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Ends in digraphs

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Abstract

In this paper a sort of end concept for directed graphs is introduced and examined. Two one-way infinite paths are called *equivalent* iff there are infinitely many pairwise disjoint paths joining them. An *end* of an undirected graph is an equivalence class with respect to this relation. For two one-way infinite directed paths U and V define: (a) $U \leqslant V$ iff there are infinitely many pairwise disjoint directed paths from U to V; (b) $U \sim V$ iff $U \leqslant V$ and $V \leqslant U$. The relation $\leqslant$ is a quasiorder, and hence $\sim$ is an equivalence relation whose classes are called *ends*. Furthermore, $\leqslant$ induces a partial order on the set of ends of a digraph. In the main section, necessary and sufficient conditions are presented for an abstract order to be representable by the end order of a digraph.

0. Introduction

In the early 1940s, Hopf [7] and Freudenthal [4] studied discrete groups with the aid of the end concept. In 1964, Halin independently reintroduced the end concept in order to study infinite graphs [5], and it turned out to be a basic and important tool in infinite graph theory. Diestel [2] and Polat [8] give each other supplementary overviews about several aspects of the subject.

In the literature so far, the end notion seems to have been used only for the investigation of *undirected* graphs. The question occurs, whether it is possible to define an analogue of the end notion for digraphs. It is the purpose of this paper to show that an analogue of 'undirected ends' for digraphs makes sense and to point out and to examine some differences to ends in undirected graphs. The end concept for digraphs introduced in Section 2 can be regarded both as a generalization of the end notion for undirected graphs as well as a refinement of certain subends of the underlying

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undirected graph. (Any graph G can be understood as a symmetric digraph by substituting each edge of G by a directed cycle of length 2, and any digraph D can be regarded as a special orientation of the underlying undirected graph.)

To state things in a more detailed fashion, call two one-way infinite paths — or briefly *rays* — of an (undirected) graph G *equivalent* iff there exist infinitely many pairwise disjoint paths joining them. An equivalence class with respect to this relation is called an *end of* G. In an undirected graph, every pair of ends can be separated by a finite subgraph. For the purpose of distinction, directed paths (resp. cycles) will be called *tracks* (resp. *circuits*). For one-way infinite tracks U and V in a digraph D let $U \leqslant_D V$ mean that there are infinitely many pairwise disjoint $U \rightarrow V$-tracks and $U \sim_D V$ mean that $U \leqslant V$ and $V \leqslant U$. Then $\leqslant$ is a quasiorder and $\sim$ is an equivalence relation on the set of all one-way infinite tracks of D. The classes with respect to $\sim$ are also called *ends*, and $\leqslant$ establishes a partial order on the set of ends of D. In Section 2 some basic results on ends in digraphs are proved and some differences to ends in undirected graphs are pointed out. For example, if two one-way infinite tracks are comparable, they cannot be separated by a finite subdigraph, but it is possible that they belong to different ends. Moreover, there are three different end-notions: ω-ends (containing only one-way infinite tracks *going* to infinity), ω^*-ends (containing only one-way infinite tracks *coming* from infinity), and composed ends (possibly containing both types of one-way infinite tracks). Furthermore, an end of a digraph belongs either completely or not at all to a strong component of D.

Under these circumstances, one may ask whether it is justified to call the $\sim$-classes 'ends', since they are not ends in the topological sense. Nevertheless, these 'directed ends' are a generalization of the 'undirected' end-notion, and they reflect the ramification structure of the one-way infinite directed paths in a natural way. However, for a full justification of this naming, a sort of 'directed topology' ought to be developed.

In Section 3, the main section of the paper, the following question is considered: Which types of abstract orders can arise on sets of ends of digraphs? An abstract poset $(X,\leqslant)$ is said to be *ω-representable* iff there is a digraph D such that the ω-end-poset of D is order isomorphic to $(X,\leqslant)$. A necessary condition for an order to be ω-representable is that every strictly increasing sequence has a supremum. Section 3 also contains a sufficient condition, but it is a slightly technical. From that sufficient condition, the fact that a *chain* is ω-representable if and only if every strictly increasing sequence has a supremum can be derived. Dual results are valid for ω^*-ends. Furthermore, some results concerning composed-end-representability are presented.

1. Preliminaries

The relations 'proper subset of' (resp. 'subset of', 'finite proper subset of', 'finite subset of') are denoted by $\subset$ (resp. $\subseteq$, $\sqsubset$, $\sqsubseteq$). For the set of natural numbers and its nth segments the following symbols shall be used: $\mathbb{N} := \{1,2,3,\ldots\}$, $\mathbb{N}_0 := \{0,1,2,\ldots\}$, $[n] := \{1,2,\ldots,n\}$, $[0] := \emptyset$. The set of all integers, rational numbers, and real numbers will be denoted by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. If X is a set, then $|X|$ shall denote the cardinality of X. Throughout this paper, the symbol a will be shorthand for $\{a\}$.

A set X with the reflexive and transitive relation $\leqslant$ (written $(X,\leqslant)$) is called a *quasiordered set* (or *quoset*), and the relation $\leqslant$ is called a *quasiorder*. $(X,\leqslant)$ is called a *partially ordered set* (or *poset*), and the relation $\leqslant$ is called a *partial order* iff the relation $\leqslant$ is also antisymmetric. A poset $(X,\leqslant)$ is called a *totally ordered set* (or *chain*) iff $\leqslant$ is also total. ω denotes the order type of $\mathbb{N}$. If $(X,\leqslant)$ is a quoset and $Y \subseteq X$, then $(Y,\leqslant)$ shall be used instead of $(Y,\leqslant|_{Y\times Y})$, and if $x,y \in X$, then $x < y$ means $x \leqslant y \wedge y \nleqslant x$. If X is a set and $R \subseteq X \times X$ is a relation on X, then let $R^{-1} := \{(x,y) \mid (y,x) \in R\}$ and $\tilde{R} := R \cap R^{-1}$. If R is an equivalence relation on X and $x \in X$, then $R(x) := \{y \in X \mid (x,y) \in R\}$. Further, let $X_R := \{R(x) \mid x \in X\}$ and $R^* \subseteq X_R \times X_R$ be defined by $R(x)\, R^*\, R(y) \;:\Leftrightarrow\; x\,R\,y$. The following is a well known order theoretic result [3, p. 62, 3.19 Satz]:

Proposition 1.1. *Let (X,R) be a quoset. Then $\tilde{R}$ is an equivalence relation on X and $\tilde{R}^*$ a partial order on $X_{\tilde{R}}$.*

If D is an arbitrary directed (resp. undirected) graph, the set of all vertices and the set of all arcs (resp. edges) of D will be denoted by V_D and A_D (resp. E_D), respectively, and it will be written $D = (V_D, A_D)$ (resp. $D = (V_D, E_D)$). The digraphs considered are assumed to have no multiple arcs, though 2-circuits may be possible, i.e. $(u,v) \in A_D$ and $(v,u) \in A_D$. C is called a *sub(di)graph of D* (in symbols: $C \subseteq D$) iff $V_C \subseteq V_D$ and $A_C \subseteq A_D$, whereby A_C only contains arcs that are incident with vertices $\in V_C$. Sets of vertices, edges, and/or arcs may be identified with sub(di)graphs. $C \sqsubseteq D$ means that C is a *finite* sub(di)graph of D. If $C \subseteq D$, then $D[C]$ denotes the induced sub(di)graph of D which has V_C as its set of vertices, and $D - C$ denotes the induced sub(di)graph of D which is spanned by the set of vertices $V_D - V_C$. D^u denotes the underlying undirected graph.

Let T be a digraph with $n+1$ vertices ($n \in \mathbb{N}_0$), $V_T = \{v_0, v_1, \ldots, v_n\}$ say. If, for each $i \in [n]$, A_T contains exactly one element of the set $\{(v_{i-1}, v_i), (v_i, v_{i-1})\}$, then T is called a *zigzag-track* (*of length n*), and v_0 and v_n are called the *rim vertices of T*. If, for each $i \in [n]$, A_T contains exactly the arc (v_{i-1}, v_i), then T is also called a *track* (*of length n*), and v_0 (resp. v_n) is called the *initial* (resp. *terminal*) *vertex of T*. A zigzag-track is called *proper* if it is not a track. Note that every single vertex can be interpreted as a (zigzag-)track and also as a circuit of length 0.

