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Higher equations of motion in $N=2$ superconformal Liouville field theory

Changrim Ahn [a,b,*], Marian Stanishkov [b,1], Michail Stoilov [c]

[a] Department of Physics, Ewha Womans University, Seoul 120-750, South Korea
[b] Institute for the Early Universe, Ewha Womans University, Seoul 120-750, South Korea
[c] Institute for Nuclear Research and Nuclear Energy (INRNE), Bulgarian Academy of Sciences (BAS), 1784 Sofia, Bulgaria

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ABSTRACT

We present an infinite set of higher equations of motion in $N=2$ supersymmetric Liouville field theory. They are in one to one correspondence with the degenerate representations and are enumerated in addition to the $U(1)$ charge ω by the positive integers m or (m,n) respectively. We check that in the classical limit these equations hold as relations among the classical fields.

In Ref. [1] it has been shown that in the Liouville field theory (LFT) an infinite set of relations holds for quantum operators. These equations relate different basic Liouville primary fields $V_\alpha(z)$ (V_α can be thought of as normal ordered exponential field $\exp(\alpha\phi)$ of the basic Liouville field ϕ). They are parameterized by a pair of positive integers (m,n) and are called conventionally "higher equations of motion" (HEM), because the first one $(1,1)$ coincides with the usual Liouville equation of motion. The equations are derived on the basis of a conjecture of the vanishing of all singular vectors, imposed by the requirement of irreducibility of the corresponding representation. They are easily verified in the classical LFT. Higher equations turn out to be useful in practical calculations. In particular, in [2–5], they were used to derive general four-point correlation function in the minimal Liouville gravity.

Similar operator valued relations have been found also for $N=1$ supersymmetric Liouville field theory (SLFT) [6] and for $SL(2,R)$ Wess–Zumino–Novikov–Witten model [7,8]. Recently it was shown in [9] that such relations hold for the boundary operators in the LFT with conformal boundary.

It is the purpose of this Letter to reveal a similar set of higher equations of motion in $N=2$ SLFT. The $N=2$ SLFT has a wide variety of applications in string theory [10–12]. This theory is quite interesting because of the fact that it has actually few properties in common with the $N=0,1$ SLFTs. For example, unlike the Liouville theories with less supersymmetry, the $N=2$ SLFT does not have a simple strong–weak coupling duality. In fact, under the change $b\to 1/b$ of the coupling constant, the $N=2$ SLFT flows to another $N=2$ supersymmetric theory as proposed in [13,14]. Another important difference between the $N=2$ SLFT and the $N=0,1$ SLFTs is the spectrum of the degenerate representations [15–17] (see also [18,19]). We will show below that the $N=2$ SLFT still possesses higher equations of motion despite these differences.

1. $N=2$ SLFT

The $N=2$ SLFT is based on the Lagrangian:

$$\mathcal{L}=\frac{1}{2\pi}\left(\partial\phi^-\bar\partial\phi^+ +\partial\phi^+\bar\partial\phi^- +\psi^-\bar\partial\psi^+ +\psi^+\bar\partial\psi^- +\bar\psi^-\partial\bar\psi^+ +\bar\psi^+\partial\bar\psi^-\right)$$
$$+i\mu b^2\psi^-\bar\psi^-e^{b\phi^+}+i\mu b^2\psi^+\bar\psi^+e^{b\phi^-}+\pi\mu^2b^2e^{b\phi^++b\phi^-} \tag{1}$$

* Corresponding author at: Department of Physics, Ewha Womans University, Seoul 120-750, South Korea.
E-mail addresses: ahn@ewha.ac.kr (C. Ahn), marian@inrne.bas.bg (M. Stanishkov), mstoilov@inrne.bas.bg (M. Stoilov).
1 On leave of absence from INRNE, BAS, Bulgaria.

