Doubly nonlinear evolution equations governed by time-dependent subdifferentials in reflexive Banach spaces

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Abstract

We prove the existence of solutions of the Cauchy problem for the doubly nonlinear evolution equation:

\[ \frac{dv(t)}{dt} + \partial V \varphi_t(u(t)) \ni f(t), \quad v(t) \in \partial H \psi(u(t)), \quad 0 < t < T, \]

where \( \partial H \psi \) (respectively, \( \partial V \varphi_t \)) denotes the subdifferential operator of a proper lower semicontinuous functional \( \psi \) (respectively, \( \varphi_t \)) explicitly depending on \( t \) from a Hilbert space \( H \) (respectively, reflexive Banach space \( V \)) into \( (-\infty, +\infty] \) and \( f \) is given. To do so, we suppose that \( V \hookrightarrow H \equiv H^* \hookrightarrow V^* \) compactly and densely, and we also assume smoothness in \( t \), boundedness and coercivity of \( \varphi_t \) in an appropriate sense, but use neither strong monotonicity nor boundedness of \( \partial H \psi \). The method of our proof relies on approximation problems in \( H \) and a couple of energy inequalities. We also treat the initial-boundary value problem of a non-autonomous degenerate elliptic–parabolic problem.

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1. Introduction

Many authors proposed various types of doubly nonlinear evolution equations and tried to prove the existence of solutions. Their results are applied to PDEs, which describe complex...
nonlinear phenomena, e.g., phase transition, dynamics of non-Newtonian fluid (see, e.g., Alt and Luckhaus [2], Barbu [4], DiBenedetto and Showalter [7], Gajewski and Skrypnik [8], Kenmochi [10], Kenmochi and Pawlow [11], Maitre and Witomski [12], Shirakawa [13]).

We deal with the following doubly nonlinear problem: Let $V$ and $V^*$ be a real reflexive Banach space and its dual space, respectively, and let $H$ be a Hilbert space whose dual space $H^*$ is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with continuous and densely defined canonical injections. Moreover, let $\varphi^t : V \to (-\infty, +\infty]$ be a proper lower semicontinuous convex functional depending on the time variable $t$, where “proper” means that $\varphi^t \not\equiv +\infty$, and let $\psi : H \to (-\infty, +\infty]$ be also proper lower semicontinuous and convex. Then we consider the Cauchy problem

$$\begin{cases}
\frac{dv}{dt}(t) + \partial V \varphi^t(u(t)) \ni f(t), & v(t) \in \partial H \psi(u(t)), \quad 0 < t < T, \\
v(0) = v_0,
\end{cases} \quad (CP)$$

where $\partial V \varphi^t : V \to 2^{V^*}$ and $\partial H \psi : H \to 2^H$ denote the subdifferential operators of $\varphi^t$ and $\psi$, respectively, $f : (0, T) \to V^*$ is given, and

$$v_0 \in R(\partial H \psi) := \{v \in \partial H \psi(u); \ u \in D(\partial H \psi)\}.$$  \hspace{1cm} (1)

For the case where $\varphi^t$ is independent of $t$, i.e., $\varphi^t \equiv \varphi$, V. Barbu [4] proved the existence of strong solutions of (CP) on $[0, T]$ with $\varphi^t$ replaced by $\varphi$ for all $v_0 \in \partial H \psi(D(\partial V \varphi))$ in the Hilbert space setting, i.e., $V = V^* = H$. The assumptions of coercivity and boundedness conditions for $\partial V \varphi : V \to 2^{V^*}$, a sufficient condition for the maximality of the sum $\partial H \psi + \partial H \varphi_H$ in $H$, where $\varphi_H$ stands for an extension of $\varphi$ onto $H$, and a compact embedding $V \hookrightarrow H$. Here, we remark that $\partial H \psi(D(\partial V \varphi)) \subset \partial H \psi(D(\varphi)) \subset R(\partial H \psi)$.

He also applied his results on (CP) to the initial-boundary value problem for the doubly nonlinear parabolic equation of the form

$$\frac{\partial v}{\partial t}(x,t) - \Delta_p u(x,t) = f(x,t), \quad v(x,t) \in \alpha(u(x,t)), \ (x,t) \in \Omega \times (0, T),$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $f$ is a given function, $\Delta_p$ stands for the so-called $p$-Laplacian given by

$$\Delta_p u(x) := \text{div}(|\nabla u(x)|^{p-2} \nabla u(x)), \quad 1 < p < +\infty,$$

and $\alpha$ is a (possibly multi-valued) maximal monotone graph in $\mathbb{R}^2$.

On the other hand, for the case where $\varphi^t$ depends on $t$, Kenmochi [10] and Kenmochi and Pawlow [11] proved the existence of strong solutions on $[0, T]$ of the Cauchy problem for the equation $dv(t)/dt + \partial H \varphi^t(Bv(t)) \ni f(t)$ with $B := \partial H \varphi^* \hookrightarrow v_0 \in \partial H \psi(D(\varphi^0))$ in the Hilbert space setting, i.e., $V = V^* = H$. They assumed a $t$-smoothness condition on $\varphi^t$ and the strong monotonicity of $B = \partial H \varphi^* : \omega |u - v|^2_H \leq (\xi - \eta, u - v)_H$ with $\omega > 0$ for all $[u, \xi], [v, \eta] \in B$; however, they did not impose any boundedness conditions on $\partial H \varphi^t$. Furthermore, they also
applied their abstract theory to elliptic–parabolic variational inequalities with time-dependent obstacles arising from modeling nonsteady flows in porous media.

Our problem has three significant features:

(i) the functional $\varphi$ depends on the time-variable $t$;
(ii) $v_0$ may not belong to $\partial H\psi(D(\varphi_0))$, since $\partial H\psi(D(\varphi_0)) \subset R(\partial H\psi)$;
(iii) $\partial H\psi$ may be degenerate or multi-valued, because neither strong monotonicity conditions nor boundedness conditions are not imposed on $\partial H\psi^*$ or $\partial H\psi$.

In this paper, we prove the existence of strong solutions of (CP) on $[0,T]$ by imposing a $t$-smoothness condition on $\varphi'$ in addition to the similar assumptions as in [4]. The method of our proof relies on some approximations of (CP) and chain rules for time-dependent subdifferential operators developed in [1,9]. To this end, we introduce approximate problems for (CP), where $\partial H\psi$ is replaced by $\partial H\phi_\varepsilon := \varepsilon I + \partial H\psi$ with the identity $I$ in $H$; then $\partial H\phi_\varepsilon$ satisfies the strong monotonicity condition. This method of approximation is different from those employed in [4,10,11]. Furthermore, a priori estimates for approximate solutions are derived from a couple of energy inequalities to obtain convergences of the approximate solutions. To do so, we employ the chain rules for time-dependent subdifferential operators.

The following sort of initial-boundary value problem falls within the scope of our abstract theory:

\begin{align*}
\begin{cases}
\partial_t v(x,t) - \text{div } a(x,t,\nabla u(x,t)) = f(x,t), & (x,t) \in \Omega \times (0,T), \\
v(x,t) \in \alpha(u(x,t)), & (x,t) \in \Omega \times (0,T), \\
u(x,t) = 0, & (x,t) \in \partial \Omega \times (0,T), \\
v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\end{align*}

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, the functions $a: \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ and $f: \Omega \times (0,T) \to \mathbb{R}$ are given, and $\alpha$ denotes a (possibly multi-valued and degenerate) maximal monotone operator in $\mathbb{R}$. Indeed, (2) can be reduced into the form of (CP) under appropriate assumptions on $a(x,t,p)$, which hold true particularly if $a(x,t,p) = k(x,t)|p|^{p-2}p$ with $k \in W^{1,1}(0,T;L^\infty(\Omega))$, i.e., $\text{div } a(x,t,\nabla u(x,t))$ coincides with the modified $p$-Laplacian: $\text{div}(k(x,t)|\nabla u(x,t)|^{p-2}\nabla u(x,t))$.

This paper is composed of seven sections. In the next section, we summarize without proofs the relevant material on subdifferential operators. Section 3 is devoted to state our main results on the existence of strong solutions for (CP). In Section 4, we construct strong solutions of approximate problems for (CP) satisfying a couple of energy inequalities. Moreover, in Section 5, we derive the convergences of those approximate solutions from the energy inequalities, and Section 6 provides the further regularity results on the solutions obtained in Section 5 for the case where $v_0 \in \partial H\psi(D(\varphi_0))$. Finally, in Section 7, we deal with (2) as a typical application of the preceding abstract theory.

2. Preliminaries

We first recall the definition of subdifferential operators. Let $\Phi(X)$ be the set of all proper lower-semicontinuous convex functionals $\phi$ from a Banach space $X$ into $(-\infty, +\infty]$, where “proper” means $\phi \neq +\infty$. Then the subdifferential $\partial_{X,X}\phi(u)$ of $\phi \in \Phi(X)$ at $u$ is given by

$$\partial_{X,X}\phi(u) := \{\xi \in X^*; \phi(v) - \phi(u) \geq \langle \xi, v - u \rangle_X \ \forall v \in D(\phi)\},$$
where \((\cdot, \cdot)_X\) denotes the duality pairing between \(X\) and \(X^*\) and \(D(\phi) := \{ u \in X; \ \phi(u) < +\infty \}\). Hence we can define the subdifferential operator \(\partial_{X, X^*}\phi : X \to 2^{X^*}; u \mapsto \partial_{X, X^*}\phi(u)\) with the domain \(D(\partial_{X, X^*}\phi) := \{ u \in D(\phi); \ \partial_{X, X^*}\phi(u) \neq \emptyset \}\). For simplicity of notation, we shall write \(\partial_X \phi\) and \((\cdot, \cdot)_X\) instead of \(\partial_{X, X^*}\phi\) and \((\cdot, \cdot)_X\), respectively, if no confusion can arise. It is well known that the graph of every subdifferential operator \(\partial_X \phi\) becomes maximal monotone in \(X \times X^*\) (see, e.g., [3] for more details of maximal monotone operators).

In particular, if \(X\) is a Hilbert space \(H\) whose dual space is identified with itself, i.e., \(H \equiv H^*\), then the subdifferential \(\partial_H \phi(u)\) of \(\phi \in \Phi(H)\) at \(u\) can be written by

\[
\partial_H \phi(u) = \{ \xi \in H; \ \phi(v) - \phi(u) \geq (\xi, v - u)_H \ \forall v \in D(\phi) \},
\]

since \((\cdot, \cdot)_H\) coincides with the inner product \((\cdot, \cdot)_H\) of \(H\); moreover, the graph of \(\partial_H \phi\) is maximal monotone in \(H \times H\). Furthermore, the Moreau–Yosida regularization \(\phi_\lambda\) of \(\phi\) is defined as follows:

\[
\phi_\lambda(u) := \inf_{v \in H} \left\{ \frac{1}{2\lambda} |u - v|^2_H + \phi(v) \right\} \forall u \in H, \ \forall \lambda > 0.
\]

The following proposition provides some useful properties of Moreau–Yosida regularizations.