One-way infinite (zigzag-)tracks are defined analogously and briefly called 1-*(zigzag-)tracks*. There are two kinds of 1-tracks: Those that go to infinity (called *ω-tracks*) and those that come from infinity (called *ω^*-tracks*). $\mathscr{T}_D(\infty)$ (resp. $\mathscr{T}_D(\omega)$, $\mathscr{T}_D(\omega^*)$) denotes the set of all 1- (resp. ω-, ω^*-) tracks of the digraph D. If U and V are both ω- or both ω^*-tracks, then U and V are said to have the *same direction*. Otherwise, U and V are said to have *opposite directions*. Every infinite subtrack of an ω- or ω^*-track T is called a *rest of T*.

Let D be a digraph and $A,B \subseteq D$. Every zigzag-track (resp. track) with one rim vertex (resp. initial vertex) $\in A$ and the other rim vertex (resp. terminal vertex) $\in B$ and having no further vertex in common with $A \cup B$ is called an *A,B-zigzag-track* (resp. an *$A \rightarrow B$-track*). B is said *to be reachable from A* (written $A \rightarrow_D B$) iff there exists an $A \rightarrow B$-track in D. $A \leftrightarrow_D B$ abbreviates the fact that A and B are mutually reachable from each other. Clearly, $(V_D, \rightarrow_D)$ is a quoset.

Let T be an arbitrary (finite or infinite) track. If $v \in V_T$, then v is said *to be on the track T*. If $V, W \subseteq T$, $V \cap W = \emptyset$, and $v \rightarrow_T w$ holds for all $v \in V$, $w \in W$, then it is said that '*V is before W on T*' or '*W is behind V on T*'.

Remark 1.2. If D is a digraph, then every (finite or infinite) path of D^u corresponds to an (equivalence) class of zigzag-tracks of D which all have the same vertices in the same arrangement. The set of all these classes is a partition of the set of all zigzag-tracks of D and can be mapped bijectively to the set of all paths of D^u, so that there is hardly any difference between running along the zigzag-tracks without consideration of the direction of the affiliated arcs and examining the underlying undirected paths.

A digraph S is called a *subdivision of D* iff S can be obtained from the digraph D by replacing every arc (v,w) of D by a $v \rightarrow w$-track of length > 0 which has (except v and w) no vertex in common with D nor with any other 'replacing track'.

In a designation of the form X_D or $X_D(Y)$ the parameter D may be suppressed whenever there is no danger of confusion.

2. Ends in digraphs

The relation $\leqslant_D$ on $\mathscr{T}_D(\infty)$ defined in the introduction is fundamental to all further investigations in this paper. In the sequel, some statements dealing with $\leqslant$ are repeated and supplemented.

Clearly, if $U \leqslant V$, then $U' \leqslant V'$ for all rests U' of U, V' of V.

The following proposition is helpful in many contexts to simplify proofs and can easily be verified:

Proposition 2.1. *Let D be a digraph. Following statements are equivalent:*
(1) $U \leqslant V$.
(2) $U \rightarrow_{D-S} V$ *for all* $S \sqsubseteq D$.

The proof of the next result is also trivial, but it shows that it makes no sense to introduce an end concept based upon 1-zigzag-tracks since transitivity cannot be expected. Of course, similar statements hold for $(\mathscr{T}(\omega), \leqslant)$ and $(\mathscr{T}(\omega^*), \leqslant)$.

Proposition 2.2. *Let D be a digraph. Then $(\mathscr{T}(\infty), \leqslant)$ is a quoset.*

Proof. Clearly, $\leqslant$ is reflexive since any vertex itself is a track. The relation $\leqslant$ is also transitive: Let U and W be 1-tracks, V be an ω-track, $U \leqslant V$, $V \leqslant W$, and $S \sqsubseteq D$. According to 2.1, there exists a $U \to V$-track T in $D - S$ and a $V \to W$-track T' in $D - (S \cup T)$. Now, $U \to_{D-S} W$ can easily be seen, as the initial vertex of T' is behind the terminal vertex of T on V. A similar construction is feasible if V is an ω^*-track. The only difference is that the tracks T, T' have to be constructed in the reverse order. □

Now, Proposition 1.1 yields that $\sim_D$ is an equivalence relation on $\mathscr{T}(\infty)$ (and also on $\mathscr{T}(\omega)$, $\mathscr{T}(\omega^*)$). Moreover, Proposition 1.1 yields that

$$\mathscr{E} \leqslant_D \mathscr{E}' :\Leftrightarrow \text{ there exist } U \in \mathscr{E},\ V \in \mathscr{E}' \text{ such that } U \leqslant V$$

is a partial order on the $\sim$-factorization of $\mathscr{T}(\infty)$ (and also of $\mathscr{T}(\omega)$, $\mathscr{T}(\omega^*)$), which will be investigated in Section 3.

Definition 2.3. An equivalence class of $\mathscr{T}(\infty)$ with respect to $\sim$ is called a *composed end* or briefly a *c-end of D*. $\Omega_D(c)$ denotes the set of all c-ends of D. If T is a one-way infinite track, then $\mathscr{E}_D(c, T)$ denotes exactly that c-end of D which contains T. *ω-ends*, *ω^*-ends*, $\Omega_D(\omega)$, $\Omega_D(\omega^*)$, $\mathscr{E}_D(\omega, T)$, and $\mathscr{E}_D(\omega^*, T)$ are defined analogously. If $\mathscr{E}$ is an arbitrary end of D, $\mathscr{E}^u$ denotes the set of all rays in D^u that exactly correspond to tracks in $\mathscr{E}$. ω- and ω^*-ends are also called *simple ends*. If there are statements valid for each of the end-notions introduced here, then the symbols Ω_D etc. may be used in a consistent way. In other words, within one statement, the symbols Ω_D etc. can be substituted uniformly by $\Omega_D(c)$, $\Omega_D(\omega)$, and $\Omega_D(\omega^*)$.

The concept of ω-ends catches the *forward* ω-ramification structure and the dual concept of ω^*-ends, the *backward* ω-ramification structure of a digraph D. On the other hand, the concept of c-ends allows a general view over the whole ω-ramification structure of D. Clearly, $|\Omega(c)| \geqslant 0 \Leftrightarrow \mathscr{T}(\infty) \neq \emptyset$, $|\Omega(\omega)| \geqslant 0 \Leftrightarrow \mathscr{T}(\omega) \neq \emptyset$, and $|\Omega(\omega^*)| \geqslant 0 \Leftrightarrow \mathscr{T}(\omega^*) \neq \emptyset$.

The following statement about the relations between composed and simple ends can easily be verified and justifies in a way the expression 'composed ends'.

Proposition 2.4. *Every simple end is completely contained in a composed end. A c-end is either an ω-end, an ω^*-end, or the union of exactly one ω-end and exactly one ω^*-end. Hence,*

$$|\Omega(\omega)| \leqslant |\Omega(c)|, \qquad |\Omega(\omega^*)| \leqslant |\Omega(c)|,$$

$$|\Omega(c)| \leqslant |\Omega(\omega)| + |\Omega(\omega^*)| \leqslant 2 \cdot |\Omega(c)|.$$

One has to be very careful in generalizing results about ends in undirected graphs to results about ends in digraphs. Halin [5] defined two rays U and V of an undirected graph G to be equivalent iff there is a ray W which meets both U and V infinitely many times. If, for $U, V \in \mathscr{T}_D(\infty)$, equivalence had been stated as

$$U \sim'_D V :\Leftrightarrow \text{ there is } W \in \mathscr{T}_D(\infty) \text{ which meets both } U \text{ and } V \text{ infinitely many times,}$$

then $\sim'$ would not necessarily have been an equivalence relation. It is not difficult to construct digraphs such that $\sim'$ is not transitive. The next proposition (compare with [5, 1.1]) shows that every counterexample contains 1-tracks of opposite directions.

Proposition 2.5. *Let D be a digraph and $U, V \in \mathscr{T}(\infty)$. Following statements are equivalent*:

(1) *If U and V have the same direction, then $U \sim' V$. Otherwise there are infinitely many pairwise disjoint circuits $\subseteq D$ such that each of them has at least one vertex in common both with U and V.*

(2) *$U \leftrightarrow_{D-S} V$ for all $S \sqsubseteq D$.*

(3) *$U \sim V$.*

Proof.

(2) $\Leftrightarrow$ (3) follows immediately from Proposition 2.1.

(1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) can be shown by straightforward inductive constructions. □

One gains no new relation by replacing $\mathscr{T}(\infty)$ by $\mathscr{T}(\omega)$ or by $\mathscr{T}(\omega^*)$ in the definition of $\sim'$. If $U, V \in \mathscr{T}(\omega)$ and $U \sim' V$, then Proposition 2.5 shows that there is an ω-track W which meets both U and V infinitely many times. Thus, it is impossible that there is $W \in \mathscr{T}(\omega^*)$ but no $W' \in \mathscr{T}(\omega)$ which meets both U and V infinitely many times. A dual statement holds if $U, V \in \mathscr{T}(\omega^*)$. Note that it would have been possible to define two rays U, V in an undirected graph G to be equivalent iff there are infinitely many pairwise disjoint cycles such that each of them has at least one vertex in common both with U and V.

The next statement can be gained by straightforward inductive constructions, too.

Proposition 2.6. *Let D be a digraph.*

(1) *If $\mathscr{E}$ is a c-end containing two 1-tracks U and V with opposite directions, then U and V have infinitely many vertices in common or D contains a subdivision of the digraph in Fig.* 1(a). *Similar statements hold for 1-tracks with the same direction in ω- and ω^*-ends.*

(2) *If $\mathscr{E}$, $\mathscr{F}$ are two different comparable ends of D, $U \in \mathscr{E}$ is an ω-track, and $V \in \mathscr{F}$ is an ω^*-track, then D contains a subdivision of the digraph in Fig.* 1(b). *Similar statements hold for other combinations of directions of U and V.*

The next result says something about the relations between ends in graphs and digraphs.

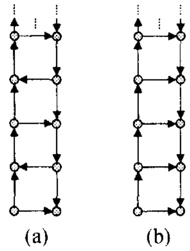

Fig. 1.

Proposition 2.7. *Let D be a digraph and $\mathscr{E}$, $\mathscr{F}$ ends of D.*

(1) *All rays $\in \mathscr{E}^u$ are contained in a single end of D^u.*

(2) *If $\mathscr{E}$ and $\mathscr{F}$ are comparable, then all rays in $\mathscr{E}^u$ and $\mathscr{F}^u$ are in the same end of D^u. Hence, every component of $(\Omega_D, \leqslant)$ belongs to one end of D^u. On the other hand, an end of D^u can contain several components of $(\Omega_D, \leqslant)$. Moreover, if $(\Omega_D(c), \leqslant)$ is a chain, D^u has exactly one end.*

(3) *All rays in $\mathscr{E}^u$ and $\mathscr{F}^u$ are in the same end of D^u if and only if there exist $U \in \mathscr{E}$ and $V \in \mathscr{F}$ such that there are infinitely many pairwise disjoint U, V-zigzag-tracks.*

(4) *Let $U, V \in \mathscr{T}_D(\infty)$. Then U^u and V^u are in the same end of D^u if and only if either $U \sim V$ or $U < V \dot{\vee} V < U$ or $U \not\leqslant V \wedge V \not\leqslant U \wedge$ there exist infinitely many pairwise disjoint U, V-zigzag-tracks.*

Proof. Remark 1.2 yields (3), (1), (2) and (4) follow immediately from (3). □

Proposition 2.7 clarifies a fundamental difference between ends in graphs and ends in digraphs: Whereas two ends $\mathscr{E}, \mathscr{E}'$ of an undirected graph G can always be separated by a finite subgraph S (that means: whenever $U \in \mathscr{E}$, $V \in \mathscr{E}'$, then rests of U and V are contained in different components of $G - S$), this is not true for digraphs. In light of this, difficulties in obtaining interesting results about ends in digraphs with the help of separation of ends can be expected. Furthermore, Proposition 2.7 shows that the end-concept of digraphs investigated here is a refinement of certain subsets of ends of undirected graphs.

The following examples (see Fig. 2) give a flavour of the differences between 'directed' and 'undirected' ends: Consider the $\mathbb{N} \times \mathbb{N}$-grid G. It has exactly one end. D_2 is an orientation of G such that D_2 has no end. On the other hand, the upwards directed $\mathbb{N} \times \mathbb{N}$-grid D_1 has two (completely independent) strictly increasing sequences of ω-ends and one ω-end — consisting of all 'diagonal' ω-tracks that are unbounded in both coordinates (the fat ω-track shows such a 'diagonal' ω-track) — as supremum of the two strictly increasing sequences. In Fig. 2, the ω-ends of D_1 are drawn as ideal points.

For the purpose of studying the relations between ends and connectivity in digraphs some new terminology is needed. A digraph D is called *strongly connected* iff $u \leftrightarrow v$

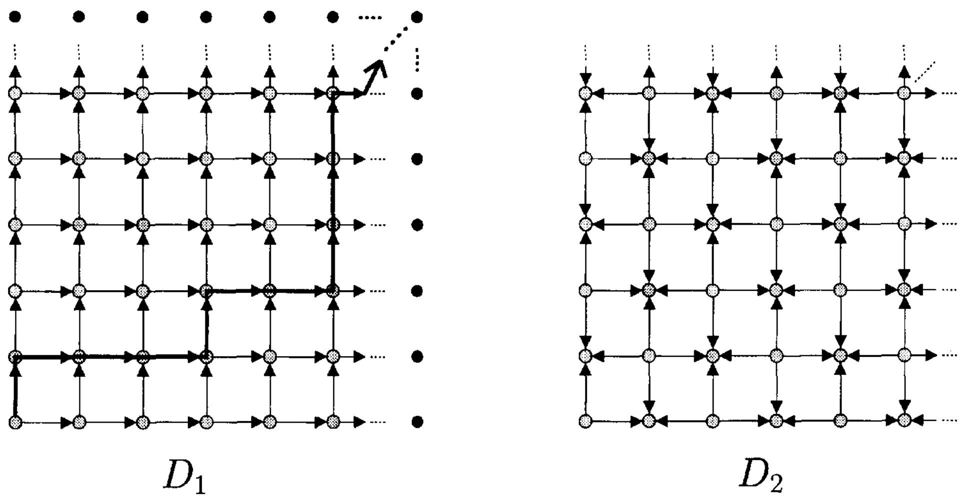

Fig. 2.

holds for all $u, v \in V_D$. A *strong component of* D is a maximal strongly connected subgraph of D. The strong components of a digraph give a partition of its vertex set.

Lemma 2.8. *Let D be a digraph, $\mathscr{E}$ an end (simple or composed), and S a strong component of D. If there exists $U \in \mathscr{E}$ such that $|U \cap S| = \infty$, then S contains a rest of every $V \in \mathscr{E}$. In other words: An end $\mathscr{E}$ belongs either completely or not at all to a strong component. In the latter case, every $U \in \mathscr{E}$ intersects with infinitely many strong components.*

Proof. Let S be a strong component of D and $U \in \mathscr{E}$ such that $|U \cap S| = \infty$. S contains a rest of U. Now, let $V \in \mathscr{E}$. Now, $v \leftrightarrow_D S$ follows from Proposition 2.5 for all but finitely many vertices v of V. Hence, S contains a rest of V. $\square$

Lemma 2.8 motivates the following definition:

Definition 2.9. A 1-track U resp. a (simple or composed) end $\mathscr{E}$ of a digraph D is called *cyclic* iff it belongs to a strong component in the sense of Lemma 2.8. Otherwise, U resp. $\mathscr{E}$ is called *acyclic*. $\mathscr{T}_D^{\mathrm{ac}}(\omega)$ (resp. $\Omega_D^{\mathrm{ac}}(\omega)$) denotes the set of all acyclic ω-tracks (resp. ω-ends) of D. $\mathscr{T}_D^{\mathrm{ac}}(\omega^*)$, $\Omega_D^{\mathrm{ac}}(c)$ etc. are defined analogously.

The following proposition deals with the inner structure of acyclic ends.

Proposition 2.10. *An acyclic composed end of the digraph D contains either exclusively ω-tracks or exclusively ω^*-tracks. Hence, there are no proper acyclic composed ends. Furthermore, if all ends of D are acyclic, then $|\Omega_D(c)| = |\Omega_D(\omega)| + |\Omega_D(\omega^*)|$.*

Proof. Let $\mathscr{E}$ be an acyclic c-end of D, $U \in \mathscr{E}$ an ω-track, and $V \in \mathscr{E}$ an ω^*-track. Let a be a vertex on U reachable from V and M the set of all vertices on U behind a. It follows, for all $u \in M$, $a \leftrightarrow u$, since V is an ω^*-track and U, $V \in \mathscr{E}$. Therefore, $U[M]$ belongs to a strong component of D, contradicting the acyclicity of $\mathscr{E}$. □

In the sequel, the relations between the end-structure of D and D^* are investigated. D^* denotes the *condensation of D* which is derived from D as follows: The vertices of D^* are exactly the strong components of D, and an arc passes in D^* from S to S' iff there is an arc (u,v) in D such that $u \in S$ and $v \in S'$. $\mathscr{S}_D$ denotes the set of all subdigraphs of D. Let $B \subseteq D$. $B(D^*)$ denotes the subdigraph of D^* having exactly those strong components of D as vertices that intersect with B and containing an arc (S,S') iff there is an arc from a vertex of S to a vertex of S' in B. Clearly,

$$\varphi^* : \mathscr{S}_D \to \mathscr{S}_{D^*}, \qquad \varphi^*(B) := B(D^*),$$

is a surjective mapping.

Proposition 2.11. *Let D be a digraph. Then* $\varphi^*|_{\mathscr{T}_D^{\mathrm{ac}}(\omega)} : \mathscr{T}_D^{\mathrm{ac}}(\omega) \to \mathscr{T}_{D^*}(\omega)$ *is a surjective mapping. A dual statement holds for* ω^**-tracks.*

Proof. Clearly, if U is an acyclic ω-track of D then $U(D^*) \in \mathscr{T}_{D^*}(\omega)$. Now, let U be an ω-track of D^*. Starting from the initial vertex, let the vertices of U (strong components of D !) be denoted by S_1, $S_2, S_3, \ldots$. For $i \in \mathbb{N}$, there exists an arc (a_i, b_i) such that $a_i \in S_i$ and $b_i \in S_{i+1}$. Furthermore, in S_{i+1}, there exists a $b_i \to a_{i+1}$-track T_i. Since $S_i \cap S_j = \emptyset$ for $i \neq j$, the union of the T_i and (a_i, b_i) $(i \in \mathbb{N})$ is an ω-track $U' \subseteq D$. Obviously, $U'(D^*) = U$. □

Lemma 2.12. *Let D be a digraph and U, V 1-tracks of D. U and V are contained in the same acyclic end of D if and only if* $U(D^*)$ *and* $V(D^*)$ *are contained in the same end of* D^*.

Proof. $\Rightarrow$: Let $\mathscr{E}$ be an acyclic ω-end of D and $U, V \in \mathscr{E}$. Every strong component of D only intersects with finitely many pairwise disjoint tracks joining U and V. (Assume $(W_i)_{i \in \mathbb{N}}$ to be a family of pairwise disjoint $U \to V$-tracks such that, for all $i \in \mathbb{N}$, W_i intersects with the strong component S, and let v a terminal vertex of one of the W_i on V. Then, of course, all vertices v' on V behind v would be reachable from S. On the other hand, U, and therefore S, would be reachable from all vertices of V. Hence, a rest of V would belong to S, a contradiction.) Therefore, if we remove finitely many strong components from D, then U and V remain mutually reachable from each other. Hence, for all $T' \sqsubseteq D^*$, $U(D^*) \leftrightarrow_{D^*-T'} V(D^*)$. Because of Proposition 2.5 $U(D^*)$ and $V(D^*)$ belong to the same ω-end of D^*. A dual argumentation holds for ω^*-tracks. Because of Proposition 2.10 there is nothing left to show.

$\Leftarrow$: If $U(D^*)$ and $V(D^*)$ are contained in the same ω-end of D^*, then, obviously, U and V are acyclic. Furthermore, for any $T \sqsubseteq D$, $T(D^*)$ is finite. Proposition 2.5 yields $U(D^*) \leftrightarrow_{D^*-T(D^*)} V(D^*)$. Now, $U \leftrightarrow_{D-T} V$ can be shown with the method in the proof of Proposition 2.11. □

Theorem 2.13. *Let D be a digraph.* $\varphi : \Omega_D^{\mathrm{ac}} \to \Omega_{D^*}$, $\varphi(\mathscr{E}) := \mathscr{E}^*$, *where* $\mathscr{E}^*$ *denotes the end of* D^* *which contains* $U(D^*)$ *for some* $U \in \mathscr{E}$, *is a bijective mapping. In other words: There exists a natural bijection between the acyclic ends of D and the ends of* D^*.

Proof. That φ is well defined follows from Lemma 2.12 '$\Rightarrow$'. The injectivity of φ follows from Lemma 2.12 '$\Leftarrow$'. The surjectivity of φ can be derived from Proposition 2.11. □

Corollary 2.14. $|\Omega_{D^*}| \leqslant |\Omega_D|$ *with equality in the finite case if and only if no strong component of D has an end.*

3. End orders

This section is dedicated to the following

Problem 1. *Which types of abstract orders can arise on sets of ends of digraphs?*

In other words, which types of abstract orders are ω- (resp. ω^*-, c-) representable? The notions ω^*- and *c-representability* are defined in the same manner as the ω-representability in the introduction. The following result gives a necessary condition for an order to be ω-representable. A (simple or composed) end of a digraph D is called *thick* iff it contains a system of infinitely many pairwise disjoint 1-tracks. Otherwise, the end is called *thin.* If $(X, \leqslant)$ is a poset, then $x \in X$ is called the *supremum* (resp. *infimum*) *of* $A \subseteq X$ iff $a \leqslant x$ (resp. $a \geqslant x$) holds for all $a \in A$ and, whenever $a \leqslant y \in X$ (resp. $a \geqslant y \in X$) holds for all $a \in A$, then $y \geqslant x$ (resp. $y \leqslant x$).

Theorem 3.1. *Let D be a digraph and* $(\mathscr{E}_i)_{i\in\mathbb{N}}$ *be a strictly increasing sequence in* $(\Omega(\omega), \leqslant)$. *Then there exists a thick* ω*-end* $\mathscr{F}$ *which is the supremum of the sequence* $(\mathscr{E}_i)_{i\in\mathbb{N}}$ *in* $(\Omega(\omega), \leqslant)$. *Moreover, D contains a subdivision of the digraph in Fig.* 3 *as subdigraph (compare with Halin's wall graph in Fig.* 4 *and Satz* 4 *of* [6]).

Remark 3.2. A dual result can be formulated for ω^*-tracks by simply substituting 'ω' by 'ω^*', 'strictly increasing' by 'strictly decreasing', and 'supremum' by 'infimum', and inverting the orientation of each arc.

Proof. Let $(\mathscr{E}_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence in $(\Omega(\omega), \leqslant)$. Choose $U_1 \in \mathscr{E}_1$ arbitrarily. If $U_i \in \mathscr{E}_i$ $(i \in \mathbb{N}_n)$ are already chosen such that the U_i $(i \in \mathbb{N}_n)$ are pairwise disjoint, then choose $U'_{n+1} \in \mathscr{E}_{n+1}$ arbitrarily. U'_{n+1} has only finitely many vertices in common with $\bigcup_{i\in\mathbb{N}_n} U_i$. Hence U'_{n+1} contains a rest U_{n+1} that has no vertex in common with $\bigcup_{i\in\mathbb{N}_n} U_i$. By induction, an infinite system of pairwise disjoint ω-tracks $(U_i)_{i\in\mathbb{N}}$

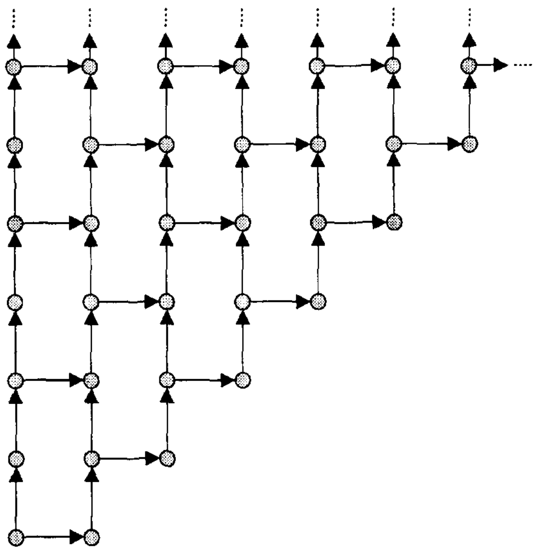

Fig. 3.

