Cut points in some connected topological spaces

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We prove that a connected topological space with endpoints has exactly two non-cut points and every cut point is a strong cut point; it follows that such a space is a COTS and the only two non-cut points turn out to be endpoints (in each of the two orders) of the COTS. A non-indiscrete connected topological space with exactly two non-cut points and having only finitely many closed points is proved homeomorphic to a finite subspace of the Khalimsky line. Further, it is shown, without assuming any separation axiom, that in a connected and locally connected topological space \( X \), for \( a, b \) in \( X \), \( S[a, b] \) is compact whenever it is closed. Using this result we show that an \( H(i) \) connected and locally connected topological space with exactly two non-cut points is a compact COTS with end points.

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1. Introduction

The concept of cut points plays a very important role in topological spaces. To study cut points a topological space is assumed to be connected and nondegenerate. The idea of a cut point in a topological space comes dates back to 1920’s [4].

The removal of a cut point from a topological space leaves the space disconnected. The real line \( \mathbb{R} \) and the Khalimsky line \( \mathbb{Z} \) are topological spaces, where every point is a cut point. In connected topological spaces, cut points have been studied by Honari and Babrampour [1]. Sometimes to study cut points some additional properties like compactness or its weaker forms and/or separation axioms like \( T_2 \), \( T_1 \), etc., are also assumed. Cut points in connected compact Hausdorff spaces have been studied by Whyburn [4]. In view of the applications of cut points (see [3]) and the fact that the many connected topological spaces used for cut points like the Khalimsky line are not \( T_1 \), the assumption of separation axioms is avoided as far as possible. In \( H(i) \) spaces no separation axioms are assumed; there are some results about cut points in \( H(i) \) connected topological spaces in [2].

In this paper, we study cut points in connected topological spaces initially without assuming any additional property. Notation, definitions and preliminaries are given in Section 2. The main results of the paper appear in Sections 3 and 4.

In Section 3, we prove that a connected space with endpoints has exactly two non-cut point and every cut point is a strong cut point; it follows that such a space is a COTS and the only two non-cut points turn out to be endpoints in each of the two orders of the COTS. Let \( X \) be a connected space and \( Y \) be a proper non-empty connected subset of \( X \) such that \( X - Y \subset \text{ct} X \). Then \( X \neq \bigcup\{S[y_i, z_i]: \ i = 1, 2, \ldots, n\} \) for any different \( y_1, y_2, \ldots, y_n \) and \( z_1, z_2, \ldots, z_n \) in \( X \). Further, it is
proved that the set of all closed points of a connected space \( X \) in which there is a proper non-empty connected subset \( Y \) that \( X - Y \) is a subset of the set of all cut points \( X \), is infinite. This strengthens Theorem 3.7 of [1] where it is assumed that the space is a cut point space. A characterization of the Khalimsky line is obtained in [1].

Here we obtain a characterization of finite connected subspaces of the Khalimsky line. A connected space having only finitely many closed points is proved to have at least two non-cut points, and such non-indiscrete space which has exactly two non-cut points is proved homeomorphic to a finite subspace of the Khalimsky line.

In Section 4, it is proved that in an \( H(i) \) connected space \( X \) where the removal of any two-point disconnected set leaves the space disconnected, given any two open or closed points \( x \) and \( y \) of \( X \), there exist two subsets \( M \) and \( N \) of \( X \) such that \( M \cup N = X \), \( M \cap N = \{x, y\} \), and both \( M \) and \( N \) are \( T_{1/2} \) \( H(i) \) COTS. Without assuming any separation axiom, we prove that in a connected and locally connected space \( X \), for \( a \) and \( b \) in \( X \), whenever \( S[a, b] \) is closed in \( X \), it is compact. Using this result, it is shown that an \( H(i) \) connected and locally connected space with exactly two non-cut points is a compact COTS with endpoints.

2. Notation, definitions and preliminaries

For notation and definitions, we shall mainly follow [2]. For completeness, we have included some of the standard notation and definitions. By a space, we shall mean a topological space containing at least two points.

