) \) is an object of \( \text{Perv}^T(\mathbb{C}^n) \). By the Riemann–Hilbert correspondence, the functor of solutions is then an equivalence of categories between \( \mathcal{D}^r_{\text{hr}} \) and \( \text{Perv}^T(\mathbb{C}^n) \).

In [6], the category \( \text{Perv}^T(\mathbb{C}^n) \) has been linearized as follows: Let \( \mathcal{C}_n \) be the category whose objects are families \( \{ \mathcal{M}_\mathbf{a} \}_{\mathbf{a} \in \Omega} \) of finitely dimensional complex vector spaces indexed by \( \Omega = \{ -1, 0 \}^n \), endowed with linear maps

\[
\mathcal{M}_\mathbf{a} \xrightarrow{u_i} \mathcal{M}_{\mathbf{a} - \mathbf{e}_i}, \quad \mathcal{M}_\mathbf{a} \xleftarrow{v_i} \mathcal{M}_{\mathbf{a} - \mathbf{e}_i}
\]
for each \( z \in \Omega \) such that \( z_i = 0 \). These linear maps are called canonical (resp., variation) maps, and they are required to satisfy the conditions:

\[
u_j v_i = v_j v_i, \quad u_i u_j = u_j u_i, \quad u_i v_j = v_j u_i \quad \text{and} \quad v_i u_i + \text{id} \text{ is invertible.}
\]

Such an object will be called an \( n \)-hypercube, the vector spaces \( \mathcal{M}_x \) will be called its vertices. A morphism between two \( n \)-hypercubes \( \{ \mathcal{M}_x \}_x \) and \( \{ \mathcal{N}_x \}_x \) is a set of linear maps \( \{ f_x : \mathcal{M}_x \rightarrow \mathcal{N}_x \}_x \), commuting with the canonical and variation maps (see [6]).

It is proved in [6] that there is an equivalence of categories between \( \text{Perv}^T(\mathbb{C}^n) \) and \( \mathcal{O}_R \). Given an object \( \mathcal{M} \) of \( \mathcal{D}_{hr}^T \), the \( n \)-hypercube corresponding to \( \mathbb{R}\text{Hom}_\mathcal{O}(\mathcal{M}, \mathcal{O}) \) is constructed as follows (see [7]):

Consider \( \mathbb{C}^n = \prod_{i=1}^n \mathbb{C}_i \), with \( \mathbb{C}_i = \mathbb{C} \) for \( 1 \leq i \leq n \), let \( K_i = \mathbb{R}^+ \subset \mathbb{C}_i \) and set \( V_i = \mathbb{C}_i \setminus K_i \). For any \( z = (z_1, \ldots, z_n) \in \Omega \) denote

\[
\mathcal{S}_z := \frac{\Gamma \prod_{i=1}^n V_i \mathcal{O}}{\sum_{a_k=-1}^1 \Gamma_{C_k} \times \prod_{a_k} V_i \mathcal{O}}.
\]

Denoting with a subscript 0 the stalk at the origin, one has:

(i) The vertices of the \( n \)-hypercube associated to \( \mathcal{M} \) are the vector spaces \( \mathcal{M}_x := \text{Hom}_{\mathcal{O}_0}(\mathcal{M}_0, S_x 0) \).

(ii) The linear maps \( u_i \) are those induced by the natural quotient maps \( \mathcal{S}_x \rightarrow \mathcal{S}_{x-e_i} \).

(iii) The linear maps \( v_i \) are the partial variation maps around the coordinate hyperplanes, i.e. for any \( \varphi \in \text{Hom}_{\mathcal{O}_0}(\mathcal{M}_0, \mathcal{S}_x) \) one has \( (v_i \circ u_i)(\varphi) = \Phi_i(\varphi) - \varphi \), where \( \Phi_i \) is the partial monodromy around the hyperplane \( x_i = 0 \).

The following is proved as well in [6, 7]:

(iv) If \( \text{CC}(\mathcal{M}) = \sum m_x \mathcal{T}_{X_x}^* \mathbb{C}^n \) is the characteristic cycle of \( \mathcal{M} \), then for all \( z \in \Omega \) one has the equality \( \dim \mathcal{M}_z = m_z \).

