Embeddings into (a)-spaces and acc spaces

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Abstract

A space $X$ is said to be an (a)-space provided that for every open cover $U$ of $X$ and every dense subspace $D$ of $X$ there exists a closed in $X$ and discrete subspace $F \subset D$ such that $St(F, U) = X$. We show that every Tychonoff space can be represented as a closed subspace of a Tychonoff (a)-space. Also we consider closed $G_\delta$-subspaces.

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A space $X$ is said to be an (a)-space [10], provided that for every open cover $U$ of $X$ and every dense subspace $Y \subset X$ there exists a closed (in $X$) discrete subspace $F \subset Y$ such that $St(F, U) = X$. This definition was motivated by the consideration of acc spaces [1,2,7–9,12,15,19,20] (a space $X$ is acc [7] provided that for every open cover $U$ of $X$ and every dense subspace $Y \subset X$ there exists a finite subset $F \subset Y$ such that $St(F, U) = X$) which form a subclass of the class of countably compact spaces due to the star characterization of countable compactness (see [4]): a Hausdorff space is countably compact iff for every open cover $U$ of $X$ there exists a finite subset $F \subset X$ such that $St(F, U) = X$. It is clear that acc is equivalent to (a) plus countable compactness. It was demonstrated in [10,11] that in many ways property (a) behaves like normality. In this paper we demonstrate that yet in one way property (a) and normality behave quite different: while normality is hereditary with respect to closed subsets, property (a) is not closed hereditary at all: all Tychonoff spaces are closed subspaces of (a)-spaces! We prove the following:
**Theorem 1.** Every Tychonoff space $X$ can be represented as a closed nowhere dense zero-set in a Tychonoff (a)-space $R(X)$. If $X$ is normal or countably paracompact then so is $R(X)$.

**Theorem 2.** Every Tychonoff countably compact space can be represented as a nowhere dense zero-set (hence as a closed $G_δ$-subspace) in a Tychonoff acc space.

Theorem 2 improves Theorem 4.2 from [9] which states that every Tychonoff countably compact space can be represented as a closed subspace of a Tychonoff acc space. It was also demonstrated in [9] that a regular closed, $G_δ$-subset in an acc space is not necessary acc. However, the following questions remain open:

**Question 3.** Characterize those Tychonoff spaces which can be represented as regular closed subsets in Tychonoff (a)-spaces.

**Question 4.** Characterize those Tychonoff countably compact spaces which can be represented as regular closed subspaces of Tychonoff acc spaces.

Before starting the constructions, we note that the problem of closed embeddings was considered in different classes of spaces:

1. Pseudocompact spaces: operation

   $$X \to βX \times (ω_1 × 1) \setminus (βX \setminus X) × \{ω_1\}$$

   embeds arbitrary Tychonoff space $X$ into a pseudocompact space as a closed subspace ([13], see [4]). Other constructions provide closed embeddings into pseudocompact spaces preserving certain topological properties [14,16,6]. Some problems concerning this subject remain open [17].

2. L. Kocinac noted that every Tychonoff space is representable as a closed subspace of a c.c.c. space.

3. The author noted that every Tychonoff space can be represented as a closed $G_δ$-subspace of a Baire space [5].

4. As it was noted before, every Tychonoff, countably compact space can be represented as a closed subspace of a Tychonoff acc space [9]. Recently, Vaughan has found an alternative proof of this fact: if $X$ is countably compact then the Alexandroff duplicate of $X$ is acc.

5. Davis showed that every topological space can be represented as a closed subspace of an AD-refinable space [3].

We start with the description of a construction of Tkačuk [18] and with the discussion of the properties of this construction. Later we use this construction in the proof of Theorem 2.
1. Some properties of Tkačuk’s construction

Let $Z$ be a topological space and let $\tau = |Z|$. Denote by $A(\tau)$ the one-point compactification of the discrete space $A$ of cardinality $\tau$: $A(\tau) = A \cup \{a\}$. Fix a disjoint family $\{A_x: x \in Z\}$ of countable infinite subsets of $A$ and put

$$T(Z) = (Z \times \{a\}) \cup Y_0 \subset Z \times A(\tau)$$

where $Y_0 = \bigcup \{\{x\} \times A_x: x \in Z\}$. Then

(1) $Z$ is closed and nowhere dense in $T(Z)$; $T(Z) \setminus Z$ is open, dense and discrete in $T(Z)$ [18].

(2) If $Z$ is compact then so is $T(Z)$ [18].

Properties (1) and (2) are true also for the well-known Alexandroff Duplicate construction; besides Tkačuk’s construction has some nicer properties which the Alexandroff Duplicate construction has not in general.

(3) Every point of $Z$ is a limit point for a convergent sequence of points of $T(Z) \setminus Z$ ($A_x$ converges to $x$).

(4) If $Z$ is countably compact then $T(Z)$ is acw [9].

(5) If $H$ is a closed subspace of $Z$ then there exists a regular closed subspace $R(H) \subset T(Z)$ such that $R(H) \cap Z = H$.

Proof. Put $R(H) = H \cup Y_H$ where $Y_H = \{\{x\} \times A_x: x \in H\}$. Then $Y_H$ is open in $Z$ and it remains to prove that $Y_H = R_H$. Let $p \in Y_H \setminus Y_H$. Then $p \in Z \times \{a\}$ since $Y_0$ is discrete. Suppose $p \notin H$. Then $((Z \times \{a\} \setminus H) \times A(\tau)) \cap T(Z)$ is an open neighbourhood of $p$ in $T(Z)$ that does not intersect $Y_H$ which is a contradiction. $\Box$

(6) There exists a decreasing sequence

$$T(Z) = T_0(Z) \supset T_1(Z) \supset \cdots \supset T_n(Z) \supset \cdots \ (n \in \omega)$$

of subspaces of $T(Z)$ such that

(6a) $\bigcap \{T_n(Z): n \in \omega\} = Z$,

(6b) $T_n(Z)$ is closed in $T(Z)$ and homeomorphic to $T(Z)$ for each $n \in \omega$.

Proof. We enumerate $A_x$ for each $x \in X$: $A_x = \{a_{xm}: m \in \omega\}$ and put $T_n(Z) = Z \cup \{a_{xm}: x \in Z$ and $m \geq n\}$. $\Box$

It is easy to show that

(6c) Each $T_n(Z)$ is homeomorphic to $T(Z)$ so that the homeomorphism restricted to $Z$ is the identity mapping. (To prove this, one needs just to reenumerate the points.)

Henceforward, in fact, we use only properties (1)–(6), so $T$ can be assumed to be any construction other than Tkačuk’s one which satisfies these properties.
2. Density tightness lemma

The density tightness of a space $X$ is the cardinal number

$$d_t(X) = \min \{ \tau : \forall \text{ dense subspace } Y \subset X \forall x \in X \exists A \subset Y$$

such that $|A| \leq \omega$ and $x \in A \}. $$

This notion was defined by Vaughan; for the countable case it implicitly appeared in [9] where the following simple lemma was proved:

**Lemma 5** [9, Lemma 1.7]. Every countably compact space with countable density tightness is acc.

In this paper we need a special case of this lemma. A subset $Y \subset X$ is $\omega$-dense in $X$ [9] provided for every $x \in X$ there exists a countable $A \subset Y$ such that $x \in A$.

**Lemma 6.** If a countably compact space $X$ contains an $\omega$-dense subspace the points of which are isolated in $X$ then $X$ is acc.

Indeed, if $J \subset X$ is a $\omega$-dense subspace of $X$ and every point of $J$ is isolated in $X$ then every dense subspace $Y \subset X$ must contain $J$ and hence $Y$ is $\omega$-dense in $X$.

3. Proof of Theorem 1

We put $R(X) = X \times (\omega + 1)$ with the topology stronger than that of a Tychonoff product: the points of $X \times \omega$ are isolated while a basic neighbourhood of a point $p = \langle x, \omega \rangle \in X \times \{\omega\}$ takes the form $U \times [n, \omega]$ where $U$ is a neighbourhood of $x$ in $X$ and $n \in \omega$. Clearly, $X$ is homeomorphic to $X \times \{\omega\}$, a closed, nowhere dense $G_\delta$-subset of $R(X)$ and $R(X)$ is a Hausdorff space. To show that it is Tychonoff, suppose that $H$ is a closed set in $R(X)$ and $p \notin H$. We are to construct a function $f : R(X) \rightarrow R$ such that $f(p) = 0$ and $f(H) = \{1\}$. Since the points of $R(X) \setminus (X \times \{\omega\})$ are isolated, only the case $p \in X \times \{\omega\}$ is interesting. In that case, there is a continuous function $f_0 : X \times \{\omega\} \rightarrow R$ such that $f_0(p) = 0$ and $f_0(H_0) = \{1\}$ where $H_0 = H \cap (X \times \{\omega\})$. Put

$$f(\langle x, n \rangle) = \begin{cases} 1 & \text{if } \langle x, n \rangle \in H, \\ f_0(\langle x, n \rangle) & \text{if } \langle x, n \rangle \notin H. \end{cases}$$

Then $f$ is the desired function.

Similar proof shows that $R(X)$ is normal if so is $X$.

Now we show that $R(X)$ is an (a)-space. Let $\mathcal{U}$ be an open cover of $R(X)$ and let $D$ be a dense subspace of $R(X)$. Then $D \supset X \times \omega$ and without loss of generality we can suppose that $D = X \times \omega$. For $n \in \omega$ we denote

$$V_n = \{ U \times [n, \omega] : U \text{ is open in } X \text{ and } U \times [n, \omega] \text{ is contained in some element of } \mathcal{U} \}$$
and

\[ U_n = \bigcup \{ U: U \times [n, \omega] \in \mathcal{V}_n \} . \]

Then \( U_0 \supset U_1 \supset U_2 \supset \cdots \) and

\[ X = \bigcup \{ U_n: n \in \omega \} . \tag{\dagger} \]

Next we put \( F_n = X \times \{ 0 \} \) for \( n = 0 \) and \( F_n = (X \setminus U_{n-1}) \times \{ n \} \) for \( 0 < n < \omega \). We denote \( F = \bigcup \{ F_n: n \in \omega \} \). We claim that \( F \) is closed and discrete in \( R(X) \). All we have to show is that an arbitrary point \( p = (x, \omega) \in X \times \{ \omega \} \) is not a limit point for \( F \). By (\dagger), there is an \( n \in \omega \) for which \( x \in U_n \) and therefore there is also an \( U \subset X \) such that \( x \in U \) and \( U \times [n, \omega] \in \mathcal{V}_n \). Then \( U \times [n+1, \omega] \) is a neighbourhood of \( p \) in \( R(X) \) that does not meet \( F \).

Now, let \( X \) be countably paracompact. We have to show that \( R(X) \) also is countably paracompact. Let \( \mathcal{O} \) be a countable open cover of \( R(X) \). For each \( O \in \mathcal{O} \) we denote

\[ \mathcal{W}_O = \{ U \subset X: U \text{ is open in } X \text{ and } U \times [n, \omega] \subset O \}. \]

Clearly, this family contains the maximal (with respect to inclusion) element \( \mathcal{W}_0 = \bigcup \{ U: U \in \mathcal{W}_0 \} \). Then \( \bigcup \{ \mathcal{W}_O: O \in \mathcal{O}, n \in \omega \} \) is a countable open cover of \( X \). There is therefore a locally finite open refinement \( \mathcal{V} \) of \( \mathcal{W} \). For each \( V \in \mathcal{V} \) we choose a \( \mathcal{W}_{O(V)} \in \mathcal{W} \) that contains \( V \). Then \( V \times [n(V), \omega] \) is contained in \( O(V) \) and \( U_0 = \{ V \times [n(V), \omega]: V \in \mathcal{V} \} \) is a locally finite (in \( R(X) \)) cover of \( X \times \{ \omega \} \) by open sets (in \( R(X) \)). Finally, \( \mathcal{U} = U_0 \cup \{ \{ q \}: q \notin \bigcup U_0 \} \) is a locally finite open refinement of \( \mathcal{O} \).

