The category of supercontinuous posets

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Received 18 March 2005
Available online 24 August 2005
Submitted by William F. Ames

Abstract

In this paper, we study the order structure—supercontinuous poset, a generalization of completely distributive lattice. The Cartesian product of supercontinuous posets and some other properties of supercontinuous posets are investigated. Also, the case of superalgebraic posets are studied and some remarks on the category of supercontinuous posets are given.

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Keywords: Supercontinuous poset; Super-compact element; Superalgebraic poset; Category

1. Introduction

A complete lattice is completely distributive if it satisfies the most general distributivity [1]. Since Raney first systematically investigated such lattices in 1950’s [2], many people have studied this structure, especially after Hofmann and Lawson discovered the close links between continuous directed complete posets and completely distributive lattices in 1970’s [3].

The main objective of this work is to study the order structure—supercontinuous poset, which is a generalization of completely distributive lattice.
In this paper, we use $\bigvee A$ or $\sup A$ to represent the supremum of $A$, and $\bigwedge A$ or $\inf A$ to represent the infimum of $A$.

For notations and concepts concerned, but not explained, please refer to [1].

**Definition 1.1.** Let $P$ be a poset. For any two elements $x$ and $y$ in $P$, we write $x \prec y$, if for any subset $A \subseteq P$ with $\bigvee A$ existing and $y \leq \bigvee A$, there exists $z \in A$ with $x \leq z$.

A poset $P$ is called supercontinuous if for each $a \in P$, $a = \bigvee \{x \in P : x \prec a\}$.

**Example 1.1.** (1) Every chain is supercontinuous. For every element $a$ in a chain $C$, $\bigvee \{x : x < a\}$ exists. Either $\bigvee \{x : x < a\} = a$, or else $\bigvee \{x : x < a\} = \inf a$. In the second case, we find that $a \prec a$. Since in a chain $x < y$ implies $x \prec y$, we conclude that in both cases $\bigvee \{x : x < a\} = a$.

Thus the posets $(Q, \preceq)$ and $(R, \preceq)$ of all rational numbers and real numbers are super-continuous. For $x, y \in Q(R)$, $x \prec y$ iff $x < y$.

(2) As we all know, a complete lattice $L$ is completely distributive if and only if it is supercontinuous (see [4]). Let $L$ be a completely distributive lattice. Then as a subposet of $L$, $L - \{1_L, 0_L\}$ is a supercontinuous poset, where $1_L$ and $0_L$ are the top and bottom elements of $L$, respectively.

(3) Given a set $X$. Let $\mathcal{P}_0(X)$ be the set of all finite subsets of $X$. Then $(\mathcal{P}_0(X), \subseteq)$ is supercontinuous. Again $\mathcal{P}_0(X)$ is generally not a complete lattice. In general, if $m$ is a cardinal, then $\mathcal{P}_m(X) = \{A \subseteq X : |A| \leq m\}$ is supercontinuous with respect to $\subseteq$. This follows from the observation that $\{x\} \prec A$ holds for every $x \in A$, $A \in \mathcal{P}_m(X)$.

(4) Let $L$ be a completely distributive lattice, and let $A$ be a lower set in $L$ (i.e. assume that $A = \downarrow A$). Moreover, assume that for every subset $D \subseteq A$ for which the least upper bound of $D$ exists in $A$, this least upper bound is also the least upper bound of $D$ in $L$ (i.e. assume that $\bigvee_A D = \bigvee_L D$ whenever $\bigvee_A D$ exists). Then $A$ is a supercontinuous poset.

(5) A non-empty subset $A$ of $P$ is relatively directed if every two elements $a, b$ in $A$ have an upper bound in $P$ (see [5]).