**Proposition 2.1.** For every \(\phi \in \Phi(H)\), the Moreau–Yosida regularization \(\phi_\lambda\) of \(\phi\) is convex and Fréchet differentiable in \(H\), and its derivative \(\partial_H (\phi_\lambda)\) coincides with the Yosida approximation \((\partial_H \phi)_\lambda\) of \(\partial_H \phi\). Furthermore, the following properties are all satisfied:

\[
\phi_\lambda(u) = \frac{1}{2\lambda} |u - J_\lambda^{\phi} u|^2_H + \phi(J_\lambda^{\phi} u) \quad \forall u \in H, \ \forall \lambda > 0, \quad (3)
\]

\[
\phi(J_\lambda^{\phi} u) \leq \phi_\lambda(u) \leq \phi(u) \quad \forall u \in H, \ \forall \lambda > 0, \quad (4)
\]

\[
\phi(J_\lambda^{\phi} u) \uparrow \phi(u) \quad \text{as } \lambda \to +0 \ \forall u \in H, \quad (5)
\]

where \(J_\lambda^{\phi}\) denotes the resolvent of \(\partial_H \phi\).

The following chain rule is often used to derive energy inequalities.

**Proposition 2.2.** Let \(\phi \in \Phi(X)\), let \(p \in (1, +\infty)\) and let \(u \in W^{1,p}(0, T; X)\) be such that \(u(t) \in D(\partial_X \phi)\) for a.a. \(t \in (0, T)\). Suppose that there exists \(g \in L^p(0, T; X^*)\) such that \(g(t) \in \partial_X \phi(u(t))\) for a.a. \(t \in (0, T)\). Then the function \(t \mapsto \phi(u(t))\) is differentiable for a.a. \(t \in (0, T)\); moreover, for every section \(f(t) \in \partial_X \phi(u(t))\),

\[
\frac{d}{dt} \phi(u(t)) = \left( f(t), \frac{du}{dt}(t) \right) \quad \text{for a.a. } t \in (0, T).
\]

Now we summarize a couple of useful properties of the Legendre–Fenchel transform \(\phi^*\) of \(\phi \in \Phi(X)\) defined by

\[
\phi^*(u) := \sup_{v \in X} \{ \langle u, v \rangle - \phi(v) \} \quad \forall u \in X^*.
\]
φ∗ ∈ Φ(X∗);  
φ∗(f) = ⟨f, u⟩ − φ(u) ∀ [u, f] ∈ ∂ξφ;  
u ∈ ∂ξφ∗(f) ∀ [u, f] ∈ ∂ξφ,

where ∂ξφ∗(f) := \{v ∈ X; φ∗(g) − φ∗(f) ≥ ⟨g − f, v⟩ ∀ g ∈ D(φ∗)\}.

3. Main results

Let V be a real reflexive Banach space and let V∗ be its dual space. Moreover, let H be a real Hilbert space whose dual space H∗ is identified with itself such that

V ↪ H ≡ H∗ ↪ V

with continuous and densely defined canonical injections. Let φ′ : V → [0, +∞] and ψ : H → [0, +∞] be such that φ′ ∈ Φ(V) and ψ ∈ Φ(H) for every t ∈ [0, T] and consider the Cauchy problem:

\[ \left\{ \begin{array}{l} \frac{dv}{dt} + ∂Vφ′(u(t)) \ni f(t), \quad v(t) ∈ ∂Hψ(u(t)), \quad 0 < t < T, \\ v(0) = v_0, \end{array} \right. \]  

(CP)

where ∂Vφ′ and ∂Hψ denote subdifferential operators of φ′ and ψ, respectively, for every t ∈ [0, T].

We are concerned with strong solutions of (CP) defined below.

Definition 3.1. A pair of functions (u, v) : [0, T] → V × H is said to be a strong solution of (CP) on [0, T] if the following (i)–(iii) hold true:

(i) v is a V∗-valued absolutely continuous function on [0, T];
(ii) u(t) ∈ D(∂Hψ) ∩ D(∂Vφ′) for a.a. t ∈ (0, T), and there exists a section g(t) ∈ ∂Vφ′(u(t)) such that

\[ \frac{dv}{dt}(t) + g(t) = f(t) \quad \text{in} \ V^*, \quad v(t) ∈ ∂Hψ(u(t)) \quad \text{for a.a.} \ t ∈ (0, T); \]  

(iii) v(t) → v₀ strongly in V* and weakly in H as t → +0.

Prior to describing our main result, we introduce the following assumptions for some p ∈ (1, +∞).

(Aφ′) There exist functions a ∈ W¹,p(0, T), b ∈ W¹,1(0, T) and a constant δ > 0 such that for every t₀ ∈ [0, T] and x₀ ∈ D(φ₀), we can take a function x : Iδ(t₀) := [t₀ − δ, t₀ + δ] ∩ [0, T] → V satisfying:

\[ \left\{ \begin{array}{l} |x(t) − x₀|_V \leq |a(t) − a(t₀)|[φ₀(x₀) + 1]^{1/p}, \\ φ′(x(t)) ≤ φ₀(x₀) + |b(t) − b(t₀)|[φ₀(x₀) + 1] \quad \forall t ∈ Iδ(t₀). \end{array} \right. \]
(A1) There exists a constant $C_1$ such that
\[ |u|^p_V \leq C_1 \{ \varphi'(u) + 1 \} \quad \forall u \in D(\varphi'), \forall t \in [0, T]. \]

(A2) There exists a constant $C_2$ such that
\[ |\xi|^p_{V^*} \leq C_2 \{ \varphi'(u) + 1 \} \quad \forall [u, \xi] \in \partial V \varphi', \forall t \in [0, T]. \]

(A3) There exists a constant $C_3$ such that
\[ \varphi'(J_\varepsilon u) \leq \varphi'(u) + \varepsilon C_3 \quad \forall \varepsilon > 0, \forall u \in D(\partial H \varphi), \forall t \in [0, T], \]
where $J_\varepsilon$ denotes the resolvent of $\partial H \psi$, i.e., $J_\varepsilon := (I + \varepsilon \partial H \psi)^{-1}$.

(A4) $V$ is compactly embedded in $H$.

Our main result reads:

**Theorem 3.2.** Let $p \in (1, +\infty)$ be fixed and assume that (A$\varphi'$), (A1)–(A4) are all satisfied. Then for all $f \in W^{1,p'}(0, T; V^*) \cap L^2(0, T; H)$ and $v_0 \in R(\partial H \psi) := \{ v \in \partial H \psi(u); u \in D(\partial H \psi) \}$,

(CP) admits a strong solution $(u, v)$ on $[0, T]$ such that
\[
\begin{aligned}
&u \in L^p(0, T; V) \cap L^\infty_{\text{loc}}((0, T); V), \\
v \in C_w([0, T); H) \cap W^{1,p'}(0, T; V^*) \cap W^{1,\infty}_{\text{loc}}((0, T); V^*). 
\end{aligned}
\]

(10)

**Remark 3.3.** The (A3) is known as a sufficient condition for the maximality of the sum $\partial H \psi + \partial H \varphi_H'$, that is, $\partial H \psi + \partial H \varphi_H' = \partial H (\psi + \varphi_H')$, where $\varphi_H'$ denotes the extension of $\varphi'$ on $H$, which will be given in (15).

**Remark 3.4.** Theorem 3.2 is also valid even if $\varphi'$ and $\psi$ are not assumed to be non-negative. Indeed, by (A1), we can always assume that $\varphi' \geq 0$ without any loss of generality. On the other hand, since there exist $\xi \in H$ and $C_0 \in \mathbb{R}$ such that $\psi(u) \geq - (\xi, u)_H + C_0$ for all $u \in H$ (see [3, Chapter II, Proposition 2.1]), we can define the non-negative function $\tilde{\psi}(u) := \psi(u) + (\xi, u)_H + C_0 \geq 0$. It then follows that $D(\tilde{\psi}) = D(\psi), D(\partial H \tilde{\psi}) = D(\partial H \psi)$, $\partial H \tilde{\psi}(u) = \partial H \psi(u) + \xi$ for all $u \in D(\partial H \psi)$. In order to prove the existence of solutions for (CP), it suffices to do so for (CP) with $\psi$ and $v_0$ replaced by $\tilde{\psi}$ and $v_0 + \xi$, respectively. Indeed, let $(u, w)$ be a strong solution on $[0, T]$ of (CP) with $\psi$ and $v_0$ replaced by $\tilde{\psi}$ and $v_0 + \xi$, respectively, and put $v(t) := w(t) - \xi$. Then observing that
\[
\begin{aligned}
v(t) &= w(t) - \xi \in \partial H \psi(u(t)) \quad \text{for a.e. } t \in (0, T), \\
dv(t)/dt &= dw(t)/dt \in f(t) - \partial V \varphi'(u(t)) \quad \text{for a.e. } t \in (0, T), \\
v(t) &= w(t) - \xi \to v_0 \quad \text{strongly in } V^* \text{ as } t \to +0 ,
\end{aligned}
\]
we can deduce that $(u, v)$ becomes a strong solution of (CP) on $[0, T]$. 
As for the case where \( v_0 \in \partial_H \psi(D(\varphi^0)) \), we have:

**Theorem 3.5.** In addition to the same assumptions as in Theorem 3.2, suppose that
\[
\psi_u \in \partial_H \psi(D(\varphi^0)) := \{ \psi_u \in \partial_H \psi(\psi) ; \ u \in D(\partial_H \psi) \cap D(\varphi^0) \}.
\]
Then the strong solution \((u, v)\) of (CP) obtained in Theorem 3.2 satisfies
\[
u \in L^\infty(0, T; V), \quad v \in W^{1, \infty}(0, T; V^*).
\]  

(11)

In particular, if \( \varphi^t \) is independent of \( t \), then we can relax the assumption (A2).

**Corollary 3.6.** Let \( p \in (1, +\infty) \) and assume that \( \varphi^t \) is independent of \( t \), i.e., \( \varphi^t \equiv \varphi \), and (A1), (A3), (A4) and the following (A2)' are satisfied with \( \varphi^t \) replaced by \( \varphi \).

(A2)' There exists a non-decreasing function \( \ell: \mathbb{R} \to [0, +\infty) \) such that
\[
\| \xi \|_{V^*} \leq \ell(\varphi(\psi)) \quad \forall [u, \xi] \in \partial V \varphi.
\]

Then for all \( f \in W^{1, p'}(0, T; V^*) \cap L^2(0, T; H) \) and \( v_0 \in \partial_H \psi(D(\varphi)) \), there exists a strong solution \((u, v)\) of (CP) on \([0, T]\) satisfying (10) and (11).