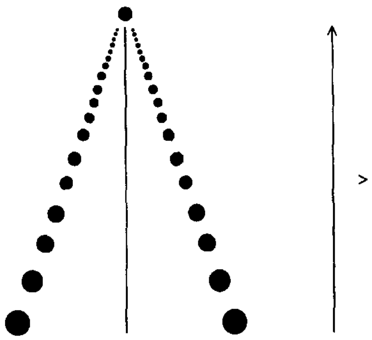

Fig. 4. The runway-poset.

with $U_i \in \mathscr{E}_i$ is obtained. Let the vertices of U_i — beginning with the initial vertex — be denoted by $u_{i,1}, u_{i,2}, u_{i,3}, \ldots$ ($i \in \mathbb{N}$). For $i \in \mathbb{N}$, $n \in \mathbb{N}$, let $U_i(n)$ denote the subtrack of U_i with the vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,n}$. The following two statements will be needed in the sequel:

(a) For $i, j \in \mathbb{N}$, $i < j$, there exist infinitely many pairwise disjoint $U_i \rightarrow U_j$-tracks because $(\mathscr{E}_i)_{i \in \mathbb{N}}$ is a strictly increasing sequence.

(b) For all $i \in \mathbb{N}$ and all $M \sqsubseteq \mathbb{N}$ with $i, i+1 \notin M$, there exist infinitely many pairwise disjoint $U_i \rightarrow U_{i+1}$-tracks each not meeting U_k for all $k \in M$.

(Otherwise there exist infinitely many pairwise disjoint $U_i \rightarrow U_{i+1}$-tracks such that each of them intersects with at least one of the U_k, $k \in M$. Since M is finite, there exists $k' \in M$ such that infinitely many pairwise disjoint $U_i \rightarrow U_{i+1}$-tracks intersect with $U_{k'}$. If $k' < i$, then $U_{k'} \geqslant U_i$, and hence $\mathscr{E}_{k'} \geqslant \mathscr{E}_i$, a contradiction. If $k' > i+1$, then $U_{i+1} \geqslant U_{k'}$, and hence $\mathscr{E}_{i+1} \geqslant \mathscr{E}_{k'}$, a contradiction as well.)

An overview of the remaining proof can be given as follows: By induction, infinitely many pairwise disjoint ω-tracks W_i ($i \in \mathbb{N}$) are constructed which belong to an ω-end $\mathscr{F}$, which turns out to be the supremum of $(\mathscr{E}_i)_{i\in\mathbb{N}}$ and whose union with the $(U_i)_{i\in\mathbb{N}}$ contains a subdivision of the digraph in Fig. 3.

Clearly, there exists a $U_1 \to U_2$-track $W_{1,1}$. For the purpose of cutting away everything that lies on a U_i 'under' $W_{1,1}$, let

$$S_1 := W_{1,1} \cup \bigcup \{U_i(n) \mid \exists i,n \in \mathbb{N}\colon\ u_{i,n} \in W_{1,1} \cap U_i\}.$$

S_1 is finite.

Because of (a), there exists a $U_1 \to U_2$-track $Y^1_{1,2}$ in $D - S_1$. Because of (b), there exists a $U_2 \to U_3$-track $W_{1,2}$ and a $U_1 \to U_2$-track $W_{2,1}$ such that $W_{1,2}$ does not meet U_1, the initial vertex of $W_{1,2}$ comes behind the terminal vertex of $Y^1_{1,2}$ on U_2, $W_{2,1}$ does not meet U_3 and $W_{1,2}$, and the terminal vertex of $W_{2,1}$ comes behind the initial vertex of $W_{1,2}$ on U_2. For the purpose of cutting away everything that lies on a U_i 'under' $W_{1,2}$, $W_{2,1}$, and $Y^1_{1,2}$, let

$$Q_2 := W_{1,2} \cup W_{2,1} \cup Y^1_{1,2}$$

and

$$S_2 := S_1 \cup Q_2 \cup \bigcup \{U_i(n) \mid \exists i,n \in \mathbb{N}\colon\ u_{i,n} \in Q_2 \cap U_i\}.$$

S_2 is finite.

Now, let $W_{1,n}$, $W_{2,n-1}, \ldots, W_{n,1}$ and $Y^k_{i,n-k+1}$ ($k \in [n-1], i < n-k+1$) and S_n already be constructed. Because of (a), there exist $U_i \to U_{n-k+2}$-tracks $Y^k_{i,n-k+2}$ ($k \in [n], i < n-k+2$) in $D - S_n$. Because of (b), there exists a $U_{n+1} \to U_{n+2}$-track $W_{1,n+1}$, a $U_n \to U_{n+1}$-track $W_{2,n}, \ldots,$ and a $U_1 \to U_2$-track $W_{n+1,1}$ such that $W_{1,n+1}$ does not meet $U_1, U_2, \ldots, U_n$, the initial vertex of $W_{1,n+1}$ comes behind the terminal vertices of the $Y^1_{i,n+1}$ on U_{n+1} ($i \in [n]$), $W_{2,n}$ does not meet $U_1, \ldots, U_{n-1}, U_{n+2}$, and $W_{1,n+1}$, the initial vertex of $W_{2,n}$ comes behind the terminal vertices of the $Y^2_{i,n}$ on U_n ($i \in [n-1]$), the terminal vertex of $W_{2,n}$ comes behind the initial vertex of $W_{1,n+1}$ on $U_{n+1}, \ldots, W_{n+1,1}$ does not meet $U_3, U_4, \ldots, U_{n+2}$, $W_{1,n+1}, \ldots, W_{n-1,3}$, and $W_{n,2}$, and the terminal vertex of $W_{n+1,1}$ comes behind the initial vertex of $W_{n,2}$ on U_2. Let

$$Q_{n+1} := \left(\bigcup_{i \in [n+1]} W_{i,n+2-i} \right) \cup \left(\bigcup_{k \in [n], i \in [n-k+1]} Y^k_{i,n-k+2} \right)$$

and

$$S_{n+1} := S_n \cup Q_{n+1} \cup \bigcup \{U_i(n) \mid \exists i,n \in \mathbb{N}\colon\ u_{i,n} \in Q_{n+1} \cap U_i\}.$$

S_{n+1} is finite.

After completing the induction, for $i,j \in \mathbb{N}$, let $W'_{i,j}$ be the union of $W_{i,j}$ with the segment of U_{j+1} between the terminal vertex of $W_{i,j}$ and the initial vertex of $W_{i,j+1}$ and set

$$W_i := \bigcup_{j \in \mathbb{N}} W'_{i,j} \quad (i \in \mathbb{N}).$$

By construction, the W_i $(i \in \mathbb{N})$ are pairwise disjoint, and, clearly, the union of the W_i with the U_i $(i \in \mathbb{N})$ contains a subdivision of the digraph in Fig. 3.

W_1 belongs to an ω-end $\mathscr{F}$ and is constructed such that, for each $i \in \mathbb{N}$, there are infinitely many $U_i \to W_1$-tracks $Y^1_{i,j}$ $(j \in \mathbb{N}\backslash\{1\})$. Hence, for all $i \in \mathbb{N}$, $U_i \leqslant W_1$, and, therefore, for all $i \in \mathbb{N}$, $\mathscr{E}_i \leqslant \mathscr{F}$. $\mathscr{F} = \mathscr{E}_i$ for an $i \in \mathbb{N}$ is impossible because $\mathscr{F} = \mathscr{E}_i < \mathscr{E}_{i+1} \leqslant \mathscr{F}$ yields a contradiction. Hence, $\mathscr{F} > \mathscr{E}_i$ for all $i \in \mathbb{N}$.

Now, let $\mathscr{E} \in \Omega(\omega)$ with $\mathscr{E} > \mathscr{E}_i$ for all $i \in \mathbb{N}$. To show that $\mathscr{F}$ is the supremum of the sequence $(\mathscr{E}_i)_{i \in \mathbb{N}}$, it remains to show that $\mathscr{E} \geqslant \mathscr{F}$. $\mathscr{E}$ contains an ω-track W. Since $\mathscr{E} > \mathscr{E}_1$ and $W_1 \cap U_1$ is finite, it is possible to find a $U_1 \to W$-track R_1 such that the initial vertex of R_1 comes behind $W_1 \cap U_1$ on U_1. Obviously, the union of R_1 with the segment of U_1 between the last vertex of $W_1 \cap U_1$ and the initial vertex of R_1 contains a $W_1 \to W$-track P_1. Let $i_1 := 1$.

For $n \in \mathbb{N}$, let the pairwise disjoint $W_1 \to W$-tracks $P_1, \dots, P_n$ already be constructed. Since the P_i $(i \in [n])$ only intersect with finitely many of the U_i $(i \in \mathbb{N})$ in D, there exists $i_{n+1} \in \mathbb{N}$ such that $U_{i_{n+1}}$ does not intersect with one of the P_i $(i \in [n])$. Because W_1 intersects with all U_i $(i \in \mathbb{N})$ and because $\mathscr{E} > \mathscr{E}_i$ $(i \in \mathbb{N})$, it is possible to find a $W_1 \to W$-track P_{n+1} in $D - (\bigcup_{i \in [n]} P_i)$, just as in the case $n = 1$.

By induction, one obtains a system of infinitely many pairwise disjoint $W_1 \to W$-tracks $(P_i)_{i \in \mathbb{N}}$. Hence, $\mathscr{E} \geqslant \mathscr{F}$.

It remains to show that F is thick. Repeating the argumentation of the last four paragraphs for each W_i $(i \geqslant 2)$ yields $W_i \in \mathscr{F}$ for all $i \in \mathbb{N}$ since suprema are uniquely determined. As the W_i are pairwise disjoint, $\mathscr{F}$ is a thick ω-end, and the proof is completed. □

The next theorem implies that the necessary condition of Theorem 3.1 is also sufficient in the class of total orders. Naturally, a poset $(X, \leqslant)$ can be regarded as a reflexive, antisymmetric, and transitive digraph $D = (X, \leqslant)$ (and vice versa). There exists a bijective mapping between the set of all strictly increasing sequences of $(X, \leqslant)$ and $\mathscr{T}_D(\omega)$: Just map $(v_i)_{i \in \mathbb{N}}$ onto that one ω-track T with vertex set $V_T = \{v_i \mid i \in \mathbb{N}\}$ such that $\to_T = \leqslant_{v_T \times v_T}$. Now, if $(X, \leqslant)$ is a chain fulfilling the condition that all strictly increasing sequences have a supremum then

(∗) two ω-tracks U, V belong to the same ω-end if and only if the corresponding strictly increasing sequences $(u_i)_{i \in \mathbb{N}}$ and $(v_i)_{i \in \mathbb{N}}$ have the same supremum $u = v$.

(If $(u_i)_{i \in \mathbb{N}}$ and $(v_i)_{i \in \mathbb{N}}$ did not have the same supremum, then $u < v$ or $v < u$ since $(X, \leqslant)$ is a chain. Without loss of generality, assume $u < v$. Then there would exist $n \in \mathbb{N}$ such that $v_j > u_i$ for all $i \in \mathbb{N}$ and $j \geqslant n$. Thus, no u_i $(i \in \mathbb{N})$ would be reachable by a v_j, $j \geqslant n$. Hence, $\mathscr{E}_D(\omega, U) < \mathscr{E}_D(\omega, V)$, a contradiction. The other direction is trivial.)

$(*)$ is not necessarily true for general posets. For example, Fig. 4 shows a poset consisting of two completely independent, strictly increasing sequences with a common supremum. From statement $(*)$, it remains only: U, V belong to the same ω-end *only if* the corresponding strictly increasing sequences have the same supremum.

The validity of $(*)$ is very convenient for constructing digraphs whose ω-end-order represents $(X, \leqslant)$. Just let each $x \in X$ that occurs as the supremum of a strictly increasing sequence be represented by that ω-end which contains the ω-tracks whose corresponding strictly increasing sequences have x as supremum. If $(*)$ is not fulfilled, it seems to be very difficult to find a digraph whose ω-end-structure represents $(X, \leqslant)$ — and, therefore, $(*)$ is postulated in Theorem 3.3. For further investigations the following remarks may be useful:

Let M be the set of all $x \in X$ that are not the supremum of any strictly increasing sequence in X. For $x \in X \setminus M$, let σ_x denote the (nonempty) set of all strictly increasing sequences of X that have supremum x. Obviously, σ_x decomposes into a family of equivalence classes $(\mathscr{S}_i^x)_{i \in I_x}$ such that the corresponding classes of ω-tracks are subsets of ω-ends $(\mathscr{E}_i)_{i \in I_x}$ of $(X, \leqslant)$, which is partially ordered corresponding to the ω-end-subordering of the $(\mathscr{E}_i)_{i \in I_x}$. One may be tempted to add arcs to weld the ω-ends $(\mathscr{E}_i)_{i \in I_x}$ together, but then at least two problems could occur: (1) Adding arcs may generate new ω-ends that have nothing to do with $(X, \leqslant)$. (2) Too many ω-ends may possibly be welded together if arcs are added globally for all suprema.

The third condition of Theorem 3.3 is required to preserve the order structure of $(X, \leqslant)$ at suprema for the ω-end-structure of the digraph constructed in the proof of Theorem 3.3.

Theorem 3.3. *Let $(X, \leqslant)$ be a poset. It is ω-representable if the following conditions are fulfilled*:

(1) *Every strictly increasing sequence in $(X, \leqslant)$ has a supremum in X.*

(2) *Whenever two strictly increasing sequences have the same supremum, the corresponding ω-tracks of $(X, \leqslant)$ belong to the same ω-end of $(X, \leqslant)$.*

(3) *If $x \in X$ is the supremum of a strictly increasing sequence in $(X, \leqslant)$, then for all $y \in X$, $y < x$, there exists a strictly increasing sequence $(z_i^{y,x})_{i \in \mathbb{N}}$ with supremum $z = x$ and $y < z_i^{y,x} < x$ for all $i \in \mathbb{N}$.*

Proof. Let M be defined as above. A digraph D can be constructed from $(X, \leqslant)$ as follows: For each $x \in M$, let W_x denote an ω-track with vertex set $\{x, x_i \mid i \in \mathbb{N}\}$ such that, for all $x, y \in M$, W_x and W_y are disjoint whenever $x \neq y$, and, for all $x \in M$, $W_x \cap (X, \leqslant) = x$. Now, let

$$A' := \{(z, x_i), (x_i, z'), (x_i, y_j) \mid x, y \in M,\ z, z' \in X,\ z < x < z', x < y, i, j \in \mathbb{N}\}$$

and

$$D := (X, \leqslant) \cup \bigcup_{x \in M} W_x \cup A'.$$

Clearly, $x \leqslant y \Leftrightarrow x \rightarrow_X y$ for all $x, y \in X$.

(A) Each $U \in \mathscr{T}_D(\omega)$ has either a rest in common with exactly one of the W_x or only finitely many vertices in common with each of the W_x $(x \in M)$. (If a track leaves W_x it can never return to W_x.) In the latter case U corresponds in a natural way to a strictly increasing sequence of $(X, \leqslant)$.

(B) $U, V \in \mathscr{T}_D(\omega)$ belong to the same ω-end of D if and only if either rests of U and V belong to the same W_x or both U and V correspond to strictly increasing sequences $(u_i)_{i \in \mathbb{N}}$, $(v_i)_{i \in \mathbb{N}}$ in $(X, \leqslant)$ that have the same supremum $u = v \in X$.

($\Leftarrow$) Trivial if rests of U and V belong to the same W_x. If both U and V correspond to strictly increasing sequences in $(X, \leqslant)$ with the same supremum, then '$\Leftarrow$' follows from (2) and the fact that all $W_x \rightarrow W_y$-arcs exist whenever $x < y$.

($\Rightarrow$) Because of (A) and (1), only three cases have to be considered:

(I) A rest of U belongs to W_x and a rest of V belongs to W_y.

(II) A rest of U belongs to W_x and V corresponds to a strictly increasing sequence $(v_i)_{i \in \mathbb{N}}$ with supremum v in $(X, \leqslant)$.

(III) Both U and V correspond to a strictly increasing sequence (with suprema u and v, respectively) in $(X, \leqslant)$.

ad (I): If $x \neq y$ then (without loss of generality) either $x < y$ or x and y are incomparable. In the former case, $U < V$ is valid, implying that $\mathscr{E}_D(\omega, U) < \mathscr{E}_D(\omega, V)$, and in the latter case, U and V are incomparable, implying that $\mathscr{E}_D(\omega, U)$ and $\mathscr{E}_D(\omega, V)$ are incomparable. In both cases, U and V do not belong to the same ω-end of D, a contradiction. Hence $x = y$.

ad (II): Again, there are three possible cases ($v = x$ is impossible since $x \in M$, $v \notin M$):

(a) $v < x$: Then $v_i < x$ for all $i \in \mathbb{N}$, and hence $V < U$ (since there is no $U \rightarrow V$-track in D and, for each $v_i \in M$, x is reachable from every vertex of W_{v_i}), implying that $\mathscr{E}_D(\omega, V) < \mathscr{E}_D(\omega, U)$. Therefore, U and V do not belong to the same ω-end, a contradiction.

(b) $v > x$: Then, because of (3), there exists a strictly increasing sequence $(z_i^{x,v})_{i \in \mathbb{N}}$ with supremum $z = v$. Because of (2), the ω-track Z with vertices $(z_i^{x,v})_{i \in \mathbb{N}}$ is end-equivalent to the ω-track V' with vertices $(v_i)_{i \in \mathbb{N}}$ (and clearly, $V' \sim V$). Hence, $Z \sim V$. Because $(x_i, z_i) \in A_D$ for all $i \in \mathbb{N}$ and there exists no $Z \rightarrow U$-track in D, $U < Z \sim V$, and hence $\mathscr{E}_D(\omega, U) < \mathscr{E}_D(\omega, V)$, a contradiction.

(c) v and x are incomparable: $v_i < x$ for all $i \in \mathbb{N}$ is not possible because, otherwise, $v \leqslant x$ since v is the supremum of $(v_i)_{i \in \mathbb{N}}$. If there exists $j \in \mathbb{N}$ such that $v_j \geqslant x$, then $v > v_j \geqslant x$, which is impossible, too. Hence, there exists $n \in \mathbb{N}$ such that v_i and x are incomparable for all $i \geqslant n$. It follows that there are no $U \rightarrow V$-tracks and only finitely many pairwise disjoint $V \rightarrow$

U-tracks in D. Hence U and V, and therefore $\mathscr{E}_D(\omega, U)$ and $\mathscr{E}_D(\omega, V)$, are incomparable, a contradiction.

Altogether, case (II) is impossible.

ad (III): Again, there are three possible cases:

(a) $u = v$: finished.

(b) Without loss of generality, $u < v$ ($v < u$ is a symmetric situation): It is possible to argue as in (II)(b) by substituting 'u_i' for 'x_i'. In particular, $\mathscr{E}_D(\omega, U) < \mathscr{E}_D(\omega, V)$, a contradiction.

(c) u and v are incomparable: $u_i < v$ for all $i \in \mathbb{N}$ is not possible because, otherwise, $u \leqslant v$ since u is the supremum of $(u_i)_{i \in \mathbb{N}}$. Symmetrically, $v_i < u$ for all $i \in \mathbb{N}$ is impossible. If there exists $j \in \mathbb{N}$ such that $u_j > v$, then $u > u_j > v$, which is impossible, too. Symmetrically, $v_j > u$ for a $j \in \mathbb{N}$ is impossible. It follows, that there exist $m, n \in \mathbb{N}$ such that u_i and v_j are incomparable for all $i \geqslant m$, $j \geqslant n$. So if U' denotes the ω-track with vertices $(u_i)_{i \in \mathbb{N}}$ and V' the ω-track with vertices $(v_i)_{i \in \mathbb{N}}$, then U′ and V′ are incomparable in $(\mathscr{T}_D(\omega), \leqslant)$. Since $U \sim U'$ and $V \sim V'$, U and V, and therefore $\mathscr{E}_D(\omega, U)$ and $\mathscr{E}_D(\omega, V)$, are incomparable, a contradiction. This completes the proof of (B).

Now, consider the mapping $f : X \to \Omega_D(\omega)$, $x \mapsto E_x$, where E_x denotes, for $x \in M$, the ω-end of D corresponding to W_x and, for $x \notin M$, the ω-end whose elements correspond to strictly increasing sequences in $(X, \leqslant)$ with supremum x.

f is well defined: Because of (B), each W_x ($x \in M$) represents exactly one ω-end of D. If $x \notin M$, then x is the supremum of a strictly increasing sequence S in $(X, \leqslant)$. There is at least one ω-track of D that corresponds to S. This, in combination with (B), guarantees the existence of exactly one ω-end fulfilling the definition of E_x in the case $x \notin M$.

Because of (A) and (1), f is surjective. That f is injective and an order isomorphism follows from the proof of (B). □

Conditions (2) and (3) of Theorem 3.3 are not necessary: There are posets not fulfilling (2) or (3) which are nevertheless ω-representable. For example, the runway-poset (Fig. 4) is ω-represented by the digraph D_1 in Fig. 2, but, obviously, it violates (2). Now, let $(X, \leqslant)$ be a poset where $X = \{x, y, x_i \mid i \in \mathbb{N}\}$, $(x_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence with supremum x and y be an element less than x but independent from each x_i, $i \in \mathbb{N}$. Now let D be constructed as in Theorem 3.3 and include the following additional arcs: (y_i, x_i) for all $i \in \mathbb{N}$. It is easy to see that $(X, \leqslant)$ is ω-represented by D in spite of the fact that $(X, \leqslant)$ does not fulfill condition (3).

Condition (1) of Theorem 3.3 characterizes those total orders that are ω-representable, since conditions (2) and (3) are always fulfilled in chains. It is possible to obtain a dual result for ω^*-tracks (see Remark 3.2).

Corollary 3.4. *Let $(X, \leqslant)$ be a chain. Then $(X, \leqslant)$ is ω-representable if and only if every strictly increasing sequence of $(X, \leqslant)$ has a supremum.*

$(X, \leqslant)$ is called a *well-ordered* (resp. *dually well-ordered*) *set* (briefly *woset* (resp. *dwoset*)) iff every nonempty subset of X contains a least (resp. greatest) element. Every (d)woset is a chain. Conditions (1), (2), and (3) are always fulfilled in posets that contain no strictly increasing sequence, which are exactly the posets in which every chain is a dwoset. Therefore:

Corollary 3.5. *Every poset that contains only dually well-ordered chains is ω-representable.*

Up to this point only ω- and ω^*-representability have been studied. It remains to investigate c-representability. Whether an abstract order is c-representable or not seems to be a somehow different problem:

- There are orders that are c- but neither ω- nor ω^*-representable, e.g. $\mathbb{Z}$ with the natural order (see Example 3.12).
- An ω-representable order $(X, \leqslant)$ may not be c-representable because each digraph whose ω-end order represents $(X, \leqslant)$ may also contain ω^*-tracks that are contained in a c-end which contains no ω-tracks.

Up to now, I was not able to find an example to illustrate the second of the above points. Therefore, the following problem may be interesting.

Problem 2. *Are there abstract orders that are ω- (resp. ω^*-) but not c-representable?*

Since composed ends contain both ω- and ω^*-tracks, it is a little bit more difficult to find a necessary condition in terms of infima/suprema for an abstract order to be c-representable. One has to draw on a more special and sophisticated looking class of orders. First of all, the following lemma is needed:

Lemma 3.6. *Let C be an (inclusion-) maximal chain in $(\Omega_D(c), \leqslant)$ and $\Omega_D^{\omega}(c)$ be the set of all c-ends of D that contain exclusively ω-tracks. If C contains a countable, upper unbounded subset, then $\Omega_D^{\omega}(c) \cap C$ is upper bounded in C. A dual statement holds for ω^*-tracks.*

Proof. Let $A = \{a_i \mid i \in \mathbb{N}\}$ be an upper unbounded, countable subset of C and $C' := \Omega_D^{\omega}(c) \cap C$. If C' would be upper unbounded in C, then, for every $i \in \mathbb{N}$, there would exist a $c_i \in C'$ such that $c_i > a_i$, i.e. $\{c_i \mid i \in \mathbb{N}\}$ would be upper unbounded. Clearly, $\{c_i \mid i \in \mathbb{N}\}$ would contain an upper unbounded strictly increasing sequence, and Theorem 3.1 would yield a contradiction to the maximality of C. $\square$

The countability of the unbounded subset of C is necessary because there exist orders in which every countable subset is bounded, e.g. any order of order type ω_1 (see [1, pp. 65–67, Theorems 17, 17′, 17″]).

The following theorem gives a sufficient condition for an order not to be c-representable. (A, B) is called a *cut of* $(X, \leqslant)$ iff $A \cap B = \emptyset$, $A \cup B = X$, and $a < b$ holds for all

$a \in A$, $b \in B$. In addition, if A and B are nonempty, then (A,B) is called a *proper cut of* $(X, \leqslant)$.

Theorem 3.7. *Let $(X, \leqslant)$ be a poset, C be a (inclusion-)maximal chain in $(X, \leqslant)$ containing an upper unbounded, countable subset, $M = M(C)$ the nonempty set of all proper cuts (A,B) of C such that A has no greatest element and B contains a lower unbounded countable subset, and $\bigcap_{(A,B)\in M} B = \emptyset$. Then $(X, \leqslant)$ is not c-representable. (A dual statement holds if C contains a lower unbounded, countable subset by changing the roles of A and B.)*

Proof. Assume C to be a maximal chain in $(X, \leqslant)$ such that C contains an upper unbounded, countable subset and $\bigcap_{(A,B)\in M} B = \emptyset$. Let D be a digraph and $f: X \to \Omega_D(c)$ be an (order) isomorphism. Then $f[C]$ is a maximal chain in $(\Omega_D(c), \leqslant)$ and contains an upper unbounded, countable subset. From Lemma 3.6 it follows that $\Omega_D^{\omega}(c) \cap f[C]$ is upper bounded. Let $f(x)$ be an upper bound of $\Omega_D^{\omega}(c) \cap f[C]$ in $f[C]$. Since $\bigcap_{(A,B)\in M} B = \emptyset$ and M is nonempty, there exists a cut $(A', B') \in M$ such that $x \notin B'$, i.e. $f(x) \notin f[B']$. It follows that $f(x) < \mathscr{E}$ for all $\mathscr{E} \in f[B']$. Since B', and hence $f[B']$, contains a lower unbounded, countable subset B'', resp. $f[B'']$, and thus also a lower unbounded strictly decreasing sequence $(x_i)_{i\in\mathbb{N}}$, resp. $(f(x_i))_{i\in\mathbb{N}}$, according to Theorem 3.1, there exists $\mathscr{F} \in f[C]$ such that $\mathscr{F}$ is the infimum of the sequence $(f(x_i))_{i\in\mathbb{N}}$. Clearly, $\mathscr{F} \in A'$ since $(f(x_i))_{i\in\mathbb{N}}$ is lower unbounded in $f[B']$, which is a contradiction because A' does not contain a greatest element. Hence, the existence of an isomorphism $f: X \to \Omega_D(c)$ is impossible. □

Example 3.8. $\mathbb{Q}$ with the natural order is not c-representable since $\mathbb{Q}$ is a countable, infinite chain, and the set of all gaps, which are characterizable with the help of cuts, is both upper and lower unbounded. Clearly, $\mathbb{Q}$ with the natural order is neither ω- nor ω^*-representable, and hence there are orders that are not at all end-representable by digraphs.

The next two theorems will show that there are orders that are c- but not ω- (resp. ω^*-) representable.

Theorem 3.9. *Every poset in which every chain is well-ordered (resp. dually well-ordered) is c-representable.*

Proof. Let $(X, \leqslant)$ be a poset such that every chain in $(X, \leqslant)$ is dually well-ordered. Then $(X, \leqslant)$ contains no strictly increasing sequence. Construct a digraph D as follows: First of all, for each $x \in X$, let D contain an ω-track W_x such that the W_x $(x \in X)$ are pairwise disjoint and the vertices of W_x — beginning with the initial vertex — are denoted by $w_{x,1}, w_{x,2}, w_{x,3}, \ldots$. Additionally, let D contain the arcs $(w_{x,n}, w_{y,n+1})$ $(n \in \mathbb{N},\ x, y \in X,\ x < y)$. With that, the construction of D is completed.

Clearly, each W_x $(x \in X)$ represents another c-end of D. Further, $x \leqslant y \Leftrightarrow \mathscr{E}_D(c, W_x) \leqslant \mathscr{E}_D(c, W_y)$ for all $x, y \in X$ because, for $x < y$, there are infinitely many $W_x \to W_y$-arcs in D, but no $W_y \to W_x$-tracks. Hence, $(\Omega_D(c), \leqslant) \simeq (X, \leqslant)$ provided that D contains no other c-ends except the $\mathscr{E}_D(c, W_x)$, $x \in X$.

Let $U \in \mathscr{T}_D(\omega)$. Then either U has a rest in common with one of the W_x or U corresponds to a strictly increasing sequence of $(X, \leqslant)$. Since $(X, \leqslant)$ contains no strictly increasing sequence, U must belong to one of the $\mathscr{E}_D(c, W_x)$. Now, let $V \in \mathscr{T}_D(\omega^*)$. V contains a vertex $w_{x,n}$. By the construction of D, there are vertices $w_{y,n-1}, \ldots, w_{z,1}$ on V, but no arc terminates at $w_{z,1}$, a contradiction. Hence, D does not contains ω^*-tracks, and the theorem is proved. □

Definition 3.10. Let $(X, \leqslant)$ and $(Y_x, \leqslant)$ $(x \in X)$ be pairwise disjoint posets. Then let $\sum_{x \in X} Y_x$, the $(X, \leqslant)$-*sum of the* $((Y_x, \leqslant))_{x \in X}$, denote the following poset: Let each $x \in X$ be substituted by Y_x, and for $y \in Y_x$, $y' \in Y_{x'}$, let $y \leqslant y'$ iff either $x = x'$ and $y \leqslant y'$ in Y_x or $x < x'$.

Theorem 3.11. *Let $(X, \leqslant)$ be a poset containing only finite chains and $(Y_x)_{x \in X}$ be a family of c-representable posets whereby X and the $(Y_x)_{x \in X}$ are pairwise disjoint. Then $\sum_{x \in X} Y_x$ is c-representable.*

Proof. Let $(D_x)_{x \in X}$ be a family of pairwise disjoint digraphs such that D_x c-represents Y_x $(x \in X)$. If one welds together the $(D_x)_{x \in X}$ in the same manner as the $(Y_x)_{x \in X}$, then it is easy to see that the resulting digraph D c-represents $\sum_{x \in X} Y_x$: Since $(X, \leqslant)$ contains only finite chains, each 1-track in D has a rest in common with a 1-track of one of the D_x $(x \in X)$. □

A version of Theorem 3.11 for ω- (resp. ω^*-) representable posets is also valid. In this version $(X, \leqslant)$ can be allowed to be a poset containing only dually well-ordered (resp. well-ordered) chains.

Example 3.12. $\mathbb{Z}$ with the natural order is c-representable (by combination of Theorem 3.9 and Theorem 3.11).

4. Outlook

A (di)graph is called *locally finite* iff every vertex is of finite degree, and *vertex-symmetric* iff its automorphism group acts transitively on its vertex set.

In this article, I have considered orders that can be end-represented by arbitrary digraphs and was not able to gain a full characterization. Maybe this is possible in smaller classes of digraphs. Especially, I expect the following problems to be interesting:

Problem 3. *Which abstract orders are end-representable by locally finite digraphs?*

Problem 4. *Which abstract orders are end-representable by vertex-symmetric digraphs?*

Additionally, I suppose that there are orders which are neither ω-representable nor fulfilling condition (2) or (3) of Theorem 3.3, but in which every strictly increasing sequence has a supremum. A suitable counterexample would increase the expectation that a simple characterization of ω-representable orders cannot be found.

This paper spots only a very close area around the 'directed end'-concept. Forthcoming articles, for example dealing with subjects like end-faithful spanning arborescences, automorphisms of infinite digraphs, and relations between the ends and the orientations of undirected graphs, will enlighten the following two important questions:

1. To what extent can results about 'undirected ends' be generalized to ends of digraphs?
2. Can 'directed ends' be productively used in the investigation of undirected infinite graphs?

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