We say the distributive law holds for all relatively directed subsets in $P$, if for any non-empty family of elements in $P \{x_{j,k} : j \in J, k \in K(j)\}$ for which $\{x_{j,k} : k \in K(j)\}$ is relatively directed for all $j \in J$, the following identity holds:

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in \prod_{j \in J} K(j)} \bigwedge_{j \in J} x_{j,f(j)}.$$  

A poset $P$ is called completely distributive if (i) every nonempty subset of $P$ has an infimum, (ii) every relatively directed set has a supremum, and (iii) the distributive law holds for all relatively directed subsets.

Every completely distributive poset is supercontinuous. Indeed, let $a \in P$ be a given element, and let $J$ be the set of all relatively directed subsets $j$ of $P$ with $\sup j \geq a$. For each $j \in J$, let $K(j) = j$. In other words, $j$ is indexing itself. Further, consider the family of elements $x_{j,k} = k$ for $j \in J$ and $k \in K(j)$. Suppose $f \in M$ where $M = \prod_{j \in J} K(j)$ and let $t = \bigwedge_{j \in J} x_{j,f(j)} = \bigwedge_{j \in J} f(j)$. Then we claim that $t \prec a$. In fact, if $A$ is a subset of $P$ with $a \leq \sup A$, then $A$ is a relatively directed subset of $P$ and hence there exists $j_0 \in J$ such that $A = j_0$. Thus $t \leq f(j_0) = f(A) \in A$. Since $\{a\} \in J$, $\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = a$. 
Also, \( \bigvee_{f \in \prod_{i \in I} K(j)} \bigwedge_{j \in I} x_j, f(j) = \bigwedge_{j \in I} \bigvee_{k \in K(j)} x_{j,k} = a \). So, \( \{ x \in P : \ x \triangleleft a \} \geq a \), hence \( a = \bigvee \{ x \in P : \ x \triangleleft a \} \).

In the sequel, we use \( \downarrow a \) to denote the set \( \{ x \in P : \ x \ll a \} \) for every \( a \in P \).

**Remark 1.1.** As was pointed in [6], with respect to inclusion relation, the set \( \varepsilon(N) = \{ A \subseteq N : |A| \leq 1 \text{ or } |A| = \infty \} \),

where \( N \) denotes the set of all natural numbers, is a supercontinuous poset, but not a continuous poset.

This is because for every \( A \in \varepsilon(N) \) and \( x \in A \), \( \{ x \} \triangleleft A \) and \( A = \bigvee \{ \{ x \} : x \in A \} \). But, \( A \ll N \) if and only if \( A \) is a singleton or \( A = \emptyset \). \( \downarrow N = \{ \{ x \} : \ x \in N \} \cup \{ \emptyset \} \) is not directed, hence \( \varepsilon(N), \subseteq \) is not a continuous poset.

However, every supercontinuous sup-semilattice \( P \) is a continuous poset. In this case, it can be easily shown that \( \downarrow a \) is directed for every \( a \in P \).

2. The Cartesian product of supercontinuous posets

In this section we investigate the construction of new supercontinuous posets from known ones by means of forming direct product (with pointwise order).

**Proposition 2.1.** Let \( \{ P_i : i \in I \} \) be a family of posets. For elements \( x = (x_i)_{i \in I} \) and \( y = (y_i)_{i \in I} \) in \( \prod_{i \in I} P_i \), \( x \triangleleft y \Rightarrow x_i \triangleleft y_i \) for all \( i \in I \).

**Proof.** Suppose first that \( x \triangleleft y \). In order to show that \( x_i \triangleleft y_i \) for all \( i \in I \), fix \( i \) and consider any subset \( A \) in \( P_i \) such that \( y_i \leq \sup A \). To every \( a \in A \) we associate the element \( \bar{a} \in \prod_{i \in I} P_i \) defined by \( \bar{a}_i = a \) and \( \bar{a}_j = y_j \) for all \( j \neq i \). The family \( \{ \bar{a} : a \in A \} \) is a subset of \( \prod_{i \in I} P_i \) and \( y \leq \sup_{a \in A} \bar{a} \). As \( x \triangleleft y \), there is some \( a \in A \) such that \( x \leq \bar{a} \), whence \( x_i \leq a \). Thus, \( x_i \triangleleft y_i \) for all \( i \in I \). \( \square \)