(v) Let \( \alpha, \beta \in \Omega \) be such that \( \alpha_i \beta_i = 0 \) for \( 1 \leq i \leq n \). For each \( j \) with \( \beta_j = -1 \) choose any \( \lambda_j \in \mathbb{C} \setminus \mathbb{Z} \), set \( \lambda = \{ \lambda_j \} \), and let \( I_{\alpha, \beta, \lambda} \) denote the left ideal in \( \mathcal{D} \) generated by \( \{ x_i | x_i = -1 \}, \{ \partial_k | x_k = \beta_k = 0 \}, \{ x_j \partial_j - \lambda_j | \beta_j = -1 \} \). Then the simple objects of the category \( \mathcal{D}_{hr}^T \) are the quotients \( \frac{\mathcal{D}}{I_{\alpha, \beta, \lambda}} \).

**Definition.** We say that an object \( \mathcal{M} \) of \( \mathcal{D}_{hr}^T \) has variation zero if the morphisms \( v_i : \mathcal{M}_{x-e_i} \rightarrow \mathcal{M}_x \) are zero for all \( 1 \leq i \leq n \) and all \( z \in \Omega \) with \( z_i = 0 \). It is easy to prove that modules with variation zero form a full abelian subcategory of \( \mathcal{D}_{hr}^T \) that will be denoted \( \mathcal{D}_{t=0}^T \).

**Remark 4.1.** (i) Let \( f = x_1^{a_1} \cdots x_n^{a_n} \), \( 0 \leq a_i \leq 1 \), be a squarefree monomial in \( \mathbb{C}[x_1, \ldots, x_n] \). From the presentation

\[
\mathcal{O}[1/f] = \frac{\mathcal{D}}{\mathcal{D}(\{ x_i \partial_i + 1 | a_i = 1 \}, \{ \partial_j | a_j = 0 \})}
\]
one can check that the $\mathcal{D}$-module $\mathcal{O}[1/f]$ has variation zero. Using a Čech complex, it follows that if $X$ is a subvariety of $\mathbb{C}^n$ defined by the vanishing of monomials, then the local cohomology modules $\mathcal{H}_X^*(\mathcal{O})$ are also modules of variation zero.

(ii) A simple object $\frac{D}{I_{a,b,l}}$ of the category $\mathcal{D}_{hr}$ has variation zero if and only if $\beta_k = 0$ for $1 \leq k \leq n$. Thus, the simple objects of $\mathcal{D}_{v=0}$ are of the form

\[ (x_i | \alpha_i = -1), (\partial_j | \alpha_j = 0) \]

This module is isomorphic to the local cohomology module $\mathcal{H}_{X_a}^{\text{var}}(\mathcal{O})$.

We have:

**Proposition 4.2.** An object $\mathcal{M}$ of $\mathcal{D}_{hr}$ has variation zero if and only if there is an increasing filtration $\{\mathcal{F}_j\}_{0 \leq j \leq n}$ of $\mathcal{M}$ by objects of $\mathcal{D}_{hr}$ and there are integers $m_{\alpha} \geq 0$ for $\alpha \in \Omega$, such that for all $0 \leq j \leq n$ one has $\mathcal{D}$-module isomorphisms

\[ \mathcal{F}_j/\mathcal{F}_{j-1} \cong \bigoplus_{\alpha \in \Omega \atop |\alpha| = j} (\mathcal{H}_{X_{\alpha}}^j(\mathcal{O}))^{\oplus m_{\alpha}}. \]

**Proof.** If $\mathcal{M}$ is an object of $\mathcal{D}_{v=0}$, then the submodules $\mathcal{F}_j$ of $\mathcal{M}$ corresponding to the hypercube:

\[ (\mathcal{F}_j)_\beta = \begin{cases} \mathcal{M}_\beta & \text{if } |\beta| \leq j, \\ 0 & \text{otherwise.} \end{cases} \]

(the canonical maps being either zero or equal to those in $\mathcal{M}$), give a filtration which satisfies the conditions of the theorem, this follows easily from the fact that the hypercube corresponding to $\mathcal{H}_{X_{\alpha}}^{\text{var}}(\mathcal{O})$ is

\[ \mathcal{H}_{X_{\alpha}}^{\text{var}}(\mathcal{O})_\delta = \begin{cases} \mathbb{C} & \text{if } \delta = \alpha, \\ 0 & \text{otherwise.} \end{cases} \]