In order to prove the existence of strong solutions of (CP), we first construct solutions of the following approximate problems for (CP) (see Section 4):

\[
\begin{aligned}
\frac{d}{dt} [\varepsilon u^\varepsilon(t) + v^\varepsilon(t)] + g^\varepsilon(t) &= f^\varepsilon(t), \quad 0 < t < T, \\
v^\varepsilon(t) &\in \partial_H \psi(u^\varepsilon(t)), \quad g^\varepsilon(t) \in \partial V \varphi'(u^\varepsilon(t)), \quad 0 < t < T, \\
\varepsilon u^\varepsilon(0) + v^\varepsilon(0) &= \varepsilon u_0 + v_0,
\end{aligned}
\]

(\(\text{CP})_\varepsilon

where \( f^\varepsilon \) is a smooth approximation of \( f \) such that \( f^\varepsilon \in C^1([0, T]; V) \), \( f^\varepsilon \to f \) strongly in \( L^{p'}(0, T; V^*) \) and weakly in \( L^2(0, T; H) \) and \( df^\varepsilon /dt \to df /dt \) weakly in \( L^{p'}(0, T; V^*) \) as \( \varepsilon \to +0 \), and \( u_0 \in D(\partial_H \psi) \) satisfies \( v_0 \in \partial_H \psi(u_0) \). Next, establishing a priori estimates for \((u^\varepsilon, v^\varepsilon)\), we obtain a strong solution \((u, v)\) of (CP) on \([0, T]\) as a limit of \((u^\varepsilon, v^\varepsilon)\) as \( \varepsilon \to +0 \) (see Section 5).

**Notation.** Let \([t \mapsto q(t)]\) denote a function which maps \( t \) to \( q(t) \). We denote by \( C \) a non-negative constant, which does not depend on the elements of the corresponding space or set and may vary from line to line. Moreover, let \( C_\varepsilon \) denote a constant which depends only on \( \varepsilon \) and may also vary from line to line.

4. Construction of approximate solutions

In this section, we construct a strong solution \((u^\varepsilon, v^\varepsilon)\) of (CP) on \([0, T]\) such that
\[
u^\varepsilon \in L^p(0, T; V) \cap L^\infty(0, T; H) \cap C_w((0, T]; V) \cap W^{1, 2}_{\text{loc}}((0, T); H),
\]
\[
u^\varepsilon \in L^\infty(0, T; H) \cap W^{1, 2}_{\text{loc}}((0, T]; V^*), \quad v^\varepsilon(T) \in \partial_H \psi(u^\varepsilon(T)),
\]
\[ \varepsilon u_\varepsilon + v_\varepsilon \in C_w([0, T]; H) \cap W^{1, p'}(0, T; V^*) \cap W^{1, \infty}_{loc}((0, T); V^*), \]

\[ g_\varepsilon \in L^p(0, T; V^*) \cap L^\infty_{loc}((0, T); V^*) \]

and

\[
\sup_{t \in [0, T]} |v_\varepsilon(t)|^2_H \leq C \left( \varepsilon \psi(u_0) + |v_0|^2_H + \int_0^T |f_\varepsilon(t)|^2_H \, dt + 1 \right), \tag{12}
\]

\[
\sup_{t \in [0, T]} \varepsilon |u_\varepsilon(t)|^2_H + \int_0^T \psi'(u_\varepsilon(t)) \, dt \leq C \left( \varepsilon |u_0|^2_H + \varepsilon \psi(u_0) + |v_0|^2_H + \int_0^T |f_\varepsilon(t)|^2_H \, dt + \psi^*(v_0) \right.

+ \int_0^T \phi'(w(t)) \, dt + \int_0^T |w(t)|^p_V \, dt + \int_0^T |f_\varepsilon(t)|^p_{V^*} \, dt + 1 \right), \tag{13}
\]

\[
\varepsilon \int_0^T \left| \frac{d u_\varepsilon}{dt}(t) \right|^2_H \, dt + \sup_{t \in [0, T]} t \phi'(u_\varepsilon(t)) \leq C \left( \varepsilon |u_0|^2_H + \varepsilon \psi(u_0) + |v_0|^2_H + \int_0^T |f_\varepsilon(t)|^2_H \, dt + \psi^*(v_0) \right.

+ \int_0^T \phi'(w(t)) \, dt + \int_0^T |w(t)|^p_V \, dt

+ \int_0^T |f_\varepsilon(t)|^p_{V^*} \, dt + \sup_{t \in [0, T]} t |f_\varepsilon(t)|^p_{V^*} + \int_0^T \left| \frac{d f_\varepsilon}{dt}(t) \right|^p_{V^*} \, dt

+ \int_0^T \left| C_2^{1/p'} |\dot{a}(t)| + |\dot{b}(t)| \right| \, dt + 1 \right) \exp \left( \int_0^T \left| C_2^{1/p'} |\dot{a}(t)| + |\dot{b}(t)| + 1 \right| \, dt \right), \tag{14}
\]

where \( \dot{a} = da/dt, \dot{b} = db/dt, \) and \( w: [0, T] \to V \) is a function such that \( w \in L^p(0, T; V) \) and \( [t \mapsto \psi'(w(t))] \in L^1(0, T) \).

To this end, we introduce the following approximate problems for \((CP)_\varepsilon:\)

\[
\begin{cases}
\frac{d}{dt} (\varepsilon u_{\varepsilon, \lambda}(t) + v_{\varepsilon, \lambda}(t)) + \partial_H \psi_{H, \lambda}(u_{\varepsilon, \lambda}(t)) = f_\varepsilon(t), & 0 < t < T, \\
v_{\varepsilon, \lambda}(t) \in \partial_H \psi(u_{\varepsilon, \lambda}(t)), & 0 < t < T, \\
\varepsilon u_{\varepsilon, \lambda}(0) + v_{\varepsilon, \lambda}(0) = \varepsilon u_0 + v_0. 
\end{cases}
\tag{CP}_{\varepsilon, \lambda}
\]
where $\partial H\varphi^I_{H,\lambda}$ denotes the Yosida approximation of $\partial H\varphi^I_H$, and $\varphi^I_H$ denotes the extension of $\varphi^I$ on $H$ given by

$$
\varphi^I_H(u) := \begin{cases} 
\varphi^I(u) & \text{if } u \in V, \\
+\infty & \text{otherwise}.
\end{cases}
$$

(15)

In the rest of this section, for abbreviation, we write $u_\lambda$ and $v_\lambda$ instead of $u_{\varepsilon,\lambda}$ and $v_{\varepsilon,\lambda}$, respectively.

Put $x_\lambda(t) := \varepsilon u_\lambda(t) + v_\lambda(t)$. Then $(CP)_{\varepsilon,\lambda}$ is equivalent to

$$
\begin{cases}
\frac{dx_\lambda(t)}{dt} + \partial H\varphi^I_{H,\lambda}(\varepsilon I + \partial H\psi)^{-1}(x_\lambda(t)) = f_\varepsilon(t) & \text{in } H, \ 0 < t < T, \\
x_\lambda(0) = x_{0,\varepsilon} := \varepsilon u_0 + v_0,
\end{cases}
$$

$(CP)'_{\varepsilon,\lambda}$

where $I$ denotes the identity in $H$. Since the mapping $t \mapsto \partial H\varphi^I_{H,\lambda}(u)$ is continuous on $[0, T]$ for each $u \in H$ (see Lemma 4.2), the mapping $u \mapsto \partial H\varphi^I_{H,\lambda}(u)$ is Lipschitz continuous in $H$ for each $t \in [0, T]$, and the mapping $(\varepsilon I + \partial H\psi)^{-1}: H \to H$ is Lipschitz continuous with Lipschitz constant $1/\varepsilon$, $(CP)'_{\varepsilon,\lambda}$ possesses a unique strong solution $x_\lambda \in C^1([0, T]; H)$ on $[0, T]$ (see [5, Theorem 1.4]). Hence $u_\lambda = (\varepsilon I + \partial H\psi)^{-1}x_\lambda$ and $v_\lambda = x_\lambda - \varepsilon u_\lambda$ belong to $W^{1,\infty}(0, T; H)$, $u_\lambda(+0) = u_0$ and $v_\lambda(+0) = v_0$.

Now, we shall establish a couple of a priori estimates to imply the convergences of $u_\lambda$ and $v_\lambda$ as $\lambda \to +0$. First, by multiplying $(CP)_{\varepsilon,\lambda}$ by $v_\lambda(t)$, we obtain

$$
\varepsilon \frac{d}{dt} \psi(u_\lambda(t)) + \frac{1}{2} \frac{d}{dt} |v_\lambda(t)|_H^2 + \left(\partial H\varphi^I_{H,\lambda}(u_\lambda(t), v_\lambda(t))\right)_H
= (f_\varepsilon(t), v_\lambda(t))_H \leq \frac{1}{2} |f_\varepsilon(t)|_H^2 + \frac{1}{2} |v_\lambda(t)|_H^2.
$$

(16)

Just as in [5, Theorem 4.4], we have the following lemma, whose proof will be given in the end of this section.

**Lemma 4.1.** Suppose that (A3) is satisfied. Then it follows that

$$
(\partial H\varphi^I_{H,\lambda}(u), v)_H \geq -C_3 \quad \text{for all } [u, v] \in \partial H\psi.
$$

(17)

Thus we can derive that $(\partial H\varphi^I_{H,\lambda}(u_\lambda(t)), v_\lambda(t))_H \geq -C_3$. Therefore, integrating (16), we have

$$
\varepsilon \psi(u_\lambda(t)) + \frac{1}{2} |v_\lambda(t)|_H^2 \leq \varepsilon \psi(u_0) + \frac{1}{2} |v_0|^2_H + C_3 t + \frac{1}{2} \int_0^t |f_\varepsilon(\tau)|_H^2 d\tau + \frac{1}{2} \int_0^t |v_\lambda(\tau)|_H^2 d\tau.
$$

Hence, Gronwall’s inequality yields

$$
\sup_{t \in [0, T]} |v_\lambda(t)|_H^2 \leq C \left\{ \varepsilon \psi(u_0) + |v_0|^2_H + \int_0^T |f_\varepsilon(\tau)|_H^2 d\tau + C_3 \right\}.
$$

