JOURNAL OF FUNCTIONAL ANALYSIS 92, 448–467 (1990)

# Reflexivity Properties of $T \oplus 0$

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*Communicated by the Editors*

Received July 5, 1989

A construction is given of a reflexive operator $T$ acting on a separable Hilbert space $\mathscr{H}$ with the property that the direct sum $T \oplus 0$ fails to be reflexive. This construction is then used to provide solutions to several other problems which have been studied concerning the direct-sum splitting of operator algebras, Scott Brown's technique, the theory of bitriangular operators, and parareflexivity. 

## INTRODUCTION

In the early 1970s a number of results were obtained and some natural questions were raised concerning the invariant subspace properties, and more generally the reflexivity properties, of certain bounded linear transformations acting on a complex infinite dimensional Hilbert space. In this article we address one of these questions. We give a construction of a reflexive operator $T$ acting on a Hilbert space $\mathscr{H}$ with the property that for a second Hilbert space $\mathscr{K}$ of dimension at least unity the direct sum $T \oplus 0$ of the operator $T$ and the zero transformation of $\mathscr{K}$, acting on the direct sum Hilbert space $\mathscr{H} \oplus \mathscr{K}$, fails to be reflexive. Our solution, together with our methods, are then employed to provide answers to some other questions which have been studied. One of these concerns the direct-sum splitting of operator algebras. Another relates to Scott Brown's technique for constructing invariant subspaces for operators. Additional results relate to parareflexivity and to the theory of bitriangular operators recently developed by Davidson and Herrero [7].

This research was partially supported by NSF Grants *DMS–8744359 and †DMS–8800130.



0022-1236/90 $3.00

An operator $T$ is called *reflexive* if the closure in the weak operator topology of the set of polynomials $\mathscr{P}(T)$ in the operator $T$ is completely determined by the set of closed subspaces of the underlying Hilbert space which are invariant under $T$. One of the questions that had been posed early (Deddens [8]) was whether the direct sum $A \oplus B$ of reflexive operators $A$ and $B$ acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively, is necessarily reflexive. The answer is known to be no by a recent result of the second author [17]. Before this the answer was known to be yes in many cases (e.g., [3, 9, 12, 15]) in which additional hypotheses were placed on one or both summands. However, if $B = \lambda I$ for some scalar $\lambda$, and in particular, if $B = 0$ (the present case) the question remained unsettled and could not be dealt with by the methods of [17]. This special case had been isolated, and studied, years before by others. (In particular, L. Curnutt [6] obtained some interesting partial results.) Thus the setting and techniques of the present study are quite different from those in [17].

We note that an early article in the literature [11] asserted that the direct sum of two reflexive operators, one of which is algebraic, is reflexive. However, there is a gap in the proof as observed by the second author of the present paper and other researchers. The present article shows that this gap cannot be filled.

We wish to thank M. Lambrou for pointing out a mistake in the original version of our proof of Lemma 3.5.

## 1. Preliminaries

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of bounded linear operators acting on $\mathscr{H}$. For a linear subspace $\mathscr{S}$ of $\mathscr{B}(\mathscr{H})$, let $\operatorname{Ref} \mathscr{S} = \{B \in \mathscr{B}(\mathscr{H}) : Bx \in [\mathscr{S}x], x \in \mathscr{H}\}$. (Here $[\cdot]$ denotes closed linear span.) The subspace $\mathscr{S}$ is called *reflexive* if $\mathscr{S} = \operatorname{Ref} \mathscr{S}$. If $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$ is a set of operators we let $\operatorname{Lat} \mathscr{A}$ denote the lattice of all closed subspaces of $\mathscr{H}$ that are invariant under each element of $\mathscr{A}$, and if $\mathscr{L}$ is a set of closed subspaces of $\mathscr{H}$ we let $\operatorname{Alg} \mathscr{L}$ denote the algebra of all operators that leave invariant each element of $\mathscr{L}$. If $\mathscr{A}$ is a unital algebra it is easily verified that $\operatorname{Ref} \mathscr{A} = \operatorname{Alg} \operatorname{Lat} \mathscr{A}$.

If $A \in \mathscr{B}(\mathscr{H})$, then $\mathscr{W}(A)$ will denote the closure in the weak operator topology of $\mathscr{B}(\mathscr{H})$ of the set $\mathscr{P}(A)$ of polynomials in $A$, and $\mathscr{W}_0(A)$ will denote the weakly closed principal ideal generated by $A$. Thus $\mathscr{W}_0(A)$ is the closure in the weak operator topology of the linear span of the positive powers of $A$, and it may happen that $\mathscr{W}_0(A) = \mathscr{W}(A)$. So the operator $A$ is reflexive if $\mathscr{W}(A) = \operatorname{Alg} \operatorname{Lat} A = \operatorname{Ref} \mathscr{W}(A)$. The ideal $\mathscr{W}_0(A)$ can be reflexive as a subspace of $\mathscr{B}(\mathscr{H})$ even if it is a proper ideal. Also, we will use the notation $\mathscr{A}(A)$ to denote the ultraweak (weak*, $\sigma$-weak) closure of

$\mathscr{P}(A)$, and $\mathscr{A}_0(A)$ for the ultraweakly closed linear span of $\{A^n : n \geq 1\}$. More generally, if $\mathscr{S}$ is a set of operators then $\mathscr{W}_0(\mathscr{S})$ and $\mathscr{W}(\mathscr{S})$ will denote the weakly closed algebras generated by $\mathscr{S}$ and $\mathscr{S} \cup \{I\}$, respectively. Frequently in this paper $\mathscr{S}$ will be a set of commuting idempotents. In this case $\mathscr{W}_0(\mathscr{S})$ will be just the weakly closed linear span of the elements of $\mathscr{S}$.

As usual, we write $\{\mathscr{S}\}'$ for the commutant in $\mathscr{B}(\mathscr{H})$ of a set $\mathscr{S}$ of operators. For $x$, $y \in \mathscr{H}$ we use the tensor notation $x \otimes y$ to denote the rank-1 operator $z \to \langle z, y \rangle x, z \in \mathscr{H}$. For $1 \leq n < \infty$ we write $\mathbb{F}_n(\mathscr{H})$ to denote the set of operators in $\mathscr{B}(\mathscr{H})$ of rank no greater than $n$.

We will have occasion to use aspects of the theory of dual algebras and the duality theory of subspaces of $\mathscr{B}(\mathscr{H})$ as developed in [2, 3, 13]. We briefly outline the aspects we use. $\mathscr{B}(\mathscr{H})$ can be identified as the dual of the ideal $C_1(\mathscr{H})$ of trace-class operators via the pairing $\langle f, A \rangle = \mathrm{tr}(fA) = \mathrm{tr}(Af)$, for $f \in C_1(\mathscr{H})$ and $A \in \mathscr{B}(\mathscr{H})$. The ultraweak topology (or $\sigma$-weak topology) of $\mathscr{B}(\mathscr{H})$ coincides with the $w^*$ topology under this identification. It is known that a $w^*$ closed linear subspace $\mathscr{S}$ of $\mathscr{B}(\mathscr{H})$ is reflexive in the sense defined above if and only if its preannihilator $\mathscr{S}_\perp$ in $C_1(\mathscr{H})$ is generated as a closed subspace of $C_1(\mathscr{H})$ by rank-1 operators. Also, a $w^*$ closed subspace $\mathscr{S}$ is closed in the weak operator topology if and only if $\mathscr{S}_\perp$ is generated by finite-rank operators. If a $w^*$ closed subspace $\mathscr{S}$ has the property that $C_1(\mathscr{H}) = \mathscr{S}_\perp + \mathbb{F}_1(\mathscr{H})$ then $\mathscr{S}$ is called *elementary*. More generally, if $C_1(\mathscr{H}) = \mathscr{S}_\perp + \mathbb{F}_n(\mathscr{H})$ for some $1 \leq n < \infty$ then $\mathscr{S}$ is called $n$-elementary. An elementary reflexive subspace $\mathscr{S}$ is hereditarily reflexive in the sense that every $w^*$ closed linear subspace contained in $\mathscr{S}$ is also reflexive. If $\mathscr{S}$ is weakly closed and $n$-elementary for some $n$ then the relative weak operator topology concides with the relative $w^*$ topology on $\mathscr{S}$. As is standard procedure, we define an operator $A$ to have a particular property if $\mathscr{W}(A)$ has that property.