Conversely, assume $\mathcal{M}$ is an object of $\mathcal{D}_{hr}$ endowed with such a filtration $\{\mathcal{F}_j\}_{0 \leq j \leq n}$. For all $1 \leq j \leq n$, we have exact sequences

\[ (s_j): 0 \rightarrow \mathcal{F}_{j-1} \rightarrow \mathcal{F}_j \rightarrow \mathcal{F}_j/\mathcal{F}_{j-1} \rightarrow 0. \]
And from them we get corresponding exact sequences in the category of \( n \)-hypercubes, which in turn give sequences of \( C \)-vector spaces:

\[
0 \longrightarrow [\mathcal{F}_j/\mathcal{F}_{j-1}]_{\delta} \longrightarrow [\mathcal{F}_j]_{\delta} \longrightarrow [\mathcal{F}_{j-1}]_{\delta} \longrightarrow 0
\]

for \( \delta \in \Omega \) with \( \delta = 0 \) (recall the functor of solutions is contravariant). From the description of the \( n \)-hypercube corresponding to \( \mathcal{H}_{X}^{|\mathcal{X}|}(C) \) given above, it follows that we only have to consider those vertices \( \delta \in \Omega \) with \( |\delta - \varepsilon_i| = j \). We have \([\mathcal{F}_j/\mathcal{F}_{j-1}]_{\delta} = [\mathcal{F}_{j-1}]_{\delta - \varepsilon_i} = 0 \), so by induction all variations vanish, as was to be proved. \( \square \)

If \( M \) is a \( A_n(C) \)-module, then \( \mathcal{M}^{an} := \mathcal{C} \otimes_R M \) has a natural \( \mathcal{D} \)-module structure. This allows to define a functor:

\[
(-)^{an} : \text{Mod}_{A_n(C)} \rightarrow \text{Mod}_{\mathcal{D}},
\]

\[
M \mapsto \mathcal{M}^{an},
\]

\[
f \mapsto \text{id} \otimes f.
\]

If \( M \) is an \( \varepsilon \)-straight module, it can be endowed with a functorial \( A_n(C) = R\langle \partial_1, \ldots, \partial_n \rangle \)-module structure extending its \( R \)-module structure as follows: If \( b \in \mathbb{Z}^n \) and \( m \in M_b \), then \( \partial_i \cdot m := b_i \chi_i^{-1}m \) (see [16, Remark 2.14]).

Since the morphism \( R \rightarrow \mathcal{C} \) is flat, one has isomorphisms \( \mathcal{C} \otimes_R H^l_{\mathcal{X}}(R) \cong \mathcal{H}^l_{\mathcal{X}}(\mathcal{C}) \) for all \( \alpha \in \Omega, \ l \geq 0 \). From this fact, together with Propositions 3.1 and 4.2, it follows that if \( M \) is a \( \varepsilon \)-straight module then \( \mathcal{M}^{an} \) is an object of \( \mathcal{D}^{T \equiv 0} \). The main result of this section is the following:

**Theorem 4.3.** The functor

\[
(-)^{an} : \varepsilon \rightarrow \text{Str} \rightarrow \mathcal{D}^{T \equiv 0}
\]

is an equivalence of categories.

We will prove first the following lemma (which in particular gives the fully faithfulness of \( (-)^{an} \)):

**Lemma 4.4.** Let \( M, N \) be \( \varepsilon \)-straight modules. For all \( i \geq 0 \), we have functorial isomorphisms

\[
* \text{Ext}^i_R(M, N) \cong \text{Ext}^i_{\mathcal{D}^{T \equiv 0}}(\mathcal{M}^{an}, \mathcal{N}^{an}).
\]
Proof. It has been proved by Yanagawa that the category of straight modules has enough injectives. It will follow from our proof that the category $\mathcal{D}_{t=0}$ has enough injectives as well, so that both Ext functors are defined and can be computed using resolutions.

By induction on the length we can suppose that $M$ and $N$ are simple objects, i.e. $M = H^{|\beta|}_{\mathfrak{p}\mathfrak{a}}(R)$ and $N = H^{[|\beta|]}_{\mathfrak{p}\mathfrak{a}}(R)$. Recall that the minimal *-injective resolution of $N$ is

$$0 \to H^{[|\beta|]}_{\mathfrak{p}\mathfrak{a}}(R) \to \ast E_{R}(R/\mathfrak{p}_{\beta})(-\varepsilon) \to \bigoplus_{i|\beta_i=0} \ast E_{R}(R/\mathfrak{p}_{\beta-i\varepsilon})(-\varepsilon) \to \cdots. \quad (2)$$

If $x \in \Omega$, set $\mathcal{E}^{\times} := (\ast E_{R}(R/\mathfrak{p}_{\mathfrak{a}})(-\varepsilon))^a_n$. We claim that for all $x \in \Omega$, $\mathcal{E}^{\times}$ is an injective object of $\mathcal{D}_{t=0}$. Using the description of *-injective envelopes in [10, 3.1.5] (or, alternatively, the equivalence of categories proved in [16, 2.8]) one can see that, for all $x \in \Omega$, there are isomorphisms

$$\ast E_{R}(R/\mathfrak{p}_{\mathfrak{a}})(-\varepsilon) \cong \sum_{x_i=-1}^{1} R[1_{x_1 \cdots x_n}].$$