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where $(\phi^\pm, \psi^\mp)$ are the components of a chiral $N=2$ supermultiplet, b is the coupling constant and μ is the cosmological constant. It is invariant under the $N=2$ superconformal algebra:

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n}, \\
&\left[L_m, G_r^\pm\right] = \left(\frac{m}{2} - r\right)G^\pm_{m+r}, \qquad \left[J_n, G_r^\pm\right] = \pm G^\pm_{n+r}, \\
&\left\{G_r^+, G_s^-\right\} = 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}, \\
&[L_m, J_n] = -nJ_{m+n}, \qquad [J_m, J_n] = \frac{c}{3}\delta_{m+n},
\end{aligned} \tag{2}$$

where $L_m, G_r^\pm$ and J_n are the modes of the corresponding conserved currents, the stress–energy tensor $T(z)$, the super-current $G(z)$ and the $U(1)$ current $J(z)$, and the central charge is:

$$c = 3 + \frac{6}{b^2}.$$

These are the left handed generators, there are in addition the right handed ones $\bar{L}_n$, $\bar{J}_n$, $\bar{G}_r^\pm$ closing the same algebra.

The basic objects are the primary fields (normal ordered exponents):

$$N_{\alpha,\bar{\alpha}} = e^{\alpha\phi^+ + \bar{\alpha}\phi^-},$$

the corresponding states being annihilated by the positive modes. These are the primary fields in the Neveu–Schwartz (NS) sector with r, s in (2) half-integer. There are in addition also Ramond (r, s — integer) primary fields $R_{\alpha,\bar{\alpha}}$ but we will not be concerned with them in this Letter. The conformal dimension and the $U(1)$ charge of the primary fields are:

$$\Delta_{\alpha,\bar{\alpha}} = -\alpha\bar{\alpha} + \frac{1}{2b}(\alpha + \bar{\alpha}), \qquad \omega = \frac{1}{b}(\alpha - \bar{\alpha}). \tag{3}$$

Among the primary fields there is a series of degenerate fields of the $N=2$ SLFT. They are characterized by the fact that at certain level of the corresponding conformal family a new primary field (i.e. annihilated by all positive modes) appears. Such fields can be divided in three classes (see e.g. [18]).

Class I degenerate fields are given by

$$\begin{aligned}
&N^\omega_{m,n} = N_{\alpha^\omega_{m,n}, \bar{\alpha}^\omega_{m,n}}, \\
&\alpha^\omega_{m,n} = \frac{1-m}{2b} + (\omega - n)\frac{b}{2}, \\
&\bar{\alpha}^\omega_{m,n} = \frac{1-m}{2b} - (\omega + n)\frac{b}{2}
\end{aligned} \tag{4}$$

m, n are positive integers. $N^\omega_{m,n}$ is degenerate at level mn and relative $U(1)$ charge zero. The irreducibility of the corresponding representations is assured by imposing the null-vector condition $D^\omega_{m,n}N^\omega_{m,n} = 0$, $\bar{D}^\omega_{m,n}N^\omega_{m,n} = 0$, where $D^\omega_{m,n}$ is a polynomial of the generators in (2) of degree mn and has $U(1)$ charge zero. It is normalized by choosing the coefficient in front of $(L_{-1})^{mn}$ to be 1. Let us give some examples of the corresponding null-operators:

$$\begin{aligned}
D^\omega_{1,1} &= L_{-1} - \frac{1}{2}b^2(1+\omega)J_{-1} + \frac{1}{\omega - 1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}, \\
D^\omega_{1,2} &= L^2_{-1} + b^2 L_{-2} - b^2(1+\omega)L_{-1}J_{-1} + \frac{b^2}{2}\left(1 + \omega - b^2(2+\omega)\right)J_{-2} \\
&\quad + \frac{b^4}{4}\omega(\omega+2)J^2_{-1} + \frac{2}{\omega - 2}L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - \frac{b^2\omega}{\omega - 2}J_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&\quad - \frac{b^2}{2}G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} + \frac{b^2}{2}\frac{\omega+2}{\omega-2}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}, \\
D^\omega_{2,1} &= L^2_{-1} + \frac{1}{b^2}L_{-2} - b^2(1+\omega)L_{-1}J_{-1} + \frac{1}{2}\left(b^2(1+\omega) - \omega - 2\right)J_{-2} \\
&\quad + \frac{1}{4}\left(b^4(\omega+1)^2 - 1\right)J^2_{-1} + \frac{2b^4\omega}{b^4(\omega-1)^2 - 1}L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&\quad - \frac{b^2 + b^6(\omega^2 - 1)}{b^4(\omega-1)^2 - 1}J_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - \frac{b^4(\omega+1) + b^2 - 2}{2 + 2b^2(\omega - 1)}G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} \\
&\quad + \frac{2 - b^2 + b^4(\omega-1)(1 + b^2(\omega+1))}{2(b^4(\omega-1)^2 - 1)}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}.
\end{aligned} \tag{5}$$

