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Embeddings into (a)-spaces and acc spaces

M.V. Matveev

Chair of Higher Mathematics, N.E. Bauman State Technical University, 2-ja Baumanskaja 5, Moscow 107005, Russia

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Abstract

A space X is said to be an (a)-space provided that for every open cover $\mathcal{U}$ of X and every dense subspace D of X there exists a closed in X and discrete subspace $F \subset D$ such that $\mathrm{St}(F, \mathcal{U}) = X$. We show that every Tychonoff space can be represented as a closed subspace of a Tychonoff (a)-space. Also we consider closed G_δ-subspaces. © 1997 Elsevier Science B.V.

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A space X is said to be an (a)-space [10], provided that for every open cover $\mathcal{U}$ of X and every dense subspace $Y \subset X$ there exists a closed (in X) discrete subspace $F \subset Y$ such that $\mathrm{St}(F, \mathcal{U}) = X$. This definition was motivated by the consideration of acc spaces [1,2,7–9,12,15,19,20] (a space X is acc [7] provided that for every open cover $\mathcal{U}$ of X and every dense subspace $Y \subset X$ there exists a finite subset $F \subset Y$ such that $\mathrm{St}(F, \mathcal{U}) = X$) which form a subclass of the class of countably compact spaces due to the star characterization of countable compactness (see [4]): a Hausdorff space is countably compact iff for every open cover $\mathcal{U}$ of X there exists a finite subset $F \subset X$ such that $\mathrm{St}(F, \mathcal{U}) = X$. It is clear that acc is equivalent to (a) plus countable compactness. It was demonstrated in [10,11] that in many ways property (a) behaves like normality. In this paper we demonstrate that yet in one way property (a) and normality behave quite different: while normality is hereditary with respect to closed subsets, property (a) is not closed hereditary at all: all Tychonoff spaces are closed subspaces of (a)-spaces! We prove the following:

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Theorem 1. *Every Tychonoff space X can be represented as a closed nowhere dense zero-set in a Tychonoff (a)-space $R(X)$. If X is normal or countably paracompact then so is $R(X)$.*

Theorem 2. *Every Tychonoff countably compact space can be represented as a nowhere dense zero-set (hence as a closed G_δ-subspace) in a Tychonoff acc space.*

Theorem 2 improves Theorem 4.2 from [9] which states that every Tychonoff countably compact space can be represented as a closed subspace of a Tychonoff acc space. It was also demonstrated in [9] that a regular closed, G_δ-subset in an acc space is not necessary acc. However, the following questions remain open:

Question 3. *Characterize those Tychonoff spaces which can be represented as regular closed subsets in Tychonoff (a)-spaces.*

Question 4. *Characterize those Tychonoff countably compact spaces which can be represented as regular closed subspaces of Tychonoff acc spaces.*

Before starting the constructions, we note that the problem of closed embeddings was considered in different classes of spaces:

(1) Pseudocompact spaces: operation

$$X \to \beta X \times (\omega_1 \times 1) \setminus (\beta X \setminus X) \times \{\omega_1\}$$

embeds arbitrary Tychonoff space X into a pseudocompact space as a closed subspace ([13], see [4]). Other constructions provide closed embeddings into pseudocompact spaces preserving certain topological properties [14,16,6]. Some problems concerning this subject remain open [17].

(2) L. Kočinac noted that every Tychonoff space is representable as a closed subspace of a c.c.c. space.

(3) The author noted that every Tychonoff space can be represented as a closed G_δ-subspace of a Baire space [5].

(4) As it was noted before, every Tychonoff, countably compact space can be represented as a closed subspace of a Tychonoff acc space [9]. Recently, Vaughan has found an alternative proof of this fact: if X is countably compact then the Alexandroff duplicate of X is acc.

(5) Davis showed that every topological space can be represented as a closed subspace of an AD-refinable space [3].

We start with the description of a construction of Tkačuk [18] and with the discussion of the properties of this construction. Later we use this construction in the proof of Theorem 2.

1. Some properties of Tkačuk's construction

Let Z be a topological space and let $\tau = |Z|$. Denote by $A(\tau)$ the one-point compactification of the discrete space A of cardinality τ; $A(\tau) = A \cup \{a\}$. Fix a disjoint family $\{A_x\colon x \in Z\}$ of countable infinite subsets of A and put

$$T(Z) = \big(Z \times \{a\}\big) \cup Y_0 \subset Z \times A(\tau)$$

where $Y_0 = \bigcup\{\{x\} \times A_x\colon x \in Z\}$. Then

(1) Z is closed and nowhere dense in $T(Z)$; $T(Z) \setminus Z$ is open, dense and discrete in $T(Z)$ [18].

(2) If Z is compact then so is $T(Z)$ [18].

Properties (1) and (2) are true also for the well-known Alexandroff Duplicate construction; besides Tkačuk's construction has some nicer properties which the Alexandroff Duplicate construction has not in general.

(3) Every point of Z is a limit point for a convergent sequence of points of $T(Z) \setminus Z$ (A_x converges to x).

(4) If Z is countably compact then $T(Z)$ is acc [9].