From these isomorphisms it is easy to compute the $n$-hypercube corresponding to $\mathcal{E}^{\times}$, namely one has

$$\mathcal{E}^{\times} = \begin{cases} \mathbb{C} & \text{if } \gamma_i \leq x_i \text{ for all } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

The map $u_i : \mathcal{E}^{\times}_{\gamma} \to \mathcal{E}^{\times}_{\gamma-i\varepsilon}$ is the identity if $\mathcal{E}^{\times}_{\gamma} \cong \mathbb{C} \cong \mathcal{E}^{\times}_{\gamma-i\varepsilon}$ and it is zero otherwise. The exactness of the functor $\text{Hom}_{\mathcal{D}}(-, \mathcal{E}^{\times})$ on $\mathcal{D}_{t=0}$ can now be proved by passing to the category $\mathcal{C}^n$ of $n$-hypercubes, where the assertion reduces to a simple question of linear algebra that is left to the reader.

From flatness of $R \to \mathcal{O}$ and the injectivity of the $\mathcal{E}^{\times}$ proved above, it follows that one has the following injective resolution of $\mathcal{M}^{a_{n}} = \mathcal{H}_{\mathfrak{p}\mathfrak{a}}^{[|\beta|]}(\mathcal{O})$ in $\mathcal{D}_{t=0}$:

$$0 \to \mathcal{H}_{\mathfrak{p}\mathfrak{a}}^{[|\beta|]}(\mathcal{O}) \to \mathcal{E}^{\beta} \to \bigoplus_{i|\beta_i=0} \mathcal{E}^{\beta-i\varepsilon} \to \cdots. \quad (3)$$

Let $K_{1}^{\bullet}$ be the complex obtained applying $\ast \text{Hom}_{\mathcal{D}}(M, -)$ to resolution (2) and let $K_{2}^{\bullet}$ be the one obtained applying $\text{Hom}_{\mathcal{D}}(\mathcal{H}^{[|\beta|]}_{\mathfrak{p}\mathfrak{a}}(\mathcal{O}), -)$ to (3). We have an injection $K_{1}^{\bullet} \hookrightarrow K_{2}^{\bullet}$ and we want to show that it is an isomorphism. We have

$$\ast \text{Hom}_{\mathcal{D}}(H^{|\beta|}_{\mathfrak{p}\mathfrak{a}}(R), \ast \text{E}_{R}(R/\mathfrak{p}_{\mathfrak{a}})(-\varepsilon)) = \begin{cases} \mathbb{C} & \text{if } x = \gamma, \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(this can be seen taking the positively graded parts and using [16, 2.8], as done before in similar situations). The same equality holds replacing the left-hand side in (4) by
this is easily proved considering the corresponding $n$-hypercubes. It follows that $K_1^* \cong K_2^*$, and then we are done. ∎

**Proof of Theorem 4.3.** By Lemma 4.4 the functor $(-)^{am}$ is fully faithful, so it remains to prove that it is dense. Let $\mathcal{N}$ be an object of $\mathcal{D}^T_{v=0}$, let $\mathcal{N}' \subseteq \mathcal{N}$ be a submodule such that $\mathcal{N}'' := \mathcal{N}' / \mathcal{N}'$ is simple. By induction on the length, there are $\mathcal{E}$-straight $R$-modules $M'$ and $M''$ such that $\mathcal{N}' \cong (M')^{am}$ and $\mathcal{N}'' \cong (M'')^{am}$. The extension $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ corresponds to an element $\xi$ of $\text{Ext}^1_{\mathcal{D}^T_{v=0}}(\mathcal{N}'', \mathcal{N}')$. Let $0 \to M' \to M \to M'' \to 0$ be an extension such that its class in $*\text{Ext}^1_R(M'', M')$ maps to $\xi$ via the isomorphism of Lemma 4.4. One can check that $\mathcal{N} \cong M^{am}$, and then the theorem is proved. ∎

**Remark.** The category of $\mathcal{E}$-straight modules, regarded as a subcategory of the category of $\mathbb{Z}^n$-graded modules, is closed under extensions [16, Lemma 2.10]. However, note that the category $\mathcal{D}^T_{v=0}$, regarded as a subcategory of $\mathcal{D}^T_{hr}$, is not.

**Acknowledgments**

The authors thank the referee for pointing out a mistake in a previous version of this paper and for a number of valuable comments which have notably improved its readability.

**References**