**Remark 2.1.** The converse of Proposition 2.1 is not valid in general, and a product of supercontinuous posets needs not to be supercontinuous, as the following counterexample shows. Let \( P = \mathbb{R} \times \mathbb{R} \), and let \( x = (c, d) \) and \( y = (a, b) \) with \( c < a \) and \( d < b \). Then \( x_i \triangleleft y_i \) for \( i = 1, 2 \). However, the relation \( x \triangleleft y \) does not hold: if we define \( M = \{ (a, d - \frac{1}{m}) : n = 1, 2, 3, \ldots \} \cup \{ (c - \frac{1}{m}, b) : m = 1, 2, 3, \ldots \} \), then \( \sup M = y \), however none of the elements of \( M \) is larger than or equal to \( (c, d) \). Actually, this shows that for every given \( (a, b) \in \mathbb{R} \times \mathbb{R} \) there is no \( (c, d) \in \mathbb{R} \times \mathbb{R} \) so that \( (c, d) \triangleleft (a, b) \). Therefore \( \mathbb{R} \times \mathbb{R} \) is not supercontinuous.

**Proposition 2.2.** If \( \{ P_i : i \in I \} \) is a family of supercontinuous posets with least element \( 0 \), then the Cartesian product \( \prod_{i \in I} P_i \) is also a supercontinuous poset. For elements \( x = (x_i)_{i \in I} \) and \( y = (y_i)_{i \in I} \) in \( \prod_{i \in I} P_i \) the \( \triangleleft \) relation is given by

\( \ x \triangleleft y \quad \text{iff} \quad x_i \triangleleft y_i \quad \text{for all} \ i \in I \ \text{and} \ |\{ i \in I : x_i \neq 0 \}| \leq 1. \)
Proof. Let \( y \in \prod_{i \in I} P_i \) be given. Fix an index \( i_0 \) and an element \( a \in P_{i_0} \) with \( a \prec y_{i_0} \). Define \( x(a, i_0) \in \prod_{i \in I} P_i \) by
\[
x(a, i_0) = \begin{cases} a, & i = i_0, \\ 0, & i \neq i_0. \end{cases}
\]
Then it is easy to verify that \( x \prec y \). Moreover, \( y \) is the least upper bound of elements of the form \( x(a, i_0) \). Hence \( \prod_{i \in I} P_i \) is supercontinuous. Every element \( x \in \prod_{i \in I} P_i \) with \( x \prec y \) is dominated by at least one of the elements of the form \( x(a, i_0) \) and therefore itself is of that form. \( \square \)

Corollary 2.1. The Cartesian product \( \prod_{i \in I} P_i \) of a family of chains \( P_i \) with least element 0 is supercontinuous.

Proposition 2.3. Every antichain is a supercontinuous poset without least element, and any product of antichain is an antichain, hence a supercontinuous poset.

Proof. By the fact that for any \( a \in P \) which is an antichain, \( \{ x \in P : x \prec a \} = \{ a \} \), every antichain is a supercontinuous poset. \( \square \)

Remark 2.2. This shows the fact that \( \prod_{i \in I} P_i \) is a supercontinuous poset does not imply that each of \( P_i \) has least element 0.

By a discrete union of posets \( M_j, j \in J \), we mean a disjoint union of the \( M_j \) such that no element in different components \( M_j \) are comparable.

Theorem 2.1. The Cartesian product \( \prod_{i \in I} P_i \) of a family of supercontinuous posets \( \{ P_i : i \in I \} \) is again supercontinuous if all the supercontinuous posets \( P_i \) are discrete unions of supercontinuous posets with least elements 0.