## 2. Some Reductions

If a Hilbert space operator $A$ is reflexive *and* elementary then since $\mathscr{W}_0(A)$ is a $w^*$ closed subspace of $\mathscr{W}(A)$ it is also reflexive. If dim $\mathscr{H} < \infty$ then every singly generated subalgebra of $\mathscr{B}(\mathscr{H})$ is elementary and hence $\mathscr{W}_0(A)$ is reflexive whenever $A$ is reflexive. A separably acting Hilbert space operator $A$ which is not elementary was first constructed by Hadwin and Nordgren in [12]. Terminology in that paper was different. However, no example was known of a non-elementary reflexive operator $A$ for which $\mathscr{W}_0(A)$ is not reflexive. Our main example in Section 3 satisfies this property. This, together with the following reduction, then answers the $T \oplus 0$ question.

**Proposition** 2.1. *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces with* $\dim \mathscr{K} \geqslant 1$. *Let $A \in \mathscr{B}(\mathscr{H})$, and let* 0 *denote the zero transformation on $\mathscr{K}$. Then*:

(i) *$A \oplus 0$ is reflexive if and only if $\mathscr{W}_0(A)$ is reflexive.*

(ii) *If $A$ is reflexive, then $A \oplus 0$ fails to be reflexive if and only if $I \notin \mathscr{W}_0(A)$ but $I \in \operatorname{Ref}(\mathscr{W}_0(A))$.*

*Proof.* We begin with (i). Suppose first that $\mathscr{W}_0(A)$ is reflexive. Let $B \in \operatorname{Ref}(\mathscr{W}(A \oplus 0))$ be arbitrary. Since $(\operatorname{Ref}(\mathscr{W}(A))) \oplus (\mathbb{C}I)$ is a reflexive algebra containing $A \oplus 0$, it follows that $B = B_1 \oplus (\lambda I)$ for some $B_1 \in \operatorname{Ref}(\mathscr{W}(A))$ and $\lambda \in \mathbb{C}$. We will show that $B_1 - \lambda I \in \mathscr{W}_0(A)$, and hence that $(B_1 - \lambda I) \oplus 0 \in \mathscr{W}_0(A) \oplus 0$. Since $\mathscr{W}(A \oplus 0) = \mathscr{W}_0(A \oplus 0) + \mathbb{C}(I \oplus I)$ and $\mathscr{W}_0(A \oplus 0) = \mathscr{W}_0(A) \oplus 0$, this will prove that $B \in \mathscr{W}(A \oplus 0)$, as required.

Fix a nonzero vector $y \in \mathscr{K}$. Let $x \in \mathscr{H}$ be arbitrary. Since $B \in \operatorname{Ref}(\mathscr{W}(A \oplus 0))$, there exists a sequence of polynomials $\{p_n\}$, depending on $x$, such that

$$\begin{aligned} \lim_n \, (p_n(A \oplus 0))(x \oplus y) &= (B_1 \oplus (\lambda I))(x \oplus y) \\ &= (B_1 x) \oplus (\lambda y). \end{aligned}$$

Since $p_n(A \oplus 0) = p_n(A) \oplus (p_n(0)\, I)$, we must have $p_n(0) \to \lambda$ and $p_n(A)\, x \to B_1(x)$. Let $q_n = p_n - p_n(0)$. Then $q_n(0) = 0$ and $q_n(A)\, x \to (B_1 - \lambda I)\, x$. Thus $(B_1 - \lambda I)\, x \in [\mathscr{W}_0(A)\, x]$. Since $x$ was arbitrary, this shows that $B_1 - \lambda I \in \operatorname{Ref}(\mathscr{W}_0(A)) = \mathscr{W}_0(A)$, as needed.

For the converse, suppose $A \oplus 0$ is reflexive. If $C \in \operatorname{Ref}(\mathscr{W}_0(A))$, then

$$\begin{aligned} C \oplus 0 \in \operatorname{Ref}(\mathscr{W}_0(A)) \oplus 0 &= \operatorname{Ref}(\mathscr{W}_0(A) \oplus 0) \\ &= \operatorname{Ref}(\mathscr{W}_0(A \oplus 0)) \subseteq \operatorname{Ref}(\mathscr{W}(A \oplus 0)) = \mathscr{W}(A \oplus 0) \\ &= \mathscr{W}_0(A \oplus 0) + \mathbb{C}(I \oplus I). \end{aligned}$$

It follows that $C \in \mathscr{W}_0(A)$, as required.

Item (ii) follows from (i). Assume that $A$ is reflexive. We have $\mathscr{W}_0(A) \subseteq \operatorname{Ref}(\mathscr{W}_0(A)) \subseteq \operatorname{Ref}(\mathscr{W}(A)) = \mathscr{W}(A) = \mathscr{W}_0(A) + \mathbb{C}I$. From this it is clear that the only way in which $A \oplus 0$, and hence $\mathscr{W}_0(A)$, can fail to be reflexive is if $I \notin \mathscr{W}_0(A)$ but $I \in \operatorname{Ref}(\mathscr{W}_0(A))$. ∎

If $A$ and $B$ are operators we say that $\mathscr{W}(A \oplus B)$ *splits* if $\mathscr{W}(A \oplus B) = \mathscr{W}(A) \oplus \mathscr{W}(B)$, and we say that $\operatorname{Lat}(A \oplus B)$ *splits* if $\operatorname{Lat}(A \oplus B) = \operatorname{Lat}(A) \oplus \operatorname{Lat}(B)$. For the special case $B = 0$, we have $\mathscr{W}(B) = \mathbb{C}I$, and $\operatorname{Lat}(B)$ is the set of all closed subspaces of $\mathscr{K}$. The splitting characterizations associated with Proposition 2.1 take the following form.

PROPOSITION 2.2. *Let* $A \in B(\mathscr{H})$. *Then*:

(i) $\mathrm{Lat}(A \oplus 0)$ *splits if and only if* $I \in \mathrm{Ref}(\mathscr{W}_0(A))$.

(ii) $\mathscr{W}(A \oplus 0)$ *splits if and only if* $I \in \mathscr{W}_0(A)$.

*Proof.* Observe that $\mathrm{Lat}(A \oplus 0)$ splits if and only if every cyclic invariant subspace of $A \oplus 0$ splits (i.e., has the form $N \oplus M$ for closed subspaces $N \subseteq \mathscr{H}$, $M \subseteq \mathscr{H}$). For $x \in \mathscr{H}, y \in \mathscr{H}$, consider $\mathscr{M}_{x \oplus y} = [\{(A \oplus 0)^n (x \oplus y) : n \geq 0\}] = [x \oplus y, \{(A^n x) \oplus 0 : n \geq 1\}]$. The subspace $\mathscr{M}_{x \oplus y}$ splits if and only if it contains $0 \oplus y$. This is true if and only if there exist polynomials $\{p_n\}$ with $p_n(0) = 0$ so that $p_n(A)\, x \to x$. It follows that $\mathrm{Lat}(A \oplus 0)$ splits if and only if $I \in \mathrm{Ref}(\mathscr{W}_0(A))$. This verifies (i). For (ii), write $\mathscr{W}(A \oplus 0) = (\mathscr{W}_0(A) \oplus 0) + \mathbb{C}(I \oplus I)$ and note that $\mathscr{W}(A \oplus 0)$ splits if and only if $I \oplus 0 \in \mathscr{W}(A \oplus 0)$, and this is true if and only if $I \in \mathscr{W}_0(A)$.

## 3. THE MAIN EXAMPLE

Fix an orthonormal basis $\{e_n\}_1^\infty$ for an infinite dimensional separable Hilbert space $\mathscr{H}$. View each operator $A \in \mathscr{B}(\mathscr{H})$ as an infinite matrix $A = (A_{jk})_{j,k \geq 1}$. Let $E_{jk}$ be the matrix unit which has a 1 as its $(j, k)$-element and all other elements 0. Let $\mathscr{M}_n = [e_1, ..., e_n]$ and let $P_n$ be the orthogonal projection onto $\mathscr{M}_n$. For each $k \geq 1$, let

$$Q_{2k-1} = P_{2k-1} + 4^k E_{2k, 2k-1}$$

and

$$Q_{2k} = P_{2k} + 4^k E_{2k, 2k+1}.$$

Set $Q_0 = 0$. Observe that each $Q_n$ is an idempotent of rank $n$ and that range $(Q_n) \subset \mathrm{range}(Q_{n+1})$. Also, if $m < n$, then $Q_m Q_n = Q_n Q_m = Q_m$. For $k \geq 1$, set $T_k = Q_k - Q_{k-1}$. Each $T_k$ is a rank-one idempotent, and $T_j T_k = 0$ if $j \neq k$. We have

$$T_1 = E_{11} + 4E_{21},$$

$$T_{2k-1} = -4^{k-1} E_{2k-2, 2k-1} + E_{2k-1, 2k-1} + 4^k E_{2k, 2k-1} \qquad \text{for} \quad k \geq 2,$$

$$T_{2k} = -4^k E_{2k, 2k-1} + E_{2k, 2k} + 4^k E_{2k, 2k+1} \qquad \text{for} \quad k \geq 1.$$

Thus $T_{2k-1}$ has nonzero entries only in the $2k-1$ column and $T_{2k}$ has nonzero entries only in the $2k$ row.

LEMMA 3.1. *The series*

$$\sum_{1}^{\infty} 4^{-n} T_n$$

*converges in norm to a compact operator T. We have*

$$\mathscr{W}_0(T)=\mathscr{W}_0(\{T_k\}_1^{\infty})=\mathscr{W}_0(\{Q_k\}_1^{\infty}).$$

*Proof.* For $k\geqslant 2$ we have $\|T_{2k-1}\|^2=1+4^{2k-2}+4^{2k}$, and for $k\geqslant 1$ we have $\|T_{2k}\|^2=1+2(4^{2k})$. So $\|T_n\|\leqslant 2^{n+1}$ for all $n$. Thus $\|4^{-n}T_n\|\leqslant 2^{1-n}$, and so $\sum \|4^{-n}T_n\|<\infty$. Thus $\sum_1^\infty 4^{-n}T_n$ converges in norm to an operator $T$ which is compact, since each $T_n$ has finite rank. (In fact, $T$ is of trace class.) Also, $T\in\mathscr{W}_0(\{T_k\}_1^{\infty})$.

Since the operators $T_n$ are idempotents with $T_nT_m=0$, $n\neq m$, we have for each $k\geqslant 1$,

$$(4T)^k=T_1+\sum_{2}^{\infty}(4^{1-n})^k\,T_n.$$

Since $k(n-1)\geqslant k+n-2$ for $n\geqslant 2$, $k\geqslant 1$, we have

$$\left\|\sum_{2}^{\infty}(4^{1-n})^k\,T_n\right\|\leqslant\sum_{2}^{\infty}(4^{1-n})^k\,\|T_n\|$$
$$\leqslant(4^{1-k})\sum_{2}^{\infty}4^{1-n}\,\|T_n\|.$$

This expression tends to 0 as $k$ increases. Hence, $T_1\in\mathscr{W}_0(T)$.

Now

$$16(4T-T_1)=T_2+\sum_{3}^{\infty}4^{2-n}T_n.$$

As above, we have

$$[16(4T-T_1)]^k=T_2+\sum_{3}^{\infty}(4^{2-n})^k\,T_n,$$

and

$$\left\|\sum_{3}^{\infty}(4^{2-n})^k\,T_n\right\|\leqslant(4^{1-k})\sum_{3}^{\infty}4^{2-n}\,\|T_n\|,$$

so

$$[16(4T-T_1)]^k\to T_2\qquad\text{as}\quad k\to\infty.$$

Continuing in this way, one sees that $T_k \in \mathscr{W}_0(T)$ for all $k \geqslant 1$. Thus $\mathscr{W}_0(\{T_k\}) = \mathscr{W}_0(T)$. Since $T_k = Q_k - Q_{k-1}$ and $Q_n = \sum_1^n T_k$, we also have $\mathscr{W}_0(\{T_k\}_1^\infty) = \mathscr{W}_0(\{Q_k\}_1^\infty)$. ∎

We note that in the above lemma $T_n$ is just the Riesz idempotent for $T$ corresponding to the isolated point $\{4^{-n}\}$ of $\sigma(T)$.

In some ways the operator $T$ behaves like a cyclic compact diagonal operator. In fact it is not hard to see that $T$ is quasisimilar to such an operator. (See [1].) However, $T$ has some exceptional properties. We break the proof of our main Theorem 3.7 into five lemmas 3.2 through 3.6. In Section 4 we will show that a slight modification in the construction of $T$ yields a closely related operator $\tilde{T}$ for which $\tilde{T} \oplus 0$ *is* reflexive, unlike $T$. This operator $\tilde{T}$ has properties which answer two questions in [7]. It fails the next lemma, which points out the delicacy of that step.

LEMMA 3.2. $I \in \operatorname{Ref} \mathscr{W}_0(T)$.

*Proof.* Fix a vector $x = \sum_1^\infty x_k e_k$ in $\mathscr{H}$. We must show $x \in [\mathscr{W}_0(T)\, x]$. Since $\{P_n\}$ converges in the strong operator topology to $I$, we have $P_n x \to x$. For $k \geqslant 1$ we have

$$Q_{2k-1} x = P_{2k-1} x + 4^k x_{2k-1} e_{2k}.$$

Hence, if the sequence of numbers $\{4^k x_{2k-1}\}_{k=1}^\infty$ has 0 as a cluster point, then some subsequence of $\{Q_{2k-1} x\}$ converges to $x$, and so $x \in [\mathscr{W}_0(T)\, x]$. Thus we reduce to the case where there is a $\delta > 0$ and a positive integer $K$ so that

$$4^k\, |x_{2k-1}| \geqslant \delta \qquad \forall k \geqslant K. \tag{1}$$

For each $k \geqslant K$, consider the equation

$$cQ_{2k-1} x + dQ_{2k} x = P_{2k} x. \tag{2}$$

We will show that, for infinitely many $k$, (2) has a solution $(c_k, d_k)$. It then follows that $x \in [\mathscr{W}_0(\{Q_k\}_1^\infty)\, x]$.

We proceed by way of contradiction. Suppose that there is a $K' \geqslant K$ so that for $k \geqslant K'$, (2) has no solution. Now (2) is equivalent to the system of $2k$ scalar equations

$$\begin{aligned} cx_n + dx_n &= x_n, \qquad n < 2k, \\ c4^k x_{2k-1} + d(x_{2k} + 4^k x_{2k+1}) &= x_{2k}. \end{aligned} \tag{3}$$

By (1), $x_{2k-1} \neq 0$, so our assumption is that the pair of scalar equations

$$\begin{aligned} c + d &= 1 \\ c(4^k x_{2k-1}) + d(x_{2k} + 4^k x_{2k+1}) &= x_{2k} \end{aligned} \tag{4}$$

has no solutions. But this means that

$$4^k x_{2k-1} = x_{2k} + 4^k x_{2k+1} \qquad \text{for all} \quad k \geqslant K'. \tag{5}$$

Thus

$$x_{2k+1} = x_{2k-1} - x_{2k}/4^k,$$

so

$$x_{2k+3} = x_{2k+1} - \frac{x_{2k+2}}{4^{k+1}} = x_{2k-1} - \frac{x_{2k}}{4^k} - \frac{x_{2k+2}}{4^{k+1}}.$$

Repeating the above steps $j$ times gives

$$x_{2k+2j+1} = x_{2k-1} - \left(\frac{x_{2k}}{4^k} + \frac{x_{2k+2}}{4^{k+1}} + \cdots + \frac{x_{2k+2j}}{4^{k+j}}\right). \tag{6}$$

Fix $k \geqslant K'$ so that

$$\left(\sum_{l \geqslant k} |x_{2l}|^2\right)^{1/2} < \frac{\delta}{4}.$$

Now

$$\begin{aligned} |x_{2k+2j+1}| &\geqslant |x_{2k-1}| - \left|\sum_{l=k}^{k+j} \frac{x_{2l}}{4^l}\right| \\ &\geqslant \frac{\delta}{4^k} - \left(\sum_{l \geqslant k} |x_{2l}|^2\right)^{1/2} \cdot \left(\sum_{l \geqslant k} 4^{-2l}\right)^{1/2} \\ &\geqslant \frac{\delta}{4^k} - \left(\frac{\delta}{4}\right) \cdot \left(\frac{2}{4^k}\right) \geqslant \frac{\delta}{4^{k+1}}. \end{aligned}$$

Thus $\lim_j x_{2k+2j+1} \neq 0$, our contradiction. ∎

LEMMA 3.3. *$T$ is a reflexive operator, and $\mathscr{W}(T) = \mathscr{A}(T)$.*

*Proof.* For every $n \geqslant 1$ we have

$$TT_n = \left(\sum 4^{-k} T_k\right) T_n = 4^{-n} T_n.$$

Thus ran $T_n$ is a one-dimensional eigenspace for $T$ for the eigenvalue $4^{-n}$. Also, $Q_{2n} = T_1 + T_2 + \cdots + T_{2n}$, so that $\mathscr{M}_{2n} = \operatorname{ran} Q_{2n} = \operatorname{ran} T_1 + \operatorname{ran} T_2 + \cdots + \operatorname{ran} T_{2n}$ is in Lat $T$. Since $T|_{\mathscr{M}_{2n}}$ has $2n$ distinct eigenvalues it is similar to a cyclic diagonal operator. In particular, $T|_{\mathscr{M}_{2n}}$ is reflexive.

That is, the algebra $\mathscr{W}(T|_{\mathscr{M}_{2n}})$ is reflexive. Since $\mathscr{M}_{2n}$ is a finite dimensional invariant subspace for $T$, we have

$$\mathscr{W}(T|_{\mathscr{M}_{2n}}) = \mathscr{W}(T)|_{\mathscr{M}_{2n}} \qquad \text{and} \qquad \mathscr{W}_0(T|_{\mathscr{M}_{2n}}) = \mathscr{W}_0(T)|_{\mathscr{M}_{2n}}.$$

Moreover, since $\mathscr{W}_0(T) = \mathscr{W}_0(\{T_k\}_1^{\infty})$ and $Q_{2n} = T_1 + \cdots + T_{2n}$, it follows that $\mathscr{W}_0(T|_{\mathscr{M}_{2n}})$ contains $I_{\mathscr{M}_{2n}}$ and hence

$$\mathscr{W}_0(T|_{\mathrm{M}_{2n}}) = \mathscr{W}(T|_{\mathscr{M}_{2n}}).$$

Furthermore, since $T_k|_{\mathscr{M}_{2n}} = 0$ for $k > 2n$, we have

$$\mathscr{W}(T|_{\mathscr{M}_{2n}}) = \mathscr{W}_0(T|_{\mathscr{M}_{2n}}) = \mathscr{W}_0(\{T_k|_{\mathscr{M}_{2n}}\}_{k=1}^{2n}).$$

By the construction, we have

$$\mathscr{W}_0(\{T_k|_{\mathscr{M}_{2n}}\}_{k=1}^{2n}) = \operatorname{span}(\{T_k|_{\mathscr{M}_{2n}}\}_{k=1}^{2n}).$$

Now let $S \in \operatorname{Alg\,Lat}(T)$ be arbitrary. Then

$$S|_{\mathscr{M}_{2n}} \in \operatorname{Alg\,Lat}(T|_{\mathscr{M}_{2n}}) = \mathscr{W}_0(\{T_k|_{\mathscr{M}_{2n}}\}_{k=1}^{2n}),$$

so there exists a unique set of complex numbers $\{c_k\}_{k=1}^{2n}$ with

$$S|_{\mathscr{M}_{2n}} = \sum_{k=1}^{2n} c_k T_k|_{\mathscr{M}_{2n}} = \sum_{k=1}^{2n} c_k(Q_k - Q_{k-1})|_{\mathscr{M}_{2n}}.$$

It is easy to see that the coefficients $c_k$ do not depend on $n$. Indeed, since the operators $T_k$, and hence $T_k|_{\mathscr{M}_{2n}}$, are idempotents, and $T_i T_j = 0$ for $i \neq j$, for each $k$ the subspace ran $T_k$ is a one-dimensional invariant subspace for $S$, so is contained in an eigenspace $N_k$ for $S$. It is clear that the eigenvalue of $S$ corresponding to $N_k$ is precisely $c_k$.

The operator $T$ is contained in the *tridiagonal* CSL algebra represented by the diagram

$$\begin{pmatrix} * & & & & & \\ * & * & * & & & \\ & & * & & & \\ & & * & * & * & \\ & & & & & \ddots \end{pmatrix}$$

which is known to be reflexive. So the operator $S$ must have this general tridiagonal form. The above paragraph shows that $S$ must in fact have finer structure. Let $\mathscr{D}$ denote the linear span (no closure) of the basis vectors $\{e_j\}_1^\infty$. Then $\mathscr{D}$ is invariant under $S$, and the series

$$\sum_{k=1}^{\infty} c_k(Q_k - Q_{k-1})$$

converges pointwise on $\mathscr{D}$ to $S|_{\mathscr{D}}$. This gives the matrix form of $S$. (However, this series need not converge in the weak operator topology. In particular, if $S=I$, then Lemma 3.5 will show that the series cannot converge weakly. For the case $S=I$ one has $c_k=1$ for each $k$.)

If we write $S=(S_{jk})$, then for all $k\geqslant 1$, we have $S_{kk}=c_k$, $S_{2k,k-1}=(c_{2k-1}-c_{2k})\,4^k$ and $S_{2k,2k+1}=(c_{2k}-c_{2k+1})\,4^k$. Every other element of $(S_{jk})$ is 0. In particular, we have

$$\begin{aligned} |c_{2k-1}-c_{2k}|\,4^k &\leqslant \|S\| \\ |c_{2k}-c_{2k+1}|\,4^k &\leqslant \|S\| \end{aligned} \tag{7}$$

for each $k$. Thus

$$\sum_{l=1}^{\infty} |c_l - c_{l+1}|$$

converges, and hence the sequence $\{c_n\}$ converges. Let $\lambda = \operatorname{Lim}_n c_n$. We have

$$\lambda = c_1 - \lim_n \left( \sum_{l=1}^{n-1} (c_l - c_{l+1}) \right).$$

Let $S=S-\lambda I$. Then $\hat{S}\in \operatorname{Alg}\operatorname{Lat}(T)$, and $\hat{S}$ has the formal series

$$\sum_{k=1}^{\infty} \hat{c}_k(Q_k - Q_{k-1}),$$

pointwise convergent on $\mathscr{D}$ to $\hat{S}|_{\mathscr{D}}$, where $\hat{c}_k = c_k - \lambda$ for each $k$. From (7) we have

$$|\hat{c}_l - \hat{c}_{l+1}| \leqslant 2^{-l}\,\|\hat{S}\|$$

for all $l\geqslant 1$. So since $\operatorname{Lim} \hat{c}_l = 0$, we have

$$|\hat{c}_{2k}| = \left| \sum_{l=2k}^{\infty} (\hat{c}_l - \hat{c}_{l+1}) \right| \leqslant 2(4^{-k})\|\hat{S}\|. \tag{8}$$

For each $n$ we have

$$T_k P_{2n} = T_k \quad \text{for} \quad 1 \leqslant k \leqslant 2n-1,$$
$$T_{2n} P_{2n} = T_{2n} - 4^n E_{2n,2n+1}$$

and

$$T_k P_{2n} = 0 \quad \text{for} \quad k \geqslant 2n+1.$$

So

$$\hat{S} P_{2n} = \sum_{k=1}^{2n} \hat{c}_k T_k P_{2n}$$
$$= \left( \sum_{k=1}^{2n} \hat{c}_k T_k \right) - 4^n \hat{c}_{2n} E_{2n,2n+1}.$$

From (8), the sequence $\{4^n \hat{c}_{2n}\}$ is *bounded*, and hence the sequence $\{4^n \hat{c}_{2n} E_{2n,2n+1}\}$ converges in the weak $*$ topology to 0. Also, since $\{P_{2n}\}$ converges weak $*$ to $I$ the sequence $\{\hat{S} P_{2n}\}$ converges weak $*$ to $\hat{S}$. Hence the sequence of partial sums

$$\left\{ \sum_{k=1}^{2n} \hat{c}_k T_k \right\}$$

converges weak $*$ to $\hat{S}$. Since $T_k \in \mathscr{A}_0(T)$, $1 \leqslant k < \infty$, this shows that $\hat{S} \in \mathscr{A}_0(T)$. Hence $S \in \mathscr{A}(T)$. We have proven that $\operatorname{Alg} \operatorname{Lat}(T) = \mathscr{A}(T)$. Hence also $\mathscr{A}(T) = \mathscr{W}(T)$. ∎

We note that it follows easily from the proof of the above lemma that $\mathscr{W}(T) = \{T\}'$.