(18)
Next, we also multiply $(CP)_{\varepsilon, \lambda}$ by $u_\lambda(t)$ to get
\[
\varepsilon \frac{d}{dt} |u_\lambda(t)|_q^2_H + \left( \frac{dv_\lambda}{dt}(t), u_\lambda(t) \right)_H + \left( \partial_H \varphi'_{H, \lambda}(u_\lambda(t)), u_\lambda(t) \right)_H = \left( f_\varepsilon(t), u_\lambda(t) \right)_H
\] for a.a. $t \in (0, T)$. Since $v_\lambda(t) \in \partial_H \psi(u_\lambda(t))$, it follows from (8) that $u_\lambda(t) \in \partial_H \psi^*(v_\lambda(t))$, which together with Proposition 2.2 implies
\[
\left( \frac{dv_\lambda}{dt}(t), u_\lambda(t) \right)_H = \frac{d}{dt} \psi^*(v_\lambda(t)).
\]
On the other hand, by virtue of (A$\varphi'$), we can construct a function $w \colon [0, T] \to V$ such that $w(t) \in D(\varphi)$ for all $t \in [0, T]$, $[t \mapsto \varphi'(w(t))] \in L^1(0, T)$ and $w \in L^p(0, T; V)$. Hence, since $v_\lambda(t) \in \partial_H \varphi'_\psi(u_\lambda(t))$, it follows from (8) that $u_\lambda(t) \in \partial_H \varphi^*(v_\lambda(t))$, which together with Proposition 2.2 implies
\[
\left( \frac{dv_\lambda}{dt}(t), u_\lambda(t) \right)_H = \frac{d}{dt} \psi^*(v_\lambda(t)).
\]
Thus we have
\[
\frac{\varepsilon}{2} |u_\lambda(t)|_q^2_H + \psi^*(v_\lambda(t)) + \frac{1}{2} \int_0^t \varphi'_{H, \lambda}(u_\lambda(\tau)) d\tau \leq \frac{\varepsilon}{2} |u_0|_q^2_H + \psi^*(v_0) + C \int_0^t \left\{ \varphi'(w(\tau)) + |w(\tau)|_V^p + (f_\varepsilon(\tau), u_\varepsilon(\tau))_H + 1 \right\} d\tau. \tag{19}
\]
Here, we notice that $\psi^*(v_\lambda(t)) \geq -C (|v_\lambda(t)|_H + 1)$ (see [3, Chapter II, Proposition 2.1]). Hence, (18) and Gronwall’s inequality yield
\[
\sup_{t \in [0, T]} \varepsilon |u_\lambda(t)|_q^2_H + \int_0^T \varphi'_{H, \lambda}(u_\lambda(\tau)) d\tau \leq C_\varepsilon, \tag{20}
\]
where $C_\varepsilon$ depends on $\varepsilon$ but not on $\lambda$.

Furthermore, multiply $(CP)_{\varepsilon, \lambda}$ by $du_\lambda(t)/dt$. Then we have
\[
\varepsilon \left| \frac{du_\lambda}{dt}(t) \right|_q^2_H + \left( \frac{dv_\lambda}{dt}(t), \frac{du_\lambda}{dt}(t) \right)_H + \left( \partial_H \varphi'_{H, \lambda}(u_\lambda(t)), \frac{du_\lambda}{dt}(t) \right)_H = \left( f_\varepsilon(t), \frac{du_\lambda}{dt}(t) \right)_H.
\]
Since $\partial_H \psi$ is monotone in $H$, it follows that $(dv_\lambda(t)/dt, du_\lambda(t)/dt)_H \geq 0$. Moreover, by [1, Lemma 2.12], the $t$-smoothness condition (A$\varphi'$) implies
\[
\left| \left( \partial_H \varphi_{t,\lambda}^i(u_\lambda(t)), \frac{d u_\lambda}{d t}(t) \right)_H - \frac{d}{d t} \varphi_{t,\lambda}^i(u_\lambda(t)) \right| \\
\leq \left| \dot{a}(t) \right| \left| \partial_H \varphi_{t,\lambda}^i(u_\lambda(t)) \right|_{V^*} \left\{ \left( \varphi_{t,\lambda}^i(u_\lambda(t)) + 1 \right)^{1/p} + \left| \dot{b}(t) \right| \left( \varphi_{t,\lambda}^i(u_\lambda(t)) + 1 \right) \right\},
\]
which together with (A2) yields
\[
\left| \left( \partial_H \varphi_{t,\lambda}^i(u_\lambda(t)), \frac{d u_\lambda}{d t}(t) \right)_H - \frac{d}{d t} \varphi_{t,\lambda}^i(u_\lambda(t)) \right| \\
\leq \left\{ C_2^{1/p'} |\dot{a}(t)| + |\dot{b}(t)| \right\} \left\{ \varphi_{t,\lambda}^i(u_\lambda(t)) + 1 \right\}.
\]

(21)

Thus we can deduce that
\[
\varepsilon \left| \frac{d u_\lambda}{d t}(t) \right|_H^2 + \frac{d}{d t} \varphi_{t,\lambda}^i(u_\lambda(t)) \\
\leq \left\{ C_2^{1/p'} |\dot{a}(t)| + |\dot{b}(t)| \right\} \left\{ \varphi_{t,\lambda}^i(u_\lambda(t)) + 1 \right\} + \left( f_\varepsilon(t), \frac{d u_\lambda}{d t}(t) \right)_H.
\]

(22)

Multiplying both sides by \( t \) and integrating this, we get
\[
\varepsilon \int_0^t \left| \frac{d u_\lambda}{d \tau}(\tau) \right|_H^2 d \tau + t \varphi_{t,\lambda}^i(u_\lambda(t)) \\
\leq \left\{ C_2^{1/p'} |\dot{a}(\tau)| + |\dot{b}(\tau)| \right\} \left\{ \tau \varphi_{t,\lambda}^i(u_\lambda(\tau)) + \tau \right\} d \tau \\
+ \int_0^t \tau \left( f_\varepsilon(\tau), \frac{d u_\lambda}{d t}(\tau) \right)_H d \tau.
\]

(23)

Hence, using Young’s inequality and Gronwall’s inequality, we can derive from (20) that
\[
\varepsilon \int_0^T \left| \frac{d u_\lambda}{d \tau}(\tau) \right|_H^2 d \tau + \sup_{t \in [0,T]} \left( \varphi_{t,\lambda}^i(u_\lambda(t)) \right) \leq C_\varepsilon.
\]

(24)

Now, it follows from (20), (A1) and (A2) that
\[
\int_0^T \left| J_{t,t}^i u_\lambda(t) \right|_{V^*}^p \, dt \leq C_\varepsilon,
\]

(25)

\[
\int_0^T \left| \partial_H \varphi_{t,\lambda}^i(u_\lambda(t)) \right|_{V^*}^{p'} \, dt \leq C_\varepsilon.
\]

(26)
Moreover, recalling the equation of (CP)\(_{\varepsilon,\lambda}\), we have

\[
\int_0^T |dx_\varepsilon(t)|_{V^*}^\prime dt = \int_0^T |f_\varepsilon(t) - \partial_H \psi'_{H,\lambda}(u_\lambda(t))|_{V^*}^\prime dt \leq C_\varepsilon, \tag{27}
\]

where \(x_\varepsilon(t) = \varepsilon u_\lambda(t) + v_\lambda(t)\). Further, (18) and (20) imply

\[
\sup_{t \in [0,T]} |x_\lambda(t)|_H \leq C_\varepsilon. \tag{28}
\]

From these a priori estimates, we can take a sequence \(\lambda_n\) such that the following convergences hold true as \(\lambda_n \to +0\):

\[
v_{\lambda_n} \to v \quad \text{weakly in } L^q(0,T;H), \tag{29}
\]

\[
u_\lambda \to u \quad \text{weakly in } L^q(0,T;H), \tag{30}
\]

\[
[t \mapsto J'_{\lambda_n} u_{\lambda_n}(t)] \to u \quad \text{weakly in } L^p(0,T;V), \tag{31}
\]

\[
[t \mapsto \partial_H \psi'_{H,\lambda_n}(u_{\lambda_n}(t))] \to g \quad \text{weakly in } L^{p'}(0,T;V^*), \tag{32}
\]

\[
x_{\lambda_n} \to x \quad \text{weakly in } W^{1,p'}(0,T;V^*), \tag{33}
\]

for enough large number \(q > 1\). Here, we used the fact that (26) implies

\[
\int_0^T |u_\lambda(t) - J'_{\lambda_n} u_{\lambda_n}(t)|_{V^*}^\prime dt = \lambda^p \int_0^T |\partial_H \psi'_{H,\lambda_n}(u_{\lambda_n}(t))|_{V^*}^\prime dt \to 0
\]

as \(\lambda \to +0\). Moreover, we also have \(x = \varepsilon u + v\).

Since (A4) ensures that \(H\) is compactly embedded in \(V^*\), it follows from (28) that \(\{x_\lambda(t)\}_{\lambda \in (0,1)}\) forms a precompact subset in \(V^*\) for every \(t \in [0,T]\). Further, by (27), the function \(x_\lambda\) becomes equicontinuous in \(C([0,T]; V^*)\) for all \(\lambda \in (0,1)\). Therefore, by Ascoli’s compactness lemma,

\[
x_{\lambda_n} \to x \quad \text{strongly in } C([0,T]; V^*), \tag{34}
\]

and furthermore, (28) gives

\[
x_{\lambda_n}(T) \to x(T) \quad \text{weakly in } H. \tag{35}
\]

By (29), we can obtain

\[
\|v\|_{L^q(0,T;H)} \leq \liminf_{\lambda_n \to +0} \|v_{\lambda_n}\|_{L^q(0,T;H)} \leq \sup_{\lambda \in (0,1)} \sup_{t \in [0,T]} \|v_\lambda(t)\|_{H} T^{1/q},
\]

which together with (18) implies \(\|v\|_{L^q(0,T;H)} \leq C\), where \(C\) is independent of \(q\). From the arbitrariness of \(q\), we can deduce that \(\limsup_{q \to +\infty} \|v\|_{L^q(0,T;H)} \leq C\), which gives \(v \in L^\infty(0,T;H)\). Just as in the same way, we can also derive \(u \in L^\infty(0,T;H)\). Furthermore, since \(x \in L^\infty(0,T;H) \cap W^{1,p'}(0,T;V^*) \subset C_w([0,T];H) \cap C([0,T];V^*)\), it follows that \(x(t) \to x_{0,\varepsilon} := \varepsilon u_0 + v_0\) strongly in \(V^*\) and weakly in \(H\) as \(t \to +0\).
Now, let $\delta > 0$ be fixed. Then it follows from (24) that

$$
\varepsilon \int_{\delta}^{T} \left| \frac{du_{\lambda}(t)}{dt} \right|_{H}^{2} dt + \sup_{t \in [\delta, T]} \varphi_{H,\lambda}^{t}(u_{\lambda}(t)) \leq \frac{C_{\varepsilon}}{\delta},
$$

which together with (A1) and (A2) yields

$$
\sup_{t \in [\delta, T]} \left\{ \left| J_{\lambda}^{1} u_{\lambda}(t) \right|_{V}^{p} + \left| \partial_{H} \varphi_{H,\lambda}^{t}(u_{\lambda}(t)) \right|_{V^{*}}^{p} \right\} \leq \frac{C_{\varepsilon}}{\delta}.
$$

Hence, by (36), the function $t \mapsto u_{\lambda}(t)$ becomes equicontinuous in $C([\delta, T]; H)$ for all $\lambda \in (0, 1]$. Moreover, by (A$\varphi'$), we have the following (see the end of this section for its proof).