The second class of degenerate fields is denoted by N^ω_m and comes in two subclasses IIA and IIB:

$$\text{class IIA:} \quad N_m^\omega = N_{\alpha_m^\omega, \bar{\alpha}_m^0} \quad \omega > 0$$
$$\text{class IIB:} \quad N_m^\omega = N_{\alpha_m^0, \bar{\alpha}_m^\omega} \quad \omega < 0, \tag{6}$$

where

$$\alpha_m^\omega = \frac{1-m}{2b} + \omega b, \qquad \bar{\alpha}_m^\omega = \frac{1-m}{2b} - \omega b. \tag{7}$$

Here m is an odd positive integer number and the level of degeneracy of N_m^ω is $\frac{m}{2}$, relative charge ± 1. In this case the operator D_m^ω is a polynomial of "degree" $m/2$, the coefficient in front of $L_{-1}^{\frac{m-1}{2}} G_{-\frac{1}{2}}^\pm$ is chosen to be 1. Analogously to the class I we have to impose $D_m^\omega N_m^\omega = \bar{D}_m^\omega N_m^\omega = 0$. Here are the first examples for class IIA fields:

$$D_1^\omega = G_{-\frac{1}{2}}^+,$$
$$D_3^\omega = L_{-1} G_{-\frac{1}{2}}^+ - J_{-1} G_{-\frac{1}{2}}^+ + \left(\frac{2}{b^2} - \omega\right) G_{-\frac{3}{2}}^+,$$
$$D_5^\omega = L_{-1}^2 G_{-\frac{1}{2}}^+ + \left(\frac{4}{b^2} - \omega - 1\right) L_{-2} G_{-\frac{1}{2}}^+ - 3 L_{-1} J_{-1} G_{-\frac{1}{2}}^+ + 2 J_{-1}^2 G_{-\frac{1}{2}}^+$$
$$+ \left(\frac{5}{2} - \frac{6}{b^2} + \frac{3}{2}\omega\right) J_{-2} G_{-\frac{1}{2}}^+ + \left(1 + \frac{6}{b^2} - 2\omega\right) L_{-1} G_{-\frac{3}{2}}^+ + 4\left(\omega - \frac{3}{b^2}\right) J_{-1} G_{-\frac{3}{2}}^+$$
$$- \frac{1}{2} G_{-\frac{3}{2}}^+ G_{-\frac{1}{2}}^+ G_{-\frac{1}{2}}^- + \left(\frac{24}{b^4} - \frac{14\omega}{b^2} + 2\omega^2 - 1\right) G_{-\frac{5}{2}}^+. \tag{8}$$

The null-operators for class IIB fields are obtained from (8) by changing $G^\pm \to G^\mp$ and $\omega \to -\omega$.

A special case of class IIA (B) fields are the chiral (antichiral) fields with $m = 1$. The class II fields having $U(1)$ charge zero are classified in a separate class III fields. The simplest $m = 1$ field here represents the identity operator.

