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Polynomial identity rings as rings of functions, II

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ABSTRACT

In characteristic zero, Zinovy Reichstein and the author generalized the usual relationship between irreducible Zariski closed subsets of the affine space, their defining ideals, coordinate rings, and function fields, to a non-commutative setting, where "varieties" carry a PGL_n-action, regular and rational "functions" on them are matrix-valued, "coordinate rings" are prime polynomial identity algebras, and "function fields" are central simple algebras of degree n. In the present paper, much of this is extended to prime characteristic. In addition, a mistake in the earlier paper is corrected. One of the results is that the finitely generated prime PI-algebras of degree n are precisely the rings that arise as "coordinate rings" of "n-varieties" in this setting. For $n = 1$ the definitions and results reduce to those of classical affine algebraic geometry.

II.1. Introduction

In characteristic zero, the usual relationship between irreducible Zariski closed subsets of affine space, their defining ideals, coordinate rings, and function fields was extended in [40] to the setting of PI-algebras. The geometric objects are the *n-varieties*, certain PGL_n-invariant locally closed subsets of $(\mathrm{M}_n)^m$. Here $(\mathrm{M}_n)^m$ denotes the m-tuples of $n \times n$ matrices over the algebraically closed base field k; PGL_n acts on $(\mathrm{M}_n)^m$ by simultaneous conjugation. An irreducible n-variety X has a *PI-coordinate ring* $k_n[X]$, which is a prime, finitely generated PI-algebra over k. Moreover, up to isomorphism, every finitely generated prime PI-algebra over k arises in this way. The total ring of fractions of $k_n[X]$ is a central simple algebra of degree n, called the *central simple algebra of rational functions on X* and denoted by $k_n(X)$.

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Table II.1
The algebraic categories.

Category	*Objects*	*Morphisms*	*Relevant Theorems*
PI_n	finitely generated prime k-algebras of PI-degree n	k-algebra homomorphisms $\alpha : R \to S$ such that for any k-algebra surjection $\phi : S \to \mathrm{M}_n$, $\phi \circ \alpha$ is also surjective	II.1.1
CS_n	central simple algebras of degree n whose center is a finitely generated field extension of k	k-algebra homomorphisms (necessarily injective)	II.1.2 II.1.3 1.2 (char 0)

Table II.2
The geometric categories.

Category	*Objects*	*Morphisms*	*Relevant Theorems*
Var_n	irreducible n-varieties	regular maps of n-varieties (Def. 6.1(a))	II.1.1
$\mathcal{C}_n$	irreducible n-varieties	dominant rational maps of n-varieties (Def. II.7.1)	II.1.2
$\mathcal{D}_n$	irreducible n-varieties	PGL_n-equivariant dominant rational maps	
$\mathcal{E}_n$	irreducible PGL_n-varieties that are PGL_n-equivariantly birationally isomorphic to an n-variety	PGL_n-equivariant dominant rational maps	II.1.3
Bir_n	irreducible generically free PGL_n-varieties	PGL_n-equivariant dominant rational maps	1.2 (char 0)

In [40], we restricted attention to characteristic zero since we were primarily interested in the "rational" theory (in particular, in Theorem 1.2), where several proofs use the characteristic zero assumption in an essential way. But it is a natural question to ask to what extent the results described above are true in prime characteristic; in fact, in his MathSciNet review [43], Francesco Vaccarino termed it "unfortunate" that the positive characteristic case was not included. The main purpose of the present paper is to extend many of the results in [40] to prime characteristic, some fully, others only partially. In particular, all results in Sections 3–5 extend fully to prime characteristic (see Section II.5 below for more details and one minor exception). The main results of [40], Theorems 1.1 and 1.2, require more discussion. In addition, we fix below an error affecting Theorem 1.1.

Before continuing, we make several conventions. This article should be considered the second part of [40], whose conventions and notation we follow with one exception: the algebraically closed base field k is of arbitrary characteristic. In this spirit, the numbers of all results (and sections, etc.) in the present paper are preceded by "II.", while numbers without this prefix refer to items in [40]. Similarly, [1–30] refer to the references of [40], while the references of the present paper are numbered [31] and higher. These conventions should make it easier to simultaneously read proofs in [40] together with the relevant commentary in the present paper.

Our main results involve several algebraic and geometric categories, see Tables II.1 and II.2. The morphisms in the category PI_n were incorrectly defined in [40]; an alternate characterization of these morphisms is given in Remark II.6.6.

Each of the geometric categories, except for Var_n, is a subcategory of the one listed below it. That $\mathcal{E}_n$ is a subcategory of Bir_n follows from Lemma II.5.1. Note that the inclusion of $\mathcal{D}_n$ into $\mathcal{E}_n$ is a category equivalence. In characteristic zero, $\mathcal{C}_n = \mathcal{D}_n$ and $\mathcal{E}_n = Bir_n$; this follows from Proposition 7.3 and Lemma 8.1, which we will discuss below. It is not known if these equalities also hold in prime characteristic (cf. the open questions listed at the end of the introduction).

II.1.1. Theorem. *The functor defined by*

$$X \mapsto k_n[X],$$
$$(f : X \to Y) \mapsto \bigl(f^* : k_n[Y] \to k_n[X]\bigr)$$

is a contravariant equivalence of categories between Var_n and PI_n.

Here, as in the following theorems, $f^*(g) = g \circ f$ (as maps $X \to \mathrm{M}_n$).

Theorem II.1.1 has the same statement as Theorem 1.1; but the latter is false without the correct definition of the morphisms in the category PI_n given in Table II.1. Beyond Theorem 1.1, the only results in [40] that need to be corrected are Remark 6.2 and Lemma 6.3, see Section II.6.

In characteristic zero, every irreducible generically free PGL_n-variety is birationally isomorphic (as a PGL_n-variety) to an irreducible n-variety (Lemma 8.1). Using this, we established a category equivalence from the category Bir_n to the category CS_n (Theorem 1.2). To prove this result in prime characteristic would require answering several open questions in the affirmative (see below). Restricting attention to certain subcategories of Bir_n we are able to obtain two results in the spirit of Theorem 1.2, the first of which is proved in Section II.7.

II.1.2. Theorem. *The functor defined by*

$$X \mapsto k_n(X),$$
$$(f : X \dashrightarrow Y) \mapsto \bigl(f^* : k_n(Y) \hookrightarrow k_n(X)\bigr)$$

is a contravariant category equivalence from the category $\mathcal{C}_n$ to the category CS_n.

Note one particular fact contained in this result: Every central simple algebra in CS_n is isomorphic to $k_n(X)$ for some irreducible n-variety X (unique up to birational isomorphism of n-varieties). This is Theorem 7.8, which remains true in prime characteristic.

For an irreducible PGL_n-variety X, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ denotes the k-algebra of PGL_n-equivariant rational maps $X \dashrightarrow \mathrm{M}_n$. In characteristic zero, if X is a generically free PGL_n-variety, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n, and if X is an irreducible n-variety, the natural inclusion of $k_n(X)$ into $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is an isomorphism (Proposition 7.3). It is not known if these facts remain true in prime characteristic. We can prove that for an irreducible n-variety X, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n whose center is a finite, purely inseparable extension of the center of $k_n(X)$ (Proposition II.8.1). This enables us to prove the following result in Section II.8.

II.1.3. Theorem. *The functor defined by*

$$X \mapsto \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n),$$
$$(f : X \dashrightarrow Y) \mapsto \bigl(f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \hookrightarrow \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)\bigr)$$

is a full and faithful contravariant functor from the category $\mathcal{E}_n$ to the category CS_n.

In Section II.9, we study several basic questions regarding the algebra $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. If X is an irreducible n-variety, $k_n(X)$ embeds into $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$, and the latter is a central simple algebra of degree n, see Proposition II.8.1. Hence by Theorem 7.8, there is an irreducible n-variety Y such that $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \cong k_n(Y)$. We show that the natural embedding of $k_n(Y)$ into $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ is an isomorphism (even if the same should not be true for X), that Y is unique up to birational isomorphism of n-varieties, and that Y is birationally equivalent to X as PGL_n-varieties (though maybe not as n-varieties), see Corollary II.9.8 and Proposition II.9.5.

We also address the question for which irreducible PGL_n-varieties X, the algebra $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n. In characteristic zero, a necessary and sufficient condition is that the PGL_n-action on X is generically free, see [41, Lemma 2.8]. In prime characteristic, we give a different criterion in Corollary II.9.6: the existence of a PGL_n-equivariant dominant rational map $X \dashrightarrow Y$ for some irreducible n-variety Y.

We conclude with several interrelated open questions in prime characteristic. Let X be an irreducible generically free PGL_n-variety, and set $\mathrm{R}(X) = \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$.

(1) Is X birationally isomorphic (as PGL_n-variety) to an n-variety? Equivalently, is $\mathcal{E}_n = Bir_n$? (Cf. Lemma 8.1.)
(2) Is $\mathrm{R}(X)$ a central simple algebra of degree n? (Cf. [41, Lemma 2.8], Proposition II.8.1, and Corollary II.9.6.)
(3) If X is an n-variety, is $k_n(X) = \mathrm{R}(X)$? That is, is the center of $k_n(X)$ always $k(X)^{\mathrm{PGL}_n}$? Equivalently, is $\mathcal{C}_n = \mathcal{D}_n$? (Cf. Propositions 7.3 and II.8.1.)
(4) Is every central simple algebra in CS_n isomorphic to $\mathrm{R}(X)$ for some X (or for some irreducible n-variety X)? (Cf. Theorem 7.8.)