**Lemma 4.2.** Let $\{\varphi_{t}\}_{t \in [0, T]} \subset \Phi(V)$ and let $u$ be a function from $[0, T]$ into $V$. Suppose that (A$\varphi'$) holds. Let $t, s \in [0, T]$ be such that $|t - s| < \delta$, where $\delta$ is given in (A$\varphi'$). Then it follows that

$$
\left| J_{\lambda}^{1} u(t) - J_{\lambda}^{s} u(s) \right|_{H}^{2} \leq |u(t) - u(s)|_{H}^{2} + 2|a(t) - a(s)||\left[ u(t) - J_{\lambda}^{1} u(t) \right]|_{V^{*}}^{p} \left( \varphi^{s} \left( J_{\lambda}^{s} u(s) \right) + 1 \right)^{1/p} + 2|s - J_{\lambda}^{s} u(s)\left|_{V^{*}}^{p} \left( \varphi^{s}(J_{\lambda}^{s} u(s)) + 1 \right)^{1/p} \\
+ 2\lambda |b(t) - b(s)| \left[ \varphi^{s}(J_{\lambda}^{s} u(t)) + \varphi^{s}(J_{\lambda}^{s} u(s)) + 2 \right],
$$

where $J_{\lambda}^{1}$ denotes the resolvent of $\partial_{H} \varphi_{H}^{t}$.

Hence the function $t \mapsto J_{\lambda}^{1} u_{\lambda}(t)$ also becomes equicontinuous in $C([\delta, T]; H)$. Thus, by (37), (A4) and Ascoli’s lemma, we can take a subsequence $\lambda_{n}^{\delta}$ of $\lambda_{n}$ depending on $\delta$ such that

$$
\sup_{t \in [\delta, T]} \left| J_{\lambda_{n}^{\delta}}^{1} u_{\lambda_{n}^{\delta}}(t) - u(t) \right|_{H} \to 0 \quad \text{as} \quad \lambda_{n}^{\delta} \to +0.
$$

Moreover, by (36) and (37), we can also verify that $u \in L^{\infty}(\delta, T; V) \cap W^{1,2}(\delta, T; H)$ and $g \in L^{\infty}(\delta, T; V^{*})$. Furthermore, noting that

$$
\sup_{t \in [\delta, T]} \left| u_{\lambda}(t) - J_{\lambda}^{1} u_{\lambda}(t) \right|_{H}^{2} \leq 2\lambda \sup_{t \in [\delta, T]} \varphi_{H,\lambda}^{t}(u_{\lambda}(t)) \leq \frac{2\lambda C_{\varepsilon}}{\delta} \to 0 \quad \text{as} \quad \lambda \to +0,
$$

we deduce that

$$
u_{\lambda_{n}^{\delta}} \to u \quad \text{strongly in} \quad C([\delta, T]; H).
$$

Thus, since $v_{\lambda_{n}^{\delta}}(t) \in \partial_{H} \psi(u_{\lambda_{n}^{\delta}}(t))$, by [3, Chapter II, Lemma 1.3], we can deduce from (29) and (39) that $\tilde{v}(t) \in \partial_{H} \psi(u(t))$ for a.a. $t \in (\delta, T)$. Particularly, (35) and (39) imply $v(T) \in \partial_{H} \psi(u(T))$. From the arbitrariness of $\delta$, we conclude that
\[ u \in L_{\text{loc}}^\infty((0, T]; V) \cap W_{\text{loc}}^{1,2}((0, T]; H) \subset C_w((0, T]; V), \quad g \in L_{\text{loc}}^\infty((0, T]; V^*), \]
\[ v(t) \in \partial_H \psi(u(t)) \quad \text{for a.a. } t \in (0, T). \]

Moreover, we have \( x \in W_{\text{loc}}^{1,\infty}((0, T]; V^*) \) and \( v \in W_{\text{loc}}^{1,2}((0, T]; V^*). \)

Since (30) and (A1) imply
\[
\begin{align*}
\int_0^t (f_\varepsilon(\tau), u_{\lambda_n}(\tau))_H d\tau &\to \int_0^t (f_\varepsilon(\tau), u(\tau))_H d\tau \\
&\leq C \left( \int_0^T |f_\varepsilon(\tau)|_{V^*}^p \ d\tau + 1 \right) + \frac{1}{4} \int_0^t \varphi^\tau(u(\tau)) \ d\tau,
\end{align*}
\]
we can derive (12) and (13) by passing to the limit in (18) and (19), respectively. By (A1), we observe that

\[
\begin{align*}
\int_0^t \left( f_\varepsilon(\tau), \frac{du_\lambda}{dt}(\tau) \right)_H d\tau &= t(f_\varepsilon(t), u_\lambda(t))_H - \int_0^t (f_\varepsilon(\tau), u_\lambda(\tau))_H d\tau - \int_0^t \left( \frac{df_\varepsilon}{d\tau}(\tau), u_\lambda(\tau) \right)_H d\tau \\
&= t(f_\varepsilon(t), J^T_\lambda u_\lambda(t))_H + \lambda t(f_\varepsilon(t), \partial_H \psi^T_{H,\lambda}(u_\lambda(t)))_H \\
&\quad - \int_0^t (f_\varepsilon(\tau), J^T_\lambda u_\lambda(\tau))_H d\tau - \lambda \int_0^t \left( f_\varepsilon(\tau), \partial_H \psi^T_{H,\lambda}(u_\lambda(\tau)) \right)_H d\tau \\
&\quad - \int_0^t \left( \frac{df_\varepsilon}{d\tau}(\tau), J^T_\lambda u_\lambda(\tau) \right)_H d\tau - \lambda \int_0^t \left( f_\varepsilon(\tau), \partial_H \psi^T_{H,\lambda}(u_\lambda(\tau)) \right)_H d\tau \\
&\leq C \left\{ t |f_\varepsilon(t)|_{V^*}^p + \int_0^t |f_\varepsilon(\tau)|_{V^*}^p d\tau + \int_0^t \left| \frac{df_\varepsilon}{d\tau}(\tau) \right|_{V^*}^p d\tau + 1 \right\} \\
&\quad + \lambda t(f_\varepsilon(t), \partial_H \psi^T_{H,\lambda}(u_\lambda(t))) - \lambda \int_0^t \left( f_\varepsilon(\tau), \partial_H \psi^T_{H,\lambda}(u_\lambda(\tau)) \right) d\tau \\
&\quad - \lambda \int_0^t \left( \frac{df_\varepsilon}{d\tau}(\tau), \partial_H \psi^T_{H,\lambda}(u_\lambda(\tau)) \right) d\tau + \frac{1}{2} \varphi^T_{H,\lambda}(u_\lambda(t)) + \int_0^t (1 + \tau) \varphi^T_{H,\lambda}(u_\lambda(\tau)) d\tau.
\end{align*}
\]

Thus, applying Gronwall’s inequality and letting \( \lambda_n \to +0 \) in (23), we can derive (14) from (13).
Finally, we shall prove that $g(t) \in \partial_V \phi_t(u(t))$ for a.a. $t \in (0, T)$. To do so, we define the functional $\phi^\varepsilon \in \Phi(H)$ by

$$\phi^\varepsilon(u) := \varepsilon \frac{|u|^2}{2} + \psi(u) \quad \forall u \in H.$$  

Then it can be easily seen that $\partial_H \phi^\varepsilon = \varepsilon I + \partial_H \psi$ and $x_\lambda(t) \in \partial_H \phi^\varepsilon(u_\lambda(t))$. We see that

$$\int_0^T \langle \partial_H \phi^\varepsilon_H(u_\lambda(t)), J^H_\lambda u_\lambda(t) \rangle dt = \int_0^T \langle \partial_H \phi^\varepsilon_H(u_\lambda(t)), u_\lambda(t) \rangle dt - \lambda \int_0^T |\partial_H \phi^\varepsilon_H(u_\lambda(t))|^2_H dt$$

$$\leq \int_0^T \langle f^\varepsilon(t), u_\lambda(t) \rangle dt - \int_0^T \left( \frac{dx_\lambda}{dt}(t), u_\lambda(t) \right)_H dt$$

$$= \int_0^T \langle f^\varepsilon(t), u_\lambda(t) \rangle dt - (\phi^\varepsilon)^*(x_\lambda(T)) + (\phi^\varepsilon)^*(x_{0, \varepsilon}).$$

Then, noting that $\liminf_{\lambda_n \to +0}(\phi^\varepsilon)^*(x_{\lambda_n}(T)) \geq (\phi^\varepsilon)^*(x(T))$, we can deduce that

$$\limsup_{\lambda_n \to +0} \int_0^T \langle \partial_H \phi^\varepsilon_H(u_{\lambda_n}(t)), J^H_{\lambda_n} u_{\lambda_n}(t) \rangle dt$$

$$\leq \int_0^T \langle f^\varepsilon(t), u(t) \rangle dt - (\phi^\varepsilon)^*(x(T)) + (\phi^\varepsilon)^*(x_{0, \varepsilon}).$$

Here, we claim that

$$(\phi^\varepsilon)^*(x(T)) - (\phi^\varepsilon)^*(x_{0, \varepsilon}) \geq \int_0^T \left( \frac{dx}{dt}(t), u(t) \right) dt.$$  

To prove this, we prepare the following lemma (see the end of this section for its proof).