2. Norms of the null-states

Let us now consider, for a further use, the norms of the states created by applying the null-operators on primary states $|\alpha\rangle$. As explained above, such sates should vanish at $\alpha = \alpha_M^\omega$. Taking the first terms in the corresponding Taylor expansion, we define:

$$r_M^\omega = \partial_\alpha \langle \alpha, \bar{\alpha} | D_M^{\omega\dagger} D_M^\omega | \alpha, \bar{\alpha} \rangle |_{\alpha = \alpha_M^\omega, \bar{\alpha} = \bar{\alpha}_M^\omega},$$
$$\bar{r}_M^\omega = \partial_{\bar{\alpha}} \langle \alpha, \bar{\alpha} | D_M^{\omega\dagger} D_M^\omega | \alpha, \bar{\alpha} \rangle |_{\alpha = \alpha_M^\omega, \bar{\alpha} = \bar{\alpha}_M^\omega} \tag{9}$$

for both classes of representations, $M = m$ or (m, n), where D_M^ω is the corresponding null-operator and $D_M^{\omega\dagger}$ is defined as usual through $L_n^\dagger = L_{-n}$, $J_n^\dagger = J_{-n}$, $(G_r^\pm)^\dagger = G_{-r}^\mp$.

One can compute "by hand" the first few r's. With the use of the explicit form of the null-operators (5) we find for the class I fields:

$$r_{1,1}^\omega = \frac{1}{b} \frac{(1+b^2)(1+\omega)}{(-1+\omega)},$$
$$r_{1,2}^\omega = \frac{-2}{b} \frac{(1-b^2)(1+b^2)(1+2b^2)(2+\omega)}{(-2+\omega)},$$
$$r_{1,3}^\omega = \frac{12}{b} \frac{(1-2b^2)(1-b^2)(1+b^2)(1+2b^2)(1+3b^2)(3+\omega)}{(-3+\omega)},$$
$$r_{2,1}^\omega = \frac{2}{b^5} \frac{(1-b^2)(1+b^2)(2+b^2)(-1+b^2+b^2\omega)(1+b^2+b^2\omega)}{(-1-b^2+b^2\omega)(1-b^2+b^2\omega)},$$
$$r_{3,1}^\omega = \frac{12}{b^9} \frac{(2-b^2)(1-b^2)(1+b^2)(2+b^2)(3+b^2)(1+\omega)(-2+b^2+b^2\omega)(2+b^2+b^2\omega)}{(-1+\omega)(-2-b^2+b^2\omega)(2-b^2+b^2\omega)}$$

and $\bar{r}_{m,n}^\omega = r_{m,n}^\omega$ for all the examples above. Based on these expression we propose for the general form of $r_{m,n}^\omega$:

$$r_{m,n}^\omega = \bar{r}_{m,n}^\omega = \prod_{l=1-m}^{m} \prod_{k=1-n}^{n} \left(\frac{l}{b} + kb\right) \prod_{l=1-m,\ \mathrm{mod}\ 2}^{m-1} \left(\frac{l-(n+\omega)b^2}{l+(n-\omega)b^2}\right). \tag{10}$$

Similarly, from (8) we have for the class IIA:

$$\bar{r}_1^\omega = 2\left(\frac{1}{b} - \omega b\right),$$
$$\bar{r}_3^\omega = \frac{2}{b^5}\left(2-b^2\omega\right)\left(3-b^2\omega\right)\left(2-b^2-b^2\omega\right),$$
$$\bar{r}_5^\omega = \frac{8}{b^9}\left(3-b^2\omega\right)\left(4-b^2\omega\right)\left(5-b^2\omega\right)\left(3-b^2-b^2\omega\right)\left(4-b^2-b^2\omega\right),$$
$$\bar{r}_7^\omega = \frac{72}{b^{13}}\left(4-b^2\omega\right)\left(5-b^2\omega\right)\left(6-b^2\omega\right)\left(7-b^2\omega\right)\left(4-b^2-b^2\omega\right)\left(5-b^2-b^2\omega\right)\left(6-b^2-b^2\omega\right),$$
$$r_m^\omega = 0, \quad m = 1,3,5,7.$$

These expressions can be fitted in a general form of r_m^ω and $\bar{r}_m^\omega$:

$$r_m^\omega = 0,$$
$$\bar{r}_m^\omega = 2\Gamma^2\left(\frac{m+1}{2}\right)b^{1-m}\prod_{l=\frac{m+1}{2}}^{m}\left(\frac{l}{b} - b\omega\right)\prod_{l=\frac{m+1}{2}}^{m-1}\left(\frac{l}{b} - b(\omega+1)\right). \tag{11}$$

For the class IIB fields one obtains $\bar{r}_m^\omega = 0$ and r_m^ω is as $\bar{r}_m^\omega$ in (11) with the change $\omega \to -\omega$.

3. Logarithmic fields and HEM

Let us now introduce the so-called logarithmic fields. They are defined as:

$$N'_{\alpha,\bar{\alpha}} = \partial_\alpha N_{\alpha,\bar{\alpha}}, \qquad \bar{N}'_{\alpha,\bar{\alpha}} = \partial_{\bar{\alpha}} N_{\alpha,\bar{\alpha}}.$$

One can introduce also the logarithmic primary fields corresponding to degenerate fields by:

$$N_M'^{\omega} = N'_{\alpha,\bar{\alpha}}|_{\alpha=\alpha_M^\omega,\bar{\alpha}=\bar{\alpha}_M^\omega}, \qquad \bar{N}_M'^{\omega} = \bar{N}'_{\alpha,\bar{\alpha}}|_{\alpha=\alpha_M^\omega,\bar{\alpha}=\bar{\alpha}_M^\omega} \tag{12}$$

where M is (m,n) for class I and M is m for class II fields respectively. The basic statement about the fields (12) is that

$$\tilde{N}_M^\omega = \bar{D}_M^\omega D_M^\omega N_M'^{\omega}, \qquad \tilde{\bar{N}}_M^\omega = \bar{D}_M^\omega D_M^\omega \bar{N}_M'^{\omega} \tag{13}$$

with D_M^ω, $\bar{D}_M^\omega$ as in (5), (8) are again primary. The proof of this statement goes along the same lines as for $N = 0,1$ SLFT [1,6] so we will not repeat it here.

Comparing the dimension and $U(1)$ charge for class I fields: $\tilde{\Delta}_{m,n} = \Delta_{m,n} + mn$, $\tilde{\omega} = \omega$ we conclude that the fields (13) are proportional to $N_{m,-n}^\omega$. Thus, we arrive at the higher equations of motion (HEM) for the class I fields:

$$\bar{D}_{m,n}^\omega D_{m,n}^\omega N_{m,n}'^{\omega} = B_{m,n}^\omega N_{m,-n}^\omega, \qquad \bar{D}_{m,n}^\omega D_{m,n}^\omega \bar{N}_{m,n}'^{\omega} = \bar{B}_{m,n}^\omega N_{m,-n}^\omega. \tag{14}$$

For class IIA (B) the dimension of the resulting primaries in (13) is $\tilde{\Delta}_m^\omega = \Delta_m^\omega + \frac{m}{2}$, the $U(1)$ charges are $\tilde{\omega} = \omega + 1$ ($\tilde{\omega} = \omega - 1$) respectively, and the HEMs in this case are:

$$\bar{D}_m^\omega D_m^\omega N_m'^{\omega} = B_m^\omega N_m^{\omega\pm1}, \qquad \bar{D}_m^\omega D_m^\omega \bar{N}_m'^{\omega} = \bar{B}_m^\omega N_m^{\omega\pm1}. \tag{15}$$

Computation of $B_{m,n}^\omega$ ($\bar{B}_{m,n}^\omega$) and B_m^ω ($\bar{B}_m^\omega$) is the final goal of this Letter. HEMs (14) and (15) are to be understood in an operator sense, i.e. they should hold for any correlation function. Here we will insert them into the simplest one-point function on the so-called Poincaré disk [20]. In this case we have:

$$\langle B_1|\bar{D}_M^\omega D_M^\omega N_M'^{\omega}\rangle = \langle B_1|\tilde{N}_M^\omega\rangle, \qquad \langle B_1|\bar{D}_M^\omega D_M^\omega \bar{N}_M'^{\omega}\rangle = \langle B_1|\tilde{\bar{N}}_M^\omega\rangle.$$

The boundary state $\langle B_1|$ corresponds to the identity boundary conditions on the Poincaré disc. It enjoys $N = 2$ superconformal invariance:

$$\langle B_1|\bar{G}_r^\pm = -i\langle B_1|G_{-r}^\mp = -i\langle B_1|\left(G_r^\pm\right)^\dagger, \qquad \langle B_1|\bar{L}_n = \langle B_1|(L_n)^\dagger, \qquad \langle B_1|\bar{J}_n = \langle B_1|(J_n)^\dagger$$

(so-called A-type boundary conditions, see e.g. [21]).

With the definition of r's in (9) the HEMs (14) and (15) take the form:

$$r_{m,n}^\omega U_1(m,n;\omega) = B_{m,n}^\omega U_1(m,-n;\omega),$$
$$\bar{r}_{m,n}^\omega U_1(m,n;\omega) = \bar{B}_{m,n}^\omega U_1(m,-n;\omega) \tag{16}$$

for class I, and

$$r_m^\omega U_1(m,\omega) = iB_m^\omega U_1(m,\omega\pm1),$$
$$\bar{r}_m^\omega U_1(m,\omega) = i\bar{B}_m^\omega U_1(m,\omega\pm1) \tag{17}$$

for class II. Here U_1 is the one-point function for "identity boundary conditions" of the corresponding field. In (17) the factor i's appear because the class II null-operators are fermionic, and $+$ $(-)$ refers to class IIA (IIB).

The one-point function on the Poincaré disk for identity boundary conditions in $N=2$ SLFT was obtained in [18] and has a general form:

$$U_1(\alpha,\bar{\alpha}) = \Gamma\big(b^{-2}\big)(\pi\mu)^{-\frac{1}{b}(\alpha+\bar{\alpha})}\frac{\Gamma(1-\alpha b)\Gamma(1-\bar{\alpha}b)}{\Gamma(-\frac{\alpha+\bar{\alpha}}{b}+\frac{1}{b^2})\Gamma(2-b(\alpha+\bar{\alpha}))}.$$

With the specific values (4) the ratio of one-point functions of class I fields then is:

$$\frac{U_1(m,n;\omega)}{U_1(m,-n;\omega)} = (\pi\mu)^{2n}\frac{\gamma(1+m-nb^2)}{\prod_{k=-n}^{n-1}(\frac{m}{b^2}+k)\prod_{l=-m}^{m}(l+nb^2)}\frac{\gamma(\frac{1-m}{2}+(n-\omega)\frac{b^2}{2})}{\gamma(\frac{1-m}{2}-(n+\omega)\frac{b^2}{2})}\prod_{l=1-m,\ \mathrm{mod}\ 2}^{m-1}\left(\frac{l+(n-\omega)b^2}{l-(n+\omega)b^2}\right)$$

and for the HEM coefficient we obtain:

$$\begin{aligned}B^{\omega}_{m,n} &= \bar{B}^{\omega}_{m,n} = r^{\omega}_{m,n}\frac{U_1(m,n;\omega)}{U_1(m,-n;\omega)}\\ &= (\pi\mu)^{2n}b^{1+2n-2m}\gamma\big(m-nb^2\big)\frac{\gamma(\frac{1-m}{2}+(n-\omega)\frac{b^2}{2})}{\gamma(\frac{1-m}{2}-(n+\omega)\frac{b^2}{2})}\prod_{l=1-m}^{m-1}\prod_{k=1-n}^{n-1}\left(\frac{l}{b}+kb\right),\end{aligned} \tag{18}$$

where we impose that $(k,l)=(0,0)$ is excluded in the product.

Analogously for class IIA fields:

$$\frac{U_1(m,\omega)}{U_1(m,\omega+1)} = \pi\mu b\frac{\prod_{l=\frac{m+1}{2}}^{m-1}(\frac{l}{b}-b(\omega+1))}{\prod_{l=\frac{m+1}{2}}^{m}(\frac{l}{b}-b\omega)}$$

and

$$\begin{aligned}&B^{\omega}_{m}=0,\\ &\bar{B}^{\omega}_{m} = -i\bar{r}^{\omega}_{m}\frac{U_1(m,\omega)}{U_1(m,\omega+1)} = -2\pi i\mu b^{2-m}\Gamma^2\left(\frac{m+1}{2}\right)\prod_{l=\frac{m+1}{2}}^{m-1}\left(\frac{l}{b}-b(\omega+1)\right)^2.\end{aligned} \tag{19}$$

For class IIB B and $\bar{B}$ are exchanged and ω is replaced by $-\omega$. Equalities (18) and (19) are the main results of this Letter.

4. Classical limit

In the classical limit $b\to 0$: $b\phi\to\varphi$, $\beta\psi\to\psi$, $\pi\mu b^2\to M$ the Lagrangian $\mathcal{L}\to\frac{1}{2\pi b^2}\mathcal{L}$. The corresponding equations of motion are given by

$$\begin{aligned}&\bar{\partial}\psi^{\pm} = -iM\bar{\psi}^{\mp}e^{\varphi^{\pm}}, \qquad \partial\bar{\psi}^{\pm} = iM\psi^{\mp}e^{\varphi^{\pm}},\\ &\partial\bar{\partial}\varphi^{\pm} = iM\psi^{\pm}\bar{\psi}^{\pm}e^{\varphi\mp} + M^2e^{\varphi^{+}+\varphi^{-}}.\end{aligned} \tag{20}$$

The holomorphic currents

$$\begin{aligned}&T = -\partial\varphi^{-}\partial\varphi^{+} - \frac{1}{2}\big(\psi^{-}\partial\psi^{+}+\psi^{+}\partial\psi^{-}\big) + \frac{1}{2}\big(\partial^2\varphi^{+}+\partial^2\varphi^{-}\big),\\ &S^{\pm} = -i\sqrt{2}\big(\psi^{\pm}\partial\varphi^{\pm}-\partial\psi^{\pm}\big), \qquad J = \partial\varphi^{+}-\partial\varphi^{-}-\psi^{-}\psi^{+},\end{aligned} \tag{21}$$

are conserved by $\bar{\partial}T=\bar{\partial}S^{\pm}=\bar{\partial}J=0$ on the equations of motion and similarly for the antiholomorphic ones. One has to introduce also the generators of $N=2$ supersymmetry $G^{\pm}$ and $\bar{G}^{\pm}$:

$$\begin{aligned}&G^{\pm}\varphi^{\mp} = i\sqrt{2}\psi^{\pm}, \qquad G^{\pm}\varphi^{\pm}=0\\ &\bar{G}^{\pm}\varphi^{\mp} = i\sqrt{2}\bar{\psi}^{\pm}, \qquad \bar{G}^{\pm}\varphi^{\pm}=0\end{aligned} \tag{22}$$

obeying the algebra:

$$\begin{aligned}&\big\{G^{+},G^{-}\big\}=2\partial, \qquad \big\{G^{\pm},G^{\pm}\big\}=\big\{\bar{G}^{\pm},\bar{G}^{\pm}\big\}=0,\\ &\big\{\bar{G}^{+},\bar{G}^{-}\big\}=2\bar{\partial}, \qquad \{G,\bar{G}\}=0.\end{aligned} \tag{23}$$