(5) If H is a closed subspace of Z then there exists a regular closed subspace $R(H) \subset T(Z)$ such that $R(H) \cap Z = H$.

Proof. Put $R(H) = H \cup Y_H$ where $Y_H = \{\{x\} \times A_x\colon x \in H\}$. Then Y_H is open in Z and it remains to prove that $\overline{Y}_H = R_H$. Let $p \in \overline{Y}_H \setminus Y_H$. Then $p \in Z \times \{a\}$ since Y_0 is discrete. Suppose $p \notin H$. Then $((Z \times \{a\} \setminus H) \times A(\tau)) \cap T(Z)$ is an open neighbourhood of p in $T(Z)$ that does not intersect Y_H which is a contradiction. □

(6) There exists a decreasing sequence

$$T(Z) = T_0(Z) \supset T_1(Z) \supset \cdots \supset T_n(Z) \supset \cdots \quad (n \in \omega)$$

of subspaces of $T(Z)$ such that

(6a) $\bigcap\{T_n(Z)\colon n \in \omega\} = Z$,

(6b) $T_n(Z)$ is closed in $T(Z)$ and homeomorphic to $T(Z)$ for each $n \in \omega$.

Proof. We enumerate A_x for each $x \in X$: $A_x = \{a_{xm}\colon m \in \omega\}$ and put $T_n(Z) = Z \cup \{a_{xm}\colon x \in Z \text{ and } m \geqslant n\}$. □

It is easy to show that

(6c) Each $T_n(Z)$ is homeomorphic to $T(Z)$ so that the homeomorphism restricted to Z is the identity mapping. (To prove this, one needs just to reenumerate the points.)

Henceforward, in fact, we use only properties (1)–(6), so T can be assumed to be any construction other than Tkačuk's one which satisfies these properties.

2. Density tightness lemma

The density tightness of a space X is the cardinal number

$$d_t(X) = \min\{\tau\colon \forall \text{ dense subspace } Y \subset X\ \forall x \in X\ \exists A \subset Y \text{ such that } |A| \leqslant \omega \text{ and } x \in \overline{A}\}.$$

This notion was defined by Vaughan; for the countable case it implicitly appeared in [9] where the following simple lemma was proved:

Lemma 5 [9, Lemma 1.7]. *Every countably compact space with countable density tightness is acc.*

In this paper we need a special case of this lemma. A subset $Y \subset X$ is ω-dense in X [9] provided for every $x \in X$ there exists a countable $A \subset Y$ such that $x \in \overline{A}$.

Lemma 6. *If a countably compact space X contains an ω-dense subspace the points of which are isolated in X then X is acc.*

Indeed, if $J \subset X$ is a ω-dense subspace of X and every point of J is isolated in X then every dense subspace $Y \in X$ must contain J and hence Y is ω-dense in X.

3. Proof of Theorem 1

We put $R(X) = X \times (\omega + 1)$ with the topology stronger than that of a Tychonoff product: the points of $X \times \omega$ are isolated while a basic neighbourhood of a point $p = \langle x, \omega \rangle \in X \times \{\omega\}$ takes the form $U \times [n, \omega]$ where U is a neighbourhood of x in X and $n \in \omega$. Clearly, X is homeomorphic to $X \times \{\omega\}$, a closed, nowhere dense G_δ-subset of $R(X)$ and $R(X)$ is a Hausdorff space. To show that it is Tychonoff, suppose that H is a closed set in $R(X)$ and $p \notin H$. We are to construct a function $f : R(X) \to R$ such that $f(p) = 0$ and $f(H) = \{1\}$. Since the points of $R(X) \setminus (X \times \{\omega\})$ are isolated, only the case $p \in X \times \{\omega\}$ is interesting. In that case, there is a continuous function $f_0 : X \times \{\omega\} \to R$ such that $f_0(p) = 0$ and $f_0(H_0) = \{1\}$ where $H_0 = H \cap (X \times \{\omega\})$. Put

$$f(\langle x, n \rangle) = \begin{cases} 1 & \text{if } \langle x, n \rangle \in H, \\ f_0(\langle x, \omega \rangle) & \text{if } \langle x, n \rangle \notin H. \end{cases}$$

Then f is the desired function.

Similar proof shows that $R(X)$ is normal if so is X.

Now we show that $R(X)$ is an (a)-space. Let $\mathcal{U}$ be an open cover of $R(X)$ and let D be a dense subspace of $R(X)$. Then $D \supset X \times \omega$ and without loss of generality we can suppose that $D = X \times \omega$. For $n \in \omega$ we denote

$$\mathcal{V}_n = \{U \times [n, \omega] : U \text{ is open in } X \text{ and } U \times [n, \omega] \text{ is contained in some element of } \mathcal{U}\}$$

and

$$U_n = \bigcup\{U\colon\ U \times [n,\omega] \in \mathcal{V}_n\}.$$

Then $U_0 \supset U_1 \supset U_2 \supset \cdots$ and

$$X = \bigcup\{U_n\colon\ n \in \omega\}. \tag{\dagger}$$

Next we put $F_n = X \times \{0\}$ for $n = 0$ and $F_n = (X \setminus U_{n-1}) \times \{n\}$ for $0 < n < \omega$. We denote $F = \bigcup\{F_n\colon\ n \in \omega\}$. We claim that F is closed and discrete in $R(X)$. All we have to show is that an arbitrary point $p = \langle x, \omega\rangle \in X \times \{\omega\}$ is not a limit point for F. By (†), there is an $n \in \omega$ for which $x \in U_n$ and therefore there is also an $U \subset X$ such that $x \in U$ and $U \times [n,\omega] \in \mathcal{V}_n$. Then $U \times [n+1,\omega]$ is a neighbourhood of p in $R(X)$ that does not meet F.