**Lemma 4.3.** Let $\phi \in \Phi(H)$ and let $u$ be a $V^*$-valued absolutely continuous function on $[0, T]$ such that $u(t) \in D(\partial_H \phi \cap V) := \{ u \in D(\partial_H \phi); \partial_H \phi(u) \cap V \neq \emptyset \}$ and $\phi(u(t))$ is differentiable for a.a. $t \in (0, T)$. Then it follows that

$$\frac{d}{dt} \phi(u(t)) = \begin{pmatrix} \frac{du}{dt}(t) \\ g(t) \end{pmatrix}$$  

for all $g(t) \in \partial_H \phi(u(t)) \cap V$ and a.a. $t \in (0, T)$.
Let \( \tau \in (0, T] \) be an arbitrary number. Since \( x \in W^{1,p'}(0, T; V^*) \), \( u \in L^\infty_{\text{loc}}((0, T]; V) \) and \( u(t) \in \partial_H(\phi^{\varepsilon})^*(x(t)) \) for a.a. \( t \in (\tau, T) \), by the definition of subdifferentials, we have

\[
\left( \int_0^{T-h} \left| \left( \phi^{\varepsilon} \right)^*(x(t+h)) - \left( \phi^{\varepsilon} \right)^*(x(t)) \right|^{p'} \, dt \right)^{1/p'} \leq \sup_{t \in [\tau, T]} |u(t)| V \left( \int_0^{T-h} \left| x(t+h) - x(t) \right|^{p'} \, dt \right)^{1/p'} \leq \sup_{t \in [\tau, T]} |u(t)|_V Ch
\]

for all \( h \in (0, T-\tau) \) (see [5, Proposition A.7]). Thus the function \( t \mapsto (\phi^{\varepsilon})^*(x(t)) \) becomes absolutely continuous on \([\tau, T]\). Therefore, by Lemma 4.3,

\[
(\phi^{\varepsilon})^*(x(T)) - (\phi^{\varepsilon})^*(x(\tau)) = \int_\tau^T \left( \frac{dx}{dt}(t),u(t) \right)_H \, dt
\]

for every \( \tau > 0 \). Recalling that \( x(\tau) \to x_{0,\varepsilon} \) weakly in \( H \) as \( \tau \to +0 \), \( x \in W^{1,p'}(0, T; V^*) \) and \( u \in L^p(0, T; V) \), we can derive that

\[
(\phi^{\varepsilon})^*(x(T)) - (\phi^{\varepsilon})^*(x_{0,\varepsilon}) \geq (\phi^{\varepsilon})^*(x(T)) - \lim inf_{\tau \to +0} (\phi^{\varepsilon})^*(x(\tau))
\]

\[
= \limsup_{\tau \to +0} \int_{\tau}^T \left( \frac{dx}{dt}(t),u(t) \right)_H \, dt = \int_0^T \left( \frac{dx}{dt}(t),u(t) \right)_H \, dt,
\]

which proves the claim. Hence

\[
\limsup_{\lambda_n \to +0} \int_0^T \left( \partial_H \varphi^I_{H,\lambda_n}(u_{\lambda_n}(t)), J^I_{\lambda_n} u_{\lambda_n}(t) \right) dt \leq \int_0^T \left( f^\varepsilon(t) - \frac{dx}{dt}(t), u(t) \right) dt = \int_0^T \left( g(t), u(t) \right) dt.
\]

By [3, Chapter II, Lemma 1.3] and [9, Proposition 1.1], we conclude that \( g(t) \in \partial_V \varphi^I(u(t)) \) for a.a. \( t \in (0, T) \). Furthermore, using the monotonicity of \( \partial_V \varphi^I \), we have

\[
\limsup_{\lambda_n \to +0} \int_0^T \left( \partial_H \varphi^I_{H,\lambda_n}(u_{\lambda_n}(t)), J^I_{\lambda_n} u_{\lambda_n}(t) \right) dt = \int_0^T \left( g(t), u(t) \right) dt,
\]

so (41) yields

\[
\int_0^T \left( g(t), u(t) \right) dt \leq \int_0^T \left( f^\varepsilon(t), u(t) \right) dt - (\phi^{\varepsilon})^*(x(T)) + (\phi^{\varepsilon})^*(x_{0,\varepsilon}),
\]

which will be used in the next section to derive the convergence of \( g_\varepsilon \) as \( \varepsilon \to +0 \).
We end this section with proofs of Lemmas 4.1–4.3.

**Proof of Lemma 4.1.** We claim that
\[
\psi(J^t_\lambda u) \leq \psi(u) + C_3\lambda \quad \text{for all } u \in D(\psi) \text{ and } \lambda > 0,
\] (46)
where \(J^t_\lambda\) denotes the resolvent of \(\partial_H \psi^t_H\). Indeed, by the definition of subdifferentials,
\[
\psi(\partial_H \psi^t_H)(J^t_\lambda u) \leq \psi(u) + \langle \partial_H \psi^t_H(J^t_\lambda u), J^t_\lambda u - u \rangle_H
\]
\[
= \psi(u) - \lambda \langle \partial_H \psi^t_H(J^t_\lambda u), \partial_H \psi^t_H(J^t_\lambda u) \rangle_H.
\]
Hence, since \(\partial_H \psi^t_H(J^t_\lambda u) \in \partial_H \psi^t_H(J^t_\lambda u)\), (A3) implies
\[
\langle \partial_H \psi^t_H(J^t_\lambda u), \partial_H \psi^t_H(J^t_\lambda u) \rangle_H = \frac{1}{\varepsilon} \langle \psi^t_H(J^t_\lambda u), \partial_H \psi^t_H(J^t_\lambda u) \rangle_H
\]
\[
\geq -\frac{1}{\varepsilon} \{\psi^t_H(J^t_\lambda u) - \psi^t_H(J^t_\lambda u)\} \geq -C_3.
\]
Thus
\[
\psi(J^t_\lambda u) \leq \psi(u) + C_3\lambda.
\] (47)
Let \(\varepsilon \to +0\) in (47). Then we can deduce that \(J^t_\lambda u \in D(\psi)\) for all \(u \in D(\psi)\), and obtain (46). Now, (17) follows immediately from (46).

**Proof of Lemma 4.2.** Since \(J^s_\lambda u(s) \in D(\psi^s)\), by (A\(\psi^s\)), we can take \(v_t \in D(\psi^t)\) such that
\[
\left| v_t - J^s_\lambda u(s) \right|_V \leq \frac{1}{p} \left| a(t) - a(s) \right| \{\psi^s(J^s_\lambda u(s)) + 1\}^{1/p},
\]
\[
\phi^t(v_t) \leq \phi^s(J^s_\lambda u(s)) + \left| b(t) - b(s) \right| \{\psi^s(J^s_\lambda u(s)) + 1\}.
\]
Hence, by the definition of subdifferentials, we have
\[
\langle u(t) - J^s_\lambda u(t), J^s_\lambda u(s) - J^s_\lambda u(t) \rangle_H
\]
\[
= \lambda \left[ \langle \partial_H \psi^s_H(\lambda u(s)), v_t - J^s_\lambda u(t) \rangle_H + \langle \partial_H \psi^s_H(\lambda u(s)), J^s_\lambda u(s) - v_t \rangle_H \right]
\]
\[
\leq \lambda \left[ \phi^s(J^s_\lambda u(s)) - \phi^s(J^s_\lambda u(s)) + \left| b(t) - b(s) \right| \{\psi^s(J^s_\lambda u(s)) + 1\} \right]
\]
\[
+ \left| \partial_H \psi^s_H(\lambda u(s)) \right| \left| a(t) - a(s) \right| \{\psi^s(J^s_\lambda u(s)) + 1\}^{1/p}.
\]
(48)
Moreover, we also have
\[
\langle u(s) - J^s_\lambda u(s), J^s_\lambda u(t) - J^s_\lambda u(s) \rangle_H
\]
\[
\leq \lambda \left[ \phi^t(J^s_\lambda u(t)) - \phi^s(J^s_\lambda u(s)) + \left| b(t) - b(s) \right| \{\phi^t(J^s_\lambda u(t)) + 1\} \right]
\]
\[
+ \left| \partial_H \psi^s_H(\lambda u(s)) \right| \left| a(t) - a(s) \right| \{\phi^t(J^s_\lambda u(t)) + 1\}^{1/p}.
\]
(49)
Therefore it follows from (48) and (49) that

\[
(u(s) - J^s_\lambda u(s) - u(t) + J^t_\lambda u(t), J^t_\lambda u(t) - J^s_\lambda u(s))_H \leq \lambda|b(t) - b(s)|\left\{\varphi'\left(J^t_\lambda u(t)\right) + \varphi^s(J^s_\lambda u(s)) + 2\right\} + |a(t) - a(s)|\left\{|u(s) - J^s_\lambda u(s)|_V^*\left\{\varphi'(J^s_\lambda u(s)) + 1\right\}^{1/p} + |u(t) - J^t_\lambda u(t)|_V^*\left\{\varphi^s(J^s_\lambda u(s)) + 1\right\}^{1/p}\right\}.
\]  

(50)

Consequently, noting that

\[
(u(s) - J^s_\lambda u(s) - u(t) + J^t_\lambda u(t), J^t_\lambda u(t) - J^s_\lambda u(s))_H \geq \frac{1}{2}\left|J^t_\lambda u(t) - J^s_\lambda u(s)\right|^2_H - \frac{1}{2}\left|u(t) - u(s)\right|^2_H.
\]  

we can derive (38) from (50). □

**Proof of Lemma 4.3.** We can choose

\[ I := \{t \in (0, T); \phi(u(\cdot)) \text{ and } u \text{ are differentiable at } t, \text{ and } u(t) \in D(\partial_H \phi \cap V) \} \]

such that \(|[0, T] \setminus I| = 0\). Now, let \(t_0 \in I\) be fixed and let \(h^+_n\) be a sequence in \((0, +\infty)\) such that \(t_0 + h^+_n \in [0, T]\) and \(h^+_n \to +0\). Moreover, let \(g(t_0) \in \partial_H \phi(u(t_0)) \cap V\). Then we have

\[ \phi(u(t_0 + h^+_n)) - \phi(u(t_0)) \geq (g(t_0), u(t_0 + h^+_n) - u(t_0))_H. \]

Dividing both sides by \(h^+_n > 0\) and letting \(h^+_n \to +0\), we see that

\[ \frac{d}{dt}\phi(u(t_0)) \geq \left\langle \frac{du}{dt}(t_0), g(t_0) \right\rangle. \]

Repeating the same argument with a sequence \(h^-_n \in (-\infty, 0)\), we can obtain the inverse inequality. □

5. **Proof of Theorem 3.2**

In this section, we shall derive convergences of solutions \((u_\varepsilon, v_\varepsilon)\) of (CP)\(_\varepsilon\) as \(\varepsilon \to +0\). To this end, we employ (12)–(14) and the boundedness of \(f_\varepsilon\) in \(W^{1,p'}(0, T; V^*) \cap L^2(0, T; H)\) for \(\varepsilon \in (0, 1]\) to get

\[
\sup_{t \in [0, T]} \left\{|v_\varepsilon(t)|_H^2 + \varepsilon|u_\varepsilon(t)|_H^2 + t\varphi'(u_\varepsilon(t))\right\} + \int_0^T \varphi'(u_\varepsilon(t)) \, dt \leq C.
\]  

(51)
Moreover, (A1) and (A2) imply
\[ \int_0^T |u_\varepsilon(t)|_V^p \, dt + \sup_{t \in [0,T]} t |u_\varepsilon(t)|_V^p \leq C, \] (52)

\[ \int_0^T |g_\varepsilon(t)|_{V^*}^p \, dt + \sup_{t \in [0,T]} t |g_\varepsilon(t)|_{V^*}^p \leq C, \] (53)

which together with \((CP)_\varepsilon\) implies
\[ \int_0^T \left| \frac{dx_\varepsilon}{dt}(t) \right|_{V^*}^p \, dt + \sup_{t \in [0,T]} t \left| \frac{dx_\varepsilon}{dt}(t) \right|_{V^*}^p \leq C. \] (54)

Therefore we can take a sequence \(\varepsilon_n\) such that
\[ v_{\varepsilon_n} \to v \quad \text{weakly in } L^q(\mathbb{R}^+; H), \] (55)

\[ u_{\varepsilon_n} \to u \quad \text{weakly in } L^p(\mathbb{R}^+; V), \] (56)

\[ g_{\varepsilon_n} \to g \quad \text{weakly in } L^{p'}(\mathbb{R}^+; V^*), \] (57)

\[ x_{\varepsilon_n} = \varepsilon_n u_{\varepsilon_n} + v_{\varepsilon_n} \to v \quad \text{weakly in } W^{1,p'}(\mathbb{R}^+; V^*) \] (58)

for enough large number \(q > 1\). By (A4) and Ascoli's lemma,
\[ x_{\varepsilon_n} = \varepsilon_n u_{\varepsilon_n} + v_{\varepsilon_n} \to v \quad \text{strongly in } C([0, T]; V^*). \] (59)

Furthermore, just as in the last section, we can also verify that \(u \in L^\infty_{loc}((0, T]; V)\) and \(v \in C_w([0, T]; H) \cap W^{1,\infty}_{loc}((0, T]; V^*)\).