II.2. Invariant-theoretic background for prime characteristic

Donkin [38] proved Procesi's Conjecture [21, p. 308/309]: The ring of invariants

$$C_{m,n} = k\big[(\mathrm{M}_n)^m\big]^{\mathrm{PGL}_n}$$

for the action of PGL_n on $(\mathrm{M}_n)^m$ by simultaneous conjugation is the algebra generated by the elements

$$c_s(X_{i_1} X_{i_2} \cdots X_{i_r}),$$

where the X_i are generic matrices and c_s denotes the coefficient in degree $n - s$ of the characteristic polynomial. Note that Donkin describes the generators as the functions

$$(x_1, \ldots, x_m) \mapsto \mathrm{tr}\Big(x_{i_1} x_{i_2} \cdots x_{i_r}, \bigwedge\nolimits^s\big(k^n\big)\Big).$$

But it is easy to see that this function is the function $(\mathrm{M}_n)^m \to k$ induced by $c_s(X_{i_1} X_{i_2} \cdots X_{i_r})$ by verifying that $\mathrm{tr}(A, \bigwedge^s(k^n)) = c_i(A)$ for every $A \in \mathrm{M}_n$. (Prove this first for diagonalizable A and then use a density argument.)

Other proofs of Procesi's Conjecture are given in [45,37,35]. We remark that the ring of invariants $C_{m,n}$ has been studied extensively; for work in prime characteristic see, e.g., [44,46,36].

We denote by $Q_{m,n}$ the affine variety with coordinate ring $C_{m,n}$. Since PGL_n is reductive, the latter ring is affine, and the natural map

$$\pi : (\mathrm{M}_n)^m \to Q_{m,n} \tag{II.2.1}$$

is a categorical quotient map. In particular, π is surjective, π maps closed PGL_n-stable subsets to closed subsets, and $C_{m,n}$ separates disjoint closed PGL_n-stable subsets of $(\mathrm{M}_n)^m$. These are all well-known facts, see [39, Section 13.2, and in particular Theorem 13.2.4] or [34, Theorem 6.1 on p. 97 (see the definition on p. 94)]. Separation of disjoint closed invariant subsets is also proved in [16, Corollary A.1.3, p. 151].

As usual, $T_{m,n}$ is the trace ring of the generic matrix ring

$$G_{m,n} = k\{X_1, \ldots, X_m\}$$

(which is generated by the m generic $n \times n$ matrices $X_1, \dots, X_m$). In prime characteristic, $T_{m,n}$ is generated over $G_{m,n}$ by adjoining all coefficients of the characteristic polynomial of each element of $G_{m,n}$. So $T_{m,n}$ is a central extension of $G_{m,n}$. As is well known, $T_{m,n}$ is affine and finite over its center.

We can think of the elements of $T_{m,n}$ as PGL_n-equivariant regular functions $(\mathrm{M}_n)^m \to \mathrm{M}_n$. This allows us to identify $C_{m,n}$ with the center of $T_{m,n}$, as we did in characteristic zero: As usual, $T_{m,n}$ is a subalgebra of $n \times n$ matrices over a large polynomial ring. Since $T_{m,n}$ has PI-degree n, its center is contained in the center of that matrix ring, so consists of scalar matrices. The center of $T_{m,n}$ thus consists of PGL_n-equivariant scalar-valued functions, i.e., PGL_n-invariant functions $(\mathrm{M}_n)^m \to kI_n$, where I_n denotes the identity matrix in M_n. Identifying k with kI_n, we see that the center of $T_{m,n}$ embeds into $C_{m,n}$. But by Donkin's result, the center of $T_{m,n}$ contains the generators of $C_{m,n}$, so that the embedding of the center of $T_{m,n}$ into $C_{m,n}$ is actually an isomorphism.

In characteristic zero, Procesi proved that $T_{m,n}$ consists precisely of the regular PGL_n-equivariant maps $(\mathrm{M}_n)^m \to \mathrm{M}_n$, see [21, Theorem 2.1]. This is also true in prime characteristic. Indeed, using Donkin's result, Procesi's proof goes through nearly literally: one can still use the nondegeneracy of the trace form. But $\bar{f}$ is now a polynomial in Donkin's generators. Note that $\bar{f}$ is "homogeneous in X_{i+1} of degree 1". Now c_s applied to a monomial M in $X_1, \dots, X_{i+1}$ involving r copies of X_{i+1} is homogeneous of degree rs. Since $c_1 = \mathrm{tr}$, one can still write $\bar{f}$ exactly as in the second displayed equation of Procesi's proof, and the rest of the proof goes through. (To make the degree argument precise: Recall that $G_{m,n}$ and $T_{m,n}$ are contained in the $n \times n$ matrices over the commutative polynomial ring in the entries of the generic matrices. Define a degree function by giving each entry of X_{i+1} degree 1 and all other variables degree 0. Then $\bar{f}$ is homogeneous of degree 1, and $c_s(M)$ is homogeneous of degree rs.)

II.3. Preliminaries on trace rings

If R is a prime PI-ring of PI-degree n, its total ring of fractions is a central simple algebra of degree n, and the trace ring $T(R)$ of R is obtained from R by adjoining to R all coefficients of the reduced characteristic polynomial of every element of R.

We will repeatedly use the well-known fact, due to Amitsur, that

$$T\big(T(R)\big) = T(R). \tag{II.3.1}$$

This follows immediately from [31]: Denote by Z the center of $T(R)$. Then every element of $T(R)$ is a Z-linear combination of elements of R. By [31], the coefficients of the characteristic polynomial of such a linear combination are polynomial expressions (with coefficients in Z) in the coefficients of the characteristic polynomials of certain elements of R, so belong to $T(R)$, implying (II.3.1).

We will also need the following result:

II.3.2. Theorem *(Amitsur–Small). Let $\phi : R \to S$ be a surjective ring homomorphism between prime PI-rings of the same PI-degree n. Then ϕ extends uniquely to a homomorphism $\phi' : T(R) \to T(S)$ that preserves characteristic polynomials; i.e., $\phi'(c_s(r)) = c_s(\phi(r))$ for all $r \in R$ and $1 \leqslant s \leqslant n$. It follows immediately that also ϕ' is surjective. Moreover, if $R = T(R)$ (e.g., if $R = T_{m,n}$), then $S = T(S)$ and ϕ preserves characteristic polynomials.*

This is [3, Theorem 2.2]. (Note the typo in the statement of this result: "into" should be "onto". Note also that the trace ring of R is in [3] denoted by $T(R)R$.)

II.4. Remarks to Section 2

Everything in Section 2 extends to prime characteristic if one uses the facts and the modified definitions from Sections II.2 and II.3. Note that the statement of Lemma 2.6, though correct in prime characteristic, can be strengthened in the natural way, see Lemma II.4.3.

2.1 was addressed in Section II.2. Note that $C_{m,n}$ has different generators in prime characteristic.

Proposition 2.3 is also true in prime characteristic; since it is of fundamental importance for this paper, we restate it formally. We denote by $U_{m,n}$ the following PGL_n-stable dense open subset of $(\mathrm{M}_n)^m$:

$$U_{m,n} = \{(a_1, \dots, a_m) \in (\mathrm{M}_n)^m \mid a_1, \dots, a_m \text{ generate } \mathrm{M}_n \text{ as } k\text{-algebra}\}.$$

Note that

$$\mathrm{Stab}_{\mathrm{PGL}_n}(x) = 1 \quad \forall x \in U_{m,n}. \tag{II.4.1}$$

Here $\mathrm{Stab}_{\mathrm{PGL}_n}(x)$ denotes the group-theoretic stabilizer of x in PGL_n. Recall the categorical quotient map $\pi : (\mathrm{M}_n)^m \to Q_{m,n}$ from (II.2.1).

II.4.2. Proposition. *Proposition* 2.3 *is true in arbitrary characteristic. That is*:

(a) *If $x \in U_{m,n}$ then $\pi^{-1}(\pi(x))$ is the PGL_n-orbit of x.*
(b) *PGL_n-orbits in $U_{m,n}$ are closed in $(\mathrm{M}_n)^m$.*
(c) *π maps closed PGL_n-invariant sets in $(\mathrm{M}_n)^m$ to closed sets in $Q_{m,n}$.*
(d) *$\pi(U_{m,n})$ is Zariski open in $Q_{m,n}$.*
(e) *If Y is a closed irreducible subvariety of $Q_{m,n}$ then $\pi^{-1}(Y) \cap U_{m,n}$ is irreducible in $(\mathrm{M}_n)^m$.*

Proof. Given (a) (which we prove below), (b) follows immediately. Part (c) was addressed in Section II.2. Concerning (d) and (e): the proofs of the corresponding parts of Proposition 2.3 go through without changes.

(a) This is an adaptation of part of the proof of [4, (12.6)]. Let $x \in U_{m,n}$, and let $\phi = \phi_x$ be the corresponding surjective (thus simple) representation $G_{m,n} \to \mathrm{M}_n$, $p \mapsto p(x)$. Let $x' \in \pi^{-1}(\pi(x))$, and let $\phi' = \phi_{x'} : G_{m,n} \to \mathrm{M}_n$ be the corresponding (potentially not surjective) representation. We have to show that x' belongs to the PGL_n-orbit of x. To do this, we will twice replace x' by another element in its own PGL_n-orbit, and then prove $x' = x$.

Note that the representations ϕ and ϕ' preserve characteristic polynomials. (For ϕ', this follows, e.g., from the fact that, using the notation in 2.4, ϕ' can be extended to the homomorphism $\mathrm{M}_n(k[x_{i,j}^{(h)}]) \to \mathrm{M}_n$ given by evaluating the indeterminates $x_{i,j}^{(h)}$ at the corresponding entries of the m-tuple $x' \in (\mathrm{M}_n)^m$.) Since $\pi(x) = \pi(x')$, $f(x) = f(x')$ for all $f \in C_{m,n}$. Let $p \in G_{m,n}$, and denote by $c_s(p)$ the coefficient in degree $n-s$ of the characteristic polynomial of p. Since $c_s(p) \in C_{m,n}$,

$$c_s(\phi(p)) = \phi(c_s(p)) = c_s(p)(x) = c_s(p)(x') = \phi'(c_s(p)) = c_s(\phi'(p)).$$

Consequently, for all $p \in G_{m,n}$, $\phi(p)$ and $\phi'(p)$ have the same trace and the same characteristic polynomial; we will use this repeatedly below.

Since ϕ is onto, there is a $z \in G_{m,n}$ such that $\phi(z)$ is a diagonal matrix with distinct eigenvalues. Since $\phi'(z)$ has the same characteristic polynomial as $\phi(z)$, it is in the PGL_n-orbit of $\phi(z)$ in M_n. Replacing x' by some other element in its PGL_n-orbit (and, of course, also changing $\phi' = \phi_{x'}$), we may assume that $\phi(z) = \phi'(z)$ is a diagonal matrix with distinct eigenvalues. It is possible to express the elementary matrix units $e_{i,i}$ as polynomials (with coefficients in k) in this diagonal matrix. Hence there exist $z_{i,i} \in G_{m,n}$ such that $\phi(z_{i,i}) = \phi'(z_{i,i}) = e_{i,i}$ for all $i = 1, \dots, n$.

Since ϕ is onto, we can choose, for $i, j \in \{1, \dots, n\}$ with $i \neq j$, elements $\tilde{z}_{i,j}$ in $G_{m,n}$ such that $\phi(\tilde{z}_{i,j}) = e_{i,j}$. For $i \neq j$, set $z_{i,j} = z_{i,i}\tilde{z}_{i,j}z_{j,j}$. Then $\phi(z_{i,j}) = e_{i,j}$ for all i and j. Now for $i \neq j$, $\phi'(z_{i,j}) = e_{i,i}\phi'(\tilde{z}_{i,j})e_{j,j} = \alpha_{i,j}e_{i,j}$ for certain scalars $\alpha_{i,j} \in k$. Now $\phi(z_{i,j}z_{j,i}) = e_{i,i}$ and $\phi'(z_{i,j}z_{j,i}) = \alpha_{i,j}\alpha_{j,i}e_{i,i}$ have the same trace, i.e., $1 = \alpha_{i,j}\alpha_{j,i}$. In particular, $\alpha_{i,j} \neq 0$. Hence $\gamma = e_{1,1} + \alpha_{1,2}e_{2,2} + \dots + \alpha_{1,n}e_{n,n}$ is in GL_n.

Set $\phi'' = \gamma\phi'\gamma^{-1}$. Then one checks readily that $\phi''(z_{i,i}) = e_{i,i}$ for $i = 1, \dots, n$, and that $\phi''(z_{1,j}) = e_{1,j}$ for $j = 2, \dots, n$. Replacing x' by $\gamma x'\gamma^{-1}$ (and thus ϕ' by $\phi'' = \gamma\phi'\gamma^{-1}$), we may assume that $\alpha_{1,j} = 1$ for $j = 2, \dots, n$. Since $\alpha_{i,j}\alpha_{j,i} = 1$, also $\alpha_{j,1} = 1$. So for all i and j,

$$\phi'(z_{1,i}) = e_{1,i} \quad \text{and} \quad \phi'(z_{j,1}) = e_{j,1}.$$

Finally, let $p \in G_{m,n}$ be arbitrary. Say $\phi(p) = (\beta_{i,j}), \phi'(p) = (\beta'_{i,j}) \in \mathrm{M}_n$. Then for all i and j,

$$\beta'_{i,j} = \mathrm{tr}\big(\beta'_{i,j}e_{1,1}\big) = \mathrm{tr}\big(e_{1,i}\phi'(p)e_{j,1}\big) = \mathrm{tr}\big(\phi'(z_{1,i}pz_{j,1})\big)$$
$$= \mathrm{tr}\big(\phi(z_{1,i}pz_{j,1})\big) = \mathrm{tr}\big(e_{1,i}\phi(p)e_{j,1}\big) = \mathrm{tr}(\beta_{i,j}e_{1,1}) = \beta_{i,j},$$

so that $\phi'(p) = \phi(p)$. Thus $\phi' = \phi$ and $x' = x$, concluding the proof of (a). □

2.4 was addressed in Section II.2.
2.5–2.10 carry over to prime characteristic. We make two remarks:
Lemma 2.6 remains true in prime characteristic as stated, but parts (a) and (b) can be strengthened as follows:

II.4.3. Lemma. *Let J be a prime ideal in $\mathrm{Spec}_n(T_{m,n})$, i.e., a prime ideal of $T_{m,n}$ such that $T_{m,n}/J$ has PI-degree n.*

(a) *The natural projection $\phi : T_{m,n} \to T_{m,n}/J$ preserves characteristic polynomials, and $T_{m,n}/J$ is the trace ring of $G_{m,n}/(J \cap G_{m,n})$.*
(b) *For every $p \in J$, $c_i(p) \in J$ for $i = 1, \ldots, n$. In particular, $\mathrm{tr}(p) \in J$.*

Proof. (a) Since $T_{m,n}$ is its own trace ring by (II.3.1), Theorem II.3.2 first implies that ϕ preserves characteristic polynomials, and then, applying it to the natural map $G_{m,n} \to G_{m,n}/(J \cap G_{m,n})$, that $T_{m,n}/J$ is the trace ring of $G_{m,n}/(J \cap G_{m,n})$. (b) Since $\phi(c_i(p)) = c_i(\phi(p)) = c_i(0) = 0$, $c_i(p) \in J$. □

Lemma 2.10: The use of [4, Theorem (9.2)] can be avoided with a short, direct argument: If ϕ_i and ϕ_j had the same kernel, say I, then they would induce k-algebra isomorphisms $G_{m,n}/I \to \mathrm{M}_n$. It follows easily that ϕ_i and ϕ_j would then differ by an automorphism of M_n, i.e., by an element of PGL_n, a contradiction.

II.5. Remarks to Sections 3–5

Using the facts and the modified definitions from Sections II.2 and II.3, and the comments in Section II.4 regarding Section 2, the content of Sections 3–5 (definitions, results, and proofs) remain true in prime characteristic, with the possible exception of Remark 3.2. In particular, the following results remain true in prime characteristic: the weak and the strong Nullstellensatz for prime ideals (Propositions 5.1 and 5.3), and the inclusion-reversing bijections between the irreducible n-varieties in $U_{m,n}$ and $\mathrm{Spec}_n(G_{m,n})$ (resp., $\mathrm{Spec}_n(T_{m,n})$) (Theorem 5.7).
Only a few items require further comment. We begin with a lemma which is likely known.

II.5.1. Lemma. *Also in prime characteristic, n-varieties are generically free as PGL_n-varieties.*

In characteristic zero, the action of a linear algebraic group G on a G-variety X is called *generically free* if there is a dense open subset U of X such that for any $x \in U$ the group-theoretic stabilizer G_x is trivial. If $G = \mathrm{PGL}_n$ and X is an n-variety in $U_{m,n}$, then G_x is trivial for all $x \in X$, in arbitrary characteristic, see (II.4.1). So the PGL_n-action on an n-variety is generically free in characteristic zero. In prime characteristic, one has to check an additional property in order to ensure that the PGL_n-action on the n-variety X is generically free (see [32, Section 4] for the precise definition and relevant results). We will not be using Lemma II.5.1 in the sequel, except to note that the category $\mathcal{E}_n$ is a subcategory of Bir_n, as already remarked in the introduction. Therefore we will only sketch the proof. We are grateful to Zinovy Reichstein for showing it to us.

Outline of the proof. By [32, Lemma 4.2(i′)], it suffices to show that for any closed point $x \in X$, the Lie algebra of the scheme-theoretic stabilizer, which we will denote by $\tilde{G}_x$, is trivial. We may assume

$x \in X(k) \subseteq (\mathrm{M}_n)^m$ (if not, replace k by the algebraic closure of the function field of the irreducible subvariety of X corresponding to x). If $f: G \to X$ is given by $f(g) = g \cdot x = gxg^{-1}$, then $\mathrm{Lie}(\tilde{G}_x)$ is the kernel of the derivative df_1.

The derivative of the conjugation representation of GL_n on $\mathrm{M}_n = \mathrm{gl}_n$ is the adjoint representation of gl_n on itself, which takes $A \in \mathrm{gl}_n$ to $\mathrm{ad}(A): X \mapsto [A, X]$. Since the center of GL_n acts trivially on M_n, the conjugation action of GL_n on M_n descends to PGL_n, and the Lie derivative for this PGL_n-action is still $\mathrm{ad}(A): x \mapsto [A, x]$, except that we are now viewing A as an element of $\mathrm{pgl}_n = \mathrm{gl}_n / kI_n$. Similarly, for the action of PGL_n on $(\mathrm{M}_n)^m$ by simultaneous conjugation, the Lie action of $A \in \mathrm{pgl}_n$ on $(\mathrm{M}_n)^m$ is given by $x = (x_1, \dots, x_m) \mapsto ([A, x_1], \dots, [A, x_m])$. For a fixed $x \in U_{m,n}$ the kernel of this map is $\mathrm{Lie}(\tilde{G}_x)$; it is trivial, since the only matrices in M_n commuting with all x_i are the scalar matrices. So also in prime characteristic, n-varieties are generically free as PGL_n-varieties. $\square$

Remark 3.2: The first two sentences are true in prime characteristic. The last may not be – the status of Proposition 7.3 is not known in prime characteristic (see the introduction to Section II.7 and Proposition II.8.1).

Example 3.3 works in characteristic $\neq 2$ since the proof of [12, Theorem 2.9] goes through under this assumption, showing that $U_{2,2}$ is defined by the non-vanishing of $c = c(X_1, X_2)$. (Note the typo in the definition of $c(X_1, X_2)$: the lone "det" should be "tr".) Now if f were in $T_{2,2}$, then c would be an invertible element of $T_{2,2}$. Hence for any $a \in (\mathrm{M}_2)^2$, $\phi_a(c) = c(a) \neq 0$, contradicting that $c(a) = 0$ for $a \notin U_{2,2}$.

Remark 3.4: This goes through. The proof in part (b) uses the following: If ϕ is a surjective k-algebra homomorphism $T_{m,n} \to \mathrm{M}_n$, then

$$\phi\big(p(X_1, \dots, X_m)\big) = p\big(\phi(X_1), \dots, \phi(X_m)\big) \tag{II.5.2}$$

for all $p \in T_{m,n}$. It suffices to verify this for the k-algebra generators of $T_{m,n}$. It is clear for $p \in G_{m,n}$. For $p = c_s(h)$ with $h \in G_{m,n}$ it follows from the results summarized in Section II.3: By (II.3.1), $T(T_{m,n}) = T_{m,n}$, and clearly $T(\mathrm{M}_n) = \mathrm{M}_n$. Hence by Theorem II.3.2, ϕ preserves characteristic polynomials, so that (II.5.2) holds for $p = c_s(h)$ with $h \in G_{m,n}$.

Lemma 3.6: The proof of (a) uses [28, Theorem 1], which does not need characteristic zero. In (b), as in many other instances in later sections, one has to use results summarized in Section II.2. In this particular instance one needs that also in prime characteristic, C and $\overline{X}$ can be separated by a function $f \in C_{m,n}$, and that f can be thought of as a central element of $T_{m,n}$.

Corollary 5.2: Note that the proof shows that for every irreducible n-variety X, $\mathcal{I}_T(X)$ is a prime ideal of $T_{m,n}$ of PI-degree n. The same argument shows that $\mathcal{I}(X)$ is a prime ideal of $G_{m,n}$ of PI-degree n.

II.6. Remarks to Section 6

As mentioned in the introduction, Theorem 1.1 is incorrect as stated, but it can be easily corrected, and the corrected version, Theorem II.1.1, is also valid in prime characteristic. The only other results in [40] that need to be corrected are Remark 6.2 and Lemma 6.3. With these changes, all results of Section 6 are also valid in prime characteristic.

The problem stems from the definition of α_* in Definition 6.1(d). Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be n-varieties, and let $\alpha: k_n[Y] \to k_n[X]$ be a k-algebra homomorphism. Denote the generic matrices generating $G_{l,n}$ by $X_1, \dots, X_l$, and their images in $k_n[Y]$ by $\overline{X_1}, \dots, \overline{X_l}$. Set $f_i = \alpha(\overline{X_i})$. We define a regular map

$$\alpha_\#: X \to (\mathrm{M}_n)^l$$

by $\alpha_\#(x) = (f_1(x), \dots, f_l(x))$. If $\alpha_\#(X) \subseteq Y$, then $\alpha_*(x) = \alpha_\#(x)$ defines a regular map of n-varieties

$$\alpha_*: X \to Y.$$

But $\alpha_\#(X) \subseteq Y$ may not be true. One easily checks that the image of $\alpha_\#$ is contained in the set $\mathcal{Z}_0$ of zeroes of $\mathcal{I}(Y)$ in $(\mathrm{M}_n)^l$ (cf. (II.7.3)), but it is not necessarily contained in $U_{l,n}$, and thus not necessarily contained in $Y = \mathcal{Z}(\mathcal{I}(Y)) = \mathcal{Z}_0 \cap U_{l,n}$:

II.6.1. Lemma. *The following are equivalent*:

(a) $\alpha_\#(X) \subseteq Y$.
(b) *For every k-algebra surjection $\phi : k_n[X] \to \mathrm{M}_n$, the composition $\phi \circ \alpha$ is also surjective.* (*We will refer to this property as "α preserves surjections onto* M_n*".*)

Note that (b) is true if α is an isomorphism (or a surjection), and false if the image of α has PI-degree $< n$ (in which case the proof shows that $\alpha_\#(X) \cap Y = \emptyset$). One can construct examples for which $\emptyset \neq \alpha_\#(X) \cap Y \subsetneq \alpha_\#(X)$. See also Example II.6.5 below.

Proof. Recall that for $x \in X$, the representation $\phi_x : G_{m,n} \to \mathrm{M}_n$ given by $p \mapsto p(x)$ factors through a map $\bar{\phi}_x$ from $k_n[X] = G_{m,n}/\mathcal{I}(X)$ onto M_n. Conversely, any k-algebra surjection $k_n[X] \to \mathrm{M}_n$ is of the form $\bar{\phi}_x$ for some $x \in X$, cf. Remark 3.4.

Note that the image of α is generated as k-algebra by the elements $f_i = \alpha(\overline{X_i})$. Let $x \in X$. By definition, $\alpha_\#(x) = (f_1(x), \ldots, f_l(x))$. Note that $f_i(x) = (\bar{\phi}_x \circ \alpha)(\overline{X_i})$. Now $\alpha_\#(x) \in Y$ iff $\alpha_\#(x) \in U_{l,n}$ iff $f_1(x), \ldots, f_l(x)$ generate M_n iff $\bar{\phi}_x \circ \alpha$ is onto. Hence $\alpha_\#(X) \subseteq Y$ iff $\phi \circ \alpha$ is onto for all k-algebra surjections $\phi : k_n[X] \to \mathrm{M}_n$. $\square$

Consequently, Definition 6.1(d) needs to be modified to require that α preserves surjections onto M_n (parts (a)–(c) go through without change):

II.6.2. Definition. Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be n-varieties, and let $\alpha : k_n[Y] \to k_n[X]$ be a k-algebra homomorphism such that for all surjective k-algebra maps $\phi : k_n[X] \to \mathrm{M}_n$, also $\phi \circ \alpha$ is surjective. Then $\alpha_\#(X) \subseteq Y$, and $\alpha_* : X \to Y$ is the regular map of n-varieties given by $\alpha_*(x) = \alpha_\#(x)$. It is easy to check that for every $g \in k_n[Y]$, $\alpha(g) = g \circ \alpha_* : X \to Y \to \mathrm{M}_n$.

Also Remark 6.2 and Lemma 6.3 need to be modified to require that α and β preserve surjections onto M_n. With these modifications, the proof of Theorem 1.1 in [40] proves Theorem II.1.1. For the convenience of the reader we present the details.

II.6.3. Remark. Let $f : X \to Y$ be a regular map of n-varieties. Then for $x \in X$, $(f^*)_\#(x) = f(x)$, so that $(f^*)_\#(X) = f(X) \subseteq Y$. Hence the k-algebra homomorphism $f^* : k_n[Y] \to k_n[X]$ preserves surjections onto M_n. Thus $(f^*)_*$ is defined, and $(f^*)_* = f$. Moreover, if $\alpha : k_n[Y] \to k_n[X]$ is a k-algebra homomorphism that preserves surjections onto M_n, then $(\alpha_*)^* = \alpha$. Note also that $(\mathrm{id}_X)^* = \mathrm{id}_{k_n[X]}$ and $(\mathrm{id}_{k_n[X]})_* = \mathrm{id}_X$.

II.6.4. Lemma. *Let X, Y and Z be n-varieties.*

(a) *If $f : X \to Y$ and $g : Y \to Z$ are regular maps of n-varieties, then $(g \circ f)^* = f^* \circ g^*$.*
(b) *If $\alpha : k_n[Y] \to k_n[X]$ and $\beta : k_n[Z] \to k_n[Y]$ are k-algebra homomorphisms that preserve surjections onto M_n, then also $\alpha \circ \beta$ preserves surjections onto M_n, and $(\alpha \circ \beta)_* = \beta_* \circ \alpha_*$.*
(c) *X and Y are isomorphic as n-varieties if and only if $k_n[X]$ and $k_n[Y]$ are isomorphic as k-algebras.*

Proof. (a) and (b) follow directly from the definitions. The proof of part (c) is the same as for Lemma 6.3(c). $\square$

Theorem 6.4 goes through without changes, in arbitrary characteristic. It states that every finitely generated prime k-algebra of PI-degree n is isomorphic to $k_n[X]$ for some irreducible n-variety X (which is unique up to isomorphism of n-varieties by Lemma II.6.4(c)).

Proof of Theorem II.1.1. By Lemma II.6.4, the functor $\mathcal{F}$ in Theorem II.1.1 is contravariant. It is full and faithful by Remark II.6.3. Moreover, by Theorem 6.4, every object in PI_n is isomorphic to the image of an object in Var_n. Hence $\mathcal{F}$ is a contravariant equivalence of categories between Var_n and PI_n. $\square$

Lemma 6.5 goes through without changes, in arbitrary characteristic.

We conclude this section with a few remarks about the category PI_n.

II.6.5. Example. Let t be an indeterminate and let $S = \mathrm{M}_2(k[t])$. Let R be the subalgebra of S consisting of all matrices whose lower left entry is contained in $tk[t]$. Then R and S are prime PI-algebras in PI_2, but the inclusion map $\beta : R \to S$ is not a morphism in PI_2: $\varphi \circ \beta$ is not onto for the surjection $\varphi : S \to \mathrm{M}_2$ defined by $t \mapsto 0$.

Note that by Theorem 6.4, there are, for $n = 2$, irreducible n-varieties X and Y such that $R \cong k_n[Y]$ and $S \cong k_n[X]$. Then the k-algebra homomorphism $\alpha : k_n[Y] \to k_n[X]$ induced by β does not satisfy the two equivalent conditions of Lemma II.6.1.

II.6.6. Remark. Here is an alternate description of the morphisms in the category PI_n (and of the k-algebra homomorphisms α satisfying the equivalent conditions of Lemma II.6.1). For a PI-algebra R in PI_n, denote by $\mathrm{Max}_n(R)$ the maximal ideals of PI-degree n of R, i.e., the ideals I such that $R/I \simeq \mathrm{M}_n$. Now let R and S be PI-algebras in PI_n, and let $\alpha : R \to S$ be an arbitrary k-algebra homomorphism. Then α is a morphism in PI_n iff α induces a map $\mathrm{Max}_n(S) \to \mathrm{Max}_n(R)$, i.e., iff for every $I \in \mathrm{Max}_n(S)$, $\alpha^{-1}(I) \in \mathrm{Max}_n(R)$. Note that in general, $\alpha^{-1}(I)$ need not even be a prime ideal of R (e.g., for $I = \mathrm{M}_2(tk[t])$ in Example II.6.5).

II.7. The first extension of Definition 7.5

It is not known if Proposition 7.3 is true in prime characteristic. Because of this, there are two (possibly nonequivalent) ways to extend Definition 7.5 to prime characteristic, which are explored in this and the following section. The first yields a category equivalence resembling Theorem 1.2 (Theorem II.1.2), the second only a full and faithful functor (Theorem II.1.3).

Definition 7.1 goes through in prime characteristic: For an irreducible n-variety X, $k_n(X)$ is defined to be the total ring of fractions of $k_n[X]$. It is a central simple algebra of degree n, called the *central simple algebra of rational functions on* X.

Remark 7.2 and the subsequent discussion go through in prime characteristic.

Proposition 7.3 may or may not go through (cf. Proposition II.8.1 below).

Definition 7.5 goes through in prime characteristic if one strikes the second sentence of part (a); the other parts go through unchanged. Because of its importance, we restate it with this one change:

II.7.1. Definition. Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be irreducible n-varieties.

(a) A rational map $f : X \dashrightarrow Y$ is called a *rational map of n-varieties* if $f = (f_1, \ldots, f_l)$ where each $f_i \in k_n(X)$.

(b) The n-varieties X and Y are called *birationally isomorphic* or *birationally equivalent* if there exist dominant rational maps of n-varieties $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ (as rational maps of varieties).

(c) A dominant rational map $f = (f_1, \ldots, f_l) : X \dashrightarrow Y$ of n-varieties induces a k-algebra homomorphism (i.e., an embedding) $f^* : k_n(Y) \to k_n(X)$ of central simple algebras defined by $f^*(\overline{X_i}) = f_i$, where $\overline{X_i}$ is the image of the generic matrix $X_i \in G_{l,n}$ in $k_n[Y] \subset k_n(Y)$. One easily verifies that for every $g \in k_n(Y)$, $f^*(g) = g \circ f$, if one views g as a PGL_n-equivariant rational map $Y \dashrightarrow \mathrm{M}_n$.

(d) Conversely, a k-algebra homomorphism (necessarily an embedding) of central simple algebras $\alpha : k_n(Y) \to k_n(X)$ (over k) induces a dominant rational map $f = \alpha_* : X \dashrightarrow Y$ of n-varieties. This map is given by $f = (f_1, \ldots, f_l)$ with $f_i = \alpha(\overline{X_i}) \in k_n(X)$, where $\overline{X_1}, \ldots, \overline{X_l}$ are the images of the generic matrices $X_1, \ldots, X_l \in G_{l,n}$. It is easy to check that for every $g \in k_n(Y)$, $\alpha(g) = g \circ \alpha_*$, if one views g as a PGL_n-equivariant rational map $Y \dashrightarrow \mathrm{M}_n$.

Note that a rational map of n-varieties $f : X \dashrightarrow Y$ is still a PGL_n-equivariant rational map of algebraic varieties, but in prime characteristic maybe not vice versa. The map α_* defined in part (d), which is a priori a rational map $X \dashrightarrow (\mathrm{M}_n)^l$, is indeed (in arbitrary characteristic) a dominant rational map from X to Y:

II.7.2. Lemma. *Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be irreducible n-varieties. Given a nonzero k-algebra homomorphism $\alpha : k_n(Y) \to k_n(X)$, the map $\alpha_* : X \dashrightarrow (\mathrm{M}_n)^l$ induces a dominant rational map of n-varieties $X \dashrightarrow Y$ (again denoted by α_*).*

Proof. Denote the generic matrices generating $G_{l,n}$ by $X_1, \dots, X_l$, and the image of $p \in G_{l,n}$ in $k_n[Y]$ by $\bar{p}$. The map $f = \alpha_* : X \dashrightarrow (\mathrm{M}_n)^l$ is defined by $f = (f_1, \dots, f_l)$ where $f_i = \alpha(\overline{X_i})$. Let U be the PGL_n-invariant nonempty open subset of X on which f is defined. Let $x \in U$. Then for $p = p(X_1, \dots, X_l) \in G_{l,n}$,

$$p\bigl(f(x)\bigr) = p\bigl(\alpha(\overline{X_1}), \dots, \alpha(\overline{X_l})\bigr)(x) = \alpha\bigl(p(\overline{X_1}, \dots, \overline{X_l})\bigr)(x) = \alpha(\bar{p})(x). \tag{II.7.3}$$

In particular, $\mathcal{I}(Y)$ vanishes at $f(x)$, and $\alpha(\bar{p})$ is defined on U. Consequently, $f(x) \in Y$ iff $f(x) \in U_{l,n}$ iff the elements $f_i(x) = \alpha(\overline{X_i})(x)$ generate M_n iff the k-algebra homomorphism $\psi_x : k_n[Y] \to \mathrm{M}_n$ defined by $\psi_x(\bar{p}) = \alpha(\bar{p})(x)$ is onto.

Claim. *There is a nonempty open subset V of U with $f(V) \subseteq Y$.*

To prove this, let t be a nonzero central element of $k_n[Y]$ such that $A = k_n[Y][t^{-1}]$ is an Azumaya algebra of PI-degree n. Then $\alpha(t)$ is a nonzero central element of $k_n(X)$, so of the form $\alpha(t) = s_1 s_2^{-1}$ for nonzero central elements s_1 and s_2 of $k_n[X]$. Denote by V the nonempty open subset of U where both s_1 and s_2 are nonzero. Let $x \in V$. Then by Lemma 6.5, $\psi_x(t) = \alpha(t)(x) = s_1(x)s_2(x)^{-1}$ is a nonzero scalar in M_n. Hence ψ_x extends to a k-algebra homomorphism $A \to \mathrm{M}_n$ which has the same image as ψ_x, and which is onto since A is Azumaya of PI-degree n. Thus $f(V) \subseteq Y$, proving the claim.

The claim implies that we can think of f as a rational map $X \dashrightarrow Y$; we now prove that it is dominant. Since f is PGL_n-equivariant, we can replace V by a PGL_n-invariant nonempty open subset of U (and thus of X) such that $f(V) \subseteq Y$. Then the closure of $f(V)$ in Y is PGL_n-invariant, so an n-variety $Y' \subseteq Y$. Clearly $\mathcal{I}(Y) \subseteq \mathcal{I}(Y')$. Let $p \in \mathcal{I}(Y')$ and $x \in V$. Then by (II.7.3), $0 = p(f(x)) = \alpha(\bar{p})(x)$. Since V is dense in X, $\alpha(\bar{p}) = 0$, so that $\bar{p} = 0$, i.e., $p \in \mathcal{I}(Y)$. Hence $\mathcal{I}(Y) = \mathcal{I}(Y')$, so that $Y = Y'$ by Theorem 5.7. This proves that the rational map $f : X \dashrightarrow Y$ is dominant. □

Remark 7.6, Lemma 7.7, and Theorem 7.8 all go through in arbitrary characteristic. Remark 7.6 states that $(f^*)_* = f$, $(\alpha_*)^* = \alpha$, $(\mathrm{id}_X)^* = \mathrm{id}_{k_n(X)}$, and $(\mathrm{id}_{k_n(X)})_* = \mathrm{id}_X$. Lemma 7.7 states, in particular, that $(g \circ f)^* = f^* \circ g^*$ and $(\alpha \circ \beta)_* = \beta_* \circ \alpha_*$. Theorem 7.8 contains the fact that every central simple algebra in CS_n is isomorphic to $k_n(X)$ for some irreducible n-variety X (which is unique up to birational isomorphism of n-varieties).

Proof of Theorem II.1.2. Call the functor $\mathcal{F}$. Since $(g \circ f)^* = f^* \circ g^*$ (Lemma 7.7(a)), $\mathcal{F}$ is a contravariant functor. Since $\alpha = (\alpha_*)^*$ and $(f^*)_* = f$ (Remark 7.6), $\mathcal{F}$ is full and faithful. By Theorem 7.8, every object in CS_n is isomorphic to the image of an object under $\mathcal{F}$. Hence $\mathcal{F}$ is a contravariant category equivalence. □

II.8. The second extension of Definition 7.5

We can also extend Definition 7.5 by using PGL_n-equivariant rational maps of varieties instead of the above defined "rational maps of n-varieties". This will allow us to prove Theorem II.1.3. First we need to describe the basic relationship between $k_n(X)$ and $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ for an n-variety X; this result is a partial replacement for Proposition 7.3. This relationship will be further investigated in Section II.9.

II.8.1. Proposition. *Let X be an irreducible n-variety. There is a natural embedding*

$$\psi_X : k_n(X) \hookrightarrow \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n),$$

allowing us to identify $k_n(X)$ with a k-subalgebra of $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. *The latter algebra is a central simple algebra of degree n, whose center we can identify with $k(X)^{\mathrm{PGL}_n}$. Moreover, $k(X)^{\mathrm{PGL}_n}$ is a finite, purely inseparable extension of the center of $k_n(X)$, and*

$$\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) = k_n(X) \cdot k(X)^{\mathrm{PGL}_n}.$$

We begin with a lemma.

II.8.2. Lemma. *Let X be an irreducible* PGL_n*-variety. If the k-algebra* $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ *contains a central simple algebra A of degree n, then it is itself a central simple algebra of degree n, and its center is the field $k(X)^{\mathrm{PGL}_n}$.*

Proof. Since X is irreducible, $k(X)$ is a field, and $R = \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ embeds naturally into $\mathrm{RMaps}(X, \mathrm{M}_n) = \mathrm{M}_n(k(X))$. Since A and $\mathrm{M}_n(k(X))$ are central simple algebras of degree n, the latter is a central extension of the former, and thus also of R. Since $\mathrm{M}_n(k(X))$ is a prime ring, so is R. Since R contains a central simple algebra of degree n and is contained in a central simple algebra of degree n, R must have PI-degree n. Consequently, the center of R is contained in the center of $\mathrm{M}_n(k(X))$, i.e., in $k(X) \cdot I_n$ (see, e.g., [33, Lemma 4.9]).

Thus the center of R consists of the PGL_n-equivariant rational maps from X to $k = k \cdot I_n$. Hence the center of R can be identified with $k(X)^{\mathrm{PGL}_n}$, the field of PGL_n-invariant rational maps $X \dashrightarrow k$. Since a prime PI-algebra whose center is a field is simple, R is a central simple algebra of degree n. $\square$

Proof of Proposition II.8.1. Set $R = \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. The natural embedding $\psi_X : k_n(X) \hookrightarrow R$ is constructed in the discussion after Remark 7.2. Lemma II.8.2 implies that R is a central simple algebra of degree n with center $k(X)^{\mathrm{PGL}_n}$. It remains to show that $k(X)^{\mathrm{PGL}_n}$ is a finite, purely inseparable extension of the center F of $k_n(X)$.

By a theorem of Rosenlicht (see [39, Theorem 13.5.3] or [34, Theorem 6.2]), there is a PGL_n-stable dense open subset X_0 in X and an algebraic variety Y with $k(Y) = k(X)^{\mathrm{PGL}_n}$ such that the inclusion map $k(Y) \hookrightarrow k(X)$ induces a morphism $\pi : X_0 \to Y$ with the following properties: π is open and surjective, and the fibers of π are the PGL_n-orbits in X_0. Let Y' be an irreducible algebraic variety with $k(Y') = F$. Then the inclusion $F \hookrightarrow k(Y)$ includes a dominant rational map $\rho : Y \dashrightarrow Y'$.

By Proposition II.4.2(a), the elements of $C_{m,n}$ separate the PGL_n-orbits in $U_{m,n}$. As discussed in Section II.2, we can also in prime characteristic naturally identify $C_{m,n}$ with the center of $T_{m,n}$. Hence F, which is the center of the total ring of fractions of $T_{m,n}/\mathcal{I}_T(X)$ (cf. Remark 7.2), separates the PGL_n-orbits in a dense open subset of X. Thus there is a dense open subset U of X_0 such that the fibers of $\rho \circ \pi$ on U are the PGL_n-orbits in U. Hence $\rho : Y \dashrightarrow Y'$ is injective on the dense open subset $\pi(U)$ of Y. Consequently, $k(Y) = k(X)^{\mathrm{PGL}_n}$ is a finite, purely inseparable extension of $k(Y') = F$ (see, e.g., [42, Theorem 5.1.6]). $\square$

II.8.3. Definition. Let $X \subset U_{m,n}$ and $Y \subset U_{l,n}$ be irreducible n-varieties, so that $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ and $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ are central simple algebras of degree n (see Proposition II.8.1).

(a) Let $f : X \dashrightarrow Y$ be a PGL_n-equivariant dominant rational map (of varieties). Define

$$f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$$

by $f^\star(g) = g \circ f$. Then $f^\star$ is an injective k-algebra homomorphism.

(b) Conversely, let $\alpha : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ be a k-algebra homomorphism (necessarily injective). Denote the generic matrices generating $G_{l,n}$ by $X_1, \ldots, X_l$, and their images in $k_n[Y] \subseteq \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ by $\overline{X_1}, \ldots, \overline{X_l}$. Then $f = \alpha_\star$ is the PGL_n-equivariant dominant rational map $X \dashrightarrow Y$ defined by

$$f = \big(\alpha(\overline{X_1}), \ldots, \alpha(\overline{X_l})\big).$$

In this context, we denote the induced maps using a star ($\star$) instead of an asterisk ($*$) to make it possible to distinguish the maps defined here from the ones discussed in the previous section. This will be useful in Section II.9, see, e.g., Remarks II.9.1 and II.9.2.

That $\alpha_\star$ is well defined follows from the following, slightly more general result (which we will use in Section II.9), applied to the restriction of α to $k_n(Y)$.

II.8.4. Lemma. *Let X be an irreducible PGL_n-variety and $Y \subseteq U_{l,n}$ an irreducible n-variety. Let $\alpha : k_n(Y) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ be a k-algebra embedding. Define f as in Definition II.8.3(b). Then f is a PGL_n-equivariant dominant rational map $X \dashrightarrow Y$.*

Proof. The map f is clearly a PGL_n-equivariant rational map $X \dashrightarrow (\mathrm{M}_n)^l$. That it is actually a dominant rational map $X \dashrightarrow Y$ is shown as in the proof of Lemma II.7.2; the argument goes through nearly literally, with exception of the proof of the claim, which we modify as follows: By Lemma II.8.2, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n with center $k(X)^{\mathrm{PGL}_n}$. So the central element $\alpha(t)$ is a PGL_n-invariant rational map $X \dashrightarrow k \cdot I_n$. Let V be the nonempty open subset of U where $\alpha(t)$ is defined and nonzero. Then for $x \in V$, $\psi_X(t) = \alpha(t)(x)$ is again a nonzero scalar in M_n. The rest of the argument goes through unchanged. $\square$

II.8.5. Lemma. *Let X and Y be irreducible n-varieties. Let*

$$\alpha : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$$

be a k-algebra homomorphism. Then $(\alpha_\star)^\star = \alpha$. That is, $\alpha(g) = g \circ \alpha_\star$ for all $g \in \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$.

Proof. Set $\beta = (\alpha_\star)^\star$, so that by definition, $\beta(g) = g \circ \alpha_\star$. We have to show that $\beta = \alpha$. For $g = \bar{p} \in k_n[Y]$, the argument at the beginning of the proof of Lemma II.7.2 shows that $\beta(g) = \bar{p} \circ \alpha_\star = p \circ \alpha_\star = \alpha(\bar{p}) = \alpha(g)$, see (II.7.3). Hence $\beta = \alpha$ by the next lemma. $\square$

II.8.6. Lemma. *Let X and Y be irreducible n-varieties. Let*

$$\alpha, \beta : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$$

be k-algebra homomorphisms agreeing on $k_n[Y]$ (more precisely, agreeing on $\psi_Y(k_n[Y])$). Then $\alpha = \beta$.

Proof. In characteristic zero, this is clear by Proposition 7.3. So assume the characteristic of k is $p \neq 0$. By Proposition II.8.1, $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ is generated by $k_n(Y)$ and $k(Y)^{\mathrm{PGL}_n}$. Since $k_n(Y)$ is the total ring of fractions of $k_n[Y]$, it suffices to show that $\alpha(g) = \beta(g)$ for an arbitrary $g \in k(Y)^{\mathrm{PGL}_n}$. By Proposition II.8.1, there is an integer N such that g^{p^N} belongs to the center of $k_n(Y)$. Hence $\alpha(g^{p^N}) = \beta(g^{p^N})$. Since α and β map central elements to central elements, $\alpha(g)$ and $\beta(g)$ both belong to $k(X)^{\mathrm{PGL}_n}$. Let $x \in X$ such that $\alpha(g)$ and $\beta(g)$ are both defined at x. Then $\alpha(g)(x)$ and $\beta(g)(x)$ are scalars in $k = k \cdot I_n$. Now

$$\alpha\big(g^{p^N}\big)(x) = \alpha(g)^{p^N}(x) = \big[\alpha(g)(x)\big]^{p^N}.$$

Similarly, $\beta(g^{p^N})(x) = [\beta(g)(x)]^{p^N}$. Hence $[\alpha(g)(x)]^{p^N} = [\beta(g)(x)]^{p^N}$. Since the Frobenius map is injective, it follows that $\alpha(g)(x) = \beta(g)(x)$. Since this is true for all x in a dense open subset of X, $\alpha(g) = \beta(g)$. □

We now consider irreducible PGL_n-varieties that are PGL_n-equivariantly birationally isomorphic to irreducible n-varieties. Let $f : Y \dashrightarrow X$ be a PGL_n-equivariant dominant rational map between two such PGL_n-varieties. Then $f^\star(g) = g \circ f$ induces a k-algebra homomorphism

$$f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n).$$

Note that $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ and $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ are central simple algebras of degree n: For example, say X is PGL_n-equivariantly birationally isomorphic to the n-variety X'. Then $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is isomorphic to $\mathrm{RMaps}_{\mathrm{PGL}_n}(X', \mathrm{M}_n)$, which is a central simple algebra of degree n by Proposition II.8.1.

II.8.7. Proposition. *Let X and Y be irreducible PGL_n-varieties that are PGL_n-equivariantly birationally isomorphic to irreducible n-varieties. Let*

$$\alpha : \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$$

be a k-algebra homomorphism. Then there is a unique PGL_n-equivariant dominant rational map $\alpha_\star : Y \dashrightarrow X$ such that $(\alpha_\star)^\star = \alpha$.

Proof. Assume first that X and Y are n-varieties. Then we can take $\alpha_\star$ as in Definition II.8.3, and $(\alpha_\star)^\star = \alpha$ by Lemma II.8.5. Say $h : Y \dashrightarrow X$ is another PGL_n-equivariant dominant rational map such that $h^\star = \alpha$. It follows easily from Definition II.8.3 that $(h^\star)_\star = h$. Hence $h = (h^\star)_\star = \alpha_\star$, proving the uniqueness of $\alpha_\star$.

In the general case, say $f : X \dashrightarrow X'$ and $g : Y \dashrightarrow Y'$ are PGL_n-equivariant birational isomorphisms, where X' and Y' are irreducible n-varieties. Consider the k-algebra homomorphism

$$\beta = \left(g^\star\right)^{-1} \circ \alpha \circ f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}\left(X', \mathrm{M}_n\right) \to \mathrm{RMaps}_{\mathrm{PGL}_n}\left(Y', \mathrm{M}_n\right).$$

By the first case, there is a unique PGL_n-equivariant, dominant rational map $\beta_\star : Y' \dashrightarrow X'$ such that $(\beta_\star)^\star = \beta$. One sees now as in the proof of Lemma 8.5 that $\alpha_\star$ exists and is unique; the only modification needed is to replace $k_n(\_)$ by $\mathrm{RMaps}_{\mathrm{PGL}_n}(\_, \mathrm{M}_n)$. (Note the typo in the diagram in the proof of Lemma 8.5: the arrow for β_* should be reversed.) □

II.8.8. Corollary. *Let X and Y be irreducible PGL_n-varieties that are PGL_n-equivariantly birationally isomorphic to irreducible n-varieties. Let $f : X \dashrightarrow Y$ be a PGL_n-equivariant dominant rational map. Then $(f^\star)_\star = f$.*

Proof. Set $\alpha = f^\star$. Since $\alpha_\star$ is the unique such map with the property that $(\alpha_\star)^\star = \alpha$, $f = \alpha_\star = (f^\star)_\star$. □

II.8.9. Corollary. *Let X, Y and Z be irreducible PGL_n-varieties that are PGL_n-equivariantly birationally isomorphic to irreducible n-varieties.*

(a) *If $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are PGL_n-equivariant dominant rational maps then $(g \circ f)^\star = f^\star \circ g^\star$.*
(b) *Set $R = \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. Then $(\mathrm{id}_X)^\star = \mathrm{id}_R$, and $(\mathrm{id}_R)_\star = \mathrm{id}_X$.*
(c) *If $\alpha : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ and $\beta : \mathrm{RMaps}_{\mathrm{PGL}_n}(Z, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ are k-algebra homomorphisms then $(\alpha \circ \beta)_\star = \beta_\star \circ \alpha_\star$.*
(d) *X and Y are birationally isomorphic as PGL_n-varieties if and only if $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ and $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ are isomorphic as k-algebras.*

Proof. (a) and the first identity in (b) are immediate from Definition II.8.3. The second identity in (b) follows from the first and Corollary II.8.8. (c) Let $f = (\alpha \circ \beta)_\star$ and $g = \beta_\star \circ \alpha_\star$. By Proposition II.8.7 and part (a), $f^\star = \alpha \circ \beta = g^\star$. The uniqueness assertion of Proposition II.8.7 now implies $f = g$. (d) follows from (a), (b) and (c) (cf. the proof of Lemma 6.3). □

Proof of Theorem II.1.3. By the discussion before Proposition II.8.7, the assignment in Theorem II.1.3 is well defined. It is a contravariant functor by Corollary II.8.9. It is full since $\alpha = (\alpha_\star)^\star$ (Proposition II.8.7), and it is faithful since $(f^\star)_\star = f$ (Corollary II.8.8). □

II.9. More on $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$

II.9.1. Remark. The algebra embedding ψ_X in Proposition II.8.1 is natural in the following sense. Let $f : X \dashrightarrow Y$ be a dominant rational map of irreducible n-varieties. Then the following diagram commutes:

$$\begin{tikzcd} k_n(Y) \arrow[r, "\psi_Y"] \arrow[d, "f^*"] & \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \arrow[d, "f^\star"] \\ k_n(X) \arrow[r, "\psi_X"] & \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n). \end{tikzcd}$$

Here the induced k-algebra embeddings f^* and $f^\star$ were studied in Sections II.7 and II.8 (see Definitions II.7.1(c) and II.8.3(a), respectively).

II.9.2. Remark. Again, let X and Y be irreducible n-varieties, and let $\alpha : k_n(Y) \to k_n(X)$ be a k-algebra homomorphism. Then there is a unique k-algebra homomorphism β so that the following diagram commutes:

$$\begin{tikzcd} k_n(Y) \arrow[r, "\psi_Y"] \arrow[d, "\alpha"] & \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \arrow[d, dotted, "\beta"] \\ k_n(X) \arrow[r, "\psi_X"] & \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n). \end{tikzcd}$$

Indeed, let $\beta = f^\star$ (Definition II.8.3(a)), where $f = \alpha_*$ (Definition II.7.1(d), see also Lemma II.7.2). Then the diagram commutes by the previous remark since $\alpha = f^*$ (Remark 7.6). Lemma II.8.6 shows that β is unique.

II.9.3. Lemma. *Let X be an irreducible* PGL_n*-variety and Y an irreducible n-variety. Assume that there is a* PGL_n*-equivariant dominant rational map $f : X \dashrightarrow Y$. Then $g \mapsto g \circ f$ defines a k-algebra homomorphism*

$$f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n).$$

Moreover, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ *is a central simple algebra of degree n with center $k(X)^{\mathrm{PGL}_n}$.*

Proof. It is clear that $f^\star$ is a k-algebra homomorphism. Since the algebra $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ is a central simple algebra of degree n by Proposition II.8.1, so is $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ by Lemma II.8.2. □

II.9.4. Lemma. *In the situation of Lemma II.8.4:*

(a) *Let $f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ be the map induced by f (via $g \mapsto g \circ f$). Then $\alpha = f^\star \circ \psi_Y$, i.e., the following diagram commutes:*

$$\begin{tikzcd} k_n(Y) \arrow[r, "\psi_Y"] \arrow[dr, "\alpha"'] & \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \arrow[d, "f^\star"] \\ & \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n). \end{tikzcd}$$

(b) *If α is an isomorphism, so are ψ_Y and $f^\star$.*
(c) *Assume that also X is an n-variety. If ψ_X is an isomorphism (or if $\alpha(k_n(Y)) \subseteq \psi_X(k_n(X))$), then f is a dominant rational map of n-varieties.*

Proof. (a) Since $k_n(Y)$ is the total ring of fractions of $k_n[Y]$, it suffices to prove that for $p \in G_{l,n}$, $(f^\star \circ \psi_Y)(\bar{p}) = \alpha(\bar{p})$. This is true since for $x \in X$ in general position,

$$\big[f^\star\big(\psi_Y(\bar{p})\big)\big](x) = \big[\psi_Y(\bar{p})\big]\big(f(x)\big) = p\big(f(x)\big) = \alpha(\bar{p})(x),$$

where the last equality follows from (II.7.3). (b) follows from (a) since ψ_Y and $f^\star$ are embeddings. (c) is clear from the definition of f in Lemma II.8.4 (cf. Definitions II.7.1(d) and II.8.3(b)). $\square$

II.9.5. Proposition. *Let X be an irreducible PGL_n-variety such that the algebra $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n. Then there is an irreducible n-variety Y with an isomorphism*

$$\alpha : k_n(Y) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n).$$

For any such Y and α, the PGL_n-equivariant dominant rational map $f : X \dashrightarrow Y$ from Lemma II.8.4 is generically injective, and both ψ_Y and $f^\star$ are isomorphisms. In particular,

$$k_n(X) \xrightarrow[1\text{–}1]{\psi_X} \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \xrightarrow[\cong]{\alpha^{-1}} k_n(Y) \xrightarrow[\cong]{\psi_Y} \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n).$$

Proof. By Theorem 7.8, there is an irreducible n-variety Y with a k-algebra isomorphism $\alpha : k_n(Y) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. Let $f : X \dashrightarrow Y$ be the PGL_n-equivariant dominant rational map from Lemma II.8.4. Then ψ_Y and $f^\star$ are isomorphisms by Lemma II.9.4(b). It remains to be shown that f is generically injective (which we will use only once in the sequel, namely in Corollary II.9.6).

Since $f^\star : \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is an isomorphism, it maps the center isomorphically onto the center, i.e., $f^\star$ maps $k(Y)^{\mathrm{PGL}_n}$ isomorphically onto $k(X)^{\mathrm{PGL}_n}$. Denote by $\pi_X : X \dashrightarrow \overline{X}$ and $\pi_Y : Y \dashrightarrow \overline{Y}$ the rational quotients of X and Y with respect to the PGL_n-actions (see [34, Section 6.3]). Then we obtain the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\pi_X"'] & Y \arrow[d, dashed, "\pi_Y"] \\ \overline{X} \arrow[r, dashed, "\bar{f}"] & \overline{Y}. \end{tikzcd}$$

Here $\bar{f}$ is induced by the isomorphism from $k(\overline{Y}) = k(Y)^{\mathrm{PGL}_n}$ onto $k(\overline{X}) = k(X)^{\mathrm{PGL}_n}$, so is a birational isomorphism. Now let $x_1, x_2 \in X$ be in general position. If $f(x_1) = f(x_2)$, then $\pi_X(x_1) = \pi_X(x_2)$ since

$$\begin{tikzcd}
k_n(Y) \arrow[r, "\psi_Y"] \arrow[dr, "\alpha" near start] & \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n) \arrow[d, shift right, "f^\star"'] \\
k_n(X) \arrow[r, "\psi_X"'] \arrow[ur, "\beta"' near start] & \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \arrow[u, shift right, "g^\star"']
\end{tikzcd}$$

Fig. II.1. The maps in the proof of Proposition II.9.7.

$\bar{f}$ is a birational isomorphism. Hence x_1 and x_2 belong to the same PGL_n-orbit. So $x_2 = hx_1$ for some $h \in \mathrm{PGL}_n$. Then $f(x_1) = f(x_2) = hf(x_1)$. This can be true only if $h = 1$, i.e., if $x_1 = x_2$, since every point in the n-variety Y has trivial stabilizer in PGL_n, see (II.4.1). □

II.9.6. Corollary. *Let X be an irreducible PGL_n-variety. The following are equivalent*:

(a) $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ *is a central simple algebra of degree n.*
(b) *There is a PGL_n-equivariant dominant rational map $f : X \dashrightarrow Y$ for some irreducible n-variety Y.*

If so, the PGL_n-action on X has trivial stabilizers at points in general position, and Y and f can be chosen such that f is in addition generically injective.

Proof. This follows from Proposition II.9.5 and Lemma II.9.3. Recall from (II.4.1) that every point in Y has trivial stabilizer in PGL_n. Since f is PGL_n-equivariant, every $x \in X$ in general position (wherever f is defined) must have trivial stabilizer in PGL_n as well. □

II.9.7. Proposition. *Let X be an irreducible n-variety. Then there is an irreducible n-variety Y such that $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \cong k_n(Y)$. For any such Y, there is a PGL_n-equivariant dominant rational map $f : X \dashrightarrow Y$ and a dominant rational map of n-varieties $g : Y \dashrightarrow X$ which are inverse to each other as rational maps. Consequently, X and Y are birationally isomorphic as PGL_n-varieties* (*but maybe not as n-varieties*). *In addition, if also ψ_X is an isomorphism, then X and Y are birationally isomorphic as n-varieties.*

Proof. By Proposition II.8.1, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is a central simple algebra of degree n. By Theorem 7.8, there is an irreducible n-variety Y with a k-algebra isomorphism $\alpha : k_n(Y) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. Let $f : X \dashrightarrow Y$ be the PGL_n-equivariant dominant rational map from Lemma II.8.4. Then Lemma II.9.4 implies that $\alpha = f^\star \circ \psi_Y$, and that ψ_Y and $f^\star$ are isomorphisms (since α is). Moreover, if also ψ_X is an isomorphism, then f is a dominant rational map of n-varieties.

Now consider the k-algebra embedding

$$\beta = \psi_Y \circ \alpha^{-1} \circ \psi_X : k_n(X) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n).$$

Applying similar reasoning to β (and with the roles of X and Y reversed), we obtain a PGL_n-equivariant dominant rational map $g : Y \dashrightarrow X$ such that $\beta = g^\star \circ \psi_X$ (see Fig. II.1). Moreover, g is a dominant rational map of n-varieties since ψ_Y is an isomorphism. Now

$$\left(f^\star \circ g^\star\right) \circ \psi_X = f^\star \circ \beta = \left(f^\star \circ \psi_Y\right) \circ \alpha^{-1} \circ \psi_X = \mathrm{id} \circ \psi_X,$$

where id is the identity map on $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. It follows from Lemma II.8.6 that $f^\star \circ g^\star = \mathrm{id}$. Since $f^\star$ is an isomorphism, $f^\star$ and $g^\star$ are inverse isomorphisms. It follows now from Corollary II.8.9 that f and g are inverse to each other as rational maps. □

II.9.8. Corollary. *Let X be an irreducible n-variety. Then there is an irreducible n-variety Y, unique up to birational isomorphism of n-varieties, such that $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \cong k_n(Y)$. Moreover, X and Y are birationally isomorphic as PGL_n-varieties* (*but maybe not as n-varieties*).

Proof. All but the uniqueness of Y follows directly from Proposition II.9.7. Let X' be another irreducible n-variety such that $k_n(X')$ is isomorphic to $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$. Then by Proposition II.9.5, $\psi_{X'}$ is an isomorphism. Applying Proposition II.9.7 with X' instead of X, we conclude that X' and Y are birationally isomorphic as n-varieties. $\square$

II.9.9. Remark. Suppose that $f : X \dashrightarrow Y$ is a birational isomorphism of irreducible n-varieties. Then the two vertical maps f^* and $f^\star$ in Remark II.9.1 must be isomorphisms (since the assignments $f \mapsto f^*$ and $f \mapsto f^\star$ are functorial), so that ψ_X must be an isomorphism if ψ_Y is. Proposition II.9.7 implies therefore the following: If there is an irreducible n-variety X such that the embedding $\psi_X : k_n(X) \to \mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ is *not* an isomorphism, then there are irreducible n-varieties which are birationally isomorphic as PGL_n-varieties but *not* as n-varieties.

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[1] [1–30] refer to the references of [40], the first part of the present paper. Of these, we only list those also cited in this part. Afterwards, we list additional references.