If \( x \in X \) is such that \([x]\) is closed, we say that \( x \) is a closed point of \( X \). A space is called \( T_{1/2} \) if every singleton set is either open or closed. A space \( X \) is called \( H(i) \) if every open cover of \( X \) has a finite subcollection such that the closures of the members of that subcollection cover \( X \). A point \( x \) of a space \( X \) is called a cut point if there exists a separation \( X \) \(-\) \([x]\). If \( A \) \( | \) \( B \) form a separation of a space \( X \), then we say that each one of \( A \) and \( B \) is a separating set of \( X \). A separation \( A \) \( | \) \( B \) of \( X \) is used to denote the set of all cut points of a space \( X \). Let \( x \in \text{int} X \). A separation \( A \) \( | \) \( B \) of \( X \) \(-\) \([x]\) is denoted by \( A_{x}[a, b] \) \( B_{x} \). Similarly, for a connected subset \( Y \) of \( X \) \(-\) \([x]\), \( A_{x}(Y) \) is used to denote the separating subset of \( X \) \(-\) \([x]\) containing \( Y \), \( A_{x}(Y) \) is used for the set \( A_{x}(Y) \) \(-\) \([x]\). If \( Y = \{a\} \), we write \( A_{x}(a) \) for \( A_{x}(Y) \) and \( A_{x}(a) \) for \( A_{x}(Y) \). A connected space with \( x \) \(-\) \([x]\) is called a cut point space. Let \( a, b \in X \). A point \( x \in \text{int} X \) \(-\) \([a, b]\), is said to be a separating point between \( a \) and \( b \) if for some separation \( A_{x}[a, b] \) of \( X \) \(-\) \([x]\), we have \( a \in A_{x}, b \in B_{x} \) or \( b \in A_{x}, a \in B_{x} \). By Lemma 2.2(a) of [3], \( A_{x} \) is a closed point of \( X \), \( T \). Without assuming any separation axiom, we prove that in a connected and locally connected space \( X \), for \( a \) and \( b \) in \( X \), whenever \( S[a, b] \) is closed in \( X \), it is compact. Using this result, it is shown that an \( H(i) \) connected and locally connected space with exactly two non-cut points is a compact COTS with endpoints.

Lemma 2.1. Let \( X \) be a connected space and \( a, b, x, y \in X \).

(i) If \( t \in S(x, y) - S[a, b] \) and \( X - \{t\} = A_{t}(x) \cup B_{t}(y) \), then

(a) if \( a \in A_{t}(x) \), then \( b \notin B_{t}(y) \),

(b) either \([a, b] \subset A_{t}(x) \) or \([a, b] \subset B_{t}(y) \).

(ii) If \( x, y \in S[a, b] \), then \( S[x, y] \subset S[a, b] \).

Proof. Let \( t \in S(x, y) - S[a, b] \) and \( X - \{t\} = A_{t}(x) \cup B_{t}(y) \).

(i) If \( t \) does not hold, then \( t \in S[a, b] \) which is not possible.

(ii) (i) by \( t \), if \( b \notin B_{t}(y) \), then \( a \notin A_{t}(x) \). Now \( (i) \) follows using \( (i) \) as \( t \notin \{a, b\} \).

(ii) Let \( x, y \in S[a, b] \). To show \( S[x, y] \subset S[a, b] \), we assume otherwise, so there is some \( t \in S(x, y) - S[a, b] \). Then \( X - \{t\} = A_{t}(x) \cup B_{t}(y) \). Now by \( (i) \), either \([a, b] \subset A_{t}(x) \) or \([a, b] \subset B_{t}(y) \).

First assume that \([a, b] \subset A_{t}(x) \). This implies that \( y \notin \{a, b\} \). Thus \( y \in S[a, b] \) and therefore \( X - \{y\} = A_{y}(a) \cup B_{y}(b) \). By Lemma 2.2(a) of [3], \( A_{y} \) is connected; so either \( A_{y} \subset A_{y} \) or \( A_{y} \subset B_{y} \), as \( A_{y} \subset X - \{y\} \). This contradicts \([a, b] \subset A_{t}(x) \). The case \([a, b] \subset B_{t}(y) \) is similar to the case \([a, b] \subset A_{t}(x) \). \( \square \)

3. Connected topological spaces and cut points

Theorem 3.1. Let \( X \) be a connected space with endpoints. Then \( X \) has exactly two non-cut points and every cut point of \( X \) is a strong cut point.

Proof. Since \( X \) is a connected space with endpoints, there are two points \( a \) and \( b \) in \( X \) such that \( X = S[a, b] \). Using Lemma 4.1 of [2], \( X - \{a\} \) and \( X - \{b\} \) are connected sets of \( X \). So \( a \) and \( b \) are non-cut points of \( X \). As \( X = S[a, b] \) and by definition of \( S[a, b] \), every point of \( S[a, b] \) is a cut point of \( X \), so every point of \( X \) other than \( a \) and \( b \) is a cut point of \( X \). Therefore \( X \) has exactly two non-cut points namely \( a \) and \( b \). Now we prove that every point of \( S[a, b] \) is a strong cut point of \( X \). For this, let \( x \in S[a, b] \). Then \( X - \{x\} = A_{x}(a) \cup B_{x}(b) \). Since \( A_{x}(a) - \{a\} \) does not contain \( b \), \( A_{x}(a) - \{a\} \subset X - \{a, b\} = S[a, b] \). Now using Lemma 4.2(ii) of [2], \( A_{x}(a) \) is connected. Similarly \( B_{x}(b) \) is connected. This completes the proof. \( \square \)
Lemma 2.11(a) of [3] can be stated as follows.
If $X$ is connected and has two points say $a$ and $b$ such that every other point of $X$ is a strong cut point that separates $a$ and $b$, then $X$ is a COTS with $a$ and $b$ its endpoints (in each of its two orders).

Cut points are not assumed to be strong in the following theorem.

**Theorem 3.2.** Let $X$ be a connected space with endpoints. Then $X$ is a COTS where the only two non-cut points of $X$ are endpoints (in each of the two orders) of the COTS.

**Proof.** Since $X$ is a connected space with endpoints, there are two points $a$ and $b$ in $X$ such that $X = S(a, b)$. By Theorem 3.1, every cut point of $X$ is a strong cut point. Now the result follows by Lemma 2.11(a) of [3].

**Remark 3.3.** By Theorem 2.9 of [3], a COTS with at least three points is $T_{1/2}$. So by Theorem 3.2, every non-indiscrete connected space with endpoints is a $T_{1/2}$ space.

**Theorem 3.4.** Let $H$ be a subset of a connected space $X$. Let $Y$ be a proper non-empty connected subset of $H$ such that $H - Y \subseteq \operatorname{ct} X$. Then there exists a chain of proper connected sets of the form $A^*_x(Y)$ where $x \in H - Y$, union of whose members contains $H$.

**Proof.** $\subseteq = \{A^*_x(Y): x \in H - Y\}$ as a partially ordered set with respect to set inclusion has a maximal chain, say $\alpha$, by the Hausdorff Maximal Principle. As elements of a separation, members of the chain are proper subsets of $X$. Also they are connected by Lemma 2.2(a) of [3]. The union of $\alpha$, which we call $W$, is a connected subset of $X$ as $\alpha$ is a chain and members of $\alpha$ are connected. To show $H \subseteq W$, we assume otherwise, so there is some $y \in H - W$. Then $y \in H - Y$ as $Y \subseteq W$. By the given condition, $y \in \operatorname{ct} X$. Therefore $W \subseteq \{x \in H - \{y\} = A_y(Y) \cup B_y$. Since $W$ is connected and $y \in W$, so $W \subseteq A_y(Y)$. Thus we get a chain $\alpha \cup \{A_y^*(Y)\}$ in $\subseteq$ containing $\alpha$ properly. This contradicts $y \notin W$, showing that $H \subseteq W$. Hence $\alpha$ is a required chain.

In many of the following results, we assume that $Y$ is a connected proper subset of a connected space $X$ such that $X - Y \subseteq \operatorname{ct} X$. Note that this cannot occur if $X$ is a connected space with endpoints and any other points. For if $X - Y \subseteq \operatorname{ct} X$ then the endpoints must be in $Y$, thus all others are as well, since by Theorem 3.1 they separate the endpoints.

**Corollary 3.5.** Let $X$ be a connected space and $Y$ be a proper non-empty connected subset of $X$ such that $X - Y \subseteq \operatorname{ct} X$. Then there exists an infinite chain of proper connected sets of the form $A^*_x(Y)$ covering $X$.

**Proof.** Take $H = X$ in Theorem 3.4. Since no $A^*_x(Y) = X$, and our chain covers $X$, the chain must be infinite.

**Lemma 3.6.** Let $X$ be a connected space and $x \in \operatorname{ct} X$. Then $(A^*_x)^-\subseteq \operatorname{ct} X$ is regularly closed.

**Proof.** By Lemma 3.1 of [2], $\{x\}$ is either open or closed. If $\{x\}$ is open, then $B_x$ is closed and so $A^*_x = X - B_x$ is open. Therefore in this case, $(A^*_x)^-\subseteq \operatorname{ct} X$ is regularly closed. If $\{x\}$ is closed, then each of $A_x$ and $B_x$ is open, and so $A^*_x$ is closed and $(A^*_x)^- = A^*_x$. Thus $(A^*_x)^- = A^*_x = (A_x)^-$. Therefore $(A^*_x)^-\subseteq \operatorname{ct} X$ is regularly closed.

**Lemma 3.7.** Let $X$ be a connected space and $Y$ a proper non-empty connected subset of $X$ such that $X - Y \subseteq \operatorname{ct} X$. Let $x \in X - Y$ be such that $\{x\}$ is open. Then $(A^*_x(Y))^\subseteq \operatorname{ct} X$ for some $t \in X - Y$, $t$ a closed point of $X$.

**Proof.** As $\{x\}$ is open, $A_x(Y)$ is closed, $(A^*_x(Y))^\subseteq \operatorname{ct} X$ is closed, and $(A^*_x(Y))^\subseteq \operatorname{ct} X = (A_x(Y)) \cup \{x\}$ = $(A_x(Y)) \cup \{x\}$. If $(\{x\})^\subseteq \operatorname{ct} X$ is not possible as $\{x\}$ is open. Let $t \in (\{x\})^\subseteq \operatorname{ct} X$, $t$ is either open or closed by Lemma 3.1 of [2]. If $t$ is open, then $t \cap \{x\} = \phi$ implies $t$ is not a closure point of $\{x\}$, a contradiction. So $t$ is closed. Also $A^*_x(Y) \subseteq X - \{t\} = A_t(Y) \cup B_t$. Therefore $A^*_x(Y) \subseteq A_t(Y)$, as $A^*_x(Y)$ is connected by Lemma 2.2(a) of [3]. Thus $(A^*_x(Y))^\subseteq \operatorname{ct} X = A_t(Y)$.

**Theorem 3.8.** Let $X$ be a connected space and $Y$ a proper non-empty connected subset of $X$ such that $X - Y \subseteq \operatorname{ct} X$. Then $x \in X - Y$, $(A^*_x(Y))^\subseteq \operatorname{ct} X$ is a properly regularly closed connected subset of $X$.

**Proof.** Let $x \in X - Y$. By Lemma 3.1 of [2], $\{x\}$ is either open or closed. If $\{x\}$ is closed, $B_x$ is open and so $A^*_x(Y) = X - B_x$ is closed. If $\{x\}$ is open, by Lemma 3.7, $(A^*_x(Y))^\subseteq \operatorname{ct} X$ for some closed point $t$ of $X$ such that $t \in X - Y$, $(A^*_x(Y))^\subseteq \operatorname{ct} X$ is regularly closed by Lemma 3.6; $(A^*_x(Y))^\subseteq \operatorname{ct} X$ is connected by Lemma 2.2(a) of [3].

**Lemma 3.9.** Let $X$ be a connected space. If $\subseteq$ is a chain of members of the form $A^*_x$, $x \in \operatorname{ct} X$ covering $X$, then for any $y, z$ in $X$, $y \neq z$, there is some $A^*_x$ in $\subseteq$ such that $\{y, z\} \subseteq A^*_x$. 


Proof. By the given condition, there exist \( A^*_x \) and \( A^*_y \) in \( \zeta \) such that \( y \in A^*_x \) and \( z \in A^*_y \). Since \( \zeta \) is a chain, we suppose that \( A^*_x \subset A^*_y \). Thus \( y, z \in A^*_x \). Let \( r \in S(y, z) \). If \( r \notin A^*_x \), then \( A^*_x \subset X - \{ r \} = A_r(y) \cup B_r(z) \). Since \( A^*_x \) is connected (by Lemma 2.2(a) of [3]), either \( A^*_x \subset A_r(y) \) or \( A^*_x \subset B_r(z) \) which is not possible as \( y, z \in A^*_x \). Therefore \( r \in A^*_x \). Thus \( S(y, z) \subset A^*_x \). \( \square \)

Theorem 3.10. Let \( X \) be a connected space and \( Y \) be a proper non-empty connected subset of \( X \) such that \( X - Y \subset c_t X \). Then \( X \neq \bigcup \{ S[y, z] : i = 1, 2, \ldots, n \} \) for any distinct \( y_1, y_2, \ldots, y_n \) and \( z_1, z_2, \ldots, z_n \) in \( X \).

Proof. By Corollary 3.3, there is an infinite chain \( \zeta \) consisting of proper connected sets of the form \( A^*_x(Y) \) covering \( X \). Let \( y_1, y_2, \ldots, y_n \) and \( z_1, z_2, \ldots, z_n \) in \( X \). Using Lemma 3.9, there exist \( A^*_x(Y), A^*_y(Y), \ldots, A^*_z(Y) \) in \( \zeta \) such that \( S[y_i, z_i] \subset A^*_x(Y), i = 1, 2, \ldots, n \). This implies that \( \bigcup \{ S[y_i, z_i] : i = 1, 2, \ldots, n \} \subset \bigcup \{ A^*_x(Y) : i = 1, 2, \ldots, n \} \). Since \( \zeta \) is a chain, we know that for some \( j \in \{ 1, 2, \ldots, n \}, \bigcup \{ A^*_x(Y) : i = 1, 2, \ldots, n \} \subset A^*_y(Y) \).

Thus \( \bigcup \{ S[y_i, z_i] : i = 1, 2, \ldots, n \} \subset A^*_y(Y) \). Hence \( X \neq \bigcup \{ S[y_i, z_i] : i = 1, 2, \ldots, n \} \) as \( X \neq A^*_y(Y) \). \( \square \)

Lemma 3.11. Let \( X \) be a connected space and \( Y \) a proper non-empty connected subset of \( X \) such that \( X - Y \subset c_t X \). Then there exists an infinite chain of proper connected sets of the form \( A^*_x(Y) \) where \( x \in X - Y \), \( x \) a closed point of \( X \), covering \( X \).

Proof. Let \( x \in X - Y \subset c_t X \). Since \( Y \subset X - \{ x \} = A_x \cup B_x \) and \( Y \) is connected, either \( Y \subset A_x \) or \( Y \subset B_x \). We suppose that \( Y \subset A_x \). This implies that \( B_x \subset c_t X \). Using Lemma 3.5 of [1], \( B_x \) contains a closed point of \( X \). Now \( \zeta = \{ A^*_x(Y) : x \in X - Y, x \) a closed point of \( X \} \) is a non-empty partially ordered set under set inclusion. By the Hausdorff Maximal Principle, there exists a maximal chain \( \alpha \) in \( \zeta \). Members of the chain are connected by Lemma 2.2(a) of [3]. \( W \) the union of members of \( \alpha \) is a connected subset of \( X \) as \( \alpha \) is a chain and members of \( \alpha \) are connected. Let \( y \in X - W \). Then \( y \in X - Y \) as \( Y \subset W \). By the given condition, \( y \in c_t X \). Therefore \( W \subset X - \{ y \} = A_y(Y) \cup B_y \). Since \( W \) is connected and \( Y \subset W \), so \( W \subset A_y(Y) \). By Lemma 3.5 of [1], \( B_y \) contains a closed point say \( t \) of \( X \). We have \( W \subset X - \{ t \} = A_t(Y) \cup B_t \). This implies that \( W \subset A_t(Y) \). Thus we get a chain of \( A \cup \{ A^*_x(Y) \} \) in \( \zeta \) containing \( \alpha \) properly. This proves that \( X = W \). \( \square \)

Using Lemma 3.11, we have the following result which does not assume the space to be a cut point space and thus strengthens Theorem 3.7 of [1].

Theorem 3.12. Let \( X \) be a connected space and \( Y \) a proper non-empty connected subset of \( X \) such that \( X - Y \subset c_t X \). Then the set of closed points of \( X \) is infinite.

Proof. By Lemma 3.11, there exists an infinite chain of proper connected sets of the form \( A^*_x(Y) \) where \( x \in X - Y \), \( x \) a closed point of \( X \), covering \( X \). If the set of closed points of \( X \) is finite, then \( X \) is equal to one member of the chain which is not possible as members of the chain of \( X \) is infinite. Hence the set of closed points of \( X \) is infinite. \( \square \)

Corollary 3.13. Let \( X \) be a connected space. If the set of closed points of \( X \) is finite, then \( X \) has at least two non-cut points.

Proof. Suppose to the contrary. Then \( X - c_t X \) has at most one element. There exists some \( x \in X \) such that \( X - \{ x \} \subset c_t X \). Taking \( Y = \{ x \} \) in Theorem 3.12, we get a contradiction. \( \square \)

Lemma 3.14. Let \( X \) be a connected space and \( x \in c_t X \). If the set of closed points of \( X \) is finite, then \( A^*_x \) has at least two non-cut points in \( A^*_x \).

Proof. Since \( x \in c_t X \), using Lemma 3.1 of [2], either \( A_x \) is closed or \( A^*_x \) is closed. Therefore the set of closed points of \( A^*_x \) is finite using the given condition. Thus \( A^*_x \) has at least two non-cut points in \( A^*_x \) by Lemma 2.2(a) of [3] and Corollary 3.13. \( \square \)

Lemma 3.15. Let \( X \) be a connected space and \( x \in c_t X \). If the set of closed points of \( X \) is finite, then \( A_x \) contains a non-cut point of \( X \).

Proof. By Lemma 3.14, \( A^*_x \) has at least two non-cut points in \( A^*_x \). Now the result follows by Lemma 3.8 of [2]. \( \square \)

Theorem 3.16. Let \( X \) be a connected space such that \( c_t X \neq \phi \). If the set of closed points of \( X \) is finite, then there is a two point disconnected subspace \( \{ a, b \} \) of \( X \) such that \( X - \{ a, b \} \) is connected.

Proof. Let \( x \in c_t X \). By Lemma 3.14, \( A^*_x \) has at least two non-cut points in \( A^*_x \). Let \( a \) be a non-cut point of \( A^*_x \) other than \( x \). Then \( A^*_x \setminus \{ a \} \) is connected and \( a \in A_x \). Similarly, there exists \( b \in B_x \) such that \( A^*_x \setminus \{ b \} \) is connected. Now \( A^*_x \setminus \{ a \} \cup (B^*_x \setminus \{ b \}) \) is connected. Therefore \( X - \{ a, b \} \) is connected. By an application of Lemma 3.1 of [2], \( \{ a, b \} \) is disconnected. \( \square \)

Theorem 3.17. Let \( X \) be a non-indiscrete connected space with exactly two non-cut points say \( a \) and \( b \). If the set of closed points of \( X \) is finite, then \( X \) is homeomorphic to a finite subspace of the Khalimsky line.
Proof. We can suppose that \( cT \neq \phi \). Let \( x \in X - \{a, b\} \). By Lemma 3.15, each one of \( A_x \) and \( B_x \) contains a non-cut point of \( X \). These non-cut points have to be \( a \) and \( b \), because \( X - ctX = \{a, b\} \). So \( a \in A_x \) and \( b \in B_x \) or conversely. This implies that \( x \in S(a, b) \), hence \( X = S(a, b) \). By Theorem 3.2, \( X \) is a COTS where the only two non-cut points of \( X \) are endpoints (in each of the two orders) of the COTS. Let \( K \), the set of closed points of \( X \) contain \( k \) elements. Suppose that \( X - K \) contains at least \( k + 2 \) elements, \( x_1, x_2, x_3, \ldots, x_{k+2} \); these can be written as \( x_1 < x_2 < x_3 < \cdots < x_{k+1} < x_{k+2} \), in an order \( < \) of the COTS \( X \). By the given condition, each \( \{x_i\} \), \( 1 \leq i \leq k + 2 \), is open in \( X \) by Theorem 2.5 of [3]. Since each \( x_i \), \( 1 \leq i \leq k + 1 \), has a successor, using Lemma 2.8(b) and (c) of [3], \( x_i \) has an immediate successor, say \( y_i \), such that \( x_i < y_i < x_{i+1} \) and \( \{y_i\} \) is closed in \( X \); so \( y_i \in K \). This shows that \( K \) contains at least \( k + 1 \) elements. Now \( X - K \) being finite, \( X \) is homeomorphic to a finite subspace of the Khalimsky line by 2.10 of [3]. \( \Box \)

4. \( H(i) \) and compact COTS

The following theorem strengthens Theorem 3.17 of [2].

Theorem 4.1. Let \( X \) be an \( H(i) \) connected space such that the removal of any two-point disconnected set leaves the space disconnected. Then given any two open or closed points \( x \) and \( y \) of \( X \), there exist two subset \( M \) and \( N \) of \( X \) such that \( M \cup N = X \), \( M \cap N = \{x, y\} \), and both \( M \) and \( N \) are \( T_{1/2} \) \( H(i) \) COTS.

Proof. Let \( H[K] \) be a separation of \( X - \{x, y\} \). By Corollary 3.13 of [2], \( x \) and \( y \) are non-cut points of \( X \); so both \( X - \{x\} \) and \( X - \{y\} \) are connected. Since \( H \) is a separating subset of \( (X - \{y\}) - \{x\} \) as well as \( (X - \{x\}) - \{y\} \), by Lemma 2.2(a) of [3], \( H \cup \{x\} \) and \( H \cup \{y\} \) are connected; note that their intersection \( i.e. \) is non-empty. Therefore their union \( H \cup \{x, y\} \) is connected. \( H \) being a separating subset of \( X - \{x, y\} \) is open and closed in \( X - \{x, y\} \). Since each of \( \{x\} \), \( \{y\} \), is either open or closed in \( X \), using Lemma 3.6(iii) of [2] inductively, \( H \cup \{x, y\} \) is \( H(i) \). Similarly, \( K \cup \{x, y\} \) is \( H(i) \) and connected. Let \( M = H \cup \{x, y\} \) and \( N = K \cup \{x, y\} \). Then \( M \cap N = X \), \( M \cap N = \{x, y\} \), and both \( M \) and \( N \) are \( H(i) \) and connected. Clearly \( x \) and \( y \) are non-cut points of \( M \) as well as \( N \). We claim that \( x \) and \( y \) are the only non-cut points of \( M \) as well as \( N \). If we suppose to the contrary, then there are two cases:

Case I. There exist \( p \in H \), \( H \) a non-cut point of \( M \), and \( q \in K \), \( K \) a non-cut point of \( N \). Then \( M - \{p\} \) and \( N - \{q\} \) are connected. Therefore \( X - (\{p, q\} = (M - \{p\}) \cup (N - \{q\}) \) is connected which is a contradiction to the given condition as \( \{p, q\} \) is a two-point disconnected set.

Case II. Only one of \( M \) and \( N \) has a non-cut point different from \( x \) and \( y \). We suppose that \( M \) has exactly two non-cut points and there is a point \( q \) in \( K \) such that \( q \) is a non-cut point of \( N \). Then \( N - \{q\} \) is connected. Let \( p \in H \). Then \( p \) is a cut point of \( M \), so \( M - \{p\} \) is disconnected. Let \( A \cap B \) be a separation of \( M - \{p\} \) in \( M \). Using Theorem 4.3 of [2], both \( A \) and \( B \) are connected. Using Theorem 3.14 of [2], we can suppose that \( x \in A \) and \( y \in B \). Therefore \( N \cup A \) is connected. This implies that \( X - \{p, q\} = (N \cup A) \cup B \) is connected, which is a contradiction to the given condition as \( \{p, q\} \) is a two-point disconnected set.

A contradiction in both cases proves our claim. Now, by Theorem 4.4 of [2], both \( M \) and \( N \) are COTS. Since each of \( M \) and \( N \) contains at least three points, by Theorem 2.9 of [3], both \( M \) and \( N \) are \( T_{1/2} \). This completes the proof. \( \Box \)

Lemma 4.2. Let \( X \) be a connected and \( x \in ctX \). Let \( p \in X \) be such that \( p \notin (A_x)^0 \). Then \( x \in V \) for every connected subset \( V \) of \( X \), containing \( p \) such that \( V \cap (A_x)^0 \neq \phi \).

Proof. Suppose \( x \notin V \) for some connected subset \( V \) of \( X \) containing \( p \) such that \( V \cap (A_x)^0 \neq \phi \). Since by Lemma 3.1 of [2], \( A_x \cap (A_x)^0 \subset A_x^* \), \( V \cap A_x \neq \phi \). Therefore \( V \subset A_x \) as \( V \) is a connected subset of \( X - \{x\} = A_x \cup B_x \). Thus \( p \in A_x \). By Lemma 3.1 of [2], \( p \in (A_x)^0 \), which is a contradiction to the given condition. This contradiction proves the lemma. \( \Box \)

Lemma 4.3. Let \( X \) be a connected and locally connected space. Let \( K = S[a, b] \) be closed in \( X \). Let \( M \) be a non-empty subset of \( S(a, b) \). Let \( \Omega = \{A: A = (A_x(a))^0 \} \) for some separating set \( A_x(a) \) of \( X - \{x\} \), \( x \in M \). Let \( T = \bigcup \{A: A \in \Omega \} \). Then

(i) \( (X - T) \cap d(T) \) is a non-empty subset of \( K \).

(ii) For \( p \in (X - T) \cap d(T) \) and an open covering \( \zeta \) of \( K \) in \( X \), there exist \( x \in M \), a member \( H \) of \( \zeta \), and a connected subset \( V \) of \( X \) containing both \( p \) and \( x \) such that \( V \subset H \).

Proof. (i) As elements of a separation, \( b \notin A \) for every \( A \in \Omega \), so \( b \notin T \). Thus \( T \) is not closed in \( X \) as \( X \) is connected and \( T \) is open. Therefore there exists \( p \in X - T \) such that \( p \) is a limit point of \( T \). This proves that \( (X - T) \cap d(T) \) is non-empty. To show \( (X - T) \cap d(T) \subset K \), let \( p \in (X - T) \cap d(T) \). If \( p \notin K \), \( p \) is a limit point of \( K \) as \( K \) is closed. Using local connectedness of \( X \), there exists a connected neighborhood \( G \) of \( p \) such that \( G \cap K = \phi \). Now \( G \cap (A_x^0)^0 \neq \phi \) for some
separating set $A_y(a)$, for some $y \in M$ as $p$ is a limit point of $T$. Since $p \in X - T$, $p \notin (A_y(a))^\circ$. Using Lemma 4.2, $y \in G$, which is not possible as $G \cap K = \emptyset$, showing that $(X - T) \cap d(T) \subseteq K$.
(ii) Let $p \in (X - T) \cap d(T)$ and $\varsigma$ be an open covering of $K$ in $X$. By (i), $p \in K$. $\varsigma$ being an open covering of $K$ in $X$, we get a member of $H$ of $\varsigma$ containing $p$. Local connectedness of $X$ gives a connected neighborhood $V$ of $p$ such that $V \subset H$. Since $p \in X - T$ and $p$ is a limit point of $T$, $V \cap (A_y(a))^\circ \neq \emptyset$ for some separating set $A_y(a)$ of $X - \{x\}, x \in M$. By Lemma 4.2, $x \in V$, and the proof is complete. □

Whyburn in [5] proved that in a $T_1$ connected and locally connected space, for $a$ and $b$ in $X$, $S[a, b]$ which is closed in $X$, is compact. We prove this result without using any separation axiom.

**Theorem 4.4.** Let $X$ be a connected and locally connected space. Then for $a$ and $b$ in $X$, $S[a, b]$ is compact whenever it is closed.

**Proof.** We assume that $S(a, b)$ is non-empty. To prove, $K = S[a, b]$ is compact, let $\varsigma$ be an open covering of $K$ in $X$.

Let $K^* = \{x \in K : S(a, x)\}$ is covered by a finite subcollection of $\varsigma$. We prove that $b \in K^*$. Suppose to the contrary.

We claim that $K^* = S[a, p]$ for some $p \in K^*$. If $K^* = \{a\}$, then $K^* = S[a, a]$. So we suppose that $K^* \neq \{a\}$. Then $K^* \neq \{a\}$ is non-empty. Taking $M = K^* - \{a\}$ in Lemma 4.3, $T = \bigcup ((A_y(a))^\circ; x \in K^* - \{a\})$, and there exist $p \in (X - T) \cap K, x \in K^* - \{a\}$, a member $H$ of $\varsigma$ and a connected set $V$ containing $p$ and $x$ such that $V \subset H$. $S[a, x]$ is covered by a finite subcollection, say $1^{\circ}$, of $\varsigma$. Let $\varsigma^2 = \varsigma^1 \cup \{H\}$. We prove that $p \in K^*$. Let $y \in S(a, p) - (S[a, x] \cup \{p\}), (X - \{y\} = A_y(a) \cup B_y(p)$. Since $y \in S(a, p) - S[a, x]$ and $a \in A_y(a)$, by Lemma 2.1(iii), $x \in A_y(a)$. Therefore $V \cap A_y(a) \neq \emptyset$; also $V \cap B_y(p) \neq \emptyset$. If $y \notin V, V \subset A_y(a)$ or $V \subset B_y(p)$ as $V$ is connected. Thus $y \in V$ and so $y \in H$. This proves that $S[a, p]$ is covered by members of $\varsigma^2$. Therefore $p \in K^*$. Now using Lemma 2.1(i), we see that $S[a, p] \subseteq K^*$. Let $x \in K^* - \{a\}$. As $p \notin T$, so $p \notin (A_y(a))^\circ$. Using Lemma 3.1 of [2], $p \notin A_y(a)$. So $p \in B_y(a)$. This shows that $x \in S[a, p]$. Thus $K^* = S[a, p]$. This proves our claim.

Now $K - K^* \neq \emptyset$ because then $K = K^* \cup \{p\}$ is covered by finitely many members of $\varsigma$. Let $T = \bigcup \{B : B = (B_y(a))^\circ$ for some separating set $B_y(a)$ of $X - \{x\}, x \in K - (K^* \cup \{b\})\}$. Using Lemma 4.3, there exist $q \in (X - T^\circ) \cap K, y \in K - (K^* \cup \{b\})$, a member $N$ of $\varsigma$ and a connected set $W$ containing $q$ and $y$ such that $W \subset N$. Suppose $S[a, q]$ is covered by a finite subcollection of $\varsigma$. Let $z \in S[a, q] - (S[a, y] \cup \{y\}), X - \{z\} = A_{z}(a) \cup B_{z}(b)$. Since $z \in S[a, y] - S[a, q]$ and $q \in A_{z}(a)$, by Lemma 2.1(ii), $q \in A_{z}(a)$. Therefore $W \cap A_{z}(a) \neq \emptyset$, also $W \cap B_{z}(b) \neq \emptyset$. If $z \notin W, W \subset A_{z}(a)$ or $W \subset B_{z}(b)$ as $W$ is connected. Thus $z \in W$ and so $y \in N$. This proves that $S[a, y]$ is covered by a finite subcollection of $\varsigma$, which is a contradiction as $y \in K - (K^* \cup \{b\})$. Thus $S[a, q]$ is not covered by any finite subcollection of $\varsigma$ and so $q \in K - (K^* \cup \{b\})$. As $q \in X - T^\circ$, for every $x \in K - (K^* \cup \{b\})$, we have $q \notin (B_y(a))^\circ$ for any separating set $B_y(a)$ of $X - \{x\}$. This implies using Lemma 3.1 of [2] that for every $x \in K - (K^* \cup \{b\}), q \notin B_y(a)$ for any separating set $B_y(a)$ of $X - \{x\}$. Therefore for every $x \in K - (K^* \cup \{q, q\}), q \notin A_y(a)$ for every separating set $A_y(a)$ of $X - \{x\}$. This implies that $S[a, q] \subseteq (K^* \cup \{q\})$. Thus $S[a, q]$ is covered by a finite subcollection of $\varsigma$. Thus $q \in K^*$ which is a contradiction as $q \in K - (K^* \cup \{b\})$. The contradiction proves that $b \in K^*$. Hence $K$ is compact. □

**Theorem 4.5.** Let $X$ be an $H(i)$ connected and locally connected space with exactly two non-cut points. Then $X$ is a compact COTS with end points.

**Proof.** By Theorem 3.14 of [2], $X = S[a, b]$ for some $a$ and $b$ in $X$. Now by Theorems 4.4 and 3.2, $X$ is a compact COTS with end points. □

**Remark 4.6.** Let $X$ be a $T_1$ separable $H(i)$ connected and locally connected space with exactly two non-cut points. Then $X$ is homeomorphic with the unit interval.

**Proof.** By Theorem 4.5, $X$ is a compact COTS with end points. $X$ being a $T_1$ COTS is Hausdorff, using Theorem 2.9 of [3]. Since a separable compact Hausdorff connected and locally connected space with exactly two non-cut points is homeomorphic with the unit interval (see [5]), $X$ is homeomorphic with the unit interval. □

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**References**