Proof. Suppose for any \( i \in I \), \( P_i = \bigsqcup_{j \in J(i)} P_{ij} \), the discrete union of a family of supercontinuous posets \( \{ P_{ij} : j \in J(i) \} \) with least elements 0. Let \( x \in \prod_{i \in I} P_i \) and \( i \in I \). Then there exists \( j(x, i) \in J(i) \) such that \( x_i \in P_{i, j(x, i)} \).

With similar method to Proposition 2.2, one can prove that for elements \( x = (x_i)_{i \in I} \) and \( y = (y_i)_{i \in I} \) in \( \prod_{i \in I} P_i \),
\[
x \prec y \quad \text{iff} \quad \text{for all } i \in I, \quad j(x, i) = j(y, i), \quad x_i \prec y_i \text{ in } P_{i, j(x, i)} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \text{and} \quad |\{ i \in I : x_i \neq 0_{i, j(x, i)} \}| \leq 1.
\]
From this characterization of \( \prec \) relation in \( \prod_{i \in I} P_i \), we have \( \vee \{ x \in \prod_{i \in I} P_i : x \prec y \} = y \) for all \( y \in \prod_{i \in I} P_i \). So \( \prod_{i \in I} P_i \) is supercontinuous. \( \square \)

3. Some properties of supercontinuous posets

In this section we study some properties of supercontinuous posets.
Let \( \bar{D}(P) \) denote the set of all lower subsets of a poset \( P \) for which the supremum exists.

**Proposition 3.1.** Let \( P \) be a poset. The following conditions are equivalent:

1. \( P \) is a supercontinuous poset;
2. for each \( a \in P \), the set \( \{ x \in P : x \prec a \} \) is the smallest lower set \( D \) in \( \bar{D}(P) \) with \( a \leq \sup D \);
3. for each \( a \in P \), there is a smallest lower set \( D \) in \( \bar{D}(P) \) with \( a \leq \sup D \);
4. the sup map \( r = (D \rightarrow \sup D) : \bar{D}(P) \rightarrow P \) has a left adjoint.

These conditions imply

(5) the sup map \( r = (D \rightarrow \sup D) : \bar{D}(P) \rightarrow P \) preserves all existing infima.

If \( P \) is a complete semilattice or a complete lattice, then all five conditions are equivalent.

**Proof.** (1) \( \Rightarrow \) (2): For each \( a \in P \), \( \{ x \in P : x \prec a \} \) is a lower set. Since

\[
a = \bigvee \{ x \in P : x \prec a \},
\]

this set has a supremum and therefore belongs to \( \bar{D}(P) \). Let \( D \in \bar{D}(P) \) be any lower set with \( a \leq \sup D \). If \( x \prec a \), by the definition of \( \prec \) relation, there exists \( z \in D \) with \( x \leq z \).

Thus \( x \in D \) and hence \( \{ x \in P : x \prec a \} \subseteq D \). So (2) holds.

Condition (2) trivially implies (3).

(3) \( \Rightarrow \) (1): Let \( M_0 \) be the smallest lower set so that \( a \leq \sup M_0 \). Then \( x \prec a \) for all \( x \in M_0 \). Indeed, if \( N \) is any subset with \( a \leq \sup N \), then also \( a \leq \sup \uparrow N \). Since \( M_0 \) is the smallest lower set with this property, we find that \( M_0 \subseteq \uparrow N \). Hence \( x \leq n \) for some \( n \in N \).

Thus (1), (2) and (3) are equivalent.

(3) iff (4): The map \( r \) has a left adjoint iff \( \min r^{-1}(\uparrow a) \) exists for all \( a \in P \). But \( \min r^{-1}(\uparrow a) \) is precisely the smallest element of \( \{ D : D \in \bar{D}(P) \text{ with } a \leq \sup D \} \).

Condition (4) trivially implies (5).

(5) \( \Rightarrow \) (4): If \( P \) is a complete semilattice and if \( B \subseteq P \) is a bounded lower set, then \( B \) has a least upper bound, and hence \( B \in \bar{D}(P) \). Now let \( a \in A \) be given, and let

\[
X(a) = \{ A \in \bar{D}(P) : a \leq \sup A \}.
\]

Since \( \downarrow a \in X(a) \), the set \( X(a) \) is non-empty and \( \bigcap X(a) \subseteq \downarrow a \). So \( \bigcap X(a) \) is a bounded lower set and therefore \( \bigcap X(a) \in \bar{D}(P) \). We define

\[
I(a) = \bigcap X(a).
\]

If \( r \) preserves infima, we find that

\[
r(I(a)) = a \quad \text{for all } a \in P,
\]

\[
l(r(A)) \subseteq A \quad \text{for all } A \in \bar{D}(P),
\]

and therefore \( I \) is the left adjoint of \( r \). □
Let $S$ be a poset. Adjoin an identity by forming $S^I = S \cup \{1\}$ with an element $1 \not\in S$ and $x \leq 1$ for all $x \in S$.

Let $P$ and $Q$ be posets. Define the following five kinds of “disjoint” sums:

1. (Disjoint sum) $P \sqcup Q$, the disjoint union of $P$ and $Q$ (with the obvious partial ordering: the elements $x \in P$ and $y \in Q$ are incomparable);
2. (Coalesced sum) $P \oplus Q$, the disjoint sum $P \sqcup Q$ with the bottom elements identified, if they have them;
3. (Separated sum) $P + Q = (P \sqcup Q)_\perp$, that is, the disjoint sum with a new bottom element adjoined;
4. $P +_1 Q = (P \oplus Q)^1$;
5. $P +_2 Q = P \oplus Q$ with the 1 elements identified, if they have them.

**Proposition 3.2.** Let $P$ and $Q$ be supercontinuous posets, then $P \sqcup Q$, $P \oplus Q$ and $P + Q$ are also supercontinuous posets.

**Proof.** (1) Let $a \in P \sqcup Q$, $a \in P$ or $a \in Q$. If $a \in P$, then $\{x \in P \sqcup Q : x \lhd_{P \sqcup Q} a\} = \{x \in P : x \lhd_P a\}$. So

$$\bigvee_{P \sqcup Q} \{x \in P \sqcup Q : x \lhd_{P \sqcup Q} a\} = \bigvee_{P \sqcup Q} \{x \in P : x \lhd_P a\}
= \bigvee_P \{x \in P : x \lhd_P a\} = a.$$

If $a \in Q$, we also have $\bigvee_{P \sqcup Q} \{x \in P \sqcup Q : x \lhd_{P \sqcup Q} a\} = a$.

(2) Suppose both $P$ and $Q$ have bottom elements. If $a = 0_{P \oplus Q}$, then $\{x \in P \oplus Q : x \lhd_{P \oplus Q} a\} = \emptyset$. Hence, $\bigvee_{P \oplus Q} \{x \in P \oplus Q : x \lhd_{P \oplus Q} a\} = 0_{P \oplus Q}$. Otherwise, without loss of generality, let $a \in P - \{0_P\}$, then $\{x \in P \oplus Q : x \lhd_{P \oplus Q} a\} = \{x \in P \oplus Q : x \lhd_P a\} \cup \{0_{P \oplus Q}\}$. So

$$\bigvee_{P \oplus Q} \{x \in P \oplus Q : x \lhd_{P \oplus Q} a\} = \bigvee_{P \oplus Q} \{x \in P \oplus Q : x \lhd_P a\} \cup \{0_{P \oplus Q}\}
= \bigvee_P \{x \in P : x \lhd_P a\} = a.$$

(3) Let $a \in (P + Q) - \{0_{P + Q}\}$. Then $a \in P$ or $a \in Q$. If $a \in P$, then

$$\{x \in P + Q : x \lhd_{P + Q} a\} = \{x \in P : x \lhd_P a\} \cup \{0_{P + Q}\}.$$

Similarly, if $a \in Q$, then

$$\{x \in P + Q : x \lhd_{P + Q} a\} = \{x \in Q : x \lhd_Q a\} \cup \{0_{P + Q}\}.
\square$$