Now, it also follows that
\[ \varepsilon_n u_{\varepsilon_n}(t) \to 0 \quad \text{strongly in } H, \text{ uniformly for all } t \in [0, T], \] (60)

which together with (59) implies
\[ v_{\varepsilon_n}(t) \to v(t) \quad \text{strongly in } V^*, \text{ uniformly for all } t \in [0, T]. \] (61)

Particularly, we get, by (51),
\[ v_{\varepsilon_n}(T) \to v(T) \quad \text{weakly in } H. \] (62)
It follows from (54) and (59) that

\[ |v(t) - v_0|_{V^*} \leq |v(t) - \left\{ \varepsilon_n u_{\varepsilon_n}(t) + v_{\varepsilon_n}(t) \right\}|_{V^*} + |\varepsilon_n u_0 + v_0|_{V^*} \]

\[ \leq \sup_{\tau \in [0, T]} |v(\tau) - \left\{ \varepsilon_n u_{\varepsilon_n}(\tau) + v_{\varepsilon_n}(\tau) \right\}|_{V^*} + C T^{1/p} + |\varepsilon_n u_0|_{V^*} \]

\[ \rightarrow C T^{1/p} \text{ as } \varepsilon_n \rightarrow +0, \]

which implies that \( v(t) \rightarrow v_0 \) strongly in \( V^* \) and weakly in \( H \) as \( t \rightarrow +0 \).

We see that

\[ \int_0^T (v_{\varepsilon_n}(t), u_{\varepsilon_n}(t))_H \, dt \rightarrow \int_0^T (v(t), u(t))_H \, dt = \int_0^T (v(t), u(t))_H \, dt. \]

Thus [3, Chapter II, Lemma 1.3] ensures that \( v(t) \in \partial_H \psi(u(t)) \) for a.a. \( t \in (0, T) \).

To prove that \( g(t) \in \partial_V \psi'(u(t)) \) for a.a. \( t \in (0, T) \), we recall (45), that is,

\[ \int_0^T (g_\varepsilon(t), u_\varepsilon(t))_H \, dt \leq \int_0^T (f_\varepsilon(t), u_\varepsilon(t))_H \, dt - (\phi^\varepsilon)(x_\varepsilon(T)) + (\phi^\varepsilon)^*(x_{0, \varepsilon}), \]

where \( x_\varepsilon(T) = \varepsilon u_\varepsilon(T) + v_\varepsilon(T) \in \partial_H \phi^\varepsilon(u_\varepsilon(T)) \) and \( x_{0, \varepsilon} = \varepsilon u_0 + v_0 \in \partial_H \phi^\varepsilon(u_0) \). Here, we notice that

\[ (\phi^\varepsilon)^*(x_\varepsilon(T)) = (x_\varepsilon(T), u_\varepsilon(T))_H - \phi^\varepsilon(u_\varepsilon(T)) \]

\[ = \frac{\varepsilon}{2} |u_\varepsilon(T)|_H^2 + |v_\varepsilon(T)|_H \psi(u_\varepsilon(T)) \geq \psi^*(v_\varepsilon(T)). \]

Further we see

\[ (\phi^\varepsilon)^*(x_{0, \varepsilon}) = \frac{\varepsilon}{2} |u_0|_H^2 + (v_0, u_0)_H - \psi(u_0) \rightarrow (v_0, u_0)_H - \psi(u_0) = \psi^*(v_0). \]

Therefore we can derive

\[ \limsup_{\varepsilon_n \rightarrow 0} \int_0^T (g_{\varepsilon_n}(t), u_{\varepsilon_n}(t))_H \, dt \leq \int_0^T (f(t), u(t))_H \, dt - \psi^*(v(T)) + \psi^*(v_0). \]

Just as in the last section, we can verify that \( \psi^*(v(\cdot)) \) is absolutely continuous on \( (0, T] \). Hence, repeating the same argument as in (43) and (44), we obtain

\[ \limsup_{\varepsilon_n \rightarrow 0} \int_0^T (g_{\varepsilon_n}(t), u_{\varepsilon_n}(t))_H \, dt \leq \int_0^T \left( f(t) - \frac{dv}{dt}(t), u(t) \right) \, dt = \int_0^T (g(t), u(t))_H \, dt, \]
which together with [3, Chapter II, Lemma 1.3] and [9, Proposition 1.1] ensures that \( g(t) \in \partial \psi^j(u(t)) \) for a.a. \( t \in (0, T) \). This completes our proof. \( \square \)

6. Proofs of Theorem 3.5 and Corollary 3.6

This section is concerned with the case \( v_0 \in \partial H \psi(D(\phi^0)) \), where we can establish all the a priori estimates obtained in the case of \( v_0 \in R(\partial H \psi) \) without multiplying the equation in \((\text{CP})_{\varepsilon, \lambda}^j\) by \( u_{\varepsilon, \lambda}(t) \). Let \( u_0 \in D(\phi^0) \cap D(\partial H \psi) \) be such that \( v_0 \in \partial H \psi(u_0) \).

**Proof of Theorem 3.5.** Recall (22) and integrate this to get

\[
\varepsilon \int_0^t \left| \frac{du_{\varepsilon, \lambda}}{d\tau}(\tau) \right|^2_H d\tau + \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(t)) \leq \varphi_{H, \lambda}^0(u_0) + \int_0^t \left\{ C_2^{1/p'} |\dot{a}(\tau)| + |\dot{b}(\tau)| \right\} \{ \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(\tau)) + 1 \} d\tau + \int_0^t \left( f_\varepsilon(\tau), \frac{du_{\varepsilon, \lambda}}{d\tau}(\tau) \right)_H d\tau. \tag{63}
\]

Thus, since \( \varphi_{H, \lambda}^0(u_0) \leq \varphi^0(u_0) < +\infty \), Young’s inequality and Gronwall’s inequality yield

\[
\varepsilon \int_0^T \left| \frac{du_{\varepsilon, \lambda}}{d\tau}(\tau) \right|^2_H d\tau + \sup_{t \in [0, T]} \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(t)) \leq C_\varepsilon. \tag{64}
\]

Therefore we can take a sequence \( \lambda_n \) such that \( du_{\varepsilon, \lambda_n}/dt \rightarrow du_{\varepsilon}/dt \) weakly in \( L^2(0, T; H) \) and \([t \mapsto J_{\lambda_n}^j u_{\varepsilon, \lambda_n}(t)] \rightarrow u_\varepsilon \) weakly in \( L^q(0, T; V) \) as \( \lambda_n \rightarrow +0 \) for enough large \( q \). We can also derive \( u_\varepsilon \in L^\infty(0, T; V) \).

Just as in (40), we can derive

\[
\int_0^t \left( f_\varepsilon(\tau), \frac{du_{\varepsilon, \lambda}}{d\tau}(\tau) \right)_H d\tau \\
\leq C \left\{ |f_\varepsilon(0)|_{V^*} + \int_0^t \frac{|df_\varepsilon|}{d\tau}(\tau)_{V^*} d\tau + 1 \right\} - \{ f_\varepsilon(0), u_0 | + \lambda \langle f_\varepsilon(t), \partial H \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(t)) \rangle \}
- \lambda \int_0^t \left( \frac{df_\varepsilon}{d\tau}(\tau), \partial H \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(\tau)) \right) d\tau + \frac{1}{2} \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(t)) + \int_0^t \varphi_{H, \lambda}^j(u_{\varepsilon, \lambda}(\tau)) d\tau. \tag{65}
\]

Hence, combining this with (63), using Gronwall’s inequality and passing to the limit, we can obtain
\[ \varepsilon \int_0^t \left| \frac{du_\varepsilon}{d\tau}(\tau) \right|^2_H d\tau + \frac{1}{2} \varphi'(u_\varepsilon(t)) \]
\[ \leq C \left( \varphi^0(u_0) + \int_0^T \left\{ C_2^{1/\rho'} |\dot{\varphi}(\tau)| + |\dot{\varphi}(\tau)| \right\} d\tau + \sup_{\tau \in [0,T]} \left| f_\varepsilon(t) \right|_{V^*}^{\rho'} + \left| f_\varepsilon(0), u_0 \right| \]
\[ + \int_0^T \left| \frac{df_\varepsilon}{d\tau}(\tau) \right|_{V^*}^{\rho'} d\tau + 1 \right) \exp \left( \int_0^T \left\{ C_2^{1/\rho'} |\dot{\varphi}(\tau)| + |\dot{\varphi}(\tau)| + 1 \right\} d\tau \right). \tag{66} \]

Hence, we can derive all the convergences obtained in the case of \( v_0 \in R(\partial H \psi) \), and moreover, we have \( u \in L^\infty(0, T; V) \), \( g \in L^\infty(0, T; V^*) \) and \( v \in W^{1,\infty}(0, T; V^*) \).

Now, we proceed to the proof of Corollary 3.6.

**Proof of Corollary 3.6.** In the proof of Theorem 3.5 described above, (A2) is used only to derive (21) and a priori estimates for \( \partial H \varphi_{H,\lambda}^{t}(u_\varepsilon,\lambda(t)) \) and \( g_\varepsilon(t) \).