For the class IIA fields only the chiral fields, $N^{\omega}_1=e^{\omega b\phi^{+}}$, has a classical limit. Their HEMs take the form:

$$\bar{G}^{+}_{-\frac{1}{2}}G^{+}_{-\frac{1}{2}}\phi^{+}N^{\omega}_1=0, \qquad \bar{G}^{+}_{-\frac{1}{2}}G^{+}_{-\frac{1}{2}}\phi^{-}N^{\omega}_1 = B^{\omega}_1N^{\omega+1}_1,$$

where $B^{\omega}_1=-2\pi i\mu b$ can be read from (19). In the classical limit along with the analogous HEMs for class IIB antichiral fields with $\omega=0$, these become:

$$\bar{G}^{\pm} G^{\pm} \varphi^{\mp} = -2iMe^{\varphi^{\pm}}.$$

Together with (22) and the algebra (23) these relations encode the equations of motion (20).

From the class I fields only the series $N^{\omega}_{1,n}$ has a classical limit, the simplest "classical null-operators" being:

$$D^{\omega(cl)}_{1,1} = \partial - \frac{1}{2}(\omega+1)J + \frac{1}{\omega-1}G^{+}G^{-},$$
$$D^{\omega(cl)}_{1,2} = \partial^2 - (\omega+1)J\partial - \frac{1}{2}(\omega+2)\partial J + \frac{1}{4}\omega(\omega+2)J^2 + \frac{2}{\omega-2}G^{+}G^{-}\partial - \frac{\omega}{\omega-2}JG^{+}G^{-} - \frac{1}{2}S^{-}G^{+} + \frac{1}{2}\frac{\omega+2}{\omega-2}S^{+}G^{-}.$$

It is easy to check, using the algebra (23) and the explicit form of the currents (21), that the classical expressions of the corresponding null-vector conditions is:

$$D^{\omega(cl)}_{1,1} e^{(\frac{1}{2}(\omega-1)\varphi^{+} - \frac{1}{2}(\omega+1)\varphi^{-})} = 0,$$
$$D^{\omega(cl)}_{1,2} e^{(\frac{1}{2}(\omega-2)\varphi^{+} - \frac{1}{2}(\omega+2)\varphi^{-})} = 0.$$

The same is of course true also for $\bar{D}^{\omega(cl)}_{1,1}$, $\bar{D}^{\omega(cl)}_{1,2}$. Then, with the help of (22) and the equations of motion (20), we find that the classical HEMs then take the form:

$$\bar{D}^{\omega(cl)}_{1,1} D^{\omega(cl)}_{1,1} \varphi^{\pm} e^{(\frac{1}{2}(\omega-1)\varphi^{+} - \frac{1}{2}(\omega+1)\varphi^{-})} = \frac{\omega+1}{\omega-1} M^2 e^{(\frac{1}{2}(\omega+1)\varphi^{+} - \frac{1}{2}(\omega-1)\varphi^{-})},$$
$$\bar{D}^{\omega(cl)}_{1,2} D^{\omega(cl)}_{1,2} \varphi^{\pm} e^{(\frac{1}{2}(\omega-2)\varphi^{+} - \frac{1}{2}(\omega+2)\varphi^{-})} = -2\frac{\omega+2}{\omega-2} M^4 e^{(\frac{1}{2}(\omega+2)\varphi^{+} - \frac{1}{2}(\omega-2)\varphi^{-})}.$$

This is in a perfect agreement with (14) if we take into account that the classical limit, $b \to 0$, of $B^{\omega}_{1,n} = \bar{B}^{\omega}_{1,n}$ from (18) is:

$$B^{\omega}_{1,n} \to (-1)^{n+1} \frac{\omega+n}{\omega-n} n!(n-1)! b^{-1} \left(\pi \mu b^2\right)^{2n}.$$

To conclude, we presented relations among primary fields, the higher equations of motion, in $N = 2$ supersymmetric Liouville field theory. We stress that, since in general the null-vectors of this theory are unknown, our results (18) and (19) should be understood as a proposal. Also, we were concerned in this Letter with primary fields from the NS sector only. Since the Ramond sector in $N = 2$ SLFT is not very different, in particular the degenerate fields fall into the same classes, we expect that very similar HEMs hold for them too.

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