4. Proof of Theorem 2

Let \( Z \) be a Tychonoff countably compact space. Consider the following subspace \( S(Z) \) of the product \( T(Z) \times (\omega + 1) \):

\[ S(Z) = \bigcup \{ T_n(Z) \times \{ n \}: n \in \omega \} \cup \{ Z \times \{ \omega \} \} \]

(see property (6) for the definition of the subspaces \( T_n(Z) \)). Clearly \( Z \times \{ \omega \} \sim Z \) is a nowhere dense zero-set in \( S(Z) \). To show that \( S(Z) \) is countably compact we need to show that \( S(Z) \) is closed in the product \( T(Z) \times (\omega + 1) \). Suppose \( p \in \overline{S(Z)} \setminus S(Z) \). Since, for each \( n \), \( T_n(Z) \) is closed in \( T(Z) \) and \( T_n(Z) \times \{ n \} \) is clopen in \( S(Z) \) we conclude that \( p \in T(Z) \times \{ \omega \} \) and hence \( p \in (T(Z) \setminus \{ n \}) \times \{ \omega \} \). By (6a) there exists an integer \( n \) such that \( p \in (T(Z) \setminus T_n(Z)) \times \{ \omega \} \). Then \( (T(Z) \setminus T_n(Z)) \times [n, \omega] \) is a neighbourhood of \( p \) in \( T(Z) \times (\omega + 1) \) that does not intersect \( S(Z) \), a contradiction. Being a closed subspace of a countably compact space \( S(Z) \) also is countably compact. Now, we have to show that \( S(Z) \) is acc. For each \( n \in \omega \) put \( J_n = (T_n(Z) \setminus Z) \times \{ n \} \). Then the subspace \( J = \bigcup \{ J_n: n \in \omega \} \) is dense in \( S(Z) \) and consists of points isolated in \( S(Z) \). By Lemma 6 it remains to check that \( J \) is \( \omega \)-dense in \( S(Z) \). Let \( x \in S(Z) \). If \( x \in T_n(Z) \times \{ n \} \) for some \( n \in \omega \) then \( x \in A \) for some countable subset \( A \subset J_n \) since
by (6c) $T_n(Z) \times \{n\}$ is homeomorphic to $T(Z)$ (where $J_n$ takes the role of $Y_0$). So let $x = \langle p, \omega \rangle \in Z \times \{\omega\}$. By previous observation, for each $n \in \omega$ there is a countable subset $A_n \subset J_n$ such that $\langle p, n \rangle \in \bar{A}_n$. Then for the set $A = \bigcup \{A_n: n \in \omega\}$ we have $|A| = \omega$, $A \subset J$ and $x \in \bar{A}$.

5. Final remarks

One can see from the proof of Theorem 1 that normality and countable paracompactness are not the only properties that are preserved by the operation $\mathcal{H}$. For example, paracompactness is preserved too. I have chosen normality and countable paracompactness because of my interest to Dowker and (a)-Dowker spaces (a space $X$ is (a)-Dowker [10] provided $X$ is an (a)-space while $X \times (\omega + 1)$ is not). Since normality and countable paracompactness are closed-hereditary properties, we obtain the following corollary from Theorem 1:

**Corollary.** Every Dowker space can be embedded as a closed subspace into a Dowker (a)-space.

Note that every normal, (a)-Dowker space is a Dowker space [10].

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References

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