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Infiniteness of double coset collections in algebraic groups

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Abstract

Let G be a linear algebraic group defined over an algebraically closed field. The double coset question addressed in this paper is the following: Given closed subgroups X and P, is the double coset collection $X\backslash G/P$ finite or infinite? We limit ourselves to the case where X is maximal rank and reductive and P parabolic. This paper presents a criterion for infiniteness which involves only dimensions of centralizers of semisimple elements. This result is then applied to finish the classification of those X which are spherical subgroups. Finally, excluding a case in F_4, we show that if $X\backslash G/P$ is finite then X is spherical or the Levi factor of P is spherical. This places great restrictions on X and P for $X\backslash G/P$ to be finite. The primary method is to descend to calculations at the finite group level and then to use elementary character theory.

Keywords: Algebraic groups; Finite groups of Lie type; Double cosets; Spherical subgroups; Finite orbit modules

1. Introduction

Given an algebraic group G we wish to classify those subgroups X and P such that the double coset collection $X\backslash G/P$ is finite. All our groups are defined over an algebraically closed field and all subgroups are assumed to be closed. The collection $X\backslash G/P$ is finite if and only if the G-orbit G/P splits into finitely many X-orbits. This viewpoint makes a complete classification of all finite double coset collections appear unlikely in the near future. In this paper we will assume that G is a reductive (or simple) algebraic group,

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that P is a parabolic subgroup and that X is maximal rank and reductive. We use a technique introduced by Lawther [13] for studying a particular instance of the double coset problem. Some of our results provide necessary and sufficient conditions for a double coset collection to be finite. In this paper we pursue the application of these results to infinite collections. We intend to establish finiteness results in a later paper.

We will state the main results of the paper first with brief indications of how these results relate to earlier work in the field. We refer the reader to the article by Seitz [15] for further discussion of progress on double coset problems.

The first result provides a powerful criterion for establishing that $X\backslash G/P$ is infinite. If G is a group and $g \in G$ we write G_g for the centralizer of g in G. We write $Z(G)$ for the center of the group.

Theorem 1 (Dimension Criterion). *Let G be a reductive algebraic group, X and P closed subgroups of G with X maximal rank and P parabolic. Let L be a Levi factor of P and let $s \in X \cap L$ be a semisimple element. If $\dim Z(G_s) + \dim G_s > \dim X_s + \dim P_s$ (equivalently, if $\dim Z(G_s) + \frac{1}{2}\dim G_s - \dim X_s - \frac{1}{2}\dim L_s > 0$), then $X_s\backslash G_s/P_s$ and $X\backslash G/P$ are infinite.*

Classification of maximal rank reductive spherical subgroups

The first application of the dimension criterion is to finish the classification of maximal rank reductive spherical subgroups of each simple algebraic group G. If a Borel subgroup B of G has a dense orbit on the quotient $X \setminus G$, then we say that X is a *spherical* subgroup. Brion [3] and Vinberg [17] independently showed that X is spherical if and only if $X\backslash G/B$ is finite. The work of Krämer [12], Brundan [4], and Lawther [13] has produced a list of subgroups which are spherical in all characteristics. The maximal rank reductive subgroups on this list are given in Table 1, where we use the following conventions. We treat A_0 and B_0 as trivial groups and D_1 as a 1-dimensional torus. Inside a D_4 root system we use the convention that root subsystems labelled as $A_1 + A_1$ and A_3 are not conjugate to root subsystems labelled as D_2 and D_3 respectively. The former contain (up to conjugacy) the first node in the Dynkin diagram for D_4 (using the standard labelling, as in [2]), and the latter do not (even after conjugation). We extend these conventions to subsystems of D_n for $n \geqslant 4$. We list only the Lie type of each group, as the property of being spherical is not affected by which representative of an isogeny class is used (see Lemma 6). The notation T_i refers to an i-dimensional torus, central in X. Thus, X is a central product of factors of the indicated type. Finally, in the subgroup $A_1\widetilde{A}_1$ of G_2 the factor $\widetilde{A}_1$ denotes a subgroup with short roots (we don't use this notation for the other groups as there is no ambiguity). To classify the maximal rank reductive spherical subgroups, it suffices to classify only those subgroups which exist in all characteristics, as the others arise from isogenies or graph automorphisms, which preserve the property of being spherical (see Lemma 6).

To prove that Table 1 is complete, we introduce the following root-theoretic property which is inspired by Lawther's anti-open property (see [13]). We abbreviate the phrase "maximal rank reductive" with MRR. We say a MRR subgroup X is *generic* if a subgroup of the same type as X is defined in all characteristics. Fix a maximal torus of X and define the root systems $\Phi(X)$ and $\Phi(G)$ with respect to this maximal torus. Let $\varphi \leqslant \Phi(G)$ be

Table 1
Generic maximal rank reductive spherical subgroups

$X \leqslant G$	$X \leqslant G$
$A_n A_m T_1 \leqslant A_{n+m+1}$	$E_6 T_1 \leqslant E_7$
$B_n D_m \leqslant B_{n+m}$	$A_7 \leqslant E_7$
$A_{n-1} T_1 \leqslant B_n$	$A_1 D_6 \leqslant E_7$
$C_n C_m \leqslant C_{n+m}$	$A_1 E_7 \leqslant E_8$
$C_{n-1} T_1 \leqslant C_n$	$D_8 \leqslant E_8$
$A_{n-1} T_1 \leqslant C_n$	$A_1 C_3 \leqslant F_4$
$D_n D_m \leqslant D_{n+m}$	$B_4 \leqslant F_4$
$A_{n-1} T_1 \leqslant D_n$	$A_2 \leqslant G_2$
$D_5 T_1 \leqslant E_6$	$A_1 \widetilde{A}_1 \leqslant G_2$
$A_1 A_5 \leqslant E_6$	

a closed root subsystem. We say that X has a φ *complement* if φ is disjoint from $\Phi(X)$. This is equivalent to the existence of a generic MRR subgroup $K \leqslant G$ with $\Phi(K) = \varphi$ and $K \cap X$ a maximal torus. The adjective "long" or "short" may be applied if φ has only long or only short roots, respectively.

Theorem 2. *Let G be a simple algebraic group and X a generic MRR subgroup. The following are equivalent*:

(i) *X is spherical.*
(ii) *X appears in Table* 1.
(iii) *X has no A_2 or B_2 complement.*

In this paper we show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) (more precisely, we show $\neg$ (ii) $\Rightarrow$ $\neg$ (iii) $\Rightarrow$ $\neg$ (i)). The implication (ii) $\Rightarrow$ (i) is due to Brundan [4] and Lawther [13].

Theorem 2 applies to a group acting on the full flag variety G/B, where B is a Borel subgroup. Using the dimension criterion, we now obtain more general infiniteness results where P is any parabolic. Since Table 1 contains relatively few subgroups, the following theorem places great restrictions upon X and P for $X\backslash G/P$ to be finite. An *end node parabolic* is conjugate to a standard parabolic obtained by crossing off exactly one of the end nodes in the Dynkin diagram of G.

Theorem 3 (Spherical X or Spherical L). *Let G be a simple algebraic group, X a MRR subgroup, P a parabolic subgroup with Levi factor L. If G equals F_4 suppose that P is not an end node parabolic. If $X\backslash G/P$ is finite then X is spherical or L is spherical.*

The extra restrictions placed upon P when G equals F_4 are necessary. In a later paper we will show that $L_1\backslash F_4/P_4$ and $L_4\backslash F_4/P_1$ are finite (where P_i is conjugate to the standard parabolic obtained by crossing off the ith node of the Dynkin diagram of F_4, and L_i is its Levi factor).

Corollary 4. *If $X\backslash G/P$ is finite and P is not maximal then X is spherical.*

Remark. The theorem and the corollary give a surprisingly strong dichotomy for MRR subgroups with respect to the double coset problem. Either they are spherical, or they have an infinite number of orbits on almost all flag varieties. For instance, A_1A_5 is spherical in E_6, but T_1A_5 has an infinite number of orbits on all flag varieties E_6/P except, possibly, if P is an end node parabolic. As another example, suppose one could show that a MRR subgroup X in $\mathrm{GL}(V)$ has a finite number of orbits on flags consisting of one and two dimensional subspaces. Then X has a finite number of orbits on full flags, i.e., upon G/B where B is a Borel subgroup.

Outline of remaining sections

The outline of the rest of this paper is as follows: Section 2 includes basic results and preliminaries; Section 3 reduces the double coset question of algebraic groups to a related question about finite groups; Section 4 applies character theory to the finite groups (roughly following Lawther [13]) and obtains the Dimension Criterion; Section 5 proves Theorems 2 and 3, assuming Proposition 21; Section 6 proves Proposition 21.

2. Preliminaries

In this section we list basic results which will be used later. Many (perhaps all) of the results in this section are known to others. We list them here either for convenience, or because references are difficult to find. For standard facts and conventions regarding algebraic groups we follow [10].

Lemma 5. *Let G be a simple algebraic group. All MRR subgroups of type A_2, of the same length, are conjugate. All MRR subgroups of type B_2 are conjugate. If the rank of G is at least three then these subgroups are all Levi factors of parabolic subgroups.*

Proof. The last statement is clear. Let H and H' be two MRR subgroups, both of type B_2 or both of type A_2 of the same length. By conjugation we may assume that H and H' share a common maximal torus T. If the rank of G is two then H and H' are equal. Otherwise H and H' are Levi factors and each is generated by T and the root groups (positive and negative) corresponding to a pair of adjacent nodes in the Dynkin diagram of G. Then H and H' are conjugate by the action of the Weyl group. □

The following lemma allows us to make a variety of convenient assumptions about G. We write G^g for $g^{-1}Gg$.

Lemma 6. *Let G be a group with subgroups X and P. Let Z be the center of G, suppose that Z is contained in P and let $\varphi_1 : G \to G/Z$ be the natural map. Let K be a finite normal subgroup of G and let $\varphi_2 : G \to G/K$ be the natural map. Let $g, h \in G$. The following are equivalent*:

(i) $|X\backslash G/P| < \infty$.
(ii) $|\varphi_1(X)\backslash\varphi_1(G)/\varphi_1(P)| < \infty$.
(iii) $|\varphi_2(X)\backslash\varphi_2(G)/\varphi_2(P)| < \infty$.
(iv) $|X^g\backslash G/P^h| < \infty$.

Proof. These statements can all be proven in an elementary fashion. □

This lemma shows that the question of whether $X\backslash G/P$ is finite depends only upon the Lie type of the groups involved. In particular, it does not depend upon which elements of an isogeny class are chosen, the presence of centers, connectedness etc. We may also assume that G has simply connected derived subgroup, which eases some of the proofs. Finally, if X and P are maximal rank then we may assume that they contain a common maximal torus.

Conventions. If σ is an endomorphism of G we denote by G_σ the fixed points of σ in G. If G is a group and $g \in G$ then G_g denotes the centralizer of g in G. Finally, $G_{\sigma,g}$ denotes those points in G fixed by both σ and g. The finite groups of Lie type arise as the fixed points in G of a Frobenius morphism $\sigma: G \to G$, where G is defined over the algebraic closure $\overline{\mathbb{F}}_p$ of the field $\mathbb{F}_p$ of p elements. We refer to [5] and [16] for details. We denote the cardinality of a set S by $|S|$.

Lemma 7. *Let G be a connected reductive group with simply connected derived subgroup. Let T be a maximal torus of G.*

(i) *The center of G is contained in each maximal torus of G.* (*This does not require that G have simply connected derived subgroup.*)
(ii) *If $s \in G$ is semisimple then G_s is reductive and connected.*
(iii) *For each $s \in T$ we have $Z(G_s) \leqslant T$.*
(iv) *The set $\{G_s \mid s \in T\}$ is finite. Its size may be bounded by a constant depending only upon the root system of G.*
(v) *Fix $s \in T$. There exist $t_1, \dots, t_r \in T$ such that $\{G_t \mid t \in G,\ G_t > G_s\} = \{G_{t_i} \mid 1 \leqslant i \leqslant r\}$. Let $Z(s) = \{t \in G \mid G_t = G_s\}$. Then $Z(s)$ is an open subset of $Z(G_s)$ and its complement is the set $U = \bigcup_i Z(G_{t_i})$.*
(vi) *If S is a torus and $L = C_G(S)$ then $\{s \in S \mid L = G_s\}$ is a dense subset of S.*

Proof. Part (i) is [10, 26.2].

Part (ii) is [5, 3.5.4, 3.5.6].

Part (iii). Note that T is a maximal torus of G_s. By part (ii) we may apply part (i) to the group G_s.

Part (iv). By [5, 3.5.3] we have that G_s is generated by T, the root groups it contains and by certain elements of the Weyl group. Since the Weyl group is finite and the number of root groups is finite the number of possibilities for G_s is finite and depends only upon the root system of G.

Part (v). Apply part (iii) to show that if $G_t > G_s$ then $t \in Z(G_t) < Z(G_s) \leqslant T$. This fact, and part (iv), show that $t_1, \dots, t_r$ may be chosen in T as stated and that $U \subseteq Z(G_s)$.

Since the union is finite, U is a closed set. Given $t \in Z(G_s)$ we have the following chain of equivalent statements: $t \notin Z(s)$ if and only if $G_t > G_s$ if and only if $G_t = G_{t_i}$ for some i, if and only if $t \in Z(G_{t_i})$ for some i, if and only if $t \in U$. This shows that U is the desired complement.

Part (vi). We have, for all $t \in S$, that $G_t \geqslant L$. By an argument similar to that for part (v), one can show that the set of $t \in S$ with $G_t > L$ is a proper, closed subset of S. □

Lemma 8. *Let G be a connected reductive group and $\sigma : G \to G$ a Frobenius morphism. Then $Z(G_\sigma) = Z(G)_\sigma$. Moreover, if G has simply connected derived subgroup and $s, t \in G_\sigma$ are semisimple elements, then $G_s = G_t$ if and only if $G_{\sigma,s} = G_{\sigma,t}$.*

Proof. The first statement is in [5, 3.6.8]. For the second statement note that "$\Rightarrow$" is obvious, for "$\Leftarrow$" suppose that $G_{\sigma,s} = G_{\sigma,t}$. By Lemma 7(ii) we have that G_s and G_t are connected and reductive. By assumption, t is in $Z(G_{\sigma,s})$, which equals $Z(G_s)_\sigma$ by the first statement. This shows that t is in $Z(G_s)$ whence $G_t \geqslant G_s$. A symmetric argument shows that $G_s \geqslant G_t$. □

Lemma 9 (Rational normalizer theorem). *Let G be a connected reductive group defined over $\overline{\mathbb{F}}_p$, let $\sigma : G \to G$ be a Frobenius morphism and let P be a σ-stable parabolic subgroup. Then $N_{G_\sigma}(P_\sigma) = P_\sigma = (N_G(P))_\sigma$*

Proof. It is well known (see [10, 23.1]) that $P = N_G(P)$, which gives the second equality. For the first equality it is clear that $P_\sigma \leqslant N_{G_\sigma}(P_\sigma)$. The reverse inclusion follows from the fact that if $\widetilde{P}$ is a σ-stable parabolic subgroup with $\widetilde{P}_\sigma = P_\sigma$ then $\widetilde{P} = P$ (see [1, 4.20]). □

Corollary 10. *Let G be a connected reductive group with a parabolic subgroup P. Let σ be a Frobenius morphism of G which fixes P and let $x \in G_\sigma$. Let $(G/P)_x$ be the variety of G-conjugates of P which contain x. Then σ acts upon $(G/P)_x$ and the character value $1_{P_\sigma}^{G_\sigma}(x)$ is equal to the number of σ-fixed points on this variety.*

Proof. Let $(G/P)_\sigma$ be the σ-fixed points in the quotient G/P. Using the Lang–Steinberg Theorem [16] it is easy to show that the map $\varphi : G_\sigma/P_\sigma \to (G/P)_\sigma$ taking gP_σ to gP is an x-equivariant bijection. Together with the rational normalizer theorem this shows that we have bijections between $(G/P)_{\sigma,x}$, $(G_\sigma/P_\sigma)_x$ and $\{{}^gP_\sigma \mid g \in G_\sigma,\ x \in {}^gP_\sigma\}$. Elementary character theory shows that $1_{P_\sigma}^{G_\sigma}(x)$ equals the size of the last collection. □

Lemma 11 [14, 3.5]. *Let G be a connected algebraic group of dimension d, let $\sigma : G \to G$ be a standard qth power Frobenius map. Then $(q-1)^d \leqslant |G_\sigma| \leqslant (q+1)^d$.*

Lemma 12. *Let G be a connected reductive group with simply connected derived subgroup, let $\sigma : G \to G$ be a standard qth power Frobenius map, let $s \in G_\sigma$ be semisimple, let T be a maximal torus containing s and let $Z(s)$ and $t_1, \ldots, t_r$ be as in Lemma 7. Let c_1 and d_1 be the number of connected components and the dimension of $Z(G_s)$ respectively. Let $I \subseteq \{1, \ldots, r\}$ such that* $\dim Z(G_{t_i}) < \dim Z(G_s)$ *if and only if $i \in I$. Let $m = |I|$ and, if*

$m > 0$, let c_2 and d_2 be the maximal number of components and the greatest dimension, respectively, of the $Z(G_{t_i})$ with $i \in I$.

Then $Z(s)$ is σ-stable and

$$(q-1)^{d_1} - mc_2(q+1)^{d_2} \leqslant \left|Z(s)_\sigma\right| \leqslant c_1(q+1)^{d_1}.$$

Proof. Since s is fixed by σ it is easy to show that G_s, $Z(G_s)$, and $Z(s)$ are σ-stable.

Since $Z(s) \subseteq Z(G_s)$ we may apply Lemma 11 to get $|Z(s)_\sigma| \leqslant |Z(G_s)_\sigma| \leqslant c_1(q+1)^{d_1}$ where the second inequality is found by calculating $|Z(G_s)_\sigma|$ under the assumption that σ stabilizes each component of $Z(G_s)$.

Let $Z(G_s)^\circ$ be the identity component of $Z(G_s)$. We have that

$$\left|Z(s)_\sigma\right| \geqslant \left|\left(sZ(G_s)^\circ \cap Z(s)\right)_\sigma\right|.$$

From Lemma 7 we have a partition $Z(G_s) = Z(s) \cup U$ whence

$$\left|\left(sZ(G_s)^\circ \cap Z(s)\right)_\sigma\right| = \left|\left(sZ(G_s)^\circ\right)_\sigma\right| - \left|\left(sZ(G_s)^\circ \cap U\right)_\sigma\right|.$$

It is easy to check that

$$(q-1)^{d_1} \leqslant \left|Z(G_s)^\circ_\sigma\right| = \left|\left(sZ(G_s)^\circ\right)_\sigma\right|,$$

and that

$$\left|\left(sZ(G_s)^\circ \cap \left(\bigcup_{i\in I} Z(G_{t_i})\right)\right)_\sigma\right| \leqslant mc_2(q+1)^{d_2},$$

whence it suffices to show that $sZ(G_s)^\circ \cap U = sZ(G_s)^\circ \cap (\bigcup_{i\in I} Z(G_{t_i}))$. We prove this by showing that $sZ(G_s)^\circ \cap Z(G_{t_i})$ is empty if $\dim Z(G_{t_i}) = \dim Z(G_s)$. Let $\dim Z(G_{t_i}) = \dim Z(G_s)$. Then $Z(G_{t_i})^\circ = Z(G_s)^\circ$ and $sZ(G_s)^\circ \cap Z(G_{t_i})$ is empty or all of $sZ(G_s)^\circ$. However, by definition of the t_i, we have $s \notin Z(G_{t_i})$ so we are done. □

3. Reduction to finite groups

In this section we reduce the double coset problem in algebraic groups to double cosets in finite groups. These results seem intuitive, but use material surprisingly far from group theory.

By a reduced algebraic group scheme over $\mathbb{Z}$, we mean that the group G is defined, as a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, using a finite number of polynomials over $\mathbb{Z}$ and that $\mathbb{Z}[G]$ has no nilpotents except 0. This is the case for the simple algebraic groups, as well as their parabolic subgroups and generic MRR subgroups (see [6] or [11]). Such a group scheme has a group of points over every field. For an algebraically closed field $\Bbbk$ one may identify the group of points (of the group scheme) over $\Bbbk$ with the algebraic group (in the naive sense) over $\Bbbk$. The field $\overline{\mathbb{F}}_p$ is the algebraic closure of the field of p elements for the prime p.

Proposition 13. *Let G be a simple algebraic group scheme and let X and P be closed algebraic subgroup schemes of G, all of which are reduced over $\mathbb{Z}$. For a field $\mathbb{F}$ we denote by $G(\mathbb{F})$, $X(\mathbb{F})$, and $P(\mathbb{F})$ the group of points over $\mathbb{F}$ of G, X, and P, respectively. Let $\Bbbk$ be an algebraically closed field.*

(i) *If* $\operatorname{char}\Bbbk = 0$ *then*

$$\left|X(\Bbbk)\backslash G(\Bbbk)/P(\Bbbk)\right| < \infty \quad \Longleftrightarrow \quad \limsup_{p\to\infty}\left|X\left(\overline{\mathbb{F}}_p\right)\backslash G\left(\overline{\mathbb{F}}_p\right)/P\left(\overline{\mathbb{F}}_p\right)\right| < \infty.$$

(ii) *If* $\operatorname{char}\Bbbk = p > 0$ *then*

$$\left|X(\Bbbk)\backslash G(\Bbbk)/P(\Bbbk)\right| < \infty \quad \Longleftrightarrow \quad \left|X\left(\overline{\mathbb{F}}_p\right)\backslash G\left(\overline{\mathbb{F}}_p\right)/P\left(\overline{\mathbb{F}}_p\right)\right| < \infty.$$

Proof. Part (ii) is proven in [8]. (We view the group $X(\Bbbk) \times P(\Bbbk)$ as acting on the affine space $G(\Bbbk)$. The assumption in [8] that $X(\Bbbk) \times P(\Bbbk)$ should be reductive is not used.) It may also be proven using a model theoretic argument similar in nature to the one we give now for part (i). For basic facts about model theory we refer to the textbooks by Fried and Jarden [7] or Hodges [9].

For p equal to 0 or a prime, let ACF_p be the theory of algebraically closed fields of characteristic p. Then ACF_p is a complete theory.

For a field $\mathbb{F}$ we identify $G(\mathbb{F})$ as a set of matrices in $\mathrm{GL}_n(\mathbb{F})$ using the defining polynomials over $\mathbb{Z}$. We make similar identifications for X and P. Since G, X, and P are defined over $\mathbb{Z}$ we can express membership in $G(\mathbb{F})$, $X(\mathbb{F})$, and $P(\mathbb{F})$ with first order sentences. Let φ be the sentence which, applied to the model $\mathbb{F}$, gives $\exists g_1, \dots, g_n \in G(\mathbb{F})$, $\forall g \in G(\mathbb{F})$, $\exists x \in X(\mathbb{F})$, $\exists y \in P(\mathbb{F})$, $\exists i \in \{1, \dots, n\}$ such that $xgy = g_i$. In other words, φ applied to $\mathbb{F}$ states that $|X(\mathbb{F})\backslash G(\mathbb{F})/P(\mathbb{F})| \leqslant n$.

Suppose $X(\Bbbk)\backslash G(\Bbbk)/P(\Bbbk)$ is infinite in characteristic zero. Then φ is false in $\Bbbk$. Then $ACF_0 \vdash \neg\varphi$ by completeness. This means that we may derive $\neg\varphi$ using a finite number of steps and a finite number of axioms. In particular, only finitely many axioms which assert that $m \cdot 1 \neq 0$ are used and so there exists a prime p_0 which is greater than every m which is used in this manner. For all primes $p \geqslant p_0$ the axioms and steps which are used in the proof of $ACF_0 \vdash \neg\varphi$ may also be used to conclude $ACF_p \vdash \neg\varphi$. Therefore, for all such p we have $|X(\overline{\mathbb{F}}_p)\backslash G(\overline{\mathbb{F}}_p)/P(\overline{\mathbb{F}}_p)| > n$ whence $\limsup_{p\to\infty} |X(\overline{\mathbb{F}}_p) \backslash G(\overline{\mathbb{F}}_p)/P(\overline{\mathbb{F}}_p)| > n$.

Conversely, a similar argument shows that

$$ACF_0 \vdash \varphi \quad \Rightarrow \quad ACF_p \vdash \varphi$$

for all p sufficiently large. Therefore finiteness in characteristic 0 implies boundedness of $|X(\overline{\mathbb{F}}_p)\backslash G(\overline{\mathbb{F}}_p)/P(\overline{\mathbb{F}}_p)|$ as $p \to \infty$. $\square$

In the following lemma we often view the collection $X\backslash G/P$ as the orbits of the group $X \times P$ acting on G in the natural way.

Lemma 14. *Let G be a connected algebraic group defined over $\Bbbk = \overline{\mathbb{F}}_p$, let $\sigma : G \to G$ be a Frobenius morphism, let X and P be closed σ-stable subgroups. If $X\backslash G/P$ is infinite let $C = 1$. If $X\backslash G/P$ is finite let C be an upper bound on the number of connected components of stabilizers of $X \times P$ acting on G. Then*

$$\frac{1}{C}\limsup_{n\to\infty} |X_{\sigma^n}\backslash G_{\sigma^n}/P_{\sigma^n}| \leqslant |X\backslash G/P| \leqslant \limsup_{n\to\infty} |X_{\sigma^n}\backslash G_{\sigma^n}/P_{\sigma^n}|.$$

Proof. Suppose $\limsup_{n\to\infty} |X_{\sigma^n}\backslash G_{\sigma^n}/P_{\sigma^n}|$ is finite and less than m. We will show that $|X\backslash G/P| < m$. Let $g_1, \ldots, g_m \in G$. There is a natural number n such that $g_1, \ldots, g_m \in G_{\sigma^n}$ and $m > |X_{\sigma^n}\backslash G_{\sigma^n}/P_{\sigma^n}|$. Then at least two of $g_1, \ldots, g_m$ are in the same $(X_{\sigma^n} \times P_{\sigma^n})$-orbit, whence they are in the same $X \times P$-orbit. Since this holds for every $g_1, \ldots, g_m \in G$ we see that $|X\backslash G/P| < m$.

Suppose now that $X\backslash G/P$ is finite, let n be given and let $(X\backslash G/P)_{\sigma^n}$ be the collection of σ^n-stable $(X \times P)$-orbits. Then the Lang–Steinberg Theorem [16], applied to the action of $X \times P$ upon G, shows that $C|X\backslash G/P| \geqslant C|(X\backslash G/P)_{\sigma^n}| \geqslant |X_{\sigma^n}\backslash G_{\sigma^n}/P_{\sigma^n}|$. □

4. Character theory and the dimension criterion

Strategy and conventions

By Lemma 6 we may, and shall, assume throughout this section that G is a connected reductive group with simply connected derived subgroup. By Section 3 we may, and shall, assume that G is defined over the algebraic closure of a field of positive characteristic. Let $\sigma : G \to G$ be a qth power Frobenius morphism. We assume that X and P are closed, σ-stable subgroups. Eventually we assume that X is maximal rank reductive and P is parabolic, but we use these assumptions only as needed in the preparatory lemmas. For fixed points we will use the notation G_σ, P_s, etc as described in Section 4. Then to prove infiniteness, in all characteristics, it suffices to show that $|X_{\sigma^n}\backslash G_{\sigma^n}/P_{\sigma^n}|$ is unbounded as n approaches infinity. If G is a group, the notation $[g] \subseteq G$ means that g is an element of G and $[g]$ is its G-conjugacy class. An element denoted by s will be semisimple, and an element denoted by u will be unipotent. A sum over $[u] \subseteq G$ means the sum over representatives u of the unipotent classes of G. This preparatory material roughly follows Lawther [13], though, in most cases, he only stated those directions relevant for proving finiteness.

Lemma 15. *We assume that P is parabolic. Define an equivalence relation on semisimple elements in X_σ as follows: s and t are equivalent if $G_{\sigma,s}$ and $G_{\sigma,t}$ are conjugate under X_σ. Denote the equivalence class of s by $E(s,\sigma)$. Choose a set S_σ of representatives of these equivalence classes. Then*

$$|X_\sigma\backslash G_\sigma/P_\sigma| = \sum_{s\in S_\sigma} \sum_{[u]\subseteq X_{\sigma,s}} \frac{|E(s,\sigma)|}{|X_\sigma|} \frac{|X_{\sigma,s}|}{|X_{\sigma,s,u}|} 1_{P_\sigma}^{G_\sigma}(su).$$

Proof. Basic character theory gives

$$|X_\sigma\backslash G_\sigma/P_\sigma| = \left(1_{X_\sigma}^{G_\sigma}, 1_{P_\sigma}^{G_\sigma}\right)_{G_\sigma} = \left(1_{X_\sigma}, 1_{P_\sigma}^{G_\sigma}\right)_{X_\sigma} = \frac{1}{|X_\sigma|}\sum_{x\in X_\sigma} 1_{P_\sigma}^{G_\sigma}(x).$$

Applying the Jordan–Chevalley decomposition within the finite group X_σ we get that this last sum is equal to

$$\frac{1}{|X_\sigma|}\sum_{s\in X_\sigma}\sum_{u\in X_{\sigma,s}} 1_{P_\sigma}^{G_\sigma}(su).$$

Now we claim that $t \in E(s,\sigma)$ implies that

$$\sum_{u\in X_{\sigma,t}} 1_{P_\sigma}^{G_\sigma}(tu) = \sum_{u\in X_{\sigma,s}} 1_{P_\sigma}^{G_\sigma}(su).$$

Let $x \in X_\sigma$ with $(G_{\sigma,t})^x = G_{\sigma,s}$. The crucial step is to show that for all $u \in X_{\sigma,t}$ we have

$$1_{P_\sigma}^{G_\sigma}(tu) = 1_{P_\sigma}^{G_\sigma}\left(su^x\right).$$

Once this is done, conjugation by x shows that the sums are equal. We work at the level of algebraic groups. Given $u \in X_{\sigma,t}$, let $(G/P)_{tu}$ and $(G/P)_{su^x}$ be the varieties of conjugates of P which contain tu and su^x respectively. Let $g \in G$ such that $t \in P^g$ and let T be a maximal torus of P^g which contains t. We apply Lemma 8 to see that $(G_t)^x = G_s$. We have the following:

$$T \leqslant G_t \quad\Rightarrow\quad T^x \leqslant G_s \quad\Rightarrow\quad s \in T^x \quad\Rightarrow\quad s \in P^{gx}.$$

It is now easy to see that if $tu \in P^g$ then $su^x \in P^{gx}$. Therefore, conjugation by x gives a σ-equivariant bijection $(G/P)_{tu} \to (G/P)_{su^x}$. Taking σ-fixed points and applying Corollary 10 finishes the claim.

Using the claim we have

$$\frac{1}{|X_\sigma|}\sum_{s\in X_\sigma}\sum_{u\in X_{\sigma,s}} 1_{P_\sigma}^{G_\sigma}(su) = \frac{1}{|X_\sigma|}\sum_{s\in S_\sigma}\sum_{u\in X_{\sigma,s}} \left|E(s,\sigma)\right| 1_{P_\sigma}^{G_\sigma}(su).$$

To finish the proof we take the sum over the representatives of unipotent classes in $X_{\sigma,s}$. $\square$

Lemma 16. *We assume that X is maximal rank. Let $s \in X_\sigma$, let T be a maximal torus containing s, let $W = N_G(T)$ be the Weyl group and let $Z(s,\sigma) = \{t \in G \mid G_{\sigma,t} = G_{\sigma,s}\}$. Let $Z(s)$ and $E(s,\sigma)$ be as in Lemma* 7 *and Lemma* 15 *respectively. Then $Z(s,\sigma) = Z(s)_\sigma$ and*

$$\frac{1}{|W|}\left|s^{X_\sigma} \times Z(s,\sigma)\right| \leqslant \left|E(s,\sigma)\right| \leqslant \left|s^{X_\sigma} \times Z(s,\sigma)\right|.$$

Proof. Using Lemma 8 for one containment, it is easy to show that $Z(s)_\sigma = Z(s,\sigma)$.

The following claim finishes the proof. We have a surjective map, φ from $s^{X_\sigma} \times Z(s,\sigma)$ to $E(s,\sigma)$ taking (s^x, t) to t^x, whose fibers are bounded in size by $|W|$. Note that this map is well-defined as every element in X_σ which centralizes s also centralizes t. To see that the map is surjective, let $t \in E(s,\sigma)$ and let $x \in X_\sigma$ with $(G_{\sigma,t})^x = G_{\sigma,s}$. Then $(s^{x^{-1}}, t^x)$ is in the domain of φ and $\varphi(s^{x^{-1}}, t^x) = t$.

The remainder of the proof bounds the size of the fibers of φ. Let (s^x, t_1) be an element of the domain. We claim that

$$\varphi^{-1}(t_1^x) = \{(s^{w^{-1}x}, t_1^w) \mid w \in W\} \cap (s^{X_\sigma} \times Z(s,\sigma)).$$

It is easy to see that the set on the right is contained in $\varphi^{-1}(t_1^x)$. For opposite containment, fix $(s^y, t_2) \in \varphi^{-1}(t_1^x)$. We first show that T contains t_1, $t_2 = t_1^{xy^{-1}}$, s, and $s^{yx^{-1}}$.

Now that we know t_1, $t_2 = t_1^{xy^{-1}}$, s, $s^{yx^{-1}} \in T$, we will apply [5, 3.7.1], and the (standard) notation which appears there to involve the action of the Weyl group. Write $xy^{-1} = ut\dot{w}u'$ in the Bruhat canonical form. Since

$$t_2 = t_1^{xy^{-1}}$$

we have that $t_2 = t_1^w$. Since xy^{-1} conjugates $s^{yx^{-1}}$ to s we have that $(s^{yx^{-1}})^w = s$ and $s^y = s^{w^{-1}x}$. Therefore $(s^y, t_2) = (s^{w^{-1}x}, t_1^w)$. □

Corollary 17. *We assume that X is maximal rank and that P is a parabolic subgroup. Let S_σ and $Z(s,\sigma)$ be as in Lemmas* 15 *and* 16 *respectively. We have*

$$\frac{1}{|W|}\sum \frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{P_\sigma}^{G_\sigma}(su) \leqslant |X_\sigma \backslash G_\sigma / P_\sigma| \leqslant \sum \frac{|Z(s,\sigma)|}{|X_{\sigma,s,u}|} 1_{P_\sigma}^{G_\sigma}(su),$$

where each sum is taken over the elements $s \in S_\sigma$, and the representatives u of the unipotent classes $[u] \subseteq X_{\sigma,s}$.

Proof. Combine Lemma 15 and the bounds for $E(s,\sigma)$ just obtained in Lemma 16. □

Proof of the Dimension Criterion. It is easy to show that $X_s \backslash G_s / P_s$ is infinite (use Lemma 6 and consider quotients by $Z(G_s)$).

It remains to show that $X \backslash G/P$ is infinite. Using Section 3 it suffices show that the term in Corollary 17 corresponding to $s \in S_\sigma$, $1 = [u] \subseteq X_{\sigma,s}$ is unbounded as we replace σ with σ^n and let n approach ∞. This term is

$$\frac{1}{|W|}\frac{|Z(s,\sigma^n)|}{|X_{\sigma^n,s}|} 1_{P_{\sigma^n}}^{G_{\sigma^n}}(s).$$

It is easy to show that $1_{P_{\sigma^n}}^{G_{\sigma^n}}(s) \geqslant \frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|}$ whence this term is bounded below by

$$\frac{1}{|W|}\frac{|Z(s,\sigma^n)|}{|X_{\sigma^n,s}|}\frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|}.$$

Therefore it suffices to show that

$$\limsup_{n\to\infty}\frac{1}{|W|}\frac{|Z(s,\sigma^n)|}{|X_{\sigma^n,s}|}\frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|}=\infty.$$

Let c_1 and c_2 be the number of connected components of X_s and P_s respectively. By Lemmas 11, 12 and 16 we have

$$\limsup_{n\to\infty}\frac{1}{|W|}\frac{|Z(s,\sigma^n)|}{|X_{\sigma^n,s}|}\frac{|G_{\sigma^n,s}|}{|P_{\sigma^n,s}|}\geqslant\lim_{n\to\infty}\frac{1}{c_1c_2}\frac{(q^n)^{\dim Z(G_s)+\dim G_s}}{(q^n)^{\dim X_s+\dim P_s}}.$$

It is now easy to see that this limit is infinite. □

5. Proof of Theorems 2 and 3

Throughout this section G is a simple algebraic group, X a generic MRR subgroup, and P is a parabolic subgroup with Levi factor L. Starting with Proposition 21 we will use H for arguments which apply to both X and L.

Lemma 18. *Let $s \in X \cap L$. If either of the following holds then $X\backslash G/P$ is infinite*:

(i) *G_s is of type A_2 and X_s and L_s are tori.*
(ii) *G_s is of type B_2, X_s is a torus and L_s is of type A_1 or a torus.*

Proof. Using the dimension criterion it suffices to show that

$$\dim Z(G_s)+\frac{1}{2}\dim G_s-\dim X_s-\frac{1}{2}\dim L_s>0.$$

It is easy to check in each case that the quantity on the left is at least 1. □

Corollary 19. *If either of the following hold then $X\backslash G/P$ is infinite*:

(i) *X and L have conjugate A_2 complements.*
(ii) *X has a B_2 complement K, and for some conjugate $\widetilde{K}=K^g$ we have that $\widetilde{K}\cap L$ is a MRR subgroup which is a torus or of type A_1.*

Proof. If G has rank 2 and (i) or (ii) holds then it is easy to show that $X\backslash G/P$ is infinite by dimension.

Assume now that the rank of G is at least 3. If (i) holds let K be the A_2 complement of X. Using Lemmas 5 and 6, we may replace P by a conjugate and assume that in (i) the A_2 complements of X and L coincide, and that in (ii), we have $\widetilde{K} = K$. Since the rank of G is at least 3, we have that K is a Levi factor of a parabolic, whence is of the form $C_G(S)$ for some torus S. Apply Lemma 7, to see that there exists $s \in S$ with $G_s = K$. We are done by the previous lemma. □

Corollary 20. *If X has an A_2 or B_2 complement then X is not spherical.*

Proof. Apply Lemma 19, noting that the Levi factor of a Borel subgroup is a torus, which has every type of complement possible. □

Proposition 21. *Let H be a generic MRR subgroup of G which does not appear in Table* 1. *The following hold, and, in particular, H has an A_2 or B_2 complement in all cases.*

(i) *If G has single root length, then H has an A_2 complement.*
(ii) *If H is the Levi factor of a parabolic with non-abelian unipotent radical then H has an A_2, B_2 or G_2 complement.*
(iii) *Let G equal B_n or C_n.*
  (a) *If $G = B_n$ and $H = D_{n_1}D_{n_2}$ then H has a B_2 complement.*
  (b) *Otherwise H has an A_2 complement.*
(iv) *Let $G = B_n$. If H is a Levi factor then there exists a MRR subgroup K of type B_2 with $H \cap K$ a MRR subgroup which is either a torus or of type A_1.*
(v) *If $G = F_4$ the maximal possibilities for H are C_3T_1, $A_2\widetilde{A}_2$, B_3T_1, $A_1A_1B_2$, $\widetilde{A}_1A_3$, D_4, where $\widetilde{A}_1$ and $\widetilde{A}_2$ denote groups with short roots. The first possibility has a long A_2 complement, the next has both long and short A_2 complements, and the rest have short A_2 complements. In particular, if L is a Levi factor for a parabolic subgroup which is not an end node parabolic, then L has both long and short A_2 complements.*

The proof of this proposition is delayed until the next section.

Proof of Theorem 2. The work of Brundan [4] and Lawther [13] show that (ii) $\Rightarrow$ (i). Corollary 20 shows that (i) $\Rightarrow$ (iii). Proposition 21 shows that (iii) $\Rightarrow$ (ii). □

Proof of Theorem 3. We assume that X and L are not spherical and will show that $X\backslash G/P$ is infinite.

If $G = G_2$ then by dimension one finds that if X is non-spherical then $X\backslash G/P$ is infinite. For the remainder of the proof assume $G \neq G_2$.

Recall our convention that D_1 is a 1-dimensional torus. If $(G, X) \neq (B_n,\ D_{n_1}D_{n_2})$ then apply Proposition 21, let H_X be an A_2 complement for X and let H_L be an A_2 complement of L, of the same length as H_X (length is only an issue for F_4). If $(G, X) = (B_n,\ D_{n_1}D_{n_2})$ then apply Proposition 21, let H_X be a B_2 complement for X and let H_L be a MRR subgroup of type B_2 with $L \cap H_L$ a MRR subgroup of type A_1 or a maximal torus. Apply Lemma 5 to see that H_X and H_L are conjugate. Apply Lemma 19 to see that $X\backslash G/P$ is infinite. □

6. Proof of Proposition 21

Throughout this section we let H be a generic MRR subgroup of G and fix a maximal torus $T \leqslant H$. Let $\Phi(G)$ and $\Phi(H)$ be the root systems defined using T.

We prove parts (i) and (ii) immediately. Parts (iii)–(v) follow after Corollary 24.

Proof of Proposition 21(i). Recall that G has single root length and H is a MRR subgroup which fails to appear in Table 1. Then, by [13], H is not anti-open, that is, there exist $\alpha, \beta, \alpha + \beta \in \Phi(G) - \Phi(H)$. Let φ equal all $\mathbb{Z}$-linear combinations of α and β which are contained in $\Phi(G)$. Then φ is an A_2 complement for H. □

Proof of Proposition 21(ii). Recall that H is the Levi factor of a parabolic with non-abelian unipotent radical Q. Let α, β be roots such that the corresponding root groups U_α and U_β are contained in Q and do not commute. Let φ equal all the $\mathbb{Z}$-linear combinations of α and β which are contained in $\Phi(G)$. Then φ is an A_2, B_2 or G_2 complement for H. □

Lemma 22. *Let φ_0 be an irreducible root system in a Euclidean space $\mathbb{E}$ with inner product $(\ ,\ )$. Let φ_1 be a proper, closed subsystem of φ_0. Then $\varphi_0 - \varphi_1$ spans $\mathbb{E}$ and for each $\beta \in \varphi_1$ there exists $\alpha \in \varphi_0 - \varphi_1$ with $(\alpha, \beta) \neq 0$.*

Proof. Let n be the dimension of $\mathbb{E}$ and fix a Dynkin diagram Δ of φ_0. Given α, $\beta \in \Delta$ the path connecting α to β is the shortest such path and includes α and β. The sum over this path means the sum of each element of Δ which is contained in the path. It is easy to check that such a sum is itself a root.

For the first conclusion it suffices to show that we have n independent vectors in $\varphi_0 - \varphi_1$. Since φ_1 is a proper, closed subsystem we have that $\Delta - \varphi_1$ is non-empty. For each $\alpha \in \Delta$ let γ_α be a path connecting α to exactly one element of $\Delta - \varphi_1$. We re-index these paths so that for $i \in \{1, \dots, n\}$ the path γ_i contains a node which does not appear in $\gamma_1, \dots, \gamma_{i-1}$. For each i let β_i be the sum over γ_i. By the manner in which the paths γ_i were indexed, it is easy to see that $\beta_1, \dots, \beta_n$ are linearly independent. By the manner in which the paths were chosen, we may write each β_i as the sum of a root in φ_1 and a root outside of φ_1. This shows that β_i is not in φ_1.

For the final conclusion note that β is not orthogonal to $\mathbb{E}$, whence it is not orthogonal to $\varphi_0 - \varphi_1$. □

Corollary 23. *Let $\varphi_0 \leqslant \Phi(G)$ be an irreducible root system and let $\varphi_2 = \varphi_0 \cap \Phi(H)$. Let φ_1 be a closed subsystem of φ_0 with $\varphi_0 > \varphi_1 > \varphi_2$.*

- (i) *If φ_0 has single root length then H has an A_2 complement* (*whose length is the same as φ_0*).
- (ii) *If φ_0 is closed in $\Phi(G)$, $G = B_n$ and $\varphi_1 - \varphi_2$ contains a short root then H has a B_2 complement.*

Proof. Fix $\beta \in \varphi_1 - \varphi_2$ and assume that β is short if (ii) holds. By the previous lemma there exists $\alpha \in \varphi_0 - \varphi_1$ with $(\alpha, \beta) \neq 0$. Note that $\alpha \neq \pm\beta$ and that if $i\alpha + j\beta \in \Phi(G)$ then $i\alpha + j\beta \in \varphi_0$ (in part (i) use $(\alpha, \beta) \neq 0$). If $(\alpha, \beta) > 0$ we replace one root with its negative and assume that $(\alpha, \beta) < 0$, whence $\alpha + \beta \in \varphi_0$. Since $\alpha \notin \varphi_1$ and $\beta \in \varphi_1$ we see that $\alpha + \beta \notin \varphi_1$. Similarly, we see that $\alpha + 2\beta \notin \varphi_1$ (of course it may not even be a root) and that $2\alpha + \beta$ is not a root. Let φ equal all the $\mathbb{Z}$-linear combinations of α and β which are contained in $\Phi(G)$. If (i) holds then φ is an A_2 complement for H. If (ii) holds then φ is a B_2 complement for H since β is short and B_n has no closed subsystems of type short A_2 or G_2. □

Corollary 24. *If $\varphi_0 \leqslant \Phi(G)$ is irreducible with single root length and $\varphi_2 = \varphi_0 \cap \Phi(H)$ is submaximal in φ_0 then there exists φ_1 as in the previous corollary.*

Proof. In a root system with single root length, every root subsystem is closed. □

Proof of Proposition 21(iii)–(v). Part (iii). Recall that G equals B_n or C_n and H is a generic MRR subgroup which does not appear in Table 1.

Part (a). If $n = 2$ and H does not appear in Table 1 then H is just a torus and G itself is a B_2 complement. We now assume that $n \geqslant 3$, $G = B_n$ and $H = D_{n_1}D_{n_2}$. We assume that $n_1 \geqslant 2$. Let $\varphi_0 = \Phi(G)$ and set $\varphi_2 = \Phi(H)$. Let $\varphi_1 = \Phi(B_{n_1}D_{n_2})$. Then φ_1 contains a short root and φ_2 does not. Thus $\varphi_1 - \varphi_2$ contains a short root and we are done by part Corollary 23(ii).

Part (b). We assume that $n \geqslant 3$ and if $G = B_n$ that $H \neq D_{n_1}D_{n_2}$. If $G = B_n$ let φ_0 and φ_2 equal the long roots in $\Phi(G)$ and $\Phi(H)$ respectively. If $G = C_n$ let φ_0 and φ_2 equal the short roots in $\Phi(G)$ and $\Phi(H)$ respectively. In both cases φ_0 is of type D_n, and is irreducible since $n \geqslant 3$. The maximal subsystems of φ_0 are A_{n-1} and $D_{n_1} + D_{n_2}$. The subsystem φ_2 cannot equal D_n, A_{n-1}, or $D_{n_1} + D_{n_2}$ as this would contradict either the assumption that H is not in Table 1 or the extra restrictions on H when $G = B_n$. Therefore φ_2 is a submaximal subsystem of φ_0 and we are done by Corollary 24.

Part (iv). We have that $G = B_n$ and that H is a Levi subgroup of G. Let $\alpha_1, \ldots, \alpha_n$ be the nodes in the Dynkin diagram of G in the usual order (as in [2]) and suppose that H is described by "crossing off" certain nodes. Let $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2 + \cdots + \alpha_n$. Let φ equal all the $\mathbb{Z}$-linear combinations of α and β which are contained in $\Phi(G)$. Let K be the connected group which contains the fixed maximal torus and whose root system equals φ.

Part (v). We have that $G = F_4$. To construct all short A_2 complements, take φ_0 equal to all the short roots in $\Phi(F_4)$, thus φ_2 equals all the short roots in $\Phi(H)$. Observe that φ_0 is of type D_4. By examining each possibility for H it is easy to verify that φ_2 is submaximal in a D_4 root system and we are done by Corollary 24. To construct the long A_2 complements, one proceeds similarly with φ_0 equal to all the long roots in $\Phi(F_4)$. □

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References

[1] A. Borel, J. Tits, Groupes réductif, Publ. Inst. Hautes Études Sci. 27 (1965) 55–160.
[2] N. Bourbaki, Groupes et Algèbres de Lie, IV, V, VI, Masson, Paris, 1981.
[3] M. Brion, Quelques propriétés des espaces homogènes sphériques, Manuscripta Math. 55 (1986) 191–198.
[4] J. Brundan, Dense orbits and double cosets, in: R.W. Carter, J. Saxl (Eds.), Algebraic Groups and Their Representations, Kluwer, Dordrecht, 1998, pp. 259–274.
[5] R. Carter, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, Wiley, New York, 1985.
[6] M. Demazure, P. Gabriel, Groupes Algébriques I, North-Holland, Amsterdam, 1970.
[7] M. Fried, M. Jarden, Field Arithmetic, Springer-Verlag, Berlin, 1986.
[8] R. Guralnick, M. Liebeck, D. Macpherson, G. Seitz, Modules for algebraic groups with finitely many orbits on subspaces, J. Algebra 196 (1997) 211–250.
[9] W. Hodges, Model Theory, in: Encyclopedia Math. Appl., vol. 42, Cambridge Univ. Press, Cambridge, 1993.
[10] J. Humphreys, Linear Algebraic Groups, in: Grad. Texts in Math., