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A survey on oscillation of impulsive delay differential equations

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ABSTRACT

This paper presents a systematic development of the oscillatory results for the linear and nonlinear impulsive delay differential equations. We also discuss linearized oscillation theory, and give some applications to mathematical biology.

1. Introduction

Impulsive differential equations, that is, differential equations involving an impulse effect, appear as a natural description of observed evolution phenomena of several real-world problems. It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulation systems, do exhibit impulse effects. The monographs [1–5] are good sources for the study of impulsive differential equations and their applications. The recent survey paper [6] provides the oscillation theory of impulsive ordinary differential equations.

We recall that from the last 40 years delay differential equations have attracted a great deal of attention of researchers in mathematical, biological, and physical sciences. This is specially due to the fact that the theory of ordinary differential equations does not carry over to delay differential equations; in fact, often it needs special devices. Among the topics studied for the delay differential equations, oscillation of the solutions has been investigated the most, and complied in the monographs [7–10]. However, oscillatory results for the impulsive delay differential equations are scattered all over. Thus, in this paper we systematically arrange, and modify known oscillatory results and their proofs, for linear and nonlinear impulsive delay differential equations. Several examples illustrating how easily the theory can be applied in practice are also included.

The first investigation of the oscillation theory of impulsive differential equations was published in 1989 [11]. In this paper Gopalsamy and Zhang considered impulsive delay differential equations of the form

$$x'(t) + ax(t-\tau) = \sum_{k=1}^{\infty} b_k x(t_k^-)\delta(t-t_k), \quad t \neq t_k, \tag{1.1}$$

and

$$\begin{cases} x'(t) + p(t)x(t-\tau) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k^-) = b_k x(t_k^-), & k \in \mathbb{N} = \{1, 2, \ldots\} \end{cases} \tag{1.2}$$

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where $p \in C([0,\infty),[0,\infty))$, a is a positive real number, $\tau \geq 0$, b_k, $k = 1, 2, \ldots$, are real numbers, $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. In [11] sufficient conditions are obtained for the asymptotic stability of the zero solution of (1.1) and the existence of oscillatory solutions of (1.2). This work initiated the oscillation theory of impulsive delay differential equations. The monographs [12,13] include some oscillation results for the impulsive delay differential equations. In what follows we shall only consider impulses at fixed times; however, to the best of our knowledge, [14,15] are the only papers on the oscillation of delay differential equations with impulses at variable times.

The plan of this paper is as follows: Section 2 includes notations, definitions and some theorems needed in the later sections. In Section 3 we are concerned with the linear impulsive delay differential equations. In the last Section 4 we deal with the nonlinear impulsive delay differential equations. This section also contains linearized oscillation and some applications to models in mathematical biology.

2. Preliminaries

In this section, we introduce notations, definitions and some well-known results which are needed throughout this paper.

Let $\mathbb{R}^+ = [0,\infty)$, $J = [t_0,\infty)$ for some fixed $t_0 \geq 0$ and $\{t_k\}_{k=1}^{\infty}$ be a sequence in J such that $t_0 \leq t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. Let $i(a,b)$ denote the number of points t_k, lying in the interval (a,b). If $\{c_k\}$ is a sequence, then $\sum_{a<t_k<b} c_k$ and $\prod_{a<t_k<b} c_k$ denote the sum and product of the numbers c_k such that $t_k \in (a,b)$. If $i(a,b) = 0$, then $\sum_{a<t_k<b} c_k = 0$ and $\prod_{a<t_k<b} c_k = 1$.

By $PLC(X,Y)$ we denote the set of all functions $\psi : X \to Y$ which are continuous for $t \neq t_k$, and continuous from the left with discontinuities of the first kind at $t = t_k$. Similarly, $PLC^r(X,Y)$ is the set of functions $\psi : X \to Y$ having derivative $\psi^{(r)} \in PLC(X,Y)$. As usual $C(X,Y)$ denotes the set of continuous functions from X to Y.

Consider the first order delay differential equation having impulses at fixed moments of the form

$$\begin{cases} x'(t) = f(t, x(t), x(\tau(t))), & t \neq t_k,\ t \geq t_0, \\ \Delta x(t_k) = f_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{2.1}$$

and the initial condition

$$x(t) = \varphi(t), \quad t \in [T_{-1}, t_0], \tag{2.2}$$

where $f \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f_k \in C(\mathbb{R},\mathbb{R})$, $\varphi \in C([T_{-1},t_0],\mathbb{R})$, $\tau \in C(J,\mathbb{R})$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, $T_{-1} = \inf_{t\geq t_0}\{\tau(t)\}$, and

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-)$$

with $x(t_k^{\pm}) = \lim_{t\to t_k^{\pm}} x(t)$. For simplicity, it is usually assumed that $x(t_k) = x(t_k^-)$.

Definition 2.1. A function $x : [T_{-1},\infty) \to \mathbb{R}$ is said to be a solution of (2.1)–(2.2), if the following conditions are satisfied:

(i) $x(t)$ is absolutely continuous in each interval (t_k, t_{k+1}), $k \in \mathbb{N}$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$;
(ii) $x(t)$ satisfies the former equation of (2.1) almost everywhere in $[t_0,\infty) \setminus \{t_k\}$ and satisfies the latter equation for every $t = t_k$, $k \in \mathbb{N}$;
(iii) $x(t)$ satisfies (2.2) for $t \in [T_{-1}, t_0]$.

Let $x(t)$ be a solution of some impulsive differential equation.

Definition 2.2 (*[12]*)**.** The solution $x(t)$ is said to be regular if it is defined in some half line $[T_x,\infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$.

Definition 2.3. A real-valued function $x(t)$, not necessarily a solution, is said to be oscillatory, if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. A differential equation is called oscillatory if all its solutions are oscillatory.

For our purposes we now state some well known results for the oscillation of delay differential equations without impulses.

Consider the delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \tag{2.3}$$

where $p_i \in \mathbb{R}$ and $\tau_i \in \mathbb{R}^+$ for $i = 1, 2, \ldots, n$.

Theorem 2.1 ([8]). *Assume that $p_i, \tau_i \geq 0$ for $i = 1, 2, \ldots, n$. Then each of the following conditions is sufficient for the oscillation of all solutions of Eq. (2.3).*

(a) $\sum_{i=1}^{n} p_i\tau_i > \frac{1}{e}$;
(b) $\left(\prod_{i=1}^{n} p_i\right)^{1/n}\left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}$.

Theorem 2.2 ([8]). *Assume that $p_i, \tau_i \geq 0$ for $i = 1, 2, \ldots, n$. Then*

$$\left(\sum_{i=1}^{n} p_i\right)\left(\max_{1\leq i\leq n} \tau_i\right) \leq \frac{1}{e}$$

is sufficient for the existence of a nonoscillatory solution of Eq. (2.3), while

$$\left(\sum_{i=1}^{n} p_i\right)\left(\min_{1\leq i\leq n} \tau_i\right) > \frac{1}{e}$$

is sufficient for all solutions of Eq. (2.3) to be oscillatory.

Now consider the following delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i(t)) = 0, \quad t_0 \leq t < T, \tag{2.4}$$

and the delay differential inequalities

$$y'(t) + \sum_{i=1}^{n} q_i(t)y(t-\tau_i(t)) \leq 0, \quad t_0 \leq t < T, \tag{2.5}$$

and

$$z'(t) + \sum_{i=1}^{n} r_i(t)z(t-\tau_i(t)) \geq 0, t_0 \leq t < T, \tag{2.6}$$

where $p_i, q_i, r_i, \tau_i \in C([t_0, T), \mathbb{R}^+)$ for $i = 1, 2, \ldots, n$, and $t_0 < T \leq \infty$.

Let

$$t_{-1} = \min_{1\leq i\leq n}\left\{\inf_{t_0\leq t<T}\{t-\tau_i(t)\}\right\}.$$

Theorem 2.3 ([8]). *Assume that $q_i(t) \geq p_i(t) \geq r_i(t)$, $t_0 \leq t < T$, $i = 1, 2, \ldots, n$ and $x(t), y(t), z(t)$ are continuous solutions of (2.4)–(2.6) respectively, such that*

$$y(t) > 0, \quad t_0 \leq t < T,$$
$$z(t_0) \geq x(t_0) \geq y(t_0),$$
$$\frac{y(t)}{y(t_0)} \geq \frac{x(t)}{x(t_0)} \geq \frac{z(t)}{z(t_0)} \geq 0, \quad t_{-1} \leq t < t_0.$$

Then

$$z(t) \geq x(t) \geq y(t), \quad t_0 \leq t < T.$$

The following well known theorem will also be needed in our paper.

Theorem 2.4 (*Lebesgue's Dominated Convergence Theorem*). *Let M be a measurable set and let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in M, and for every $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ a.e. in M, where g is integrable on M. Then*

$$\lim_{n\to\infty}\int_M f_n \mathrm{d}\mu = \int_M f \mathrm{d}\mu.$$

3. Linear differential equations

In this section, we consider the oscillation of first, second and higher order impulsive linear delay differential equations. We shall also discuss some results on the generic oscillations.

3.1. First order equations

Let us consider the impulsive delay differential equation

$$\begin{cases} x'(t) + p(t)x(t-\tau) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k^-) = b_k x(t_k^-), & k \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $p \in C([0,\infty),[0,\infty))$, $\tau > 0$, b_k, $k = 1, 2, \ldots$ are real numbers, $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

It is clear that all solutions of (3.1) are oscillatory if there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $b_{n_k} \leq -1$ for all $k \in \mathbb{N}$. So, we assume $b_k > -1$ for all $k \in \mathbb{N}$.

There are many papers for the oscillation of Eq. (3.1) and its various extensions [16–25]. The following results were established in [19].

For convenience, define

$$\alpha(s,t) = \begin{cases} \prod_{s<t_k\leq t} \dfrac{1}{1+b_k}, & (s,t] \cap \{t_k\} \neq \varnothing, \\ 1, & (s,t] \cap \{t_k\} = \varnothing, \end{cases}$$
$$\beta(s,t) = \min\{\alpha(u,t) : u \in (s,t]\},$$
$$\gamma(t) = \min\{\tau, t_k - t : t_k > t\},$$
$$\overline{b}_k = \max\{0, b_k\}.$$

Theorem 3.1 ([19]). *Assume that*

$$\limsup_{t\to\infty} \beta(t-\tau, t+\gamma(t)-\tau)\alpha(t+\gamma(t)-\tau, t)\int_t^{t+\gamma(t)} p(s)\mathrm{d}s > 1. \tag{3.2}$$

Then Eq. (3.1) is oscillatory.

Proof. Let x be a nonoscillatory solution. Without loss of generality, we can suppose that $x(t) > 0$ for $t \geq t^*$, then $x(t)$ is nonincreasing on intervals of the form $(t_k, t_{k+1}]$ for $t_k > t^* + \tau$. From (3.1), we have

$$x(t+\gamma(t)) - x(t^+) + \int_t^{t+\gamma(t)} p(s)x(s-\tau)\mathrm{d}s = 0. \tag{3.3}$$

Since x is nonincreasing, we obtain

$$\inf_{t-\tau<s<t+\gamma(t)-\tau} x(s) = \min\{x(t+\gamma(t)-\tau), x(t_k) : t_k \in (t-\tau, t+\gamma(t)-\tau]\}.$$

Let $t_k < t_{k+1} < \cdots < t_m$ be impulse points in $(t-\tau, t+\gamma(t)-\tau]$, then

$$\begin{aligned} x(t_m) &= \frac{1}{1+b_m}x(t_m^+) \geq \frac{1}{1+b_m}x((t+\gamma(t)-\tau)^+), \\ x(t_{m-1}) &= \frac{1}{1+b_{m-1}}x(t_{m-1}^+) \geq \frac{1}{1+b_{m-1}}x(t_m) \\ &\geq \frac{1}{1+b_{m-1}}\frac{1}{1+b_m}x((t+\gamma(t)-\tau)^+), \\ &\ldots, \\ x(t_k) &\geq \frac{1}{1+b_k}\frac{1}{1+b_{k+1}}\cdots\frac{1}{1+b_m}x((t+\gamma(t)-\tau)^+). \end{aligned}$$

Thus,

$$\inf_{t-\tau<s<t+\gamma(t)-\tau} x(s) \geq \beta(t-\tau, t+\gamma(t)-\tau)x((t+\gamma(t)-\tau)^+), \tag{3.4}$$

and analogously

$$x((t+\gamma(t)-\tau)^+) \geq \alpha(t+\gamma(t)-\tau, t)x(t^+). \tag{3.5}$$

From (3.3)–(3.5) we have

$$x(t+\gamma(t)) + x(t^+)\left[\beta(t-\tau, t+\gamma(t)-\tau)\alpha(t+\gamma(t)-\tau, t)\int_t^{t+\gamma(t)} p(s)\mathrm{d}s - 1\right] \leq 0,$$

which contradicts (3.2). The proof of Theorem 3.1 is complete. □

Corollary 3.1. *Assume that $t_{k+1} - t_k \geq T$, $k \in \mathbb{N}$, $\tau \geq T$ and the following condition holds*

$$\limsup_{k\to\infty} \beta(t_k - \tau, t_k + T - \tau)\alpha(t_k + T - \tau, t_k) \int_{t_k}^{t_k+T} p(s)\mathrm{d}s > 1.$$

Then Eq. (3.1) is oscillatory.

Corollary 3.2. *Assume that $t_{k+1} - t_k \geq \tau$, $k \in \mathbb{N}$, and the following condition holds*

$$\limsup_{k\to\infty} \frac{1}{1+b_k} \int_{t_k}^{t_k+\tau} p(s)\mathrm{d}s > 1.$$

Then Eq. (3.1) is oscillatory.

Corollary 3.3. *Assume that $t_{k+1} - t_k \geq 2\tau$, $k \in \mathbb{N}$, and the following condition holds*

$$\limsup_{k\to\infty} \int_{t_k+\tau}^{t_k+2\tau} p(s)\mathrm{d}s > 1.$$

Then Eq. (3.1) is oscillatory.

Remark 3.1. Corollary 3.1 is the modification of the first result of Theorem 3.1 in [11] and Corollary 3.2 is the second result of Theorem 3.1 in [11].

Theorem 3.2 (*[19]*). *Assume that*

$$\limsup_{t\to\infty} \prod_{t-\tau<t_k<t} (1+\bar{b}_k) < \infty, \tag{3.6}$$

and

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\mathrm{d}s > \frac{1}{\mathrm{e}} \limsup_{t\to\infty} \prod_{t-\tau<t_k<t} (1+b_k). \tag{3.7}$$

Then Eq. (3.1) is oscillatory.

Proof. Suppose that x is an eventually positive solution, say $x(t) > 0$ for $t \geq t^*$. Since $x(t)$ is nonincreasing on the interval $(t_k, t_{k+1}]$ for $t_k > t^* + \tau$, we have

$$x(t-\tau) \geq \prod_{t-\tau\leq t_k<t} \frac{1}{1+\bar{b}_k} x(t). \tag{3.8}$$

On the other hand, from (3.7) for sufficiently large t,

$$\int_{t-\tau}^{t} p(s)\mathrm{d}s \geq c > 0,$$

where c is a constant. So, there exists a sequence $\{T_n\}$ which satisfies $\lim_{n\to\infty} T_n = \infty$ such that

$$\int_{T_n-\tau/2}^{T_n} p(s)\mathrm{d}s \geq \frac{c}{2} \quad \text{and} \quad \int_{T_n}^{T_n+\tau/2} p(s)\mathrm{d}s \geq \frac{c}{2}.$$

Let $t_k < t_{k+1} < \cdots < t_m$ be impulse points in $(T_n - \tau/2, T_n)$. Integrating (3.1) on $[T_n - \tau/2, T_n]$, we find

$$x\left(\left(T_n - \frac{\tau}{2}\right)^+\right) + b_k x(t_k) + \cdots + b_m x(t_m) \geq \int_{T_n-\tau/2}^{T_n} p(s)x(s-\tau)\mathrm{d}s.$$

Notice that

$$x\left(\left(T_n - \frac{\tau}{2}\right)^+\right) + b_k x(t_k) + \cdots + b_m x(t_m) \leq \prod_{i=k}^{m}(1+\bar{b}_i) x\left(\left(T_n - \frac{\tau}{2}\right)^+\right),$$

and hence

$$\begin{aligned}\prod_{T_n-\tau/2<t_i<T_n}(1+\bar{b}_i)x\left((T_n-\frac{\tau}{2})^+\right) &\geq \int_{T_n-\tau/2}^{T_n} p(s)x(s-\tau)\mathrm{d}s \\ &\geq \inf_{T_n-3\tau/2<s<T_n-\tau} x(s)\int_{T_n-\tau/2}^{T_n} p(s)\mathrm{d}s \\ &\geq \frac{c}{2}\beta\left(T_n-\frac{3\tau}{2},T_n-\tau\right)x\left((T_n-\tau)^+\right) \\ &\geq \frac{c}{2}\prod_{T_n-3\tau/2<t_k<T_n-\tau}\frac{1}{1+\bar{b}_k}x\left((T_n-\tau)^+\right).\end{aligned}$$

Thus

$$x\left((T_n-\tau)^+\right)\leq\frac{2}{c}\prod_{T_n-3\tau/2<t_k\leq T_n-\tau}(1+\bar{b}_k)\prod_{T_n-\tau/2<t_k<T_n}(1+\bar{b}_k)x\left(\left(T_n-\frac{\tau}{2}\right)^+\right). \tag{3.9}$$

Similarly integrating (3.1) on $[T_n, T_n+\tau/2]$ and using (3.9), we get

$$x\left((T_n-\tau)^+\right)\leq\left(\frac{2}{c}\right)^2\prod_{T_n-3\tau/2<t_k\leq T_n-\tau/2}(1+\bar{b}_k)\prod_{T_n-\tau/2<t_k<T_n+\tau/2}(1+\bar{b}_k)x(T_n^+). \tag{3.10}$$

Let

$$\sigma=\liminf_{t\to\infty}\frac{x(t-\tau)}{x(t)}.$$

From (3.6), (3.8) and (3.10), σ is finite and positive. Now from (3.1) for sufficiently large t we have

$$\frac{x'(t)}{x(t)}+p(t)\frac{x(t-\tau)}{x(t)}=0.$$

Integrating the above equality over $[t-\tau, t]$ we obtain

$$\frac{x(t-\tau)}{x(t)}\prod_{t-\tau<t_k<t}(1+b_k)\geq e\inf_{t-\tau<s<t}\frac{x(s-\tau)}{x(s)}\int_{t-\tau}^{t}p(s)\mathrm{d}s,$$

which implies that

$$\frac{1}{\mathrm{e}}\limsup_{t\to\infty}\prod_{t-\tau<t_k<t}(1+b_k)\geq\liminf_{t\to\infty}\int_{t-\tau}^{t}p(s)\mathrm{d}s,$$

but this contradicts (3.7). The proof is complete. □

Remark 3.2. The above Theorem 3.2 includes Theorem 3.2 in [11] as a special case when $t_{i+1}-t_i>\tau$, $0\leq b_i\leq M$, $i\in\mathbb{N}$.

Corollary 3.4. *Assume that*

(i) $b_k\leq 0$, $k\in\mathbb{N}$;
(ii) $\liminf_{t\to\infty}\int_{t-\tau}^{t}p(s)\mathrm{d}s>\frac{1}{\mathrm{e}}$.

Then Eq. (3.1) *is oscillatory.*

In [19] the authors also considered the constant coefficient impulsive equation of the form

$$\begin{cases}x'(t)+ax(t-\tau)=0, & t\neq t_k,\\ x(t_k^+)-x(t_k)=b_kx(t_k), & k\in\mathbb{N},\end{cases} \tag{3.11}$$

and proved the following results.

Theorem 3.3 (*[19]*)**.** *Assume that*

(i) $a\tau e<1$;
(ii) $-1<b_i\leq 0$;
(iii) $\sum_{i=1}^{\infty}b_i>-\infty$.

Then Eq. (3.11) *has a nonoscillatory solution.*

Theorem 3.4 ([19]). *Assume that*

(i) $t_{i+1} - t_i \geq T > 0, i \in \mathbb{N}$;
(ii) $b_i \leq 0, i \in \mathbb{N}$ *and* $\lim_{i\to\infty} b_i = 0$;
(iii) $a\tau e < 1$.

Then Eq. (3.11) *has a nonoscillatory solution.*

Now we consider the impulsive linear differential equation of the type

$$\begin{cases} x'(t) + px(t-\tau) = 0, & t \neq t_k, \\ \Delta x(t_k) + p_0 x(t_k - \tau) = 0, \end{cases} \tag{3.12}$$

where $p > 0$, $0 < p_0 < 1$, $\tau > 0$. We assume that the following condition holds:
(H) There exists $m \in \mathbb{N}$ such that

$$i[t-\tau, t) \equiv m, \quad t \in \mathbb{R}.$$

When we look for a positive solution of Eq. (3.12) of the form

$$x(t) = \mathrm{e}^{-\lambda t}(1-\mu)^{i[0,t)}, \quad \lambda \in \mathbb{R},\ \mu < 1,$$

we obtain the following characteristic system

$$\begin{aligned} \lambda &= p\mathrm{e}^{\lambda\tau}(1-\mu)^{-m}, \\ \mu &= p_0 \mathrm{e}^{\lambda\tau}(1-\mu)^{-m}. \end{aligned} \tag{3.13}$$

It is easy to see that the solution (λ, μ) of (3.13) satisfies

$$\mu = \frac{p_0}{p}\lambda.$$

Moreover, the system (3.13) has a solution (λ, μ) with $\mu < 1$ if and only if the characteristic equation

$$H(\lambda) \equiv -\lambda + p\mathrm{e}^{\lambda t}\left(1 - \frac{p_0}{p}\right)^{-m} = 0 \tag{3.14}$$

has a solution $\lambda \in \left(0, \frac{p}{p_0}\right)$.

Theorem 3.5 ([12]). *If the condition* (H) *holds, then the following assertions are equivalent:*

(i) *The Eq.* (3.14) *has no solution* $\lambda \in \left(0, \frac{p}{p_0}\right)$;
(ii) *The characteristic system* (3.13) *has no solution* (λ, μ) *with* $\mu < 1$;
(iii) *Each regular solution of the Eq.* (3.12) *is oscillatory.*

In [23], authors obtained explicit necessary and sufficient conditions for the oscillation of Eq. (3.12) by applying Theorem 3.5.

Theorem 3.6. *Let the condition* (H) *be satisfied. Then every solution of Eq.* (3.12) *oscillates if and only if*

$$-\lambda_1(1-\lambda_1)^m \mathrm{e}^{-\frac{p}{p_0}\lambda_1\tau} + p_0 > 0,$$

where $\lambda_1 = \dfrac{\frac{p\tau}{p_0} + m + 1 - \sqrt{\left(\frac{p\tau}{p_0} + m + 1\right)^2 - \frac{4p\tau}{p_0}}}{\frac{2p\tau}{p_0}}$.

In [20], authors considered the impulsive delay differential equation of the form

$$\begin{cases} x'(t) + \displaystyle\sum_{i=1}^{n} p_i x(t-\tau_i) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in \mathbb{N}, \end{cases} \tag{3.15}$$

where $p_i > 0$, for $i = 1, 2, \ldots, n$; $0 < \tau_1 < \tau_2 < \cdots < \tau_n$; $b_k \in \mathbb{R}$ for $k \in \mathbb{N}$.
Let the following conditions (C) be satisfied:

(a) For the case $n = 1$, there exists a positive integer m such that for $j = 1, 2, \ldots, m$; $k \in \mathbb{N}$,

$$t_{km+j} = t_j + k\tau_1, \quad \text{and} \quad b_{km+j} = b_j;$$

(b) For the case $n > 1$, the quotients τ_i/τ_1 are rational numbers for $i = 2, 3, \dots, n$, that is, there exist positive integers q_i and r_i which are coprime such that $\tau_i/\tau_1 = q_i/r_i$, and there exists a positive integer m such that for $j = 1, 2, \dots, m$; $k \in \mathbb{N}$,

$$t_{km+j} = t_j + kT, \quad \text{and} \quad b_{km+j} = b_j,$$

where $T = \tau_1/r$ and r is the least common multiple of $r_1, r_2, \dots, r_n$.

A necessary and sufficient condition for the oscillation of (3.15) is that its characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{n} p_i \xi^{\alpha_i} e^{-\lambda\tau_i} = 0 \tag{3.16}$$

has no real roots, where

$$\xi = \prod_{k=1}^{m}(1 + b_k), \qquad \eta = \prod_{k=1}^{m}(1 + b_k^+), \qquad \alpha_i = \frac{rq_i}{r_i} \quad \text{for } i = 1, 2, \dots, n.$$

Theorem 3.7 ([20]). *Assume that conditions* (C) *hold. Then the following statements are equivalent:*

(a) *Eq.* (3.15) *has a nonoscillatory solution.*
(b) *The characteristic Eq.* (3.16) *has a real root.*

From Theorems 2.1 and 3.7 the following result is immediate.

Corollary 3.5. *Assume that conditions* (C) *hold. Then each of the following conditions is sufficient for the oscillation of all solutions of* (3.15)*:*

(a) $\sum_{i=1}^{n} p_i \xi^{\alpha_i}\tau_i > \frac{1}{e}$;
(b) $\left(\prod_{i=1}^{n} p_i \xi^{\alpha_i}\right)^{1/n} \left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}$.

Similarly, from Theorems 2.2 and 3.7 the following result follows.

Corollary 3.6. *Assume that conditions* (C) *hold. Then,*

$$\left(\sum_{i=1}^{n} p_i \xi^{\alpha_i}\right) \left(\max_{1\le i\le n} \tau_i\right) \le \frac{1}{e}$$

is sufficient for the existence of a nonoscillatory solution of (3.15).

The impulsive delay differential equation of the form

$$\begin{cases} y'(t) + a(t)y(t) + p(t)y(t-\tau) = 0, & t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in \mathbb{N} \end{cases} \tag{3.17}$$

was examined in [26,27]. In [27], authors also considered the impulsive delay differential inequalities

$$\begin{cases} y'(t) + a(t)y(t) + p(t)y(t-\tau) \le 0, & t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in \mathbb{N}, \end{cases} \tag{3.18}$$

$$\begin{cases} y'(t) + a(t)y(t) + p(t)y(t-\tau) \ge 0, & t \ne t_k, \\ y(t_k^+) - y(t_k) = b_k y(t_k), & k \in \mathbb{N}, \end{cases} \tag{3.19}$$

where

(A_1) $a, p : J \to \mathbb{R}$ are locally summable functions, τ is a positive constant,
(A_2) $b_k > -1$ are constants for $k \in \mathbb{N}$.

They also dealt with the following delay differential equation and inequalities

$$x'(t) + a(t)x(t) + P(t)x(t-\tau) = 0, \quad \text{a.e. } t \ge t_0 + \tau, \tag{3.17$'$}$$

$$x'(t) + a(t)x(t) + P(t)x(t-\tau) \le 0, \quad \text{a.e. } t \ge t_0 + \tau, \tag{3.18$'$}$$

$$x'(t) + a(t)x(t) + P(t)x(t-\tau) \ge 0, \quad \text{a.e. } t \ge t_0 + \tau, \tag{3.19$'$}$$

where $P(t) = \prod_{t-\tau\le t_k<t}(1 + b_k)^{-1}p(t)$, $t \ge t_0 + \tau$, a, p, τ, and $\{b_k\}$ satisfy the same as (A_1)–(A_2).

Theorem 3.8 (*[27]*). *Assume that* (A_1)–(A_2) *hold. Then the following hold:*

(i) *Inequality* (3.18) *has no eventually positive solutions if and only if* $(3.18')$ *has no eventually positive solutions;*
(ii) *Inequality* (3.19) *has no eventually negative solutions if and only if* $(3.19')$ *has no eventually negative solutions;*
(iii) *All solutions of* (3.17) *are oscillatory if and only if all solutions of* $(3.17')$ *are oscillatory.*

Proof. Since (ii) and (iii) follow from (i), it is sufficient to prove (i).

Let $y(t)$ be an eventually positive solution of (3.18). Then there exists a $T \geq 0$ such that $y(t) > 0$ and $y(t-\tau) > 0$, $t \geq T$. Set $x(t) = \prod_{T \leq t_k < t}(1+b_k)^{-1}y(t)$. Clearly $x(t) > 0, x(t-\tau) > 0$ for $t \geq T$. Since $y(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, in view of $y(t_k^+) = (1+b_k)y(t_k)$, it follows for $t \geq T$

$$x(t_k^+) = \prod_{T \leq t_j \leq t_k} (1+b_j)^{-1} y(t_k^+) = x(t_k),$$

and for all $t_k \geq T$

$$x(t_k^-) = \prod_{T \leq t_j \leq t_{k-1}} (1+b_j)^{-1} y(t_k^-) = x(t_k).$$

Moreover, we have

$$\begin{aligned} x'(t) + a(t)x(t) + P(t)x(t-\tau) &= \prod_{T \leq t_k < t}(1+b_k)^{-1}y'(t) + a(t)\prod_{T \leq t_k < t}(1+b_k)^{-1}y(t) \\ &\quad + P(t)\prod_{T \leq t_k < t-\tau}(1+b_k)^{-1}y(t-\tau) \\ &= \prod_{T \leq t_k < t}(1+b_k)^{-1}(y'(t) + a(t)y(t) + p(t)y(t-\tau)) \\ &\leq 0, \end{aligned}$$

which implies that $x(t)$ is a positive solution of $(3.18')$.

Conversely, let $x(t)$ be an eventually positive solution of $(3.18')$ and $x(t) > 0$ and $x(t-\tau) > 0$ for $t \geq T \geq t_0$. Set $y(t) = \prod_{T \leq t_k < t}(1+b_k)x(t)$. For every $t_k \geq T$, we have

$$y(t_k^+) = \prod_{T \leq t_j \leq t_k}(1+b_j)x(t_k) \quad \text{and} \quad y(t_k) = \prod_{T \leq t_j < t_k}(1+b_j)x(t_k).$$

Thus for every $t_k \geq T, k \in \mathbb{N}$, we find $y(t_k^+) = (1+b_k)y(t_k)$. On the other hand since $x(t)$ is absolutely continuous on $[T, \infty)$, $y(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, $t_k \geq T$ and for almost everywhere $t \in [\sigma, \infty)$,

$$\begin{aligned} y'(t) + a(t)y(t) + p(t)y(t-\tau) &= \prod_{T \leq t_k < t}(1+b_k)x'(t) + a(t)\prod_{T \leq t_k < t}(1+b_k)x(t) + p(t)\prod_{T \leq t_k < t-\tau}(1+b_k)x(t-\tau) \\ &= \prod_{T \leq t_k < t}(1+b_k)(x'(t) + a(t)x(t) + P(t)x(t-\tau)) \\ &\leq 0. \end{aligned}$$

So, $y(t)$ is a positive solution of (3.18). The proof of Theorem 3.8 is complete. □

The following theorem improves and generalizes Theorem 3.2 in [11] and Theorem 5 in [26].

Theorem 3.9 (*[27]*). *Assume that* (A_1)–(A_2) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \prod_{s-\tau \leq t_k < s}(1+b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\mathrm{d}\sigma\right) \mathrm{d}s > \frac{1}{\mathrm{e}},$$

then Eq. (3.17) *is oscillatory.*

Theorem 3.10 (*[27]*). *Assume that* (A_1)–(A_2) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} \prod_{s-\tau \leq t_k < s}(1+b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\mathrm{d}\sigma\right) \mathrm{d}s > 1,$$

then Eq. (3.17) *is oscillatory.*

Theorem 3.10 generalizes Corollaries 3.2 and 3.3.

Now we introduce the following conditions:

(A_3) There exists an integer m such that $m(t_{k+1} - t_k) \geq \tau$ for all $k \in \mathbb{N}$;
(A_4) There exists a constant $M > 0$ such that $0 \leq b_k \leq M$ for all $k \in \mathbb{N}$.

The following two results are corollaries of Theorems 3.9 and 3.10, respectively.

Corollary 3.7. *Assume that* (A_1), (A_3), (A_4) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\mathrm{d}\sigma\right) \mathrm{d}s > \frac{(1+M)^m}{e},$$

then Eq. (3.17) *is oscillatory.*

Corollary 3.8. *Assume that* (A_1), (A_3), (A_4) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\mathrm{d}\sigma\right) \mathrm{d}s > (1+M)^m,$$

then Eq. (3.17) *is oscillatory.*

The following result provides a sufficient condition for the existence of a nonoscillatory solution of (3.17).

Theorem 3.11 ([27]). *Assume that* (A_1)–(A_2) *hold and* $p(t) \geq 0$ *for* $t \geq t_0$. *If there exists* $T \geq t_0$ *such that for all* $t \geq T$

$$\int_{t-\tau}^{t} \prod_{s-\tau\leq t_k<s} (1+b_k)^{-1} p(s) \exp\left(\int_{s-\tau}^{s} a(\sigma)\mathrm{d}\sigma\right) \mathrm{d}s \leq \frac{1}{\mathrm{e}},$$

then (3.17) *has a nonoscillatory solution on* $[T, \infty)$.

Impulsive differential equations with variable delays were dealt in [28–32]. Particularly, in [32], authors considered the impulsive delay differential equations

$$\begin{cases} x'(t) + p(t)x(g(t)) = 0, & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), x(g(t_k))), & k \in \mathbb{N}, \end{cases} \tag{3.20}$$

and

$$\begin{cases} x'(t) + q(t)x(t) + p(t)x(g(t)) = 0, & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), x(g(t_k))), & k \in \mathbb{N}. \end{cases} \tag{3.21}$$

We need the following conditions:

(H_1) $p \in PLC(\mathbb{R}^+, \mathbb{R}^+)$;
(H_2) $g \in C(\mathbb{R}^+, \mathbb{R}), g(t) < t, g'(t) \geq 0, \lim_{t\to\infty} g(t) = \infty$;
(H_3) $I_k \in C(\mathbb{R}^2, \mathbb{R}), uI_k(u, v) < 0$ for $uv > 0, u, v \in \mathbb{R}, k \in \mathbb{N}$;
(H_4) $q \in PLC(\mathbb{R}^+, \mathbb{R})$.

Theorem 3.12 ([32]). *Let the following hold:*

(i) *Conditions* (H_1)–(H_3) *are satisfied;*
(ii) *There exist constants* $M_k (0 < M_k < 1)$ *such that* $|I_k(u, v)| \geq M_k |u|$ *for* $u \neq 0, v \in \mathbb{R}, k \in \mathbb{N}$ *and*

$$\limsup_{t\to\infty} \prod_{g(t)\leq t_k<t} (1 - M_k) > 0;$$

(iii)

$$\liminf_{t\to\infty} \int_{g(t)}^{t} p(s)\mathrm{d}s > \frac{1}{\mathrm{e}} \limsup_{t\to\infty} \prod_{g(t)\leq t_k<t} (1 - M_k).$$

Then Eq. (3.20) *is oscillatory.*

Proof. Without loss of generality we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq t_0$. Denote

$$w(t) = \frac{x(g(t))}{x(t)}, \quad t \geq t_0.$$

From (3.20), (H_1) and (H_3), it follows that $x'(t) \leq 0$ and $\Delta x(t_k) \leq 0$ for $t, t_k \geq t_0$. Thus $x(t)$ is a nonincreasing function on $[t_0, \infty)$ and $w(t) \geq 1$ for $t \geq t_0$. We shall prove that the function w is bounded above for $t \geq t_0$. Denote

$$\limsup_{t\to\infty} \prod_{g(t)\leq t_k<t} (1 - M_k) = L.$$

From (iii), there exists $t^* \in (g(t), t)$ such that

$$\int_{g(t)}^{t^*} p(s)\mathrm{d}s \geq \frac{L}{2e} \quad \text{and} \quad \int_{t^*}^{t} p(s)\mathrm{d}s > \frac{L}{2e}.$$

Integrating (3.20) from t^* to t, we obtain

$$x(t^*) \geq -\sum_{t^*\leq t_k<t} I_k(x(t_k), x(g(t_k))) + x(g(t)) \int_{t^*}^{t} p(s)\mathrm{d}s > x(g(t))\frac{L}{2e}. \tag{3.22}$$

Similarly, integrating (3.20) from $g(t)$ to t^*, we obtain

$$x(g(t)) \geq x(g(t^*))\frac{L}{2e}.$$

From the above inequality and (3.22) it follows that

$$x(t^*) > x(g(t^*))\left(\frac{L}{2e}\right)^2.$$

Therefore the function w is bounded from above for $t \geq t_0$. Dividing (3.20) by $x(t)$ and integrating from $g(t)$ to t, we obtain from (ii)

$$\ln\left[w(t) \prod_{g(t)\leq t_k<t} (1 - M_k)\right] \geq w_0 \int_{g(t)}^{t} p(s)\mathrm{d}s,$$

where $w_0 = \liminf_{t\to\infty} w(t)$. It is clear that $1 \leq w_0 < \infty$. Thus

$$\liminf_{t\to\infty} \int_{g(t)}^{t} p(s)\mathrm{d}s \leq \frac{1}{\mathrm{e}} \limsup_{t\to\infty} \prod_{g(t)\leq t_k<t} (1 - M_k),$$

which contradicts (iii). The proof is complete. □

Theorem 3.13 ([32]). *Let the following hold:*

(i) *Conditions* (H_1)–(H_3) *are satisfied;*
(ii) *There exist constants* $L_k > 0$ *such that* $|I_k(u, v)| \geq L_k\,|v|$ *for* $uv > 0$, $u, v \in \mathbb{R}$, $k \in \mathbb{N}$*;*
(iii)

$$\limsup_{t\to\infty}\left[\int_{g(t)}^{t} p(s)\mathrm{d}s + \sum_{g(t)\leq t_k<t} L_k\right] > 1.$$

Then Eq. (3.20) *is oscillatory.*

Theorem 3.14 ([32]). *Let the following hold:*

(i) *Conditions* (H_1)–(H_4) *and condition* (ii) *of Theorem* 3.12 *are satisfied;*
(ii)

$$\liminf_{t\to\infty} \int_{g(t)}^{t} p(s)\mathrm{e}^{\int_{g(s)}^{s} q(u)\mathrm{d}u}\mathrm{d}s > \frac{1}{\mathrm{e}} \limsup_{t\to\infty} \prod_{g(t)\leq t_k<t} (1 - M_k).$$

Then Eq. (3.21) *is oscillatory.*

Theorem 3.15 ([32]). *Let the following hold:*

(i) *Conditions* (H_1)–(H_4) *and condition* (ii) *of Theorem 3.13 are satisfied;*
(ii)

$$\limsup_{t\to\infty}\left[\int_{g(t)}^{t} p(s)e^{\int_{g(s)}^{s} q(u)du}ds + \sum_{g(t)\le t_k<t} L_k e^{\int_{g(t_k)}^{t_k} q(u)du}\right] > 1.$$

Then Eq. (3.21) *is oscillatory.*

In [32], the authors also considered the nonhomogeneous impulsive differential equation

$$\begin{cases} x'(t) + p(t)x(g(t)) = b(t), & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), x(g(t_k))), & k \in \mathbb{N}, \end{cases} \tag{3.23}$$

with the following conditions:

(H_5) There exists a function $w \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that $w'(t) = b(t)$;
(H_6) There exist two sequences $\{t_k'\}_{k=1}^{\infty}, \{t_k''\}_{k=1}^{\infty} \subset \mathbb{R}^+$ and two constants $q_1 > 0, q_2 > 0$ such that $\lim_{k\to\infty} t_k' = \lim_{k\to\infty} t_k'' = \infty$, $w(t_k') = q_1 \le w(t) \le q_2 = w(t_k'')$, $k \in \mathbb{N}$, $t \in \mathbb{R}^+$.

Theorem 3.16 ([32]). *Assume that conditions* (H_5)–(H_6) *and hypotheses of Theorem 3.12 are satisfied. Then Eq.* (3.23) *is oscillatory.*

Theorem 3.17 ([32]). *Assume that conditions* (H_5)–(H_6) *and hypotheses of Theorem 3.13 are satisfied. Then Eq.* (3.23) *is oscillatory.*

The Sturmian Comparison Theory for the impulsive differential equation with deviating arguments

$$\begin{cases} x'(t) + \sum_{i=1}^{m} a_i(t)x[r_i(t)] = 0, & t \neq t_k, \\ x(t_k^+) = \alpha_k x(t_k^-), & k \in \mathbb{N} \end{cases}$$

was investigated in [33]. For these results we also refer to the book of Bainov and Simeonov [12].

In [34], authors considered the impulsive differential equation with a distributed delay

$$\begin{cases} y'(t) + \int_{-\infty}^{t} y(s)d_sR(t,s) = 0, t > t_0, & t \neq \tau_j, \\ y(\tau_j + 0) = B_j y(\tau_j), & j \in \mathbb{N}, \end{cases} \tag{3.24}$$

with the initial function

$$y(t) = \varphi(t), \quad t < t_0, \tag{3.25}$$

under the following assumptions.

(a1) $R(t,.)$ is a left continuous function of bounded variation and for each s its variation on the segment $[t_0, s]$, $P(t,s) = \mathrm{var}_{[t_0,s]}R(t,.)$ is a locally integrable function in t.
(a2) $R(t,s) = R(t,t)$, $t \le s$.
(a3) $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.
(a4) For each t_1, there exists $s_1 = s(t_1) \le t_1$ such that $R(t,s) = 0$ for $s < s_1$, $t > t_1$ and $\lim_{t\to\infty} s(t) = \infty$.
(b1) $t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ satisfy $\lim_{j\to\infty} \tau_j = \infty$.
(b2) $B_j > 0, j \in \mathbb{N}$.

They also considered the nonimpulsive differential equation

$$x'(t) + \int_{-\infty}^{t} x(s)d_sT(t,s) = 0, \quad t > t_0, \tag{3.26}$$

where

$$T(t,s) = \prod_{s\le\tau_j<t} B_j^{-1}R(t,s).$$

Theorem 3.18 ([34]). *Suppose* (a1)–(a4) *and* (b1)–(b2) *hold. Then Eqs.* (3.24)–(3.25) *is oscillatory (nonoscillatory) if and only if* (3.26)–(3.25) *is oscillatory (nonoscillatory).*

Proof. Let y be a solution of (3.24)–(3.25). Then $x(t) = \prod_{t_0 \le \tau_j < t} B_j^{-1} y(t)$ is continuous and $y(t) = \prod_{t_0 \le \tau_j < t} B_j x(t)$. From (3.24), we obtain

$$x'(t) + \int_{t_0}^{t} x(s) \prod_{s \le \tau_j < t} B_j^{-1} d_s R(t,s) = x'(t) + \int_{-\infty}^{t} x(s) d_s T(t,s) = 0.$$

Conversely, if $x(t)$ is a solution of (3.26)–(3.25), then $y(t) = \prod_{t_0 \le \tau_j < t} B_j x(t)$ is a solution of (3.24)–(3.25). Since $B_j > 0$, it is clear that x and y are oscillatory (nonoscillatory) at the same time, which completes the proof. □

Corollary 3.9. *Let a_k be locally essentially bounded functions, $h_k(t)$ be Lebesgue measurable functions, and $h_k(t) \le t$, $k \in \mathbb{N}$, $\lim_{t\to\infty} h_k(t) = \infty$. Then the equation*

$$\begin{cases} y'(t) + \sum_{k=1}^{m} a_k(t) y(h_k(t)) = 0, & t \ne \tau_j, \\ y(\tau_j + 0) = B_j y(\tau_j), & j \in \mathbb{N}, \end{cases}$$

is oscillatory (nonoscillatory) if and only if

$$x'(t) + \sum_{k=1}^{m} a_k(t) \prod_{h_k(t) \le \tau_j < t} B_j^{-1} x(h_k(t)) = 0$$

is oscillatory (nonoscillatory).

Oscillatory properties of the following impulsive delay differential equation with continuously distributed type deviating arguments

$$\begin{cases} x'(t) + a(t)x(t) + b(t)x(t-\tau) + \int_{t-\delta}^{t} b(t,s)x(s)\mathrm{d}s, & t \ne t_k,\ t \ge 0, \\ x(t_k^+) - x(t_k^-) = b_k x(t_k^-), & k \in \mathbb{N}, \end{cases}$$

have been addressed in [35].

Now we consider the impulsive delay differential equation of the type

$$x'(t) + a(t)x(t) + b(t)x([t-1]) = 0, \quad t \ne n, \tag{3.27}$$

$$x(n^+) - x(n^-) = d_n x(n), \quad n \in \mathbb{N} \cup \{0\}, \tag{3.28}$$

where $a, b : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $d_n \in \mathbb{R} - \{1\}$, $n \in \mathbb{N} \cup \{0\}$, $x(n^+) = \lim_{t\to n^+} x(t)$, $x(n^-) = \lim_{t\to n^-} x(t)$, and $[\cdot]$ denotes the greatest integer function. Recently, in [36], the authors have obtained some results on the oscillation, nonoscillation and periodicity of the solutions of Eqs. (3.27)–(3.28).

Definition 3.1. The function $x : \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ is a solution of (3.27)–(3.28) provided:

(i) $x(t)$ is continuous on $\mathbb{R}^+$ with the possible exception of the points $[t] \in \mathbb{R}^+$;
(ii) $x(t)$ is right continuous and has a left-hand limit at the points $[t] \in \mathbb{R}^+$;
(iii) $x(t)$ is differentiable and satisfies (3.27) for any $t \in \mathbb{R}^+$, with the possible exception of the points $[t] \in \mathbb{R}^+$ where one-sided derivatives exist;
(iv) $x(n)$ satisfies (3.28) for $n \in \mathbb{N} \cup \{0\}$.

The following interesting result gives the existence and uniqueness of the solutions.

Theorem 3.19 (*[36]*). *For any fixed $x_0, x_{-1} \in \mathbb{R}$ the Eq. (3.27)–(3.28) has a unique solution $x : \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ satisfying the initial conditions $x(-1) = x_{-1}$, $x(0) = x_0$. Moreover for $n \le t < n+1$, $n \in \mathbb{N}$, x has the form*

$$x(t) = \exp\left(-\int_n^t a(s)\mathrm{d}s\right)\left[y(n) - y(n-1)\int_n^t b(u)\exp\left(\int_n^u a(s)\mathrm{d}s\right)\mathrm{d}u\right],$$

where $y(n) = x(n)$ and the sequence $\{y(n)\}_{n\ge -1}$ is the unique solution of the difference equation

$$y(n+1) = \frac{1}{1-d_{n+1}} \exp\left(-\int_n^t a(s)\mathrm{d}s\right)\left[y(n) - y(n-1)\int_n^{n+1} b(u)\exp\left(\int_n^u a(s)\mathrm{d}s\right)\mathrm{d}u\right] \tag{3.29}$$

with the initial conditions

$$y(-1) = x_{-1} \quad \textit{and} \quad y(0) = x_0. \tag{3.30}$$

Remark 3.3. If $a(t) \equiv a$, $b(t) \equiv b$ and $d_n = d$ for $n \in \mathbb{N} \cup \{0\}$; a, b, d are real constants, $d \neq 1$, then we get the constant coefficient difference equation

$$(1-d)y(n+1) - e^{-a}y(n) + \frac{b}{a}(1-e^{-a})y(n-1) = 0, \quad n \geq 0. \tag{3.31}$$

If we look for a solution of (3.31) of the form $y(n) = k\lambda^n$, we find the characteristic equation

$$(1-d)\lambda^2 - e^{-a}\lambda + \frac{b}{a}(1-e^{-a}) = 0. \tag{3.32}$$

Assume that the roots of (3.32), λ_1 and λ_2, are different. Then the solution of (3.31)–(3.30) is

$$y(n) = \frac{1}{\lambda_1 - \lambda_2}\left[\lambda_1^{n+1}(x_0 - \lambda_2 x_{-1}) - \lambda_2^{n+1}(x_0 - \lambda_1 x_{-1})\right].$$

If $\lambda_1 = \lambda_2$, then

$$y(n) = \lambda_1^n[x_0(n+1) - \lambda_1 x_{-1} n].$$

Theorem 3.20 ([36]). *Let $x : \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ be a solution of the problem (3.27)–(3.28) with $x(-1) = x_{-1}$ and $x(0) = x_0$. Then the following hold:*

(α) *If the solution $\{y_n\}_{n\geq -1}$ of (3.29)–(3.30) is oscillatory, then x is also oscillatory;*
(β) *When the solution $\{y_n\}_{n\geq -1}$ of (3.29)–(3.30) is nonoscillatory, then x is nonoscillatory iff there exists a $N' \in \mathbb{N}$ such that*

$$\frac{y(n)}{y(n-1)} > \int_n^t b(u)\exp\left(\int_n^u a(s)ds\right)du, \quad n \leq t < n+1, n > N'.$$

Theorem 3.21 ([36]). *If $b(t) > 0$ and*

$$\limsup_{n\to\infty}(1-d_n)\int_n^{n+1} b(t)\exp\left(\int_{n-1}^t a(s)ds\right)dt > 1, \quad t \geq 0,$$

then all solutions of (3.29) are oscillatory.

Corollary 3.10. *Under the hypotheses of Theorem 3.21, all solutions of (3.27)–(3.28) are oscillatory.*

Theorem 3.22 ([36]). *If $1 - d_n > M > 0$, $n \in \mathbb{N} \cup \{0\}$, and*

$$\liminf_{n\to\infty}\exp\left(\int_n^{n+1} a(s)ds\right)\liminf_{n\to\infty}\int_n^{n+1} b(u)\exp\left(\int_n^u a(s)ds\right)du > \frac{1}{4M}, \tag{3.33}$$

then all solutions of (3.29) are oscillatory.

Corollary 3.11. *Under the hypotheses of Theorem 3.22, all solutions of (3.27)–(3.28) are oscillatory.*

Now consider the equation

$$\begin{cases} x'(t) + ax(t) + bx([t-1]) = 0, & t \neq n, \\ x(n^+) - x(n^-) = dx(n), & n \in \mathbb{N} \cup \{0\} \end{cases} \tag{3.34}$$

where a, b, d are real constants, $a \neq 0$, $b \neq 0$, $d \neq 1$.

Remark 3.4. If a, b and d are real constants, then condition (3.33) reduces to

$$b > \frac{ae^{-a}}{4M(e^a - 1)}. \tag{3.35}$$

If $d = 0$, then $0 < M < 1$ and condition (3.35) reduces to

$$b > \frac{ae^{-a}}{4(e^a - 1)},$$

which is a sharp condition for the corresponding nonimpulsive equation as stated in [37].

Corollary 3.12. *If $1 - d > M > 0$ and (3.35) is satisfied, then all solutions of (3.34) are oscillatory.*

Let λ_1 and λ_2 be the roots of (3.32), and when they are different we assume that $\lambda_2 < \lambda_1$.

Theorem 3.23 ([36]). *If any of the following hypotheses is satisfied, then Eq. (3.34) has nonoscillatory solutions:*

(i) $b < 0, d < 1$ *and* $x_0 - \lambda_2 x_{-1} \neq 0$;
(ii) $0 < b < ae^{-a}/4M(e^a - 1)$ *and* $0 < 1 - d < M$;
(iii) $b = ae^{-a}/4M(e^a - 1)$ *and* $1 - d = M > 0$.

Theorem 3.24 ([36]). *If* $b < 0, d < 1$ *and* $x_0 - \lambda_2 x_{-1} = 0$, *then every solution of* (3.31) *is oscillatory.*

Corollary 3.13. *Every solution of* (3.34) *is oscillatory iff either*

(i) $b > ae^{-a}/4M(e^a - 1)$ *and* $1 - d > M > 0$; *or*
(ii) $b < 0, d < 1$ *and* $x_0 = \lambda_2 x_{-1}$.

The following theorems provide sufficient conditions for the existence of periodic solutions.

Theorem 3.25 ([36]). *A necessary and sufficient condition for the solution of the Eq.* (3.34) *with the conditions* $x(-1) = x_{-1}, x(0) = x_0$ *to be periodic with period* $k, k \in \mathbb{N}$, *is* $y(k) = y(0)$ *and* $y(k-1) = y(-1)$, *where* $\{y(n)\}_{n \geq -1}$ *is the solution of* (3.31) *with the initial conditions* $y(0) = x_0, y(-1) = x_{-1}$.

Theorem 3.26 ([36]). *Assume that* $b > 0$ *and* $1 - d > M > 0$. *A necessary and sufficient condition for every oscillatory solution of* (3.34) *to be periodic with period* k *is*

$$b = \frac{ae^a(1-d)}{e^a - 1} \quad \text{and} \quad a = -\ln\left(2(1-d)\cos\frac{2\pi m}{k}\right),$$

where m *and* k *are relatively prime and* $m = 1, 2, \ldots, [(k-1)/4]$.

Theorem 3.27 ([36]). *Assume that* $b < 0, d < 1$ *and* $x_0 = \lambda_2 x_{-1}$. *A necessary and sufficient condition for every oscillatory solution of* (3.34) *to be periodic with period 2 is*

$$b = -\frac{a(1 + (1-d)e^a)}{e^a - 1}.$$

Example 3.1. Let us consider the impulsive differential equation with piecewise constant argument

$$\begin{cases} x'(t) + (\ln 2)x(t) - 2(\ln 2)x([t-1]) = 0, & t \neq n, \\ x(n^+) - x(n^-) = \dfrac{1}{2}x(n), & n \in \mathbb{N} \cup \{0\}, \\ x(0) = x_0, \qquad x(-1) = -x_0. \end{cases} \tag{3.36}$$

Here, the hypothesis (ii) of Corollary 3.13 holds. So, the solution of (3.36) is oscillatory. Moreover, since the conditions of Theorem 3.27 are satisfied, this oscillatory solution is periodic with period 2.

Example 3.2. Now consider the following equation

$$\begin{cases} x'(t) + (\ln 3)x(t) + \dfrac{\sqrt{2}}{4}(\ln 3)x([t-1]) = 0, & t \neq n, \\ x(n^+) - x(n^-) = \dfrac{6-\sqrt{2}}{6}x(t), & n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{3.37}$$

For this equation, the hypothesis (i) of Corollary 3.13 is satisfied, and for $k = 8$ the conditions of Theorem 3.26 are satisfied. Hence, every oscillatory solution of (3.37) is periodic with period 8.

3.2. Second and higher order equations

Oscillation of second order linear impulsive delay differential equations has been investigated by many authors [34,38–44]. First, we consider the following equation

$$\begin{cases} (r(t)y'(t))' - \displaystyle\sum_{i=1}^{n} p_i(t)y(t-h_i) = 0, & t \neq \tau_k, k \in \mathbb{N}, \\ \Delta y'(\tau_k) = y'(\tau_k + 0) - y'(\tau_k - 0) = \beta_k y(\tau_k), \\ \Delta y(\tau_k) = y(\tau_k^+) - y(\tau_k^-) = 0, \end{cases} \tag{3.38}$$

with the initial conditions

$$y(t) = \varphi(t), \quad t \in [-h, 0],\ h = \max\{h_i : i \in \mathbb{N}_n\},$$
$$y'(0) = \varphi'(0) = y'_0.$$

Here, $\mathbb{N}_n = \{1, 2, \ldots, n\}$; $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of positive numbers; h_i, $i \in \mathbb{N}_n$, are positive constants; $y'(\tau_k^-) = y'(\tau_k)$; $r \in PLC(\mathbb{R}^+, \mathbb{R}^+ - \{0\})$, $r(\tau_k^+) > 0$, $k \in \mathbb{N}$; $p_i \in PLC(\mathbb{R}^+, \mathbb{R}^+ - \{0\})$, $i \in \mathbb{N}_n$; $\varphi \in C^2([-h, 0], \mathbb{R})$.

The following results provide sufficient conditions for the oscillation of bounded solutions of Eq. (3.38).

Theorem 3.28 ([38]). *Let the following conditions hold:*

(i) $\lim_{t\to+\infty} R(t) = +\infty$, *where* $R(t) = \int_0^t \frac{\mathrm{d}s}{r(s)}$;
(ii) $\int^{\infty} R(s) \sum_{i=1}^n p_i(s)\mathrm{d}s = +\infty$.

Then all bounded solutions of Eq. (3.38) *either tend to zero as* $t \to +\infty$ *or oscillate.*

Theorem 3.29 ([38]). *Let the following conditions hold:*

(i) $\int_0^{\infty} \frac{\mathrm{d}t}{r(t)} = +\infty$;
(ii) $\limsup_{t\to+\infty} \frac{1}{r(t)} \int_{t-\bar{h}}^t (s - t + \bar{h}) \sum_{i=1}^n p_i(s)\mathrm{d}s > 1$,

where $\bar{h} = \min\{h_i : i \in \mathbb{N}_n\}$. *Then all bounded nontrivial solutions of Eq.* (3.38) *are oscillatory.*

In [39], the authors considered the impulsive delay differential equation

$$\begin{cases} x''(t) + \displaystyle\sum_{k=1}^m a_k(t)x(g_k(t)) = 0, \quad t \geq 0, \\ x(\tau_j) = A_j x(\tau_j^-), \qquad x'(\tau_j) = B_j x'(\tau_j^-), \quad j \in \mathbb{N}, \end{cases} \tag{3.39}$$

under the following conditions:

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \cdots$ are fixed points, and $\lim_{j\to\infty} \tau_j = \infty$;
(a2) $a_k,\ k = 1, 2, \ldots, m$, are Lebesgue measurable and locally essentially bounded functions on $\mathbb{R}^+$, $A_j, B_j \in \mathbb{R}$, $j = 1, 2, \ldots$;
(a3) $g_k : \mathbb{R}^+ \to \mathbb{R}$ are Lebesgue measurable functions, $g_k(t) \leq t$, $\lim_{t\to\infty} g_k(t) = \infty,\ k = 1, 2, \ldots, m$.

They studied nonoscillation of the Eq. (3.39) and the corresponding differential inequality, positiveness of the fundamental function, existence of a solution of a generalized Riccati inequality, comparison theorems; and obtained the following explicit conditions for the nonoscillation and oscillation. In these results $A_j > 0$, $B_j > 0$, and $a^+ = \max\{a, 0\}$.

Theorem 3.30. *Suppose for some* $t_0 > 0$, $0 < q < 1$, $r > -1$, $m > 0$, $M > 0$ *at least one of the following conditions holds:*

(i) $\sup_{t\geq t_0} \left|\prod_{t_0<\tau_j\leq t} B_j/A_j - 1\right| \leq q$, $\sup_{t\geq t_0} t^2 \sum_{k=1}^m \prod_{g_k(t)<\tau_j\leq t} A_j^{-1} a_k^+(t) \left[\frac{g_k(t)}{t}\right]^{(1-q)/2} \leq \frac{(1-q)^2}{4}$;
(ii) $mt^r \leq \sup_{t\geq t_0} \prod_{t_0<\tau_j\leq t} B_j/A_j \leq Mt^r$, $\sup_{t\geq t_0} t^2 \sum_{k=1}^m \prod_{g_k(t)<\tau_j\leq t} A_j^{-1} a_k^+(t) \leq \frac{m(1+r)^2}{4M}$.

Then Eq. (3.39) *has a positive solution for* $t > t_0$ *with a nonnegative derivative.*

Corollary 3.14. *Suppose*

$$a_k(t) \leq 0, \qquad mt^r \leq \sup_{t\geq t_0} \prod_{t_0<\tau_j\leq t} B_j/A_j \leq Mt^r,$$
$$r > -1,\ m > 0,\ M > 0.$$

Then Eq. (3.39) *has a positive solution for* $t > t_0$ *with a nonnegative derivative.*

Example 3.3. Consider the impulsive delay differential equation

$$\begin{cases} x''(t) + \dfrac{1}{2t^2} x(t - \delta) = 0, \quad t \neq j, \\ x(j) = \dfrac{j}{j+1} x(j^-), \qquad x'(j) = x'(j^-), \quad j \in \mathbb{N}. \end{cases} \tag{3.40}$$

Since

$$t \leq \prod_{t_0<\tau_j\leq t} B_j/A_j \leq t + 1 \leq t\left(1 + \frac{1}{t_0}\right),$$

Eq. (3.40) satisfies condition (ii) of Theorem 3.30 for $r = 1$, $m = 1$, $M = 1 + 1/t_0$. So, Eq. (3.40) is nonoscillatory.

Now let

$$b(t) = \sum_{k=1}^{m} \prod_{\tau_j \le t} A_j/B_j \prod_{g_k(t)<\tau_j\le t} A_j^{-1} a_k^{+}(t).$$

Theorem 3.31. *Suppose $\prod_{t_1\le\tau_j\le t} B_j/A_j \le 1$. If for $t > t_1$ there exists a positive solution of the nonimpulsive ordinary differential equation*

$$x''(t) + b(t)x(t) = 0,$$

then for $t > t_1$ there exists a positive solution of Eq. (3.39).

Theorem 3.32. *Suppose $a_k(t) \ge 0$ and there exist $M > 0$, $\delta > 0$ such that*

$$\sup_{t\ge 0} \prod_{\tau_j\le t} B_j/A_j \le M, \quad t - g_k(t) \le \delta.$$

If for some k, $k = 1, 2, \ldots, m$,

$$\int^{\infty} \prod_{\tau_j\le t} A_j/B_j \prod_{g_k(t)<\tau_j\le t} A_j^{-1} a_k(t)\mathrm{d}t = \infty, \quad \text{and} \quad \int^{\infty} \prod_{\tau_j\le t} B_j/A_j \mathrm{d}t = \infty,$$

then Eq. (3.39) is oscillatory.

Example 3.4. Consider the impulsive delay differential equation

$$\begin{cases} x''(t) + \dfrac{1}{4t^2} x(t-\delta) = 0, & t \ne j, \\ x(j) = \dfrac{j+1}{j} x(j^-), \qquad x'(j) = x'(j^-), & j \in \mathbb{N}. \end{cases} \tag{3.41}$$

Eq. (3.41) satisfies the conditions of Theorem 3.32, and hence all solutions of Eq. (3.41) are oscillatory.

Next consider the following delay differential equation without impulses

$$x''(t) + \sum_{k=1}^{m} \prod_{g_k(t)<\tau_j\le t} A_j^{-1} a_k(t) x(g_k(t)) = 0. \tag{3.42}$$

Theorem 3.33. *Suppose $a_k(t) \ge 0, A_j = B_j > 0$. Then Eq. (3.39) is oscillatory (nonoscillatory) if and only if Eq. (3.42) is oscillatory (nonoscillatory).*

For higher order linear equations we refer to papers [45,46]. In [45], the authors considered the impulsive delay differential equation

$$\begin{cases} x^{(n)}(t) + p(t)x(t-\tau) = 0, & t \ge t_0,\ t \ne t_k, \\ x^{(i)}(t_k^+) = a_k^{(i)} x^{(i)}(t_k), & i = 0, 1, \ldots, n-1, k \in \mathbb{N}, \end{cases}$$

where n is a natural number with $n \ge 2$. The authors improved the known results for the oscillation of ordinary differential equations. They separately dealt with the cases n even and n odd.

In [46], the following delay differential equations are considered

$$\begin{cases} x^{(m)}(t) + a(t)x^{(m-1)}(t) + \displaystyle\sum_{i=1}^{n} p_i(t) x(g_i(t)) = 0, & t \ge t_0,\ t \ne t_k, \\ x^{(j)}(t_k) - x^{(j)}(t_k^-) = \alpha_k x^{(j)}(t_k^-), & j = 0, 1, 2, \ldots, m-1, \end{cases} \tag{3.43}$$

and

$$y^{(m)}(t) + a(t)y^{(m-1)}(t) + \sum_{i=1}^{n} p_i(t) \prod_{g_i(t)<t_k\le t} (1+\alpha_k)^{-1} y(g_i(t)) = 0, \quad t \ge t_0, \tag{3.44}$$

where $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ are fixed points with $\lim_{k\to\infty} t_k = \infty$; $a, p_i \in C(\mathbb{R}^+, \mathbb{R})$, $i = 1, 2, \ldots, n$, are Lebesgue measurable and locally essentially bounded functions; $g_i \in C(\mathbb{R}^+, \mathbb{R})$, $i = 1, 2, \ldots, n$, are Lebesgue measurable functions, $g_i(t) \le t$ and $\lim_{t\to\infty} g_i(t) = \infty$; $\{\alpha_k\}$ is a sequence of constants and $\alpha_k > -1$. For any fixed $t_0 \ge 0$, we define $\bar{t}_0 = \min_{1\le i\le n} \inf_{t\ge t_0} g_i(t)$.

Theorem 3.34 (*[46]*).

(i) *If y is a solution of (3.44) on $[\bar{t}_0, \infty)$, then $x(t) = \prod_{t_0<t_k\leq t}(1+\alpha_k)y(t)$ is a solution of (3.43) on $[\bar{t}_0, \infty)$.*
(ii) *If x is a solution of (3.43) on $[\bar{t}_0, \infty)$, then $y(t) = \prod_{t_0<t_k\leq t}(1+\alpha_k)^{-1}x(t)$ is a solution of (3.44) on $[\bar{t}_0, \infty)$.*

The proof of Theorem 3.34 is similar to that of Theorem 3.8. We also note that from Theorem 3.34 the following result is immediate.

Theorem 3.35. *All solutions of (3.43) are oscillatory (nonoscillatory) if and only if all solutions of (3.44) are oscillatory (nonoscillatory).*

In our next result we let m to be an even integer and $g(t) = \min_{1\leq i\leq n} g_i(t)$.

Theorem 3.36 (*[46]*). *Assume that the following conditions hold:*

(A1) $p_i(t) \geq 0$, $i \in \mathbb{N}$;
(A2) $\int_{t_0}^{\infty} \frac{1}{r(s)}\mathrm{d}s = \infty$, *where* $r(t) = \exp\left(\int_0^t a(s)\mathrm{d}s\right)$;
(A3) g_i *has an absolutely continuous derivative* g_i' *on* $(\bar{t}_0, \infty)$ *and* $g_i' \geq 0$;
(A4) $\int_{t_0}^{\infty} s^{m-1}r(s)\sum_{i=1}^{n} p_i(t)\prod_{g_i(s)<t_k\leq s}(1+\alpha_k)^{-1} = \infty$;
(A5) *there exists* $G > 0$ *such that* $r(t) < G$.

Then all bounded solutions of (3.43) are oscillatory.

3.3. Generic oscillation

The concept of generic oscillation was first introduced in [47] for delay differential equations. Later, in [48–50], the authors generalized this theory to impulsive delay differential equations.

Consider the second order impulsive delay differential equation

$$\begin{cases} x''(t) + \sum_{j=1}^{m} q_j x'(t-\sigma_j) + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \quad t \geq t_0, \ t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), \qquad x'(t_k^+) - x'(t_k) = b_k x'(t_k), \quad k \in \mathbb{N}, \end{cases} \tag{3.45}$$

and the delay differential equation

$$y''(t) + \sum_{j=1}^{m} q_j\beta_j y'(t-\sigma_j) + \sum_{i=1}^{n} p_i\alpha_i y(t-\tau_i) = 0, \tag{3.46}$$

with the initial conditions

$$x(t) = \phi(t), \qquad x'(t) = \psi(t), \quad t \in [t_0 - r, t_0], \text{ and } x'(t_0) = \xi, \tag{3.47}$$

where we assume that the following conditions are satisfied:

(A_1) $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are fixed points and $\lim_{k\to\infty} t_k = +\infty$;
(A_2) $b_k \in (-1, \infty)$ are constants for $k \in \mathbb{N}$, $p_i, q_j \in \mathbb{R}$, $\alpha_i = \prod_{t-\tau_j\leq t_k<t}(1+b_k)^{-1}$,
$\beta_j = \prod_{t-\sigma_j\leq t_k<t}(1+b_k)^{-1}$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $t \geq t_0$;
(A_3) Let $r_1 = \max_{1\leq i\leq n}\{\tau_i\}$, $r_2 = \max_{1\leq j\leq m}\{\sigma_j\}$, $r = \max\{r_1, r_2\}$, $\phi, \psi \in PLC([t_0 - r, t_0], \mathbb{R})$.

In what follows we let $\overline{PC} = PLC([t_0 - r, t_0], \mathbb{R}) \times PLC([t_0 - r, t_0], \mathbb{R})$.

Definition 3.2. The solutions of (3.45), (3.47) ((3.46), (3.47)) are called generically oscillatory, if the set of all $(\phi, \psi) \in \overline{PC}$, for which the corresponding solution $x_{(\phi,\psi)}(y_{(\phi,\psi)})$ of (3.45), (3.47), ((3.46), (3.47)) is nonoscillatory, is nowhere dense in $\overline{PC}$.

Definition 3.3. The solutions of (3.45), (3.47), ((3.46), (3.47)) are called generically nonoscillatory, if the set of all $(\phi, \psi) \in \overline{PC}$, for which the corresponding solution $x_{(\phi,\psi)}(y_{(\phi,\psi)})$ of (3.45), (3.47), ((3.46), (3.47)) is oscillatory, is nowhere dense in $\overline{PC}$.

We note that, if the condition

(A_4) $b_k \equiv b$, $t_{k+1} - t_k = \omega$, $k \in \mathbb{N}$, and $\sigma_j = c_j\omega$, $\tau_i = d_i\omega$, c_j, d_i are integers for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, holds, then the characteristic equation of (3.46) is

$$\lambda^2 + \sum_{j=1}^{m} q_j\beta_j\lambda e^{-\lambda\sigma_j} + \sum_{i=1}^{n} p_i\alpha_i e^{-\lambda\tau_i} = 0. \tag{3.48}$$

Definition 3.4. It is said that the characteristic Eq. (3.48) has a real leading root x_0, if x_0 is a root of (3.48) and all other roots of (3.48) lie in the half plane $\{\operatorname{Re} z < x_0\}$. It is said that (3.48) has complex leading roots $x_0 \pm iy_0$ ($y_0 \neq 0$), if $x_0 \pm iy_0$ are roots of (3.48) and all other roots of (3.48) lie in the half plane $\{\operatorname{Re} z < x_0\}$.

In [48], the authors established that generic oscillation (generic nonoscillation) of solutions of (3.45) can be reduced to generic oscillation (generic nonoscillation) of solutions of the corresponding delay differential equation (3.46).

Theorem 3.37 (*[48]*). *Assume that* (A_1)–(A_3) *hold.*

(a) *If* $y(t, t_0, \phi)$ *is a solution of* (3.46), *then* $x(t, t_0, \phi) = \prod_{t_0 \le t_k < t}(1 + b_k)y(t, t_0, \phi)$ *is a solution of* (3.45).
(b) *If* $x(t, t_0, \phi)$ *is a solution of* (3.45), *then* $y(t, t_0, \phi) = \prod_{t_0 \le t_k < t}(1 + b_k)^{-1}x(t, t_0, \phi)$ *is a solution of* (3.46).

Corollary 3.15. *Assume that* (A_1)–(A_3) *hold. Then Eq.* (3.45) *is oscillatory if and only if Eq.* (3.46) *is oscillatory.*

Corollary 3.16. *Assume that* (A_1)–(A_3) *hold. Then Eq.* (3.45) *is generically oscillatory if and only if Eq.* (3.46) *is generically oscillatory.*

Corollary 3.17. *Assume that* (A_1)–(A_4) *hold.*

(a) *If* (3.48) *has a real leading root, then* (3.45) *is generically nonoscillatory;*
(b) *If* (3.48) *has complex leading roots, then* (3.45) *is generically oscillatory.*

Let $m = 1, \sigma_1 = 0, \beta_1 = 1$ in (3.45), then we have the following equation

$$\begin{cases} x''(t) + qx'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0, & t \ge t_0,\ t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), \quad x'(t_k^+) - x'(t_k) = b_k x'(t_k), & k \in \mathbb{N}. \end{cases} \tag{3.49}$$

Theorem 3.38. *Assume that* (A_1)–(A_4) *hold,* $q > 0, p_i \in \mathbb{R},\ i = 1, 2, \ldots, n$, *and there exists* $\beta_0 < 0$ *such that* $\beta_0^2 + q\beta_0 + \sum_{i=1}^{n} |p_i\alpha_i|\, e^{-\beta_0\tau_i} < 0$. *Then Eq.* (3.49) *is generically nonoscillatory.*

Theorem 3.39. *Assume that* (A_1)–(A_4) *hold,* $q > 0, p_i\alpha_i \ge 0,\ i = 1, 2, \ldots, n$. *Then Eq.* (3.49) *is generically nonoscillatory if and only if there exists* $\beta_0 < 0$ *such that* $\beta_0^2 + q\beta_0 + \sum_{i=1}^{n} p_i\alpha_i e^{-\beta_0\tau_i} \le 0$.

Theorem 3.40. *Assume that* (A_1)–(A_4) *hold,* $q < 0, p_i \in \mathbb{R},\ i = 1, 2, \ldots, n$, *and there exists* $\beta_0 > 0$ *such that* $\beta_0^2 + q\beta_0 + \sum_{i=1}^{n} |p_i\alpha_i|\, e^{-\beta_0\tau_i} < 0$. *Then Eq.* (3.49) *is generically nonoscillatory.*

Theorem 3.41. *Assume that* (A_1)–(A_4) *hold,* $q < 0, p_i\alpha_i \ge 0,\ i = 1, 2, \ldots, n$. *Then Eq.* (3.49) *is generically nonoscillatory if and only if there exists* $\beta_0 > 0$ *such that* $\beta_0^2 + q\beta_0 + \sum_{i=1}^{n} p_i\alpha_i e^{-\beta_0\tau_i} \le 0$.

Theorem 3.42. *If* $q = 0, n = 1$, *then*

(a) *If* $p\alpha > 0$, *Eq.* (3.49) *is oscillatory,*
(b) *If* $p\alpha < 0$, *Eq.* (3.49) *is generically nonoscillatory.*

Example 3.5 (*[48]*). Consider the equation

$$\begin{cases} x''(t) + 4x'(t) + 8x(t - 1) - \epsilon x(t - 2) = 0, & t \neq t_k, \\ x(t_k^+) = (1/2)x(t_k), \quad x'(t_k^+) = (1/2)x'(t_k), \\ t_{k+1} - t_k = 1, \quad k = 1, 2, \ldots \end{cases} \tag{3.50}$$

It can be shown that the characteristic equation of the corresponding nonimpulsive equation has complex leading roots. Thus, from Corollary 3.17, Eq. (3.50) is generically oscillatory.

4. Nonlinear differential equations

In this section, we consider the oscillation of first, second, and higher order nonlinear equations. Moreover, linearized oscillation and applications to mathematical biology are also discussed.

4.1. First order equations

Consider the impulsive differential equation and inequalities

$$\begin{cases} x'(t) + a(t)x(t) + p(t)f(x(t-h_1), x(t-h_2), \dots, x(t-h_m)) = 0, & t \neq \tau_k, \\ \Delta x(\tau_k) = b_k x(\tau_k), & k \in \mathbb{N}, \end{cases} \tag{4.1}$$

$$\begin{cases} x'(t) + a(t)x(t) + p(t)f(x(t-h_1), x(t-h_2), \dots, x(t-h_m)) \leq 0, & t \neq \tau_k, \\ \Delta x(\tau_k) = b_k x(\tau_k), & k \in \mathbb{N}, \end{cases} \tag{4.2}$$

$$\begin{cases} x'(t) + a(t)x(t) + p(t)f(x(t-h_1), x(t-h_2), \dots, x(t-h_m)) \geq 0, & t \neq \tau_k, \\ \Delta x(\tau_k) = b_k x(\tau_k), & k \in \mathbb{N}, \end{cases} \tag{4.3}$$

with the initial function

$$x(t) = \varphi(t), \quad t \in [-\overline{h}, 0],$$

where $\varphi \in C([-\overline{h}, 0], \mathbb{R})$, $\overline{h} = \max\{h_i : i \in \mathbb{N}_m\}$, $\mathbb{N}_m = \{1, 2, \dots, m\}$.

In [51], authors examined Eq. (4.1) and the inequalities (4.2)–(4.3) under following conditions:

(H1) $a \in C_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+ - \{0\})$;
(H2) $p \in C_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+ - \{0\})$;
(H3) $f \in C_{\text{loc}}(\mathbb{R}^m, \mathbb{R})$, $f(u_1, u_2, \dots, u_m)u_1 > 0$ for $u_1 \neq 0$ and $\operatorname{sgn} u_1 = \operatorname{sgn} u_2 = \cdots = \operatorname{sgn} u_m$;
(H4) There exist constants $L > 0$ and $\alpha_1, \alpha_2, \dots, \alpha_m$, $\alpha_i \geq 0$, $i \in \mathbb{N}_m$, such that $\sum_{i=1}^m \alpha_i = 1$ and

$$|f(u_1, u_2, \dots, u_m)| \geq L\, |u_1|^{\alpha_1}\, |u_2|^{\alpha_2} \cdots |u_m|^{\alpha_m};$$

(H5) There exist constants l_1 and l_2 such that $\lim_{k\to\infty}(\tau_k - kl_1) = l_2$;
(H6) There exists a constant $M > 0$ such that for any $k \in \mathbb{N}$, $0 < b_k < M$;
(H7) $\tau_{k+1} - \tau_k \geq T > \overline{h}$ for $k \in \mathbb{N}$.

Let us construct the new sequence

$$\{t_i\}_{i=1}^\infty = \{\tau_i\}_{i=1}^\infty \cup \{\tau_{is}\}_{i=1}^\infty{}_{,s=1}^{\,m},$$

where $\tau_{is} = \tau_i + h_s$, $i \in \mathbb{N}$, $s \in \mathbb{N}_m$ and $t_i < t_{i+1}$, $i \in \mathbb{N}$.

Theorem 4.1 (*[51]*). *Let the following hold:*

(i) *Conditions* (H1)–(H6)*are satisfied;*
(ii)

$$\liminf_{t\to\infty} \int_{t-h}^{t} a(s)\mathrm{d}s \geq k > 0,$$

where k is a constant, $h = \min\{h_i : i \in N_m\}$*;*
(iii)

$$\liminf_{t\to\infty} \int_{t-h}^{t} p(s)\mathrm{d}s > \frac{(1+M)^{2l}}{L\mathrm{e}^k} \max\left\{\frac{1}{\mathrm{e}}, 2(1+M)^l[(1+M)^l - 1]\right\}.$$

Then inequality (4.2) has no positive solutions.

Corollary 4.1. *Let the conditions of Theorem 4.1 hold. Then:*

1. *The inequality (4.3) has no negative solutions.*
2. *All solutions of Eq. (4.1) are oscillatory.*

Theorem 4.2 (*[51]*). *Let the following hold:*

(i) *Conditions* (H1)–(H4) *and* (H7)*are satisfied;*
(ii) $b_k > -1$, $k \in \mathbb{N}$*;*
(iii)

$$\limsup_{k\to\infty} \frac{1}{1+b_k} \int_{\tau_k}^{\tau_k+h} p(s) \left(\exp \int_{s-h}^{s} a(u)\mathrm{d}u\right) \mathrm{d}s > \frac{1}{L}.$$

Then the following hold:

1. *All solutions of Eq. (4.1) are oscillatory.*
2. *The inequality (4.2) has no positive solutions.*
3. *The inequality (4.3) has no negative solutions.*

Proof. We shall prove the Assertion 1. The proof of Assertions 2 and 3 is similar. Let x be a nonoscillatory solution of Eq. (4.1). Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_1 \geq 0$. It is clear that $x(t-h_i) > 0, i \in \mathbb{N}_m$, and $f(x(t-h_1), \ldots, x(t-h_m)) > 0$ for $t \geq t_1 + \overline{h} = t_2$. Then from (4.1) it follows that x is a nonincreasing function in the set $(t_2, \tau_s) \cup [\cup_{i=s}^{\infty}(\tau_i, \tau_{i+1})]$, where $\tau_{s-1} < t_2 < \tau_s$.

Set $z(t) = x(t) \exp\left(\int_T^t a(s)\mathrm{d}s\right)$. Then from (4.1), we obtain

$$\begin{cases} z'(t) + p(t)\exp\left(\int_T^t a(u)\mathrm{d}u\right) f\left(z(t-h_1)\exp\left(-\int_T^{t-h_1} a(u)\mathrm{d}u\right), \ldots, \right. \\ \qquad \left. z(t-h_m)\exp\left(-\int_T^{t-h_m} a(u)\mathrm{d}u\right)\right) = 0, \quad t \neq \tau_k, \\ \Delta z(\tau_k) = b_k z(\tau_k). \end{cases} \tag{4.4}$$

Integrating (4.4) from τ_k to $\tau_k + h$, $k \geq s$, we obtain

$$z(\tau_k + h) - z(\tau_k^+) + L\int_{\tau_k}^{\tau_k+h} p(s)\exp\left(\int_{s-h}^{s} a(u)\mathrm{d}u\right)\prod_{i=1}^{m} z^{\alpha_i}(s-h_i)\mathrm{d}s \leq 0. \tag{4.5}$$

Moreover, since z is nonincreasing in the interval $[s-h_i, s-h]$, $s \in [\tau_k, \tau_k + h]$, from (H4) it follows that

$$\prod_{i=1}^{m} z^{\alpha_i}(s-h_i) \geq z(s-h).$$

From (4.5) and the last inequality, we have

$$z(\tau_k + h) - z(\tau_k^+) + L\int_{\tau_k}^{\tau_k+h} p(s)\exp\left(\int_{s-h}^{s} a(u)\mathrm{d}u\right) z(s-h)\mathrm{d}s \leq 0.$$

Using impulse conditions, we find

$$\frac{1}{1+b_k}\int_{\tau_k}^{\tau_k+h} p(s)\exp\left(\int_{s-h}^{s} a(u)\mathrm{d}u\right)\mathrm{d}s \leq \frac{1}{L}.$$

The last inequality contradicts (iii). This completes the proof. □

Corollary 4.2. *Let the following hold:*

(i) *Conditions* (H1)–(H4), (H6) *and* (H7) *are satisfied;*
(ii)

$$\limsup_{k\to\infty}\int_{\tau_k}^{\tau_k+h} p(s)\exp\left(\int_{s-h}^{s} a(u)\mathrm{d}u\right)\mathrm{d}s > \frac{1+M}{L}.$$

Then the following hold:

1. *The inequality* (4.2) *has no positive solutions.*
2. *The inequality* (4.3) *has no negative solutions.*
3. *All solutions of Eq.* (4.1) *are oscillatory.*

Corollary 4.3. *Let the following hold:*

(i) *Conditions* (H1)–(H4), *and* (H7) *are satisfied;*
(ii) $b_k > -1$, $k \in \mathbb{N}$*;*
(iii)

$$\liminf_{k\to\infty}\int_{\tau_k-h}^{\tau_k} a(u)\mathrm{d}u \geq s > 0, \quad s = \text{const.};$$

(iv)

$$\limsup_{k\to\infty}\frac{1}{1+b_k}\int_{\tau_k}^{\tau_k+h} p(u)\mathrm{d}u > \frac{1}{Le^s}.$$

Then the assertions of Corollary 4.2 *are valid.*

Theorem 4.3 ([51]). *Let the following hold:*

(i) *Conditions* (H1)–(H6)*are satisfied;*
(ii)

$$\limsup_{k\to\infty}\int_{\tau_k}^{\tau_k+h} p(s)\left(\exp\int_{s-h}^{s} a(u)\mathrm{d}u\right)\mathrm{d}s > \frac{(1+M)^{2l}}{L}.$$

Then the following hold:

1. *The inequality* (4.2) *has no positive solutions.*
2. *The inequality* (4.3) *has no negative solutions.*
3. *All solutions of Eq.* (4.1) *are oscillatory.*

Corollary 4.4. *Let the following hold:*

(i) *Conditions* (H1)–(H3), *and* (H7) *are satisfied;*
(ii) $b_k > -1$, $k \in \mathbb{N}$;
(iii) $\liminf_{k\to\infty}\int_{\tau_k-h}^{\tau_k} a(u)\mathrm{d}u \geq s > 0,\ s = \text{const.}$;
(iv) $\limsup_{k\to\infty}\int_{\tau_k}^{\tau_k+h} p(s)\mathrm{d}s > \frac{(1+M)^{2l}}{Le^s}$.

Then the assertions of Theorem 4.3 *are valid.*

Eq. (4.1) together with inequalities (4.2)–(4.3) were also considered in [52] under the following conditions:

(A_1) $a, p \in (\mathbb{R}^+, \mathbb{R})$ are locally summable functions, $p \geq 0$, and $0 < h_1 < h_2 < \cdots < h_m$ are positive constants;
(A_2) $b_k \in (-1, \infty)$ are constants;
(A_3) $f \in C_{\text{loc}}(\mathbb{R}^m, \mathbb{R})$ and satisfies

$$\text{when } u_i > 0,\ i = 1, 2, \ldots, m, f(u_1, u_2, \ldots, u_m) > 0,$$
$$\text{when } u_i < 0,\ i = 1, 2, \ldots, m, f(u_1, u_2, \ldots, u_m) < 0;$$

(A_4) there exist a constant $L > 0$ and nonnegative constants $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\sum_{i=1}^{m}\alpha_i = 1$ and

$$|f(u_1, u_2, \ldots, u_m)| \geq L\prod_{i=1}^{m}|u_i|^{\alpha_i}.$$

Now consider the following nonimpulsive equation and inequalities for $t \geq t_0 + h_m$:

$$y'(t) + a(t)y(t) + P(t)\prod_{i=1}^{m}|y(t-h_i)|^{\alpha_i}\,\mathrm{sgn}\,(y(t)) = 0,\ \text{a.e.,} \tag{4.6}$$

$$y'(t) + a(t)y(t) + P(t)\prod_{i=1}^{m}|y(t-h_i)|^{\alpha_i}\,\mathrm{sgn}\,(y(t)) \leq 0,\ \text{a.e.,} \tag{4.7}$$

$$y'(t) + a(t)y(t) + P(t)\prod_{i=1}^{m}|y(t-h_i)|^{\alpha_i}\,\mathrm{sgn}\,(y(t)) \geq 0,\ \text{a.e.,} \tag{4.8}$$

where $P(t) = Lp(t)\prod_{i=1}^{m}\prod_{t-h_i\leq\tau_k<t}(1+b_k)^{-\alpha_i},\ t \geq t_0 + h_m$, and $a, p, \alpha_i,\ i = 1, 2, \ldots, m, L$ and $\{b_k\}$ satisfy (A_1), (A_2) and (A_4).

The proof of the following result is similar to that of Theorem 3.8.

Theorem 4.4. *Assume that* (A_1)–(A_4) *are satisfied. Then the following hold:*

(i) *Inequality* (4.2) *has no eventually positive solutions if inequality* (4.7) *has no eventually positive solutions.*
(ii) *Inequality* (4.3) *has no eventually negative solutions if inequality* (4.8) *has no eventually negative solutions.*
(iii) *All solutions of Eq.* (4.1) *are oscillatory if all solutions of Eq.* (4.6) *are oscillatory.*

The following result improves Theorem 4.1 and Corollary 4.1.

Theorem 4.5 ([52]). *Assume that conditions* (A_1)–(A_4) *are satisfied, and*

$$\liminf_{t\to\infty}\sum_{i=1}^{m}\alpha_i\int_{t-h_i}^{t} p(s)\exp\left(\sum_{i=1}^{m}\alpha_i\int_{s-h_i}^{s} a(r)\mathrm{d}r\right)\prod_{i=1}^{m}\prod_{s-h_i\leq\tau_k<s}(1+b_k)^{-\alpha_i}\mathrm{d}s > \frac{1}{Le}. \tag{4.9}$$

Then the following hold:

(i) *Inequality* (4.2) *has no eventually positive solutions;*
(ii) *Inequality* (4.3) *has no eventually negative solutions;*
(iii) *All solutions of Eq.* (4.1) *are oscillatory.*

Proof. We shall only prove (i). Suppose that $x(t)$ is an eventually positive solution of (4.2). By Theorem 4.4, (4.7) also has an eventually positive solution $y(t)$. Then there exists $T \geq t_0 + h_m$ such that for $t \geq T \geq t_0 + h_m$, $y(t) > 0$, $y(t-h_i) > 0,\ i = 1, 2, \ldots, m$. Set

$$z(t) = y(t)\exp\left(\int_T^t a(s)\mathrm{d}s\right), \quad t \geq T.$$

Then (4.7) reduces to

$$z'(t) + P_1(t)\prod_{i=1}^{m}[z(t-h_i)]^{\alpha_i} \leq 0, \quad \text{a.e. for } t \geq T, \tag{4.10}$$

where

$$P_1(t) = Lp(t)\exp\left(\sum_{i=1}^{m}\alpha_i\int_{t-h_i}^{t} a(s)\mathrm{d}s\right)\left(\prod_{i=1}^{m}\prod_{t-h_i\leq\tau_k<t}(1+b_k)^{-\alpha_i}\right) \geq 0. \tag{4.11}$$

By (4.9), without loss of generality, we can assume that $\alpha_m > 0$. Hence, there exists a positive constant C and $T_1 \geq T$ such that

$$\liminf_{t\to\infty}\alpha_m\int_{t-h_m}^{t} P_1(s)\mathrm{d}s > C > 0.$$

Thus for any $t \geq T_1$, there exists a $t^* > t$ such that

$$\alpha_m\int_{t^*-h_m}^{t} P_1(s)\mathrm{d}s > \frac{C}{2} \quad \text{and} \quad \alpha_m\int_{t}^{t^*} P_1(s)\mathrm{d}s \geq \frac{C}{2}. \tag{4.12}$$

Integrating (4.10) from $t - h_m$ to t, and using the monotonicity of $z(t)$, we get

$$\frac{z(t-h_m)}{z(t)} \geq \prod_{i=1}^{m}\left[\frac{z(t-h_m)}{z(t)}\right]^{\alpha}\int_{t-h_m}^{t} P_1(s)\mathrm{d}s.$$

Since $z(t-h_i) \geq z(t),\ i = 1, 2, \ldots, m$, it follows from (4.10) that for $t \geq T_1$,

$$-z'(t)(z(t))^{\alpha_m-1} \geq P_1(t)\,[z(t-h_m)]^{\alpha_m}, \text{ a.e.} \tag{4.13}$$

Integrating (4.13) from $t^* - h_m$ to t and using (4.12), we find

$$\begin{aligned} \left[z(t^*-h_m)\right]^{\alpha_m} - [z(t)]^{\alpha_m} &\geq \alpha_m\int_{t^*-h_m}^{t} P_1(s)\,[z(s-h_m)]^{\alpha_m}\,\mathrm{d}s \\ &\geq \alpha_m\,[z(t-h_m)]^{\alpha_m}\int_{t^*-h_m}^{t} P_1(s)\mathrm{d}s \\ &\geq \frac{C}{2}\,[z(t-h_m)]^{\alpha_m}. \end{aligned} \tag{4.14}$$

Next integrating (4.13) from t to t^* and using (4.12), we obtain

$$[z(t)]^{\alpha_m} - [z(t^*)]^{\alpha_m} \geq \frac{C}{2}\left[z(t^*-h_m)\right]^{\alpha_m}. \tag{4.15}$$

Therefore from (4.14) and (4.15), we have

$$[z(t)]^{\alpha_m} \geq \frac{C^2}{4}[z(t-h_m)]^{\alpha_m} \quad \text{for } t \geq T_1.$$

Let

$$u(t) = \prod_{i=1}^{m}\left[\frac{z(t-h_i)}{z(t)}\right]^{\alpha_i}, \quad t \geq T_1.$$

Then $u(t)$ is continuous on $[T, \infty)$ and

$$1 \le u(t) = \prod_{i=1}^{m} \left[\frac{z(t-h_i)}{z(t)}\right]^{\alpha_i} \le \prod_{i=1}^{m} \left[\frac{z(t-h_m)}{z(t)}\right]^{\alpha_i} = \frac{z(t-h_m)}{z(t)} \le \left(\frac{4}{C^2}\right)^{1/\alpha_m}.$$

Hence

$$\liminf_{t\to\infty} u(t) = u_0 < \infty. \tag{4.16}$$

Dividing (4.10) by $z(t)$ and integrating from $t-h_i$ to t, we obtain for $t \ge T_1$

$$\int_{t-h_i}^{t} P_1(s)u(s)\mathrm{d}s \le \ln\frac{z(t-h_i)}{z(t)}.$$

Multiplying the above inequality by α_i and summing from 1 to m, we get

$$\sum_{i=1}^{m} \alpha_i \int_{t-h_i}^{t} P_1(s)u(s)\mathrm{d}s \le \sum_{i=1}^{m} \alpha_i \ln\frac{z(t-h_i)}{z(t)} = \ln u(t), \quad t \ge T_1. \tag{4.17}$$

From (4.16) and (4.17), for any sufficiently small ϵ, $0 < \epsilon < u_0$, there exists $T_\epsilon \ge T_1$ such that for all $t \ge T_\epsilon$,

$$(u_0 - \epsilon)\sum_{i=1}^{m} \alpha_i \int_{t-h_i}^{t} P_1(s)\mathrm{d}s \le \ln u(t).$$

Taking liminf on both sides of the last inequality, we obtain

$$(u_0 - \epsilon)C_1 \le \ln u_0, \tag{4.18}$$

where $C_1 = \liminf_{t\to\infty} \sum_{i=1}^{m} \alpha_i \int_{t-h_i}^{t} P_1(s)\mathrm{d}s$. From (4.18), it follows that

$$C_1 \le \frac{\ln u_0}{u_0} \le \frac{1}{\mathrm{e}},$$

which contradicts (4.9). This completes the proof of Theorem 4.5. $\square$

Now we consider the following condition which is a special case of (A_4).

$(\mathrm{A}_4{}')$ There exist a constant $L > 0$ and nonnegative constants $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $\sum_{i=1}^{m} \alpha_i = 1$ and

$$|f(u_1, u_2, \dots, u_m)| \equiv L\prod_{i=1}^{m} |u_i|^{\alpha_i}.$$

Theorem 4.6. *Assume that* (A_1)–(A_3) *and* $(\mathrm{A}_4{}')$ *are satisfied. Then the following hold*

(i) *Inequality (4.2) has no eventually positive solutions if and only if inequality (4.7) has no eventually positive solutions.*
(ii) *Inequality (4.3) has no eventually negative solutions if and only if inequality (4.8) has no eventually negative solutions.*
(iii) *All solutions of Eq. (4.1) are oscillatory if and only if all solutions of Eq. (4.6) are oscillatory.*

Theorem 4.7. *Assume that* (A_1)–(A_3) *and* $(\mathrm{A}_4{}')$ *hold, and there exists* $T \ge t_0 + h_m$ *such that for all* $t \ge T$,

$$\sum_{i=1}^{m} \alpha_i \int_{t-h_i}^{t} p(s)\exp\left(\sum_{i=1}^{m} \alpha_i \int_{s-h_i}^{s} a(r)\mathrm{d}r\right)\prod_{i=1}^{m}\prod_{s-h_i\le\tau_k<s}(1+b_k)^{-\alpha_i}\mathrm{d}s \le \frac{1}{Le}. \tag{4.19}$$

Then Eq. (4.1) has a positive solution on $[T, \infty)$.

Proof. Consider the set W of all nonnegative continuous functions w satisfying the conditions

$$W = \{w \in C([T-h_m, \infty), \mathbb{R}^+) : 1 \le w(t) \le e \text{ for every } t \ge T - h_m\}$$

and a mapping F on W

$$(Fw)(t) = \begin{cases} \exp\left(\sum_{i=1}^{m} \alpha_i \int_{t-h_i}^{t} P_1(s)w(s)\mathrm{d}s\right), & t \ge T, \\ (Fw)(T), & T - h_m \le t < T, \end{cases} \tag{4.20}$$

where $P_1(t)$ is defined in (4.11).

Observe that $(Fw)(t) : [T-h_m, \infty) \to \mathbb{R}^+$ is continuous, and that (4.20) defines an increasing mapping $F : W \to W$. Here, the increasing character of F is considered with respect to the usual pointwise ordering in W; that is, for any $w_1, w_2 \in W$, $w_1(t) \le w_2(t)$ implies $(Fw_1)(t) \le (Fw_2)(t)$. By combining (4.19) and (4.20) we find $1 \le (Fw)(t) \le e$ for all $t \ge T-h_m$. Hence, Fw is uniformly bounded on W.

Consider the increasing sequence $\{u_n\}_{n=0}^{\infty}$ of functions on W defined by $u_0(t) \equiv 1$ and $u_n(t) = (Fu_{n-1})(t)$, $n = 1, 2, \ldots,$ and set $u(t) = \lim_{n\to\infty} u_n(t)$ pointwise on $[T-h_m, \infty)$. Then by using the Lebesgue dominated convergence theorem, from (4.20), we obtain

$$u(t) = \begin{cases} \exp\left(\sum_{i=1}^{m} \alpha_i \int_{t-h_i}^{t} P_1(s)u(s)\mathrm{d}s\right), & t \ge T, \\ (Fu)(T), & T-h_m \le t < T. \end{cases}$$

Let

$$Z(t) = \exp\left(-\int_T^t P_1(s)u(s)\mathrm{d}s\right).$$

It can be shown that $Z(t)$ is a positive solution of

$$Z'(t) + P_1(t)\prod_{i=1}^{m} |Z(t-h_i)|^{\alpha_i} \operatorname{sgn} Z(t) = 0, \quad \text{a.e. } t \ge T.$$

Now set $y(t) = Z(t)\exp\left(-\int_T^t a(s)\mathrm{d}s\right)$, $t \ge T$. Clearly, $y(t)$ is a positive solution of (4.6) on $[T, \infty)$. But then by Theorem 4.6, (4.1) has a positive solution $x(t) = \prod_{T\le\tau_k<t}(1+b_k)y(t)$, $t \ge T$. This completes the proof. □

Corollary 4.5. *Assume that* (A_4') *and the following condition hold:*

(A) $\tau_{k+1} - \tau_k = \delta > 0$, $b_k \in (-1, \infty)$, $k \in \mathbb{N}$, $a(t) \equiv a \in \mathbb{R}$, $p(t) \equiv p > 0$, $h_i > 0$ *and* $\prod_{t-h_i\le\tau_k<t}(1+b_k)^{-\alpha_i} = r_i$, $i = 1, 2, \ldots, m$, *are constants.*

Then the conclusions of Theorem 4.5 are valid if and only if

$$Lp\sum_{i=1}^{m}\alpha_i h_i \exp\left(a\sum_{i=1}^{m}\alpha_i h_i\right)\left(\prod_{i=1}^{m} r_i\right) > \frac{1}{\mathrm{e}}.$$

Example 4.1. Consider the differential equation

$$\begin{cases} x'(t) + ax(t) + p\,|x(t-h)|^{1/3}\,|x(t-2h)|^{2/3}\operatorname{sgn}(x(t)) = 0, \\ x(\tau_k^+) - x(\tau_k) = bx(\tau_k), \quad k \in \mathbb{N}, \end{cases} \tag{4.21}$$

where $p > 0$, $h > 0$, $b \in (-1, \infty)$ are constants, $\tau_k = kh$, $k \in \mathbb{N}$.

For (4.21), we have

$$\prod_{t-h\le\tau_k<t}(1+b)^{-1/3}\prod_{t-2h\le\tau_k<t}(1+b)^{-2/3} = (1+b)^{-5/3}.$$

Thus by Corollary 4.5 if

$$\frac{5}{3}ph\exp\left(\frac{5}{3}ah\right)(1+b)^{-5/3} > \frac{1}{\mathrm{e}},$$

then all solutions of (4.21) are oscillatory. If

$$\frac{5}{3}ph\exp\left(\frac{5}{3}ah\right)(1+b)^{-5/3} \le \frac{1}{\mathrm{e}},$$

then (4.21) has a nonoscillatory solution.

Finally, we remark that first order nonlinear equations have also been addressed in [53–57].

4.2. Linearization and applications to mathematical biology

In this section, we present linearized oscillation theory for impulsive delay differential equations. We shall also consider some impulsive models from mathematical biology, and provide sufficient conditions for the oscillation of their positive solutions about the steady state.

Linearized oscillation of nonimpulsive delay differential equations has been investigated in [8]. In this theory, the main goal is to show that oscillations of nonlinear equations are equivalent to those of corresponding linear equations. Linearized oscillation of impulsive delay differential equations is addressed in [58,59].

Consider the nonlinear impulsive delay differential equation

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) f_i(x(t - \tau_i(t))) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{4.22}$$

and the associated linear equations and inequalities

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{4.23}$$

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) \leq 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{4.24}$$

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) \geq 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{4.25}$$

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in \mathbb{N}. \end{cases} \tag{4.26}$$

In what follows we shall assume that some of the following conditions are satisfied.

(A_1) $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are fixed points with $\lim_{k\to\infty} t_k = \infty$,
(A_2) $p_i, \tau_i \in C(J, \mathbb{R}^+)$, $\lim_{t\to\infty}\{t - \tau_i(t)\} = \infty$, $i = 1, 2, \ldots, n$,
(A_3) $I_k(x) \in C(\mathbb{R}, \mathbb{R})$, $I_k(0) = 0$, $k \in \mathbb{N}$;
(A_4) $|I_k(x)| \leq b_k |x|$,
(A_5) $b_k \geq 0$, $k \in \mathbb{N}$, and $\prod_{1\leq k<\infty}(1 + b_k) < \infty$,
(A_6) $\int_t^\infty \sum_{i=1}^n p_i(s) ds = \infty$,
(A_7) $f_i(\mu)\mu > 0$ for $\mu \neq 0$,
(A_8) $I_k(x)$ is not decreasing, $k \in \mathbb{N}$.

For any $\sigma \geq t_0$, we shall also need to define $r_\sigma = \min_{1\leq i\leq n} \inf_{t\geq\sigma}\{t - \tau_i(t)\}$.

First we prove the following lemmas which are modeled after [58].

Lemma 4.1. *Assume that* (A_1)–(A_3) *and* (A_8) *hold. If* (4.24) *has an eventually positive solution* $x(t)$*, then* (4.23) *and* (4.25) *have eventually positive solutions* $y(t)$ *and* $z(t)$*, which satisfy* $x(t) \leq y(t) \leq z(t)$ *for sufficiently large* t.

Proof. We assume that $x(t)$ is an eventually positive solution of (4.24), then there exists a k such that $x(t) > 0$ and $t - \tau_i(t) \geq t_0$ for $t \geq t_k - r_\sigma$, $i = 1, 2, \ldots, n$. Set

$$z(t) = y(t) = x(t), \quad t_k - r_\sigma \leq t \leq t_k.$$

Then $y(t_k^+) = x(t_k^+)$ and $z(t_k^+) = x(t_k^+)$. From Theorem 2.3, we have

$$z(t) \geq y(t) \geq x(t), \quad t_k < t \leq t_{k+1},$$

and

$$y(t_{k+1}^+) = y(t_{k+1}) + I_{k+1}(y(t_{k+1})) \geq x(t_{k+1}^+).$$

Similarly, we have $z(t_{k+1}^+) \geq y(t_{k+1}^+)$. Thus, by induction,

$$z(t) \geq y(t) \geq x(t), \quad t \in (t_m, t_{m+1}], \quad \text{and} \quad z(t_m^+) \geq y(t_m^+) \geq x(t_m^+), \quad m \geq k.$$

Hence,

$$z(t) \geq y(t) \geq x(t) \quad \text{for } t \geq t_k.$$

This completes the proof of Lemma 4.1. $\square$

Lemma 4.2. *Assume that* (A_1)–(A_4) *and* (A_8) *hold. If* (4.24) *has an eventually positive solution* $x(t)$*, then* (4.26) *has an eventually positive solution* $y(t)$*, which satisfies* $y(t) \geq x(t)$*.*

The proof of Lemma 4.2 is similar to that of Lemma 4.1.

Lemma 4.3. *If* (A_1)–(A_6) *hold, then every nonoscillatory solution of* (4.22) *tends to zero as* $t \to \infty$*.*

Proof. Without loss of generality, we assume that $z(t)$ is an eventually positive solution of (4.22). Take a sequence $\{t_k^*\}$ from $\{t_j\}_{j=1}^{\infty}$ such that $I_k(z(t_k^*)) > 0$ and choose the corresponding sequence $\{b_k^*\}$ from $\{b_j\}_{j=1}^{\infty}$. Then, there exists a sufficiently large $T \geq t_0$ such that

$$z(t) > 0 \quad \text{and} \quad z'(t) \leq 0, \quad \text{for } t \geq T, \quad t \neq t_k. \tag{4.27}$$

Moreover, it can be seen that $z(t)$ is decreasing in $(t_k^*, t_{k+1}^*]$ for $t_k^* \geq T$, $k \geq m$. Hence, for $t \geq t_k^*$,

$$z(t) \leq z(t_k^{*^+}) \leq (1 + b_k^*)z(t_k^*) \leq \cdots \leq (1 + b_k^*)(1 + b_{k-1}^*) \cdots (1 + b_m^*)z(t_m^*).$$

In view of (A_5) and the last inequality, there exists a constant $M > 0$ such that $z(t) < M$ for $t \geq T$. Now we claim that $\liminf_{t\to\infty} z(t) = 0$. Otherwise, set $\liminf_{t\to\infty} z(t) = l > 0$, then there exists $T_2 \geq T_1 \geq T$ such that $t - \tau_i(t) > T_1$ for $t \geq T_2$, $i = 1, 2, \ldots, n$ and $z(t) \geq l/2$ for $t \geq T_1$. In view of $l/2 \leq z(t) \leq l$ and the continuity of $f_i(\eta)$, there exists $b > 0$ such that $f_i(z(t - \tau_i(t))) \geq b$ for $t \geq T_2$, then from (4.22), we have

$$0 = z'(t) + \sum_{i=1}^{n} p_i(t) f_i(z(t - \tau_i(t))) \geq z'(t) + b \sum_{i=1}^{n} p_i(t).$$

An integration of the above inequality from t to ∞ with $t \geq T_2$ yields

$$l - M \sum_{k \geq m} b_k^* - z(t) + b \left(\frac{l}{2}\right) \int_t^{\infty} \sum_{i=1}^{n} p_i(s) \mathrm{d}s \leq 0,$$

which in view of (A_5) and (A_7) is a contradiction, and this confirms our claim.

Now we shall prove $\limsup_{t\to\infty} z(t) = 0$. In view of (4.27) and the fact that $\liminf_{t\to\infty} z(t) = 0$, we can take subsequence $\{\xi_k\}$ from $\{t_k^*\}$ such that

$$\lim_{k\to\infty} z(\xi_k) = 0. \tag{4.28}$$

Similarly, take another subsequence $\{\eta_k^+\}$ from $\{t_k^{*^+}\}$ between ξ_k and ξ_{k+1} such that $\lim_{k\to\infty} z(\eta_k^+) = \limsup_{t\to\infty} z(t)$. Assume that b_k^* and $\overline{b}_k^*$ correspond to the moments ξ_k, η_k of impulsive effects, respectively. According to (4.22) and (4.27), it follows from

$$0 < z(\eta_k^+) \leq (1 + \overline{b}_k^*)z(\eta_k) \leq (1 + \overline{b}_k^*)z(\eta_{k-1}^+) \leq (1 + \overline{b}_k^*)(1 + \overline{b}_{k-1}^*) \cdots (1 + b_k^*)z(\xi_k)$$

and (4.28) that $\lim_{k\to\infty} z(\eta_k^+) = 0$. Therefore, we have $\lim_{t\to\infty} z(t) = 0$, which completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Assume that* (A_2) *holds, then the following two statements are equivalent:*

(i) *The equation*

$$x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0$$

has an eventually positive solution.

(ii) *There exists* $\epsilon_0 > 0$ *such that for every* $\epsilon \in [0, \epsilon_0]$*, the equation*

$$x'(t) + (1 - \epsilon) \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0$$

has an eventually positive solution.

Theorem 4.8 ([58]). *Assume that* (A_1)–(A_8) *hold, and there exists a* $\delta > 0$ *such that*

$$|f_i(\mu)| \leq |\mu| \quad \text{for } 0 < |\mu| < \delta.$$

If (4.22) *is oscillatory, then* (4.23) *is also oscillatory.*

Proof. Let $y(t)$ be an eventually positive solution of (4.23). Then there exists t_k such that $y(t) > 0$ for $t \geq t_k - r_\sigma$. According to Lemma 4.3, we have $\lim_{t\to\infty} y(t) = 0$. Set $x(t) = y(t)$, $t_k - r_\sigma \leq t \leq t_k$, then there exists a neighborhood of t_k such that $x(t) > 0$ and

$$x'(t) = -\sum_{i=1}^{n} p_i(t) f_i(x(t-\tau_i(t))) \geq -\sum_{i=1}^{n} p_i(t) x(t-\tau_i(t)),$$
$$x(t_k^+) = x(t_k) + I_k(x(t_k)) = y(t_k) + I_k(y(t_k)) = y(t_k^+).$$

By Lemma 4.1, $x(t) \geq y(t) > 0$ for sufficiently large t, which contradicts the fact that (4.22) is oscillatory. This completes the proof of Theorem 4.8. □

Theorem 4.9 ([58]). *Assume that* (A_1)–(A_8) *hold, and*

$$\liminf_{\mu\to 0} \frac{f_i(\mu)}{\mu} \geq 1, \quad i = 1, 2, \ldots, n. \tag{4.29}$$

If (4.26) *is oscillatory, then* (4.22) *is also oscillatory.*

Proof. If (4.22) has an eventually positive solution $x(t)$, then for every $\epsilon > 0$, there exists $T \geq t_0$ such that

$$f_i(x(t-\tau_i(t))) \geq (1-\epsilon) x(t-\tau_i(t)), \quad t \geq T,\ i = 1, 2, \ldots, n.$$

Then from (4.22),

$$\begin{cases} x'(t) + \sum_{i=1}^{n} (1-\epsilon) p_i(t) x(t-\tau_i(t)) \leq 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases}$$

has an eventually positive solution. By Lemma 4.2,

$$\begin{cases} x'(t) + \sum_{i=1}^{n} (1-\epsilon) p_i(t) x(t-\tau_i(t)) = 0, & t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in \mathbb{N}, \end{cases}$$

also has an eventually positive solution. According to Theorem 2 in [31],

$$x'(t) + \sum_{i=1}^{n} (1-\epsilon) p_i(t) \prod_{\sigma \leq t_k < t-\tau_i(t)} (1+b_k)^{-1} x(t-\tau_i(t)) = 0$$

has an eventually positive solution. Thus, by Lemma 4.4,

$$x'(t) + \sum_{i=1}^{n} p_i(t) \prod_{\sigma \leq t_k < t-\tau_i(t)} (1+b_k)^{-1} x(t-\tau_i(t)) = 0$$

has an eventually positive solution. Therefore, (4.26) has an eventually positive solution, which is a contradiction. This completes the proof. □

Corollary 4.6. *Assume that* (A_1)–(A_8) *hold, and*

$$\lim_{\mu\to 0} \frac{f_i(\mu)}{\mu} = 1, \quad i = 1, 2, \ldots, n.$$

Then (4.22) *is oscillatory if and only if* (4.26) *is oscillatory.*

Corollary 4.7. *Assume that* (A_1)–(A_8) *and* (4.29) *are satisfied, and* $\tau(t) = \min_{1\leq i\leq n}\{\tau_i(t)\}$. *If either*

$$\liminf_{t\to\infty} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \prod_{t-\tau_i(t) \leq t_k < t} (1+b_k)^{-1} p_i(s) \mathrm{d}s > \frac{1}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty}\int_{t-\tau(t)}^{t}\sum_{i=1}^{n}\prod_{t-\tau_i(t)\le t_k<t}(1+b_k)^{-1}p_i(s)\mathrm{d}s>1$$

holds, then (4.22) is oscillatory.

Corollary 4.8. *Assume that* (A_1)–(A_8) *and* (4.29) *are satisfied, and* $r(t)=\max_{1\le i\le n}\{\tau_i(t)\}$. *If*

$$\liminf_{t\to\infty}\int_{t-r(t)}^{t}\sum_{i=1}^{n}\prod_{t-\tau_i(t)\le t_k<t}(1+b_k)^{-1}p_i(s)\mathrm{d}s\le\frac{1}{\mathrm{e}},$$

then (4.22) has an eventually positive solution.

Example 4.2. Consider the equation

$$\begin{cases}x'(t)+tx(t-2)\mathrm{e}^{x(t-2)}=0, & t\ne t_k,\quad t\ge 2,\\ x(t_k^+)-x(t_k)=\dfrac{1}{2^k}x(t_k), & k\in\mathbb{N}.\end{cases}\tag{4.30}$$

It is easily seen that Eq. (4.30) satisfies the conditions of Corollary 4.7, and hence it is oscillatory.

Linearized oscillation theory is also examined by Berezansky and Braverman [59]. They considered the following differential equation

$$\begin{cases}x'(t)+\displaystyle\sum_{k=1}^{m}r_k(t)f_k[x(h_k(t))]=0, & t\ne\tau_j,\\ x(\tau_j)=I_j(x(\tau_j^-)), & j\in\mathbb{N}.\end{cases}\tag{4.31}$$

Unlike the results in [58], in [59] authors did not assume coefficients $r_k(t)$ and delays to be continuous. They also applied the results to impulsive equations of mathematical biology. Following their work we introduce the following conditions:

(a1) $r_k(t)\ge 0$, $k=1,2,\dots,m$, are Lebesgue measurable and locally essentially bounded functions;
(a2) $h_k:[0,\infty)\to\mathbb{R}$, $k=1,2,\dots,m$, are Lebesgue measurable functions, $h_k(t)\le t$, $\lim_{t\to\infty}h_k(t)=\infty$;
(a3) $f_k:\mathbb{R}\to\mathbb{R}$, $k=1,2,\dots,m$, are continuous functions, $xf_k(x)>0$, $x\ne 0$;
(a4) $0=\tau_0<\tau_1<\tau_2<\cdots$ are fixed points, $\lim_{j\to\infty}\tau_j=\infty$;
(a5) I_j are continuous functions satisfying $xI_j(x)>0$, $x\ne 0$, $j\in\mathbb{N}$.

Theorem 4.10 (*[59]*). *Let* (a1)–(a5) *and the following conditions hold:*

(i) *There exists a k such that*

$$\int_{t_0}^{\infty}r_k(t)=\infty,\qquad \liminf_{t\to\infty}f_k(t)>0;$$

(ii) *For sufficiently large x*

$$\left|I_j(x)\right|\le c_j\left|x\right|,\quad c_j\ge 1,\qquad \sum_{j=1}^{\infty}(c_j-1)<\infty;$$

(iii) *There exist* $\delta>0$, $a_k>0$, $d_j>0$, $k=1,2,\dots,m$, $j\in\mathbb{N}$ *such that*

$$\lim_{x\to 0}\frac{f_k(x)}{x}=a_k\quad\text{and}\quad\left|I_j(x)\right|\le d_j\left|x\right|\quad\text{for }\left|x\right|<\delta.$$

If for some ϵ, $0<\epsilon<a_k$, *all solutions of the impulsive equation*

$$\begin{cases}x'(t)+\displaystyle\sum_{k=1}^{m}(a_k-\epsilon)r_k(t)x(h_k(t))=0, & t\ne\tau_j,\\ x(\tau_j)=d_jx(\tau_j^-), & j\in\mathbb{N},\end{cases}$$

are oscillatory, then all solutions of Eq. (4.31) are also oscillatory.

Theorem 4.11 ([59]). *Let* (a1)–(a5) *and there exist* $a_k > 0$, $k = 1, 2, \ldots, m$, $d_j > 0, j \in \mathbb{N}$, *such that*

$$|f_k(x)| \geq a_k\, |x| \quad \text{and} \quad \left|I_j(x)\right| \leq d_j\, |x|\,.$$

If all solutions of the impulsive equation

$$\begin{cases} x'(t) + \sum_{k=1}^{m} a_k r_k(t) x(h_k(t)) = 0, & t \neq \tau_j, \\ x(\tau_j) = d_j x(\tau_j^-), & j \in \mathbb{N}, \end{cases}$$

are oscillatory, then all solutions of Eq. (4.31) *are also oscillatory.*

Theorem 4.12 ([59]). *Suppose that* (a1)–(a5) *hold and there exist* $M_k > 0$, $k = 1, 2, \ldots, m$, $d_j > 0, j \in \mathbb{N}$, *such that* $f_k(x) \leq M_k x$, $I_j(x) \geq d_j x$ *for any* $x > 0$. *If there exists a nonoscillatory solution of the linear impulsive equation*

$$\begin{cases} x'(t) + \sum_{k=1}^{m} M_k r_k(t) x(h_k(t)) = 0, & t \neq \tau_j, \\ x(\tau_j) = d_j x(\tau_j^-), & j \in \mathbb{N}, \end{cases} \tag{4.32}$$

then there exists an eventually positive solution of (4.31).

Theorem 4.13 ([59]). *Suppose that* (a1)–(a5) *hold and there exist* $M_k > 0$, $k = 1, 2, \ldots, m$, $d_j > 0, j \in \mathbb{N}$, *such that* $f_k(x) \geq M_k x$, $I_j(x) \leq d_j x$ *for any* $x < 0$. *If there exists a nonoscillatory solution of the linear impulsive Eq.* (4.32), *then there exists a nonoscillatory (eventually negative) solution of* (4.31).

Now we consider the impulsive logistic equation

$$\begin{cases} N'(t) = N(t) \sum_{k=1}^{m} r_k(t) \left(1 - \frac{N(h_k(t))}{K}\right), & t \neq \tau_j, \\ N(\tau_j) - K = b_j(N(\tau_j^-) - K), \end{cases} \tag{4.33}$$

with the initial condition

$$N(t) = \psi(t) \geq 0, \quad t < t_0, \qquad N(t_0) = y_0 > 0, \tag{4.34}$$

where r_k, h_k satisfy (a1) and (a2), $K > 0$ and $\psi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

Definition 4.1. A positive solution N of (4.33)–(4.34) is said to be oscillatory about K if there exists a sequence $t_n, t_n \to \infty$, such that $N(t_n) - K = 0$, $n \in \mathbb{N}$; N is said to be nonoscillatory about K if there exists $t_0 \geq 0$ such that $|N(t) - K| > 0$ for $t \geq t_0$.

Theorem 4.14 ([59]). *Suppose that* $0 < b_j \leq 1$ *and the conditions* (a1), (a2), (a4), *and the first equality of hypothesis* (i) *in Theorem* 4.10 *hold. If for sufficiently small* $\epsilon > 0$, $\delta > 0$ *all solutions of the equation*

$$\begin{cases} x'(t) + (1 - \epsilon) \sum_{k=1}^{m} r_k(t) x(h_k(t)) = 0, & t \geq t_0, t \neq \tau_j, \\ x(\tau_j) = (b_j + \delta) x(\tau_j^-), \end{cases} \tag{4.35}$$

are oscillatory, then all solutions of (4.33) *are oscillatory about* K.

Proof. Using the substitution $N(t) = K\mathrm{e}^{x(t)}$, Eq. (4.33) is transformed into the Eq. (4.31) with $f_k(x) = f(x) = \mathrm{e}^x - 1$, $I_j(x) = \ln(1 - b_j + b_j \mathrm{e}^x)$.

Now we can apply Theorem 4.10. For this, condition (ii) holds with $c_k = 1$, condition (iii) is satisfied with $a_k = 1$, and since $\lim_{x\to 0} \frac{I_j(x)}{x} = b_j$, there exists $\delta > 0$ such that $\left|\frac{I_j(x)}{x}\right| \leq (b_j + \delta)$. Thus all conditions of Theorem 4.10 are satisfied. Hence all solutions of Eq. (4.35) are oscillatory about zero. This in turn implies that all solutions of (4.33) are oscillatory about K. $\square$

Theorem 4.15 ([59]). *Suppose that* $b_j \geq 1$, $\prod_{j=1}^{\infty} b_j \leq M < \infty$, *and the conditions* (a1), (a2), (a4) *hold. If there exists a nonoscillatory solution of linear impulsive delay equation*

$$\begin{cases} x'(t) + \sum_{i=1}^{m} r_k x(h_k(t)) = 0, & t \neq \tau_j, \\ x(\tau_j) = b_j x(\tau_j^-), \end{cases}$$

then there exists a nonoscillatory about K *solution of the Eq.* (4.33).

The proof of Theorem 4.15 is similar to that of Theorem 4.14.

In [59], following generalized Lasota-Wazewska equation, which describes the survival of red blood cells, is also considered

$$\begin{cases} N'(t) = -\mu N(t) + p\mathrm{e}^{-\gamma N(h(t))}, & t \geq 0, \\ N(\tau_j) - N^* = b_j(N(\tau_j^-) - N^*), \end{cases} \tag{4.36}$$

where $\mu, p, \gamma > 0$, $h(t)$ satisfies condition (a2), and N^* satisfies the equation $N^* = \frac{p}{\mu}\mathrm{e}^{-\gamma N^*}$.

Using the transformation $N(t) = N^* + \frac{1}{\gamma}x(t)$, Eq. (4.36) takes the form

$$\begin{cases} x'(t) + \mu x(t) + \mu\gamma N^*[1 - \mathrm{e}^{-x(h(t))}] = 0, \\ x(\tau_j) = b_j x(\tau_j^-). \end{cases} \tag{4.37}$$

Clearly, Eq. (4.37) is of the form (4.31). Hence solutions of (4.36) are oscillatory about N^* if and only if all solutions of (4.37) are oscillatory about zero. Thus as a consequence of Theorem 4.10 and any one of the Theorems 4.12 and 4.13 we have the following results.

Theorem 4.16. *Suppose that $0 < b_j \leq 1$, and there exists $\epsilon > 0$ such that all solutions of the linear equation*

$$\begin{cases} x'(t) + (1-\epsilon)\mu x(t) + (1-\epsilon)\mu\gamma N^* x(h(t)) = 0, \\ x(\tau_j) = b_j x(\tau_j^-) \end{cases}$$

are oscillatory. Then all solutions of Eq. (4.36) are oscillatory about N^*.

Corollary 4.9. *Suppose that $0 < b_j \leq 1$, $\limsup_{t\to\infty}(t - h(t)) < \infty$, and*

$$\liminf_{t\to\infty} \mu\gamma N^* \int_{h(t)}^{t} \exp\{\mu(s - h(s))\} \prod_{h(s)<\tau_j\leq s} b_j^{-1}\mathrm{d}s > \frac{1}{\mathrm{e}}.$$

Then all solutions of Eq. (4.36) are oscillatory about N^*.

Theorem 4.17. *Suppose that $b_j \geq 1$, and there exists a nonoscillatory solution of the linear equation*

$$\begin{cases} x'(t) + \mu x(t) + \mu\gamma N^* x(h(t)) = 0, \\ x(\tau_j) = b_j x(\tau_j^-). \end{cases}$$

Then there exists a nonoscillatory about N^* solution of Eq. (4.36).

Corollary 4.10. *Suppose that $b_j \geq 1$ and*

$$\limsup_{t\to\infty} \mu\gamma N^* \int_{h(t)}^{t} \exp\{\mu(s - h(s))\} \prod_{h(s)<\tau_j\leq s} b_j^{-1}\mathrm{d}s < \frac{1}{\mathrm{e}}.$$

Then there exists a nonoscillatory about N^* solution of Eq. (4.36).

In [60], authors considered the food-limited equation

$$\begin{cases} N'(t) = r(t)N(t)\dfrac{K - N(h(t))}{K + \sum\limits_{i=1}^{m} p_i(t)N(g_i(t))}, \\ N(t_k^+) - N(t_k) = b_k(N(t_k) - K), \end{cases}$$

and obtained sufficient conditions for the oscillation or nonoscillation about K.

For further results on the oscillation of nonlinear impulsive equations which describe models in mathematical biology, we refer to the interesting papers [61–65].

4.3. Second and higher order equations

In recent years oscillation of second order nonlinear impulsive delay differential equations has been examined extensively [66–77]. Following [67], we consider the equation

$$\begin{cases} (a(t)(x'(t))^{\sigma})' + f(t, x(t), x(t-\tau)) = 0, & t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), & x'(t_k^+) = \widetilde{I}_k(x'(t_k)), \end{cases} \tag{4.38}$$

where $\tau > 0$, $0 < \sigma = p/q$ with p and q odd integers, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $\lim_{t\to\infty} t_k = \infty$, $t_{k+1} - t_k > \tau$.

With respect to (4.38), we assume that:

(i) $f(t,u,v)$ is continuous in $[t_0-\tau,\infty)\times\mathbb{R}\times\mathbb{R}$, $t_0\geq 0$, $uf(t,u,v)>0(uv>0)$ and $f(t,u,v)/\varphi(v)\geq p(t)(v\neq 0)$, where $p(t)$ is continuous in $[t_0-\tau,\infty)$, $p(t)\geq 0$ and $x\varphi(x)>0(x\neq 0)$, $\varphi'(x)\geq 0$;
(ii) $I_k(x)$, $\widetilde{I}_k(x)$ are continuous in $\mathbb{R}$ and there exist positive numbers c_k, c_k^*, d_k, d_k^* such that

$$c_k^*\leq\frac{I_k(x)}{x}\leq c_k,\qquad d_k\leq\frac{\widetilde{I}_k(x)}{x}\leq d_k^*;$$

(iii) $a(t)$ is continuous positive function in $[t_0-\tau,\infty)$.

Lemma 4.5. *Let $x(t)$ be a solution of Eq. (4.38). Suppose that there exists some $T\geq t_0$ such that $x(t)>0$ for $t\geq T$. If*

$$\begin{aligned}&A(t_{j+1})-A(t_j)+\frac{d_{j+1}}{c_{j+1}}(A(t_{j+2})-A(t_{j+1}))+\frac{d_{j+1}d_{j+2}}{c_{j+1}c_{j+2}}(A(t_{j+3})-A(t_{j+2}))\\&\quad+\cdots+\frac{d_{j+1}d_{j+2}\cdots d_{j+n}}{c_{j+1}c_{j+2}\cdots c_{j+n}}(A(t_{j+n+1})-A(t_{j+n}))+\cdots=+\infty,\end{aligned}\tag{4.39}$$

for some $t_j(\geq t_1)$, where $A(t)=\int_{t_0}^{t}\frac{ds}{a^{1/\sigma}(s)}$, then $x'(t_k^+)\geq 0$ and $x'(t)\geq 0$ for $t\in(t_k,t_{k+1}]$, $t_k\geq T$.

Proof. First, we will show that $x'(t_k)\geq 0$, $t_k\geq T$. If not, then there exists some j such that $t_j\geq T$, $x'(t_j)<0$ and $x'(t_j^+)=\widetilde{I}_j(x'(t_j))\leq d_jx'(t_j)<0$. Let

$$a(t_j^+)(x'(t_j^+))^\sigma=-\beta^\sigma\quad(\beta>0)\quad\text{and}\quad S(t)=a(t)(x'(t))^\sigma.$$

From (4.38), it is clear that $S(t)$ is nonincreasing in $(t_{j+i-1},t_{j+i}]$. Thus,

$$a(t_{j+1})(x'(t_{j+1}))^\sigma\leq a(t_j^+)(x'(t_j^+))^\sigma=-\beta^\sigma<0.\tag{4.40}$$

By induction, we have

$$a(t_{j+n})(x'(t_{j+n}))^\sigma\leq-(d_{j+1}d_{j+2}\cdots d_{j+n-1})^\sigma\beta^\sigma<0,\quad n\geq 2.\tag{4.41}$$

Now we claim that for $n\geq 2$,

$$\begin{aligned}x(t_{j+n})\leq{}&c_{j+1}c_{j+2}\cdots c_{j+n-1}\Bigg[x(t_j^+)-\beta(A(t_{j+1})-A(t_j))-\frac{d_{j+1}}{c_{j+1}}\beta(A(t_{j+2})-A(t_{j+1}))\\&-\cdots-\frac{d_{j+1}d_{j+2}\cdots d_{j+n-1}}{c_{j+1}c_{j+2}\cdots c_{j+n-1}}\beta(A(t_{j+n})-A(t_{j+n-1}))\Bigg].\end{aligned}\tag{4.42}$$

Since $S(t)$ is nonincreasing in $(t_{j+i-1},t_{j+i}]$, we get

$$x'(t)\leq\frac{[a(t_j^+)(x'(t_j^+))^\sigma]^{1/\sigma}}{a^{1/\sigma}(t)}.$$

Integrating the above inequality from s to t and then letting $t\to t_{j+1}$, $s\to t_j^+$, we obtain

$$\begin{aligned}x(t_{j+1})&\leq x(t_j^+)+[a(t_j^+)(x'(t_j^+))^\sigma]^{1/\sigma}(A(t_{j+1})-A(t_j))\\&\leq x(t_j^+)-\beta(A(t_{j+1})-A(t_j)).\end{aligned}\tag{4.43}$$

Similar to (4.43), we also have

$$x(t_{j+2})\leq x(t_{j+1}^+)+[a(t_{j+1}^+)(x'(t_{j+1}^+))^\sigma]^{1/\sigma}(A(t_{j+2})-A(t_{j+1})).\tag{4.44}$$

Now by condition (ii), (4.40), (4.43), (4.44), we obtain

$$\begin{aligned}x(t_{j+2})&\leq I_{j+1}x(t_{j+1})+a(t_{j+1})^{1/\sigma}\widetilde{I}_{j+1}x'(t_{j+1})(A(t_{j+2})-A(t_{j+1}))\\&\leq c_{j+1}\left[x(t_j^+)-\beta(A(t_{j+1})-A(t_j))-\frac{d_{j+1}}{c_{j+1}}\beta(A(t_{j+2})-A(t_{j+1}))\right].\end{aligned}$$

Thus (4.42) holds for $n=2$. By induction it can be shown that (4.42) holds for any $n\geq 2$. Since $x(t_k)\geq 0(t_k\geq T)$, one finds that (4.42) contradicts (4.39). Therefore, $x'(t_k)\geq 0$, $(t_k\geq T)$. By condition (ii), we have for any $t_k\geq T$, $x'(t_k^+)\geq d_k$, $x'(t_k)\geq 0$. Because $S(t)$ is nonincreasing in $(t_{j+i-1},t_{j+i}]$, we get $S(t)\geq 0$, $t\in(t_{j+i-1},t_{j+i}]$, which implies $x'(t)\geq 0$. This completes the proof. □

Remark 4.1. If $x(t)$ is eventually negative and (4.39) holds, then $x'(t_k^+) \leq 0$, $t_k \geq T$, and $x'(t) \leq 0$ for $t \in (t_{j+i-1}, t_{j+i}]$.

Theorem 4.18 ([67]). *Assume that* (4.39) *holds and there exists a positive integer k_0 such that $c_k^* \geq 1$ for $k \geq k_0$. If*

$$\int_{t_0}^{t_1} p(s)\mathrm{d}s + \frac{1}{(d_1^*)^\sigma}\int_{t_1}^{t_2} p(s)\mathrm{d}s + \frac{1}{(d_1^*d_2^*)^\sigma}\int_{t_2}^{t_3} p(s)\mathrm{d}s + \cdots + \frac{1}{(d_1^*d_2^*\cdots d_n^*)^\sigma}\int_{t_n}^{t_{n+1}} p(s)\mathrm{d}s + \cdots = +\infty, \tag{4.45}$$

then Eq. (4.38) *is oscillatory.*

Proof. Without loss of generality, we can assume $k_0 = 1$ and (4.38) has a solution $x(t)$, such that $x(t) > 0$ for $t \geq t_0$. It follows from Lemma 4.5 that $x'(t) \geq 0$ for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$. Let

$$w(t) = \frac{a(t)(x'(t))^\sigma}{\varphi(x(t-\tau))}.$$

Then $w(t_k^+) \geq 0$, $k \in \mathbb{N}$, and $w(t) \geq 0$, $t \geq t_0$. Using condition (i) and Eq. (4.38), we get

$$\begin{aligned} w'(t) &= -\frac{f(t, x(t), x(t-\tau))}{\varphi(x(t-\tau))} - \frac{a(t)(x'(t))^\sigma \varphi'(x(t-\tau))x'(t-\tau)}{\varphi^2(x(t-\tau))} \\ &\leq -p(t), \quad t \neq t_k, t_k + \tau. \end{aligned} \tag{4.46}$$

It follows from the continuity of $a(t)$, condition (ii), $c_k^* \geq 1$ and $\varphi'(x) \geq 0$ that

$$\begin{aligned} w(t_k^+) &= \frac{a(t_k^+)(x'(t_k^+))^\sigma}{\varphi(x(t_k^+ - \tau))} \leq (d_k^*)^\sigma w(t_k), \\ w(t_k^+ + \tau) &= \frac{a(t_k^+ + \tau)(x'(t_k^+ + \tau))^\sigma}{\varphi(x(t_k^+))} \leq w(t_k + \tau). \end{aligned} \tag{4.47}$$

Integrate (4.46) from s to t, let $s \to t_0^+$ and $t \to t_1$, then in view of (4.47), we get

$$w(t_1^+) \leq (d_1^*)^\sigma w(t_1) \leq (d_1^*)^\sigma \left[w(t_0^+) - \int_{t_0}^{t_1} p(s)\mathrm{d}s \right] \leq (d_1^*)^\sigma w(t_0^+) - (d_1^*)^\sigma \int_{t_0}^{t_1} p(s)\mathrm{d}s.$$

Similarly,

$$\begin{aligned} w(t_2^+) &\leq (d_2^*)^\sigma w(t_2) \leq (d_2^*)^\sigma \left[w(t_1^+ + \tau) - \int_{t_1+\tau}^{t_2} p(s)\mathrm{d}s \right] \\ &\leq \cdots \leq (d_1^*d_2^*)^\sigma w(t_0^+) - (d_1^*d_2^*)^\sigma \int_{t_0}^{t_1} p(s)\mathrm{d}s - (d_2^*)^\sigma \int_{t_1}^{t_2} p(s)\mathrm{d}s. \end{aligned}$$

Thus by induction, we have

$$w(t_n^+) \leq (d_1^*d_2^*\cdots d_n^*)^\sigma \left\{ w(t_0^+) - \int_{t_0}^{t_1} p(s)\mathrm{d}s - \frac{1}{(d_1^*)^\sigma}\int_{t_1}^{t_2} p(s)\mathrm{d}s - \cdots - \frac{1}{(d_1^*d_2^*\cdots d_{n-1}^*)^\sigma}\int_{t_{n-1}}^{t_n} p(s)\mathrm{d}s \right\}.$$

Since $w(t) \geq 0$, the last inequality contradicts (4.45). This completes the proof of Theorem 4.18. □

Theorem 4.19 ([67]). *Assume that* (4.39) *holds, and $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. If*

$$\int_{t_0}^{t_1} p(s)\mathrm{d}s + \frac{\varphi(c_1^*)}{(d_1^*)^\sigma}\int_{t_1}^{t_2} p(s)\mathrm{d}s + \frac{\varphi(c_1^*)\varphi(c_2^*)}{(d_1^*d_2^*)^\sigma}\int_{t_2}^{t_3} p(s)\mathrm{d}s + \cdots + \frac{\varphi(c_1^*)\varphi(c_2^*)\cdots\varphi(c_n^*)}{(d_1^*d_2^*\cdots d_n^*)^\sigma}\int_{t_n}^{t_{n+1}} p(s)\mathrm{d}s + \cdots = +\infty,$$

then Eq. (4.38) *is oscillatory.*

The proof of Theorem 4.19 is similar to that of Theorem 4.18.

Corollary 4.11. *Assume that* (4.39) *holds, and there exists a positive integer k_0 such that $c_k^* \geq 1$, $d_k^* \leq 1$ for $k \geq k_0$. If*

$$\int^{+\infty} p(s)\mathrm{d}s = +\infty,$$

then Eq. (4.38) *is oscillatory.*

Corollary 4.12. *Assume that (4.39) holds, and there exist a positive integer k_0 and a constant $\alpha > 0$ such that*

$$c_k^* \geq 1, \qquad \frac{1}{(d_k^*)^\sigma} \geq \left(\frac{t_{k+1}}{t_k}\right)^\alpha, \quad \text{for } k \geq k_0,$$

and

$$\int^{+\infty} t^\alpha p(t)\mathrm{d}t = +\infty.$$

Then Eq. (4.38) is oscillatory.

Example 4.3. Consider the superlinear impulsive equation

$$\begin{cases} x'' + \dfrac{1}{t^3}x^{2n-1}\left(t - \dfrac{1}{3}\right) = 0, & t \neq k, k \in \mathbb{N}, \\ x(k^+) = \left(\dfrac{k+1}{k}\right)x(k), x'(k^+) = x'(k), & k \in \mathbb{N}, \end{cases} \tag{4.48}$$

where $n \geq 2$ is a natural number, $c_k = c_k^* = (k+1)/k$, $d_k = d_k^* = 1$, $t_0 = 2/3$, $A(t) = t - 2/3$, $p(t) = 1/t^3$, $t_k = k$ and $\varphi(x) = x^{2n-1}$. It is clear that all conditions of Corollary 4.12 are satisfied for $k_0 = 1$, $\alpha = 3$. Thus every solution of Eq. (4.48)) is oscillatory.

The technique of [67] has been adapted to a variety of different types of impulsive equations [69,70,73,75].

In [72], Simeonov examined the impulsive delay differential equations of the type

$$\begin{cases} (r(t)x'(t))' + F(t, x(\tau_1(t)), \dots, x(\tau_m(t))) = 0, & t \neq t_k, \\ \Delta(r(t_k)x'(t_k)) + F_k(x(\tau_1(t_k)), \dots, x(\tau_m(t_k))) = 0. \end{cases} \tag{4.49}$$

The oscillatory properties of the solutions of (4.49) were compared with the oscillatory properties of the inequality

$$\begin{cases} [(r(t)x'(t))' + F(t, x(\tau_1(t)), \dots, x(\tau_m(t)))]\operatorname{sgn} x(t) \leq 0, & t \neq t_k, \\ [\Delta(r(t_k)x'(t_k)) + F_k(x(\tau_1(t_k)), \dots, x(\tau_m(t_k)))]\operatorname{sgn} x(t_k) \leq 0, \end{cases} \tag{4.50}$$

and with the comparison equation

$$\begin{cases} (q(t)y'(t))' + G(t, y(\sigma_1(t)), \dots, y(\sigma_m(t))) = 0, & t \neq t_k, \\ \Delta(q(t_k)y'(t_k)) + G_k(y(\sigma_1(t_k)), \dots, y(\sigma_m(t_k))) = 0. \end{cases} \tag{4.51}$$

With respect to the above equations, we need the following conditions:

(H1) $r \in PLC^1(\mathbb{R}^+, \mathbb{R}^+)$, $\int^\infty \frac{\mathrm{d}t}{r(t)} = +\infty$ and $r(t) > 0$, $r(t_k^+) > 0$ for $t, t_k \in \mathbb{R}^+$;
(H2) $\tau_i \in C(\mathbb{R}^+, \mathbb{R})$, $\lim_{t\to\infty} \tau_i(t) = +\infty$ and $\tau_i(t) \leq t$ for $t \in \mathbb{R}^+$, $i = 1, 2, \dots, m$;
(H3) $F \in C((t_{k-1}, t_k] \times \mathbb{R}^m, \mathbb{R})$, $F_k \in C(\mathbb{R}^m, \mathbb{R})$ for $k \in \mathbb{N}$ and $x_1F(t, x_1, \dots, x_m) > 0$, $x_1F_k(x_1, \dots, x_m) > 0$, $\lim_{y\to x, t\to t_{k-1}^+} F(t, y) \in \mathbb{R}$ for $x_1x_i > 0$, $i = 1, 2, \dots, m$, $t \in \mathbb{R}$, $k \in \mathbb{N}$, where $y = (y_1, \dots, y_m)$, $x = (x_1, \dots, x_m)$;
(H4) $F(t, x_1, \dots, x_m)$ and $F_k(x_1, \dots, x_m)$ are nondecreasing functions with respect to x_i $i = 1, 2, \dots, m$ for each fixed $t \in \mathbb{R}$ and $k \in \mathbb{N}$;
(H5) $q \in PLC^1(\mathbb{R}^+, \mathbb{R}^+)$, $\int^\infty \frac{\mathrm{d}t}{q(t)} = +\infty$ and $q(t) > 0$, $q(t_k^+) > 0$ for $t, t_k \in \mathbb{R}^+$;
(H6) $\sigma_i \in C(\mathbb{R}^+, \mathbb{R})$, $\lim_{t\to\infty} \sigma_i(t) = +\infty$ and $\sigma_i(t) \leq t$ for $t \in \mathbb{R}^+$, $i = 1, 2, \dots, m$;
(H7) $G \in C((t_{k-1}, t_k] \times \mathbb{R}^m, \mathbb{R})$, $G_k \in C(\mathbb{R}^m, \mathbb{R})$ for $k \in \mathbb{N}$ and $x_1G(t, x_1, \dots, x_m) > 0$, $x_1G_k(x_1, \dots, x_m) > 0$, $\lim_{y\to x, t\to t_{k-1}^+} G(t, y) \in \mathbb{R}$ for $x_1x_i > 0$, $i = 1, 2, \dots, m$, $t \in \mathbb{R}$, $k \in \mathbb{N}$, where $y = (y_1, \dots, y_m)$, $x = (x_1, \dots, x_m)$;
(H8) $G(t, x_1, \dots, x_m)$ and $G_k(x_1, \dots, x_m)$ are nondecreasing functions with respect to x_i, $i = 1, 2, \dots, m$ for each fixed$t \in \mathbb{R}$ and $k \in \mathbb{N}$.

Lemma 4.6. *Let the condition* (H1) *be satisfied, $x \in C([T, +\infty), \mathbb{R})$ and $rx' \in PLC^1 \in ([T, +\infty), \mathbb{R})$. Then the following hold:*

1. *1. If $x(t) > 0$, $(r(t)x'(t))' \leq 0$, $\Delta(r(t_k)x'(t_k)) \leq 0$ for $t \geq T$, $t \neq t_k \geq T$, then $r(t)x'(t)$ is nonincreasing and nonnegative for $t \geq T$.*
2. *If $x(t) < 0$, $(r(t)x'(t))' \geq 0$, $\Delta(r(t_k)x'(t_k)) \geq 0$ for $t \geq T$, $t \neq t_k \geq T$, then $r(t)x'(t)$ is nondecreasing and nonpositive for $t \geq T$.*

Proof. 1. Obviously $r(t)x'(t)$ is nonincreasing for $t \geq T$. Assume that there exists $T_0 \geq T$ such that $r(t)x'(t) \leq -m < 0$ for $t \geq T_0$. Integrating this inequality we obtain a contradiction:

$$0 < x(t) \leq x(T_0) - mR(t) + mR(T_0) \to -\infty \quad \text{as } t \to +\infty,$$

where $R(t) = \int^t \frac{ds}{r(s)}$. Hence, $r(t)x'(t) \geq 0$ for $t \geq T$.

2. The proof of assertion 2 is analogous. □

Theorem 4.20 (*[72]*). *Suppose that conditions* (H1)–(H8) *hold, and there exists* $T \geq 0$ *such that*

$$\begin{aligned} &r(t) \leq q(t), \quad t \geq T,\\ &\tau_i(t) \geq \sigma_i(t), \quad t \geq T,\ i = 1, 2, \ldots, m,\\ &F(t, x_1, \ldots, x_m)\operatorname{sgn} x_1 \geq G(t, x_1, \ldots, x_m)\operatorname{sgn} x_1,\\ &F_k(x_1, \ldots, x_m)\operatorname{sgn} x_1 \geq G_k(x_1, \ldots, x_m)\operatorname{sgn} x_1 \end{aligned} \tag{4.52}$$

for $x_1x_i > 0,\ i = 1, 2, \ldots, m,\ t \geq T$ *and* $k : t_k \geq T$. *Then Eq.* (4.51) *has a nonoscillatory solution if inequality* (4.50) *has a nonoscillatory solution.*

Proof. Without loss of generality we assume that inequality (4.50) has an eventually positive solution $x(t) > 0,\ t \geq T$. It follows from (H2) that there exists $T_0 \geq T$ such that $\tau_i(t) \geq T$ for $t \geq T_0,\ i = 1, 2, \ldots, m$. Then $x(\tau_i(t)) > 0,\ t \geq T_0, i = 1, \ldots, m$ and from (H3) and (4.50), the hypothesis of the first part of Lemma 4.6 is satisfied. Hence, $r(t)x'(t)$ is nonincreasing and $r(t)x'(t) \geq 0$ for $t \geq T_0$. Then it follows from (4.50) that

$$r(t)x'(t) \geq \int_t^\infty F(s, x(\tau_1(s)), \ldots, x(\tau_m(s)))ds + \sum_{t \leq t_k} F_k(x(\tau_1(t_k)), \ldots, x(\tau_m(t_k))),$$

which implies

$$x(t) \geq x(T_0) + \int_{T_0}^t \frac{1}{r(u)} \left[\int_u^\infty F(s, x(\tau_1(s)), \ldots, x(\tau_m(s)))ds + \sum_{u \leq t_k} F_k(x(\tau_1(t_k)), \ldots, x(\tau_m(t_k)))\right] du, \quad t \geq T_0.$$

Since $x(t)$ is nondecreasing, it follows from conditions (4.52) and the last inequality that

$$x(t) \geq x(T_0) + \int_{T_0}^t \frac{1}{q(u)} \left[\int_u^\infty G(s, x(\sigma_1(s)), \ldots, x(\sigma_m(s)))ds + \sum_{u \leq t_k} G_k(x(\sigma_1(t_k)), \ldots, x(\sigma_m(t_k)))\right] du. \tag{4.53}$$

Let $T_{-1} = \min_{1 \leq i \leq m} \inf\{\sigma_i(t) : t \geq T_0\}$ and B be the space of continuous functions $y : [T_{-1}, +\infty) \to \mathbb{R}$. In the set

$$Y = \{y \in B : x(T_\sigma) \leq y(t) \leq x(t),\ t > T_0;\ y(t) = x(t), T_{-1} \leq t \leq T_0\}$$

define the operator $S : Y \to B$ as follows

$$Sy(t) = \begin{cases} x(T_0) + \displaystyle\int_{T_0}^t \frac{1}{q(u)} \left[\int_u^\infty G(s, y(\sigma_1(s)), \ldots, y(\sigma_m(s)))ds\right.\\ \qquad \left. + \displaystyle\sum_{u \leq t_k} G_k(y(\sigma_1(t_k)), \ldots, y(\sigma_m(t_k)))\right] du, \quad t > T_0,\\ x(t), \quad T_{-1} \leq t \leq T_0. \end{cases}$$

Using (4.53) it is obtained that $SY \subseteq Y$. Define the functions $y_0(t) \equiv x(t_0),\ t \geq T_{-1}$ and $y_n(t) = Sy_{n-1}(t),\ t \geq T_{-1}, n \in \mathbb{N}$. By induction we conclude that

$$y_n(t) = x(t), T_1 \leq t \leq T_0 \quad \text{and} \quad x(T_0) \leq y_n(t) \leq y_{n-1}(t) \leq x(t), \quad t \geq T_0, n \in \mathbb{N}.$$

Therefore there exists the finite limit $\lim_{n \to +\infty} y_n(t) = y(t)$ for each $t \geq T_{-1}, y(t) = x(t), T_{-1} \leq t \leq T_0$ and $x(T_0) \leq y(t) \leq x(t), t > T_0$.

Now in view of the Lebesgue dominated convergence theorem we have $y \in Y$ and $y(t) = Sy(t),\ t \geq T_{-1}$. Finally, a direct verification shows that $y(t)$ is a nonoscillatory solution of Eq. (4.51). □

Theorem 4.21 (*[72]*). *Suppose that conditions* (H1)–(H8) *and* (4.52) *hold. Then Eq.* (4.49) *is oscillatory if Eq.* (4.51) *is oscillatory.*

Theorem 4.22 (*[72]*). *Suppose that conditions* (H1)–(H4) *hold. Then the followings assertions are equivalent:*

1. *Eq.* (4.49) *has a nonoscillatory solution.*
2. *Inequality* (4.50) *has a nonoscillatory solution.*

Theorem 4.23 ([72]). *Suppose that conditions* (H1)–(H4) *hold and the condition*

$$\int^{\infty} |F(t, c, \ldots, c)|\,dt + \sum_{k}^{\infty} |F_k(c, \ldots, c)| = +\infty \tag{4.54}$$

is satisfied for each constant $c \neq 0$. Then Eq. (4.49) is oscillatory.

Proof. Without loss of generality we assume that Eq. (4.49) has an eventually positive solution $x(t) > 0,\ t \geq T$. Using the same arguments as in Theorem 4.20, we obtain

$$r(t)x'(t) \geq 0, \quad t \geq T_0. \tag{4.55}$$

Let $T_{-1} = \min_{1\leq i\leq m} \inf\{\tau_i(t) : t \geq T_0\}$. Since $x(t)$ is a nondecreasing function, we have for $t \geq T_0,\ i = 1, 2, \ldots, m$ that $x(\tau_i(t)) \geq x(T_{-1}) \equiv c > 0$. Therefore

$$\begin{cases} (r(t)x'(t))' + F(t, c, \ldots, c) \leq 0, \quad t \geq T_0,\ t \neq t_k, \\ \Delta(r(t_k)x'(t_k)) + F_k(c, \ldots, c) \leq 0, t_k \geq T_0. \end{cases}$$

Integrating the last inequality and using (4.54), we get

$$\lim_{t\to\infty} r(t)x'(t) = -\infty,$$

which contradicts (4.55). □

Oscillation properties of higher order nonlinear differential equations have been addressed in [78–80]. Particularly, in [78], authors considered the third order nonlinear differential equation

$$\begin{cases} y'''(t) + f(t, y(t), y(g(t))) = 0, \quad t \neq t_k, \\ \Delta y''(t_k) + f_k(y(t_k), y(g(t_k))) = 0, \quad k \in \mathbb{N}, \\ \Delta y(t_k) = \Delta y'(t_k) = 0, \quad k \in \mathbb{N}, \end{cases} \tag{4.56}$$

assuming that the following conditions hold:

(H1) $g \in C((0, \infty), (0, \infty)), g(t) \leq t, g'(t) \geq 0, \lim_{t\to\infty} g(t) = +\infty$;
(H2) $f \in C((0, \infty) \times \mathbb{R}^2, \mathbb{R}), uf(t, u, v) > 0$, for $uv > 0$; $f_k \in C(\mathbb{R}^2, \mathbb{R}), uf_k(u, v) > 0$ for $uv > 0, u, v \in \mathbb{R}, k \in \mathbb{N}$;
(H3) $|f(t, u_1, v_1)| \leq |f(t, u_2, v_2)|, |f_k(u_1, v_1)| \leq |f_k(u_2, v_2)|, k \in \mathbb{N}$ for $|u_1| \leq |u_2|, |v_1| \leq |v_2|, u_1u_2 > 0, v_1v_2 > 0$.

Theorem 4.24. *Let* (H1)–(H3) *hold. If*

$$\int^{\infty} t^2 |f(t, c, c)|\,dt + \sum_{k=1}^{\infty} t_k^2 |f_k(c, c)| < +\infty$$

for some constant $c \neq 0$, then Eq. (4.56) has a bounded nonoscillatory solution.

Theorem 4.25. *Let* (H1)–(H3) *hold. If there exists a point $T \geq 0$ such that*

$$\int_T^{\infty} (t - T)^2 |f(t, c, c)|\,dt + \sum_{t_k\geq T} (t_k - T)^2 |f_k(c, c)| = +\infty$$

for some constant $c \neq 0$, then each bounded solution $y(t)$ of the Eq. (4.56) either oscillates or

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y''(t) = 0.$$

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