**Example 3.1.** Let $P$ and $Q$ be supercontinuous posets, then $P +_1 Q$ and $P +_2 Q$ need not be supercontinuous.

(1) Let $P = Q = [0, 1]$. Then both $P$ and $Q$ are supercontinuous. Let $a \in P - \{0_P, 1_P\} \subseteq (P +_2 Q) - \{0_{P +_2 Q}, 1_{P +_2 Q}\}$, put $B = Q - \{0_Q, 1_Q\}$, then $\bigvee_{P +_2 Q} B = 1_{P +_2 Q}$. But for
any \( b \in B, a \not\leq b \). Thus, \( a \triangleleft 1 \) is not right. Similarly, we have for all \( b \in Q - \{0_Q, 1_Q\} \), \( b \triangleleft_{P+2Q} 1_{P+2Q} \) is not right. So \( \{x \in P + 2Q: x \triangleleft_{P+2Q} 1_{P+2Q}\} = \{0_{P+2Q}\} \), whence \( P + 2Q \) is not supercontinuous.

(2) Let \( P = Q = [0, 1) \). Then both \( P \) and \( Q \) are supercontinuous. Then

\[
P + 1 = P^1 + 2Q_1.
\]

By what we have proved above, \( P + 1 \) is not supercontinuous.

**Definition 3.1.** [1] Let \( P \) be a poset. A projection operator (shortly projection) is an idempotent, monotone self-map \( p: P \rightarrow P \).

**Lemma 3.1.** For a projection \( p \) on a poset \( P \), consider its image \( p(P) \) in \( P \) with the induced ordering. Then the following properties hold:

1. If \( X \) is a subset of \( p(P) \) which has a supremum in \( P \), then \( X \) has a supremum in \( p(P) \) and \( \sup p(P) X = p(\sup P X) \).
2. If, in addition, \( p \) has a right adjoint, then \( p(P) \) is closed in \( P \) for existing suprema, i.e., every subset \( A \subseteq p(P) \) that has a supremum in \( P \) also has a supremum in \( p(P) \) and \( \sup p(P) A = \sup P A \).

**Proof.** (1) See [1].

(2) As \( p \) has a right adjoint, \( p \) preserves existing suprema. If \( A \subseteq p(P) \) has a supremum in \( P \), then by (1), \( \sup p(P) A = p(\sup P A) = \sup P p(A) = \sup P A \), which finishes the proof.

**Proposition 3.3.** Let \( P \) be a supercontinuous poset and \( p: P \rightarrow P \) a projection which has a right adjoint. Then the image \( p(P) \) with the order induced from \( P \) is a supercontinuous poset, too. For \( x, y \in p(P) \), we have \( x \triangleleft_{p(P)} y \) iff there exists \( a \in P \) such that \( x \leq p(a) \) and \( a \triangleleft P y \).

**Proof.** From Lemma 3.1, we know that \( p(P) \) is closed in \( P \) under passing to suprema. Let \( y \in p(P) \) be given. As \( P \) is supercontinuous, \( y = \bigvee P \{a \in P: a \triangleleft P y\} \). As \( p \) has a right adjoint, \( p \) is a left adjoint and preserves existing suprema. Then

\[
\bigvee p(P) \{p(a): a \in P, a \triangleleft P y\} = p(y) = y.
\]

Thus, for the supercontinuity of \( p(P) \), it suffices to prove that \( p(a) \triangleleft_{p(P)} y \) whenever \( a \triangleleft P y \). For this, let \( a \) be an element of \( P \) such that \( a \triangleleft P y \). Consider any subset \( B \subseteq p(P) \) such that \( \bigvee p(P) B \) exists and \( y \leq \bigvee p(P) B = \bigvee P B \). As \( a \triangleleft P y \), we find a \( b \in B \) such that \( a \leq b \). Then \( p(a) \leq p(b) = b \) by the monotonicity and idempotency of \( p \). This shows that \( p(a) \triangleleft_{p(P)} y \). For the second part of the claim, let \( x, y \in p(P) \) be such that \( x \triangleleft_{p(P)} y \). As \( y = \bigvee P \{a \in P: a \triangleleft P y\} = \bigvee p(P) \{a \in P: a \triangleleft P y\} \), by the above proof, there is an \( a \in P \) with \( a \triangleleft P y \) such that \( x \leq p(a) \). The converse has already been shown in the first part of the proof.
4. Superalgebraic posets