Now, let X be countably paracompact. We have to show that $R(X)$ also is countably paracompact. Let $\mathcal{O}$ be a countable open cover of $R(X)$. For each $O \in \mathcal{O}$ we denote

$$\mathcal{W}_{On} = \{U \subset X\colon\ U \text{ is open in } X \text{ and } U \times [n,\omega] \subset O\}.$$

Clearly, this family contains the maximal (with respect to inclusion) element $W_{On} = \bigcup\{U\colon\ U \in \mathcal{W}_{On}\}$. Then $\bigcup\{W_{On}\colon\ O \in \mathcal{O},\ n \in \omega\}$ is a countable open cover of X. There is therefore a locally finite open refinement $\mathcal{V}$ of $\mathcal{W}$. For each $V \in \mathcal{V}$ we choose a $W_{O(V)n(V)} \in \mathcal{W}$ that contains V. Then $V \times [n(V),\omega]$ is contained in $O(V)$ and $\mathcal{U}_0 = \{V \times [n(V),\omega]\colon\ V \in \mathcal{V}\}$ is a locally finite (in $R(X)$) cover of $X \times \{\omega\}$ by open sets (in $R(X)$). Finally, $\mathcal{U} = \mathcal{U}_0 \cup \{\{q\}\colon\ q \notin \bigcup\mathcal{U}_0\}$ is a locally finite open refinement of $\mathcal{O}$.

4. Proof of Theorem 2

Let Z be a Tychonoff countably compact space. Consider the following subspace $S(Z)$ of the product $T(Z) \times (\omega + 1)$:

$$S(Z) = \bigcup\{T_n(Z) \times \{n\}\colon\ n \in \omega\} \cup \{Z \times \{\omega\}\}$$

(see property (6) for the definition of the subspaces $T_n(Z)$). Clearly $Z \times \{\omega\} \sim Z$ is a nowhere dense zero-set in $S(Z)$. To show that $S(Z)$ is countably compact we need to show that $S(Z)$ is closed in the product $T(Z) \times (\omega + 1)$. Suppose $p \in \overline{S(Z)} \setminus S(Z)$. Since, for each n, $T_n(Z)$ is closed in $T(Z)$ and $T_n(Z) \times \{n\}$ is clopen in $S(Z)$ we conclude that $p \in T(Z) \times \{\omega\}$ and hence $p \in (T(Z) \setminus Z) \times \{\omega\}$. By (6a) there exists an integer n such that $p \in (T(Z) \setminus T_n(Z)) \times \{\omega\}$. Then $(T(Z) \setminus T_n(Z)) \times [n,\omega]$ is a neighbourhood of p in $T(Z) \times (\omega + 1)$ that does not intersect $S(Z)$, a contradiction. Being a closed subspace of a countably compact space $S(Z)$ also is countably compact. Now, we have to show that $S(Z)$ is acc. For each $n \in \omega$ put $J_n = (T_n(Z) \setminus Z) \times \{n\}$. Then the subspace $J = \bigcup\{J_n\colon\ n \in \omega\}$ is dense in $S(Z)$ and consists of points isolated in $S(Z)$. By Lemma 6 it remains to check that J is ω-dense in $S(Z)$. Let $x \in S(Z)$. If $x \in T_n(Z) \times \{n\}$ for some $n \in \omega$ then $x \in \overline{A}$ for some countable subset $A \subset J_n$ since by (6c) $T_n(Z) \times \{n\}$ is homeomorphic to $T(Z)$ (where J_n takes the role of Y_0). So let $x = \langle p, \omega \rangle \in Z \times \{\omega\}$. By previous observation, for each $n \in \omega$ there is a countable subset $A_n \subset J_n$ such that $\langle p, n \rangle \in \overline{A}_n$. Then for the set $A = \bigcup\{A_n\colon n \in \omega\}$ we have $|A| = \omega$, $A \subset J$ and $x \in \overline{A}$.

5. Final remarks

One can see from the proof of Theorem 1 that normality and countable paracompactness are not the only properties that are preserved by the operation R. For example, paracompactness is preserved too. I have chosen normality and countable paracompactness because of my interest to Dowker and (a)-Dowker spaces (a space X is (a)-Dowker [10] provided X is an (a)-space while $X \times (\omega+1)$ is not). Since normality and countable paracompactness are closed-hereditary properties, we obtain the following corollary from Theorem 1:

Corollary. *Every Dowker space can be embedded as a closed subspace into a Dowker (a)-space.*

Note that every normal, (a)-Dowker space is a Dowker space [10].

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