Hence, as for the case where \( \varphi^t \) is independent of \( t \), i.e., \( \varphi^t \equiv \varphi \), we can derive (21) with \( \varphi^t = \varphi \), \( a = b \equiv 0 \) without using (A2), so we obtain \( \sup_{t \in [0,T]} |\partial H \varphi_{H,\lambda}^{t}(u_\varepsilon,\lambda(t))|_H \leq C_\varepsilon \). Furthermore, we can also verify \( \sup_{t \in [0,T]} |\partial H \varphi_{H,\lambda}^{t}(u_\varepsilon,\lambda(t))|_H \leq C_\varepsilon \) by using \( (A2)' \) instead of (A2). The rest of proof runs as before. \( \square \)

7. Application to PDE

In this section, we give a typical example of PDE to which our abstract theory can be applied.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) and let \( \alpha \) be a (possibly multivalued and degenerate) maximal monotone operator in \( \mathbb{R} \) such that \( \alpha(0) \ni 0 \). Now, we consider the non-autonomous elliptic–parabolic problem:

\[ \begin{aligned}
\frac{dv}{dt}(x,t) - \text{div} \ a(x,t, \nabla u(x,t)) &= f(x,t), \quad (x,t) \in \Omega \times (0, T), \\
v(x,t) &\in \alpha(u(x,t)), \quad (x,t) \in \Omega \times (0, T),
\end{aligned} \tag{67} \]

where \([(x,t, p) \mapsto a(x,t, p)]\) is a Carathéodory function from \( \Omega \times [0, T] \times \mathbb{R}^N \) into \( \mathbb{R}^N \), i.e., measurable in \( x \) and continuous in \( (t, p) \), satisfying

(H1) There exists a Carathéodory function \( A: \Omega \times [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} \) such that \( A(x,t,p) \) is convex and Fréchet differentiable in \( p \) and \( a(x,t,p) \) coincides with the Fréchet derivative \( \partial_{p} A(x,t,p) \) of \( A(x,t,p) \) for a.e. \( x \in \Omega \) and all \( t \in [0, T] \),

and \( f: \Omega \times (0, T) \rightarrow \mathbb{R} \) is a given function. Moreover, we impose the following on (67):

\[ \begin{aligned}
u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (0, T), \\
u(x,0) &= v_0(x), \quad x \in \Omega.
\end{aligned} \tag{68, 69} \]

Here, we are concerned with weak solutions defined below.
Definition 7.1. A pair of functions \((u, v) : \Omega \times (0, T) \to \mathbb{R}^2\) is said to be a weak solution of the initial-boundary value problem \{\{(67), (68), (69)\} on \([0, T]\) if the following (70)–(72) are all satisfied:

\[
v(x, t) \in \alpha(u(x, t)) \quad \text{for a.e. } (x, t) \in \Omega \times (0, T),
\]

(70)

\[
\left\{ \frac{dv}{dt}(\cdot, t), \phi \right\}_{W_0^{1,p}} + \int_\Omega a(x, t, \nabla u(x, t)) \cdot \nabla \phi(x) \, dx = \int_\Omega f(x, t) \phi(x) \, dx
\]

for a.e. \(t \in (0, T)\) and all \(\phi \in W_0^{1,p}(\Omega)\),

(71)

\[
v(\cdot, t) \to v_0 \quad \text{strongly in } W^{-1,p'}(\Omega) \text{ and weakly in } L^2(\Omega) \text{ as } t \to +0.
\]

(72)

Now, we introduce the following assumptions for some \(p \in (1, +\infty)\) and \(m \in L^1(\Omega)\).

(H2) There exist \(b \in W^{1,1}(0, T)\) and \(\delta > 0\) such that

\[
A(x, t, p) - A(x, s, p) \leq |b(t) - b(s)| \left\{ A(x, s, p) + m(x) \right\} \quad \text{for a.e. } x \in \Omega
\]

and all \(p \in \mathbb{R}^N\) and \(t, s \in [0, T]\) satisfying \(|t - s| < \delta\).

(H3) There exists a constant \(C_4\) such that

\[
|p|^p \leq C_4 A(x, t, p) + m(x) \quad \text{for a.e. } x \in \Omega \text{ and all } (t, p) \in [0, T] \times \mathbb{R}^N.
\]

(H4) There exists a constant \(C_5\) such that

\[
|a(x, t, p)|^{p'} \leq C_5 A(x, t, p) + m(x) \quad \text{for a.e. } x \in \Omega \text{ and all } (t, p) \in [0, T] \times \mathbb{R}^N.
\]

(H5) If \(|p| \leq |q|\), then \(A(x, t, p) \leq A(x, t, q)\) for a.e. \(x \in \Omega\) and all \(t \in [0, T]\).

Remark 7.2. We note that (H4) implies that \(A(x, t, p) \leq C(|p|^p + A(x, t, 0) + m(x))\) for a.e. \(x \in \Omega\) and all \((t, p) \in [0, T] \times \mathbb{R}^N\). Indeed, we see

\[
A(x, t, p) \leq A(x, t, 0) + a(x, t, p) \cdot p \leq A(x, t, 0) + A(x, t, p)/2 + C(|p|^p + m(x)),
\]

which implies the claim.

Remark 7.3. Set \(a(x, t, p) = k(x, t)|p|^{p-2}p\) for a given function \(k(x, t) \in W^{1,1}(0, T; L^\infty(\Omega))\) and suppose that \(k(x, t) \geq k_0 > 0\) for some positive constant \(k_0\). Then, setting \(A(x, t, p) = p^{-1}k(x, t)|p|^p\), we can easily verify that (H1) and (H3)–(H5) are all satisfied. Moreover, we can also infer (H2) from the fact that

\[
A(x, t, p) - A(x, s, p) \leq \frac{1}{p} |k(x, t) - k(x, s)| |p|^p \leq \frac{1}{p} |b(t) - b(s)| |p|^p,
\]

where \(b(t) = \int_0^t |\partial k(\cdot, \tau)/\partial \tau|_{L^\infty(\Omega)} \, d\tau\). In particular, if \(k \equiv 1\), then \(\text{div } a(x, t, \nabla u(x))\) coincides with \(\Delta_p u(x)\), where \(\Delta_p\) stands for the well-known \(p\)-Laplace operator given by \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\).

To verify the existence of solutions for the initial-boundary value problem \{\{(67), (68), (69)\}\}, we set \(V = W_0^{1,p}(\Omega)\) and \(H = L^2(\Omega)\) with the norms \(|u|_V := |
\nabla u|_{L^p(\Omega)}\) and \(|u|_H := |u|_{L^2(\Omega)}\).
Here, we note that every maximal monotone operator in \( \mathbb{R} \) becomes cyclic monotone, so there exists a primary function \( A \in \Phi(\mathbb{R}) \) of \( \alpha \), i.e., \( \partial_{\mathbb{R}} A = \alpha \). Hence we put
\[
\phi_t(u) = \int_{\Omega} A(x, t, \nabla u(x)) \, dx,
\]
(73)
\[
\psi(u) = \begin{cases} \int_{\Omega} A(u(x)) \, dx & \text{if } A(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}
\]
(74)

It then follows that \( \phi_t \in \Phi(V) \) and \( \psi \in \Phi(H) \) (see [6]), and moreover, we can rewrite the initial-boundary value problem \{(67), (68), (69)\} to the abstract Cauchy problem (CP). Indeed, \( \partial_V \phi_t(u) \) coincides with \( -\text{div} \, a(\cdot, t, \nabla u(\cdot)) \) with the boundary condition (68) in the sense of distribution, and \( \partial_H \psi(u) \) coincides with \( \alpha(u(\cdot)) \) in \( L^2(\Omega) \). Furthermore, by Remark 7.2, we have \( D(\phi_t) = D(\partial_V \phi_t) = V \).

Remark 7.4. As for the case where \( A \) is unbounded below, since \( A \in \Phi(\mathbb{R}) \), we can take \( \xi_0, C_0 \in \mathbb{R} \) such that \( \tilde{A}(s) := A(s) + \xi_0 s + C_0 \geq 0 \) for all \( s \in \mathbb{R} \). Then \( \tilde{\alpha}(s) := \partial_{\mathbb{R}} \tilde{A}(s) = \alpha(s) + \xi_0 \). It suffices to prove the existence of solutions for \{\( \phi_t \), (68), (69)\} with \( \alpha \) and \( v_0 \) replaced by \( \tilde{\alpha} \) and \( v_0 + \xi \), respectively (see also Remark 3.4).

Now, we shall check \( (A\phi_t), (A1)–(A4) \) to apply the preceding abstract theory to the initial-boundary value problem. Let \( t_0 \in [0, T] \) and \( x_0 \in D(\phi_{t_0}) \) be fixed. Then, by (H2), we put \( x(t) \equiv x_0 \) to get
\[
\phi_t(x(t)) - \phi_{t_0}(x_0) \leq |b(t) - b(t_0)| \left\{ \int_{\Omega} A(x, t_0, x_0(x)) \, dx + |m|_{L^1} \right\}
\]
for all \( t \in I_{\delta}(t_0) \), which implies \( (A\phi_t) \). Furthermore, by (H3), it is obvious that \( (A1) \) holds true. Now, let \( [u, g] \in \partial_V \phi_t \). Then we see, by (H4),
\[
\langle g, z \rangle = \int_{\Omega} a(x, t, \nabla u(x)) \cdot \nabla z(x) \, dx \leq \left( \int_{\Omega} \left\{ C_5 A(x, t, \nabla u(x)) + m(x) \right\} \, dx \right)^{1/p'} |z|_V \quad \forall z \in V,
\]
which yields \( (A2) \). Furthermore, since \( j_\lambda := (1 + \lambda \alpha)^{-1} : \mathbb{R} \to \mathbb{R} \) is non-expansive, we get \( |\nabla j_\lambda u(x)| \leq |\nabla u(x)| \), so \( (A3) \) follows immediately from (H5). Moreover, if \( 2N/(N + 2) < p \), then \( V \) is embedded compactly in \( H \), i.e., \( (A4) \) holds. Consequently, by Theorems 3.2 and 3.5, we have:

Theorem 7.5. Let \( p \in (2N/(N + 2), +\infty) \) and suppose that \( (H1)–(H5) \) hold. Then for all \( f \in W^{1,p'}(0, T; W^{-1,p'}(\Omega)) \cap L^2(0, T; L^2(\Omega)) \) and
\[
v_0 \in \left\{ w \in L^2(\Omega); \text{ there exists } u_0 \in L^2(\Omega) \text{ such that } w(x) \in a(u_0(x)) \text{ for a.e. } x \in \Omega \right\},
\]
the initial-boundary value problem \{(67), (68), (69)\} admits at least one weak solution \((u, v)\) satisfying
\[ u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty_{\text{loc}}((0, T]; W_0^{1,p}(\Omega)), \]
\[ v \in C_w([0, T]; L^2(\Omega)) \cap W^{1,p'}(0, T; W^{-1,p'}(\Omega)) \cap W^{1,\infty}_{\text{loc}}((0, T]; W^{-1,p'}(\Omega)). \]

In particular, if
\[ v_0 \in \{ w \in L^2(\Omega); \text{there exists } u_0 \in W_0^{1,p}(\Omega) \text{ such that } w(x) \in \alpha(u_0(x)) \text{ for a.e. } x \in \Omega \}, \]
then the solution \((u, v)\) satisfies \( u \in L^\infty(0, T; W_0^{1,p}(\Omega)) \) and \( v \in W^{1,\infty}(0, T; W^{-1,p'}(\Omega)) \).

References