Definition 4.1. In any poset $P$, an element $k$ is called super-compact iff $k \prec k$, i.e., whenever $A$ is a subset of $P$ such that $\sup A$ exists and $k \leq \sup A$, then $k \leq a$ for some $a \in A$. The set of all super-compact elements is denoted by $SK(P)$.

Remark 4.1. In [7], the super-compact elements are called “hyper-compact” elements. In [8], they also appear under the name “completely join-prime elements.”

Definition 4.2. A poset $P$ is called superalgebraic if and only if for all $x \in P$, $x = \sup(\downarrow x \cap SK(P))$.

Although in a poset every super-compact element is compact, a superalgebraic poset need not be an algebraic poset.

Example 4.1. (1) The example in Remark 1.1 is a superalgebraic poset. This follows from the fact that $\{x\} \ll \{x\}$ for all $x \in N$. However, this particular poset is not an algebraic poset.

(2) Chains like $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$ are supercontinuous posets that are not superalgebraic.

Proposition 4.1. In a poset $P$, the following statements are equivalent:

(1) $P$ is superalgebraic;
(2) $P$ is supercontinuous, and $x \lessdot y$ iff there is a $k \in SK(P)$ with $x \leq k \leq y$.

In particular, every superalgebraic poset is supercontinuous and every superalgebraic lattice is a supercontinuous lattice, i.e. a completely distributive lattice.

Proof. (1) $\Rightarrow$ (2): Assume (1) holds and $x, y \in P$. If $x \lessdot y$, then since $y = \bigvee(\downarrow y \cap SK(P))$. By (1), there is a $k \in \downarrow y \cap SK(P)$ with $x \leq k$. Hence $x \leq k \leq y$ with $k \in SK(P)$. Conversely, if there is a super-compact element $k$ with $x \leq k \leq y$, then $x \leq k \lessdot k \leq y$, whence $x \lessdot y$ by the properties of $\ll$ relation.

The supercontinuity of $P$ follows immediately from the definitions.

(2) $\Rightarrow$ (1): Assume (2) holds and let $y \in P$. Then $y = \sup\{x \in P; x \lessdot y\}$. As, by (2), for every $x \lessdot y$, there is a super-compact element $k$ such that $x \leq k \leq y$. We conclude that $y = \sup(\downarrow y \cap SK(P))$. □

Let $\mathcal{D}(P)$ denote the set of all lower sets of a poset $P$ ordered by inclusion.

Proposition 4.2. Let $P$ be a poset. Then

(1) $\mathcal{D}(P)$ is a superalgebraic lattice whose super-compact elements are the principle ideals.
(2) The principal ideal map is an isomorphism $x \mapsto \downarrow x$: $P \to SK(\mathcal{D}(P))$. 
Proof. (1) Principal ideals are super-compact. In fact, let \( \downarrow x \subseteq \bigcup D \) where \( D \subseteq D(P) \).

Then, there exists \( D \in D \) such that \( x \in D \), whence \( \downarrow x \subseteq D \). So \( \downarrow x \preceq \downarrow x \). Conversely, let \( A \) be super-compact in \( D(P) \), then as \( A \) is the union of the set of principle ideals \( \downarrow x, x \in A \), it follows that \( A = \downarrow x \) for some \( x \in A \). Thus \( D(P) \) is a superalgebraic poset. It is a superalgebraic lattice as follows from the fact it is a complete lattice.

(2) The proof of (2) is clear. \( \square \)

Let \( P \) be a poset and \( S \subseteq P \). We use \( \tilde{D}(S) \) to denote the set of all lower sets \( A \subseteq S \) for which the supremum \( \sup A \) exists in \( P \).

Proposition 4.3. Let \( P \) be a superalgebraic poset and \( S = SK(P) \). Then the map

\[
\phi : x \mapsto \downarrow x \cap S : P \to \tilde{D}(S)
\]

is an isomorphism.

Proof. To prove that

\[
\phi : x \mapsto \downarrow x \cap S : P \to \tilde{D}(S)
\]

is bijective, we claim that

\[
\sup : A \mapsto \sup A : \tilde{D}(S) \to P
\]

is the inverse of this map. Since \( \sup(\downarrow x \cap S) = x \), it suffices to show that \( \downarrow (\sup A) \cap S = A \) for each \( A \in \tilde{D}(S) \). Since \( \sup \) is clear, we must show \( \subseteq \). Let \( k \in \downarrow (\sup A) \cap S \), that is, \( k \preceq k \preceq \sup A \). Thus there exists an \( a \in A \) such that \( k \preceq a \). Since \( A \) is a lower set in \( S \), we have \( k \in A \). The two maps are clearly order-preserving, whence \( \phi \) is an isomorphism. \( \square \)

Proposition 4.4. If \( \{P_i : i \in I\} \) is a family of superalgebraic posets with least element \( 0 \), then the Cartesian product \( \prod_{i \in I} P_i \) is also a superalgebraic poset; the same holds for Cartesian product of superalgebraic lattices.

Proof. From Proposition 2.2, an element \( (x_i)_{i \in I} \) of the product is super-compact iff \( x_i \in SK(P_i) \) for all \( i \in I \) and \( |\{i \in I : x_i \neq 0\}| \leq 1 \). Since every factor \( P_i \) is superalgebraic, every element of \( \prod_{i \in I} P_i \) is the supremum of such elements. \( \square \)

5. Some remarks on the category of supercontinuous posets

Definition 5.1. Let \( SCP \) be the category whose objects are the supercontinuous posets, and morphisms are mappings \( f : P \to Q \) which have a right adjoint, i.e.,

\[
\text{Ob}(SCP) = \{P : P \text{ is a supercontinuous poset}\},
\]

\[
\text{Mor}(SCP) = \{f : P \to Q : f \text{ has a right adjoint}\}.
\]

Hence, \( SCP \) is a concrete category.
Remark 5.1. (1) The category $\text{SCP}$ is not small. This is because $\text{Ob}(\text{SCP})$ is not a set, but a proper class. But it is locally small since it is a concrete category.

(2) The category $\text{SCP}$ is not a discrete category. In fact, let $L$ be a poset which is also a non-trivial complete lattice. Then $f : L \to L \in \text{hom}(L, L)$ iff $f$ preserves suprema. For given $x_0 \in L$, the constant map with value $x_0$ belongs to $\text{hom}(L, L)$. Again, $1_L \in \text{hom}(L, L)$. Thus $|\text{hom}(L, L)| \neq 1$. So $\text{SCP}$ is not discrete.

(3) The category $\text{SCP}$ is not a connected category. In this case there is only one map $g : \mathbb{R} \to \{0\}$, and this map does not have any adjoints, since $\mathbb{R}$ has neither a smallest nor a largest element. Therefore there is no map $f : \{0\} \to \mathbb{R}$ with an adjoint.

We denote by $\text{SCP}^*$ the full subcategory of supercontinuous posets with a top element 1 and a bottom element 0.

Remark 5.2. The trivial poset $\{0\}$ is both initial and terminal in the category $\text{SCP}^*$.

Remark 5.3. As what we have pointed in Remark 5.1(3), $\{0\}$ is not terminal in $\text{SCP}$.

Acknowledgment

We thank the referee for his valuable comments for improvement.

References