LEMMA 3.4. *Let $\mathscr{H} = \oplus_j \mathscr{H}_j$ be the countable direct sum of Hilbert spaces $\{\mathscr{H}_j\}$. Suppose that $f_j \in \mathbb{F}_1(\mathscr{H}_j)$, $j = 1, 2, ...$, and that $\sum \| f \|_j < \infty$. Then there is an operator $f \in \mathbb{F}_1(\mathscr{H})$ such that $P_j f |_{\mathscr{H}_j} = f_j$ for each $j$, where $P_j$ is the projection of $\mathscr{H}$ onto $\mathscr{H}_j$.*

*Proof.* For each $j$, write $f_j = w_j \otimes v_j$, where $\| w_j \| = \| v_j \| = \| f_j \|^{1/2}$. Then $w_j, v_j \in \mathscr{H}_j$. Let $w = \oplus_j w_j$ and $v = \oplus_j v_j$. Then $w$, $v \in \mathscr{H}$, since $\| w \|^2 = \| v \|^2 = \sum \| f_j \| < \infty$. Let $f = w \otimes v$. Then $P_j f P_j = P_j w \otimes P_j v$ so that $P_j f |_{\mathscr{H}_j} = w_j \otimes v_j = f_j$, as desired. ∎

LEMMA 3.5. *$T$ is 2-elementary. The relative weak operator topology coincides with the relative weak $*$ topology on $\mathscr{W}(T)$.*

*Proof.* We must show that $C_1(\mathscr{H})=\mathscr{W}(T)_\perp+\mathbb{F}_2(\mathscr{H})$. Let $f=(f_{jk})\in C_1(\mathscr{H})$ be arbitrary. Then

$$\operatorname{tr}(Q_{2k-1}f)=\sum_{j=1}^{2k-1} f_{jj}+4^k f_{2k-1,2k} \qquad \forall k\geq 1 \tag{9}$$

and

$$\operatorname{tr}(Q_{2k}f)=\sum_{j=1}^{2k} f_{jj}+4^k f_{2k+1,2k} \qquad \forall k\geq 1.$$

Since each of the diagonals of a trace class operator is absolutely summable, we have

$$\sum_{k=1}^{\infty} |f_{kk}|<\infty, \qquad \sum_{k=1}^{\infty} |f_{2k-1,2k}|<\infty,$$

and

$$\sum_{k=1}^{\infty} |f_{2k+1,2k}|<\infty.$$

Let $g=(g_{ij})$ be the operator defined in terms of its coordinate elements by $g_{11}=\sum_{j=1}^{\infty} f_{jj}$, and

$$\left.\begin{aligned} g_{4k-1,4k}&=f_{4k-1,4k}-\frac{(\sum_{j=4k}^{\infty} f_{jj})}{4^{2k}}\\ g_{4k+1,4k}&=f_{4k+1,4k}-\frac{(\sum_{j=4k+1}^{\infty} f_{jj})}{4^{2k}}\end{aligned}\right\}\quad \text{for} \quad k=1,2,...,$$

with $g_{ij}=0$ otherwise. Then $g\in C_1(\mathscr{H})$. Similarly, define $h=(h_{ij})\in C_1(\mathscr{H})$ by setting

$$h_{4k-3,4k-2}=f_{4k-3,4k-2}-\frac{(\sum_{j=4k-2}^{\infty} f_{jj})}{4^{2k-1}}$$

$$h_{4k-1,4k-2}=f_{4k-1,4k-2}-\frac{(\sum_{j=4k-1}^{\infty} f_{jj})}{4^{2k-1}}$$

for $k=1,2,...$, and setting all other coordinate elements equal to 0. This choice of $g$ and $h$ yields, using (9), that $\operatorname{tr}(f)=\operatorname{tr}(g+h)$ and that $\operatorname{tr}(Q_l f)=\operatorname{tr}(Q_l(g+h))$ for all $l\geq 1$. Since $\mathscr{W}(T)=\mathscr{A}(T)$ and since $\mathscr{A}(T)$ is the weak $*$ topology closed linear span of $I$ together with $\{Q_l\colon 1\leq l<\infty\}$, it follows that $f-(g+h)\in\mathscr{W}(T)_\perp$. Thus the problem is reduced to verifying that $g+h\in\mathscr{W}(T)_\perp+\mathbb{F}_2(\mathscr{H})$.

For $k\geq 1$ let $R_k$ be the orthogonal projection onto $[e_{4k-1}, e_{4k}, e_{4k+1}, e_{4k+2}]$. Let $k_0=\operatorname{proj}[e_1,e_2]$. From the construction of $g$ we see that $R_k g R_k$ has rank 1 for each $k$, and, moreover, $g=\sum_{k=1}^{\infty} R_k g R_k$ with convergence in trace class norm. We have

$$\|g\|_1=\sum_{k=1}^{\infty} \|R_k g R_k\|_1.$$

By Lemma 3.4 there is a rank one operator $G$ with $R_k G R_k = R_k g R_k$ for all $k$. Since $g$ has nonzero coordinate elements only in the columns 1, 4, 8, 12, ... we may assume that $G$ also has this property by multiplying $G$ on the right by $\text{proj}[e_1, e_4, e_8, e_{12}, ...]$ if necessary. Then Eqs. (9) show that $G-g$ annihilates each $Q_l$ (and has trace 0) so $G-g \in \mathscr{W}(T)_\perp$.

We must also construct an operator $H \in \mathbb{F}_1$ with $H-h \in \mathscr{W}(T)_\perp$. For $k \geqslant 0$ let $S_k$ be the projection onto $[e_{4k+1}, e_{4k+2}, e_{4k+3}, e_{4k+4}]$. From the construction of $h$, the operators $S_k h S_k$ have rank 1 and we have

$$h = \sum_{k=0}^{\infty} S_k h S_k, \qquad \|h\|_1 = \sum_{k=0}^{\infty} \|S_k h S_k\|.$$

Apply Lemma 3.4 to obtain an operator $H \in \mathbb{F}_1$ with $S_k H S_k = S_k h S_k$, $k = 0, 1, 2, ...$. Since $h$ has nonzero coordinate elements only in the columns 2, 6, 10, ... we may assume that $H$ also has this property. As above, Eq. (9) yield that $H - h \in W(T)_\perp$.

We have $g + h = (g - G + h - H) + (G + H) \in \mathscr{W}(T)_\perp + \mathbb{F}_2$. Thus $\mathscr{W}(T)$ is 2-elementary. Hence the weak operator and weak $*$ topologies coincide on $\mathscr{W}(T) = \mathscr{A}(T)$ (cf. [2 or 3]). ∎

LEMMA 3.6. *$\mathscr{W}_0(T)$ does not contain $I$.*

*Proof.* By Lemma 3.5, $\mathscr{W}_0(T)$ is the weak $*$ closure of the linear span of the idempotents $\{Q_k\}_{k=1}^{\infty}$. Let $h = (h_{ij})$ be the operator defined in terms of its coordinate elements by

$$h_{kk} = 2^{-k}$$

$$h_{2k-1,2k} = -4^{-k}\left(\sum_{j=1}^{2k-1} 2^{-j}\right)$$

$$h_{2k+1,2k} = -4^{-k}\left(\sum_{j=1}^{2k} 2^{-j}\right)$$

for all $k \geqslant 1$ and all other elements 0. Then $h \in C_1(\mathscr{H})$, since it is supported on finitely many (three) diagonals and each diagonal is absolutely summable. It is easily verified using Eqs. (9) that $\text{tr}(Q_k h) = 0$ for all $l \geqslant 1$. So $h \in \mathscr{W}_0(T)_\perp$. Also, $\text{tr}(h) = 1$. This shows that $I \notin \mathscr{W}_0(T)$. ∎

THEOREM 3.7. *$T$ is reflexive, but $T \oplus 0$ is not reflexive.*

*Proof.* It was shown in Lemma 3.3 that $T$ is reflexive. On the other hand, Lemmas 3.2 and 3.6 show that $\mathscr{W}_0(T)$ is not reflexive, so by Proposition 2.1, $T \oplus 0$ is not reflexive.

*Remark.* Since the operator 0 is trivially reflexive and elementary, Theorem 3.7 shows that the direct sum of two reflexive operators need not be reflexive even under the hypothesis that one is elementary and the other is 2-elementary. We note that the direct sum of two reflexive elementary operators *is* reflexive [12]. Also, $T \oplus 0$ is the first known example of a *non-reflexive* operator for which the Hilbert space on which it acts is the closed linear span of eigenvectors for the operator.

We now discuss briefly some additional questions which can be answered using the above sequence of lemmas.

In [8], J. Deddens raised the following question: If $\mathrm{Lat}(A \oplus B) = \mathrm{Lat}(A) \oplus \mathrm{Lat}(B)$, must $\mathscr{W}(A \oplus B) = \mathscr{W}(A) \oplus \mathscr{W}(B)$? (See also the article of Conway and Wu [5].) The answer is no.

COROLLARY 3.8. $\mathrm{Lat}(T \oplus 0) = \mathrm{Lat}\, T \oplus \mathrm{Lat}\, 0$, *but* $\mathscr{W}(T \oplus 0) \neq \mathscr{W}(T) \oplus \mathscr{W}(0)$.

*Proof.* This is immediate from Lemmas 3.2 and 3.6 together with Proposition 2.2.

*Remark.* An operator $A$ was given in [17, Example 1] with the property that $\mathscr{W}(A)$ is properly contained in $\{A\}' \cap \mathrm{Alg}\,\mathrm{Lat}(A)$. The operator $T \oplus 0$ shares this property. This follows from the fact that $\mathrm{Alg}\,\mathrm{Lat}(T \oplus 0) = \mathscr{W}(T) \oplus \mathbb{C}I \subseteq \{T \oplus 0\}'$ together with the fact that $\mathscr{W}(T \oplus 0)$ is properly contained in $\mathrm{Alg}\,\mathrm{Lat}(T \oplus 0)$. The operator $A$ in [17] was an extension of a triangular operator but was not triangular. The operator $T \oplus 0$ *is* triangular, and is in fact bitriangular, as is easily seen. (An operator $A$ is called *bitriangular* [7] if both $A$ and $A^*$ are triangular with respect to perhaps different orthonormal bases for $\mathscr{H}$.) We note that $T$ is triangular with respect to the ordering $\{e_2, e_1, e_4, e_3, e_6, e_5, ...,\}$ of the basis $\{e_n\}$ and $T^*$ is triangular with respect to the ordering $\{e_1, e_3, e_2, e_5, e_4, ...\}$. So $T$ is bitriangular, hence $T \oplus 0$ is bitraingular.

The above results yield the first known example of a reflexive operator $A$ and an operator $B$ contained in $\mathscr{W}(A) = \mathrm{Alg}\,\mathrm{Lat}(A)$ for which $\mathrm{Lat}(A) = \mathrm{Lat}(B)$ and yet for which $\mathscr{W}(B)$ is *properly* contained in $\mathscr{W}(A)$.

COROLLARY 3.9. $\mathrm{Lat}(T \oplus 0) = \mathrm{Lat}(T \oplus (2I))$ *and* $T \oplus 0 \in \mathscr{W}(T \oplus (2I))$, *but* $\mathscr{W}(T \oplus 0)$ *is properly contained in* $\mathscr{W}(T \oplus (2I))$. *Also,* $T \oplus (2I)$ *is reflexive.*

*Proof.* We have $\sigma(T) = \{4^{-k} : k \geqslant 1\} \cup \{0\}$, so $\sigma(T)$ and $\sigma(2I)$ are contained in disjoint disks. Thus (see [5], for example) we have $\mathscr{W}(T \oplus (2I)) = \mathscr{W}(T) \oplus \mathscr{W}(2I)$ and $\mathrm{Lat}(T \oplus (2I)) = \mathrm{Lat}(T) \oplus \mathrm{Lat}(2I)$. Since the direct sum of two reflexive algebras is reflexive, this shows that $T \oplus (2I)$ is reflexive and that $T \oplus 0 \in \mathscr{W}(T \oplus (2I))$. Corollary 3.8 now completes the proof. ∎

We next give an application of our work to the theory of dual algebras. (See [3, 4].) A *dual algebra* $A$ is a weak* closed unital subalgebra of $\mathscr{B}(\mathscr{H})$. The dual algebra generated by an operator $A$ is $\mathscr{A}(A)$. The key idea in the Scott Brown technique for constructing invariant subspaces for $A$ can be described as follows: One attempts to find a nonzero weak* continuous complex homomorphism $\Lambda : \mathscr{A}(A) \to \mathbb{C}$ with the additional property that $\Lambda$ is *spatial*. This means that there exist vectors $x$ and $y$ in $\mathscr{H}$ with the property that $\Lambda(B) = \langle Bx, y \rangle$ for all $B \in \mathscr{A}(A)$. The difficulty usually lies in proof of spatiality of a known $\Lambda$. If this can be done, then $[(\ker \Lambda)\, x]$ is a proper invariant subspace for $A$. So general results proving spatiality of functionals $\Lambda$ can yield invariant subspace results. Up to this time no example delimiting the theory was known of a non-spatial weak* continuous complex homomorphism of a singly generated dual algebra. Next we show that our example $T$ accomplishes this.

COROLLARY 3.10. *There is a weak *-continuous complex homomorphism of $\mathscr{W}(T)$ which is not spatial.*

*Proof.* Let $h$ be the trace class operator constructed in the proof of Lemma 3.6. Define a linear functional $\Lambda$ on $\mathscr{W}(T)$ by $\Lambda(A) = \operatorname{tr}(Ah)$, $A \in \mathscr{W}(T)$. Then $\Lambda$ is weak *-continuous, and ker $\Lambda = \mathscr{W}_0(T)$, which is a maximal ideal in $\mathscr{W}(T)$. Also, $\Lambda(I) = 1$. So $\Lambda$ is a homomorphism. Suppose, by way of contradiction, that $\Lambda$ is spatial. Then there are vectors $x, y \in \mathscr{H}$ with $\Lambda(A) = \langle Ax, y \rangle$ $\forall A \in \mathscr{W}(T)$. By Lemma 3.2, $I \in \operatorname{Ref}(\mathscr{W}_0(T))$, so there exists a sequence $\{B_n\} \subset \mathscr{W}_0(T)$ with $B_n x \to x$. We have $\langle B_n x, y \rangle = \Lambda(B_n) = 0$, so $\langle x, y \rangle = \operatorname{Lim} \langle B_n x, y \rangle = 0$. However, $\langle x, y \rangle = \Lambda(I) = 1$, a contradiction. ∎

The next corollary answers a question that had been posed several years ago (personal communication) to the first author by J. Erdös.

COROLLARY 3.11. *There is a linear subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ which is weak * generated by rank one operators yet for which* ref($\mathscr{S}$) *is not weak * generated by rank one operators.*

*Proof.* Let $T$ be the operator considered above and let $\mathscr{S} = \mathscr{W}_0(T)$. Then $\mathscr{S}$ is weak * generated by the rank one operators $\{T_n : n \geqslant 1\}$, and ref $\mathscr{S} = \mathscr{W}(T)$. If $A$ is any operator in $\mathscr{W}(T)$ then, as in the proof of Lemma 3.3, $A$ has a representation as a formal series $\sum c_n T_n$ which is pointwise convergent on a dense $\mathscr{W}(T)$-invariant domain $\mathscr{D}$. From the construction it follows that if $A$ is of rank 1 then $A$ is a scalar multiple of $T_n$ for some $n$, so $A \in \mathscr{W}_0(T)$. Since $\mathscr{W}(T) \neq \mathscr{W}_0(T)$, the conclusion follows. ∎

## 4. Some Related Results

The focus of Section 3 was to show that the operator $T$ constructed in that section has the property that it is reflexive yet $T\oplus 0$ fails to be reflexive. We next show that $T\oplus 0$ is, in an essential way, *nearly* reflexive.

If $\mathscr{A}$ is a unital subalgebra of $\mathscr{B}(\mathscr{H})$, let $\mathrm{Lat}_{1/2}\,\mathscr{A}$ denote the lattice of invariant operator ranges for $\mathscr{A}$. Thus $M\in \mathrm{Lat}_{1/2}\,\mathscr{A}$ if $M=\operatorname{ran} S$ for some $S\in\mathscr{B}(\mathscr{H})$ and $AM\subseteq M$ for all $A\in\mathscr{A}$. Given a family $\mathscr{L}$ of operator ranges let $\mathrm{Alg}\,\mathscr{L}$ denote the algebra of operators in $\mathscr{B}(\mathscr{H})$ that leave every element of $\mathscr{L}$ invariant. An algebra $\mathscr{A}$ is said to be *parareflexive* if $\mathscr{A}=\mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathscr{A}$. We note that parareflexive algebras are typically *not* closed in any operator topology. For example, if $S$ is an operator which is not algebraic, then the algebra of entire functions of $S$ is parareflexive [10]. In fact, Ong in [16] posed the question of whether weakly closed unital parareflexive algebras are always reflexive. For the operator $T\oplus 0$ of Section 3 with the second direct summand space one-dimensional, we will show that $\mathscr{A}=\mathscr{W}(T\oplus 0)$ is parareflexive. Since $\mathscr{A}$ is not reflexive, this yields a counterexample to Ong's question.

Theorem 4.1. *Let $T$ be the operator constructed in Section 3. Then $\mathscr{W}(T\oplus 0)$ acting on $\mathscr{H}\oplus\mathbb{C}$ is parareflexive.*

*Proof.* We have

$$\mathscr{W}(T\oplus 0)\subset \mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathscr{W}(T\oplus 0)$$
$$\subset \mathrm{Alg}\,\mathrm{Lat}(T\oplus 0)\subset \mathscr{W}(T)\oplus\mathbb{C}I.$$

Also, $\mathscr{W}(T\oplus 0)$ has codimension one in $\mathscr{W}(T)\oplus\mathbb{C}I$. Thus to show that $\mathscr{W}(T\oplus 0)$ is parareflexive, it suffices to show that $\mathrm{Alg}\,\mathrm{Lat}_{1/2}\,\mathscr{W}(T\oplus 0)\neq \mathrm{Alg}\,\mathrm{Lat}(T\oplus 0)$. Since

$$\mathrm{Alg}\,\mathrm{Lat}(T\oplus 0)=\mathrm{Alg}\,\mathrm{Lat}(T)\oplus\mathbb{C}I,$$

it is enough to find an $M$ in $\mathrm{Lat}_{1/2}(T\oplus 0)$ so that $M$ does not split.

Consider the operator $V$ in $\mathscr{W}(T)$ with formal series

$$V=\sum_{n=1}^{\infty}\frac{1}{2^n}T_n.$$

From the description of $\mathscr{W}(T)$ in the proof of Lemma 3.3, we see that $V=(V_{jk})$ satisfies $V_{kk}=2^{-k}$, $V_{2k,2k-1}=(2^{-2k+1}-2^{-2k})\cdot 2^{2k}=1$, and $V_{2k,2k+1}=(2^{-2k}-2^{-2k-1})\cdot 2^{2k}=\frac{1}{2}$, $k=1,2,\ldots$. All other coordinate entries of $(V_{ij})$ are 0.

Let $x=\sum_{n=1}^{\infty} n^{-1}e_{2n}$. Then $x$ is not in ran $V$, for solving the equation $V(\sum b_n e_n)=x$ would give $b_n=2^{2n}/n$. Let $M$ be the (unclosed) linear span of $x\oplus 1$ and ran $V\oplus 0$. Then $M$ does not split, since $x$ is not in the range of $V$. It remains to show that $M\in \mathrm{Lat}_{1/2}\,\mathscr{W}(T\oplus 0)$.

Define $A\in\mathscr{B}(\mathscr{H}\oplus\mathbb{C})$ as follows: for all $z\in\mathbb{C}$, let $A(0\oplus z)=0$. Let $A(x\oplus 0)=x\oplus 1$. Let $U$ be an isometry from $\{x\}^{\perp}$ onto $\mathscr{H}$, and for $y\in\{x\}^{\perp}$, let $A(y\oplus 0)=(VUy)\oplus 0$. Then $\operatorname{ran} A=\mathbb{C}(x\oplus 1)+(\operatorname{ran} V)\oplus 0=M$. This shows that $M$ is an operator range.

We now show that $M$ is invariant for $\mathscr{W}(T\oplus 0)$. If $S\in\mathscr{W}(T\oplus 0)$, then $S=S_1\oplus\lambda$ for some $S_1\in\mathscr{W}(T)$ and $\lambda\in\mathbb{C}$. For each element $Vy$ of ran $V$, we have $(S_1\oplus\lambda)(Vy\oplus 0)=S_1 Vy\oplus 0=VS_1 y\oplus 0\in \operatorname{ran} V\oplus 0\in M$. That is, $S(\operatorname{ran} V\oplus 0)\subseteq M$. To show that $S(x\oplus 1)\in M$, note that $S-\lambda I=(S_1-\lambda I)\oplus 0\in\mathscr{W}_0(T\oplus 0)$, so $S_1-\lambda I\in\mathscr{W}_0(T)$. From the proof of Lemma 3.3 it follows that $S_1-\lambda I$ has the formal series $\sum_{n=1}^{\infty} c_n T_n$ for some sequence $\{c_n\}$ of numbers such that $\{2^n|c_n|\}$ is bounded. Hence the series $\sum_{n=1}^{\infty}(2^{2n}c_{2n}/n)\,e_{2n}$ converges in norm to a vector $w\in\mathscr{H}$. A computation yields $(S_1-\lambda I)\,x=\sum_{n=1}^{\infty}(c_{2n}/n)\,e_{2n}=Vw\in\operatorname{ran} V$. So

$$(S_1\oplus\lambda)(x\oplus 1)=(S_1-\lambda I)\,x\oplus 0+\lambda(x\oplus 1)\in M,$$

as required. The proof is complete. ∎

Next, we consider a slight modification of the example of Section 3. This leads to answers to two questions appearing in the paper [7] of Davidson and Herrero. In this paper the authors show that each bitriangular operator $A$ is quasisimilar to a direct sum of Jordan blocks. This leads to information on the invariant and hyperinvariant subspaces of $A$.

We begin with a lemma. This should be compared with Proposition 2.1 and Lemma 3.2.

LEMMA 4.2. *Suppose that $\{G_n\}_{n=1}^{\infty}$ is a sequence of rank one idempotents which is algebraically orthogonal (that is, $G_nG_m=0$ if $m\neq n$) and such that $[\{\operatorname{ran} G_n : n\geqslant 1\}]=\mathscr{H}$. Then $I\in\operatorname{Ref}\mathscr{W}_0(\{G_n\}_{n=1}^{\infty})$ if and only if for each $M\in\operatorname{Lat}(\{G_n\})$ we have $M=[\{\operatorname{ran} G_n : \operatorname{ran} G_n\subseteq M\}]$.*

*Proof.* We have $I\in\operatorname{Ref}\mathscr{W}_0(\{G_n\})$ if and only if for each $x\in\mathscr{H}$ we have $x\in[\mathscr{W}_0(\{G_n\})\,x]=[\{G_nx : G_nx\neq 0\}]$. Thus if $M_x$ denotes the cyclic subspace for $\mathscr{W}(\{G_n\})$ generated by $x$, then $I\in\operatorname{Ref}\mathscr{W}_0(\{G_n\})$ if and only if for each $x$, $M_x=[\{G_nx : G_nx\neq 0\}]$. This proves the lemma. ∎

Now we modify our main example. Let $Q_0=0$, and for $k\geqslant 1$, set

$$\tilde{Q}_{2k-1}=P_{2k-1}+2^{2k-1}E_{2k,2k-1}$$

and

$$\tilde{Q}_{2k}=P_{2k}+2^{2k}E_{2k,2k+1}.$$

Let $\tilde{T}_n = \tilde{Q}_n - \tilde{Q}_{n-1}$ for $n \geqslant 1$, and let $\tilde{T} = \sum_{n=1}^{\infty} 4^{-n} \tilde{T}_n$. Note that $\tilde{Q}_{2k-1}$ has been obtained from $Q_{2k-1}$ by changing the coefficient of $E_{2k,2k-1}$ from $2^{2k}$ to $2^{2k-1}$, and $\tilde{Q}_{2k} = Q_{2k}$. We leave to the reader the verification that Lemmas 3.1 and 3.3 carry over with only trivial modifications. Thus we have

LEMMA 4.3. *The operator $\tilde{T}$ is reflexive and $\mathscr{A}(\tilde{T}) = \mathscr{W}(\tilde{T}) = \{\tilde{T}\}'$.*

Let $\phi_1 = e_1 + 2e_2$, $\phi_{2n-1} = -2^{2n-2} e_{2n-2} + e_{2n-1} + 2^{2n-1} e_{2n}$ for $n > 1$, and $\phi_{2n} = e_{2n}$ for $n \geqslant 1$. Also let $\psi_{2n-1} = e_{2n-1}$ and $\psi_{2n} = -2^{2n-1} e_{2n-1} + e_{2n} + 2^{2n} e_{2n+1}$ for $n \geqslant 1$. Then $\tilde{T}_n = \langle \cdot, \psi_n \rangle \phi_n = \phi_n \otimes \psi_n$ for $n \geqslant 1$.

LEMMA 4.4.

$$[\{\phi_{2n-1}\}_{n=1}^{\infty} \cup \{\psi_{2n}\}_{n=1}^{\infty}] \neq \mathscr{H}.$$

*Proof.* We will construct a vector $x = \sum_1^{\infty} x_n e_n$ in $\mathscr{H}$, $x \neq 0$, so that $x \perp \phi_{2n-1}$ and $x \perp \psi_{2n}$ for all $n \geqslant 1$. It is necessary and sufficient that such a vector satisfy

$$x_1 + 2x_2 = 0 \tag{1}$$

$$-2^{2n-1} x_{2n-1} + x_{2n} + 2^{2n} x_{2n+1} = 0, \qquad n \geqslant 1 \tag{2}$$

$$-2^{2n-2} x_{2n-2} + x_{2n-1} + 2^{2n-1} x_{2n} = 0, \qquad n \geqslant 2. \tag{3}$$

Set $x_1 = 1$ and apply (1) to obtain $x_2 = -\frac{1}{2}$. Then iteratively apply (2) and (3) to determine the remainder of the sequence. An argument by induction shows that for all $n \geqslant 1$ both $|x_{2n-1}|$ and $|x_{2n}|$ are less than or equal to $(1 - 2^{-n})\, 2^{2-n}$. Thus $(x_n) \in l^2$, and thus the series for $x$ converges in $\mathscr{H}$. ∎

We are now in a position to provide an answer to Problem 6.10 of [7]. This problem asks if $[\{\ker(S - \lambda)^n : n \geqslant 1,\ \lambda \in \Gamma\}]$ and $[\{\ker(S^* - \bar{\lambda})^n : n \geqslant 1,\ \lambda \in \mathbb{C} \backslash \Gamma\}]$ are complementary (necessarily orthogonal) subspaces for every bitriangular operator $S$ and every subset $\Gamma$ of $\mathbb{C}$. The authors showed that many bitriangular operators have this strong structural property. The operator $\tilde{T}$ which we have constructed is clearly bitriangular (as is $T$) and $\sigma_p(\tilde{T}) = \{4^{-n} : n \geqslant 1\}$. A computation shows that for all $k \geqslant 1$, $\ker(\tilde{T} - 4^{-n})^k = \ker(\tilde{T} - 4^{-n}) = \operatorname{ran} \tilde{T}_n = [\phi_n]$, and $\ker(\tilde{T}^* - 4^{-n})^k = \ker(\tilde{T}^* - 4^{-n}) = \operatorname{ran} \tilde{T}_n^* = [\psi_n]$. Thus Lemma 4.4, with $\Gamma = \{4^{-(2n-1)} : n \geqslant 1\}$, shows that this problem has a negative answer.

LEMMA 4.5. *If $x \in [\{\phi_{2n-1} : n \geqslant 1\} \cup \{\psi_{2n} : n \geqslant 1\}]^{\perp}$, then $x \perp [\mathscr{W}_0(\tilde{T}) x]$.*

*Proof.* The condition $x \perp \psi_{2n}$ implies $\tilde{T}_{2n} x = 0$ while $x \perp \phi_{2n-1}$ implies $\tilde{T}_{2n-1}^* x = 0$. Thus for all $n \geqslant 1$, either $\tilde{T}_n x = 0$ or $\tilde{T}_n^* x = 0$, hence $\langle \tilde{T}_n, x, x \rangle = \langle x, \tilde{T}_n^* x \rangle = 0$, $n \geqslant 1$. Thus $x \perp [\mathscr{W}_0(\tilde{T})\, x]$. ∎

Lemma 4.5 together with Proposition 2.1 shows that $\tilde{T}\oplus 0$ *is* reflexive, unlike the operator $T\oplus 0$. This indicates that the condition in Section 3 is quite delicate.

We can now answer another problem posed in [7]. Problem 6.11 asks if every hyperinvariant subspace $M$ of a bitriangular operator $S$ satisfies $M=[\{M\cap\ker(S-\lambda)^n : n\geqslant 1, \lambda\in\sigma_p(S)\}]$. Since $I\notin \operatorname{Ref}\mathscr{W}_0(\tilde{T})$, Lemma 4.2 shows that $[\{\operatorname{ran}\tilde{T}_n : \operatorname{ran}\tilde{T}_n\subseteq M]\neq M$ for some $M\in\operatorname{Lat}\tilde{T}$. Since $\{\tilde{T}\}'=\mathscr{W}(\tilde{T})$, $M$ is hyperinvariant. Finally, $\operatorname{ran}\tilde{T}_n=\ker(\tilde{T}-4^{-n})$.

We note that the results of this section are closely related to some interesting results of Marcus [14]. In fact his Example 3.1° provides another answer to Problem 6.10 of [7].

While the results and proofs in this article are for a specially constructed single operator, they really concern properties of a special unbounded Boolean algebra of idempotents (or a special biorthogonal system of vectors). It is possible that our present results may become absorbed in a more general theory. With this in mind, some natural questions arise. If $\{T_n'\}_1^\infty$ is an algebraically orthogonal sequence of idempotents in $\mathscr{B}(\mathscr{H})$, is $\mathscr{W}(\{T_n'\}_1^\infty)$ always reflexive? Is it always true that $\mathscr{W}(\{T_n'\})=\mathscr{A}(\{T_n'\})$? Does the relative weak operator topology always coincide with the relative weak $*$ topology on $\mathscr{W}(\{T_n'\})$? When is $I$ contained in $\mathscr{W}_0(\{T_n'\})$? If $I\notin\mathscr{W}_0(\{T_n'\})$, when is $I\in\operatorname{Ref}\mathscr{W}_0(\{T_n'\})$?

*Note added in proof.* We wish to point out that the construction in this paper can be used to answer three additional open questions. We give only brief outlines of the solutions:

1. If $A$ and $B$ are reflexive operators, is $A\otimes B$ reflexive? The answer is no. Let $A$ be our main example $T$ and let $B$ be a rank one projection on a two-dimensional Hilbert space. Then $A\otimes B$ is equivalent to $T\oplus 0$, which is not reflexive.

2. Consider the following question of W. E. Longstaff: if $L$ is a completely distributive subspace lattice, is the algebra generated by the rank one operators in Alg $\mathscr{L}$ dense in Alg $\mathscr{L}$ in the strong operator topology? The reader may wish to consult S. Argyros, M. Lambrou, and W. E. Longstaff ("Atomic Boolean Subspace Lattices and Applications to the Theory of Bases," Memoirs AMS, to appear) for definitions, references, and a discussion of the above question and related questions. The answer to this question is no—even for an atomic Boolean subspace lattice. Let $\mathscr{L}$ be the subspace lattice generated by $\{\operatorname{ran}T_n : n\geqslant 1\}$. Lemma 4.2 shows that $\mathscr{L}$ is a complete atomic Boolean algebra, so in particular, $\mathscr{L}$ is completely distributive. Then Lemma 3.6 shows that $I$ is not in the strong operator closure of the algebra generated by the rank one operators in Alg $\mathscr{L}$.

3. At the end of this paper we raised this question. If $\{T_n'\}_1^\infty$ is an algebraically orthogonal sequence of idempotents in $\mathscr{B}(\mathscr{H})$, is $\mathscr{W}(\{T_n\}_1^\infty)$ always reflexive? The answer is no. Consider the family $\{T_n\oplus 0\}_1^\infty\cup\{0\oplus T_n\}_1^\infty$ in $\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$. Denote this family by $\mathscr{P}$. It is not hard to see that $I\oplus 0\in\operatorname{Ref}\mathscr{W}(\mathscr{P})$ but $I+0\notin\mathscr{W}(\mathscr{P})$; thus, $\mathscr{W}(\mathscr{P})$ is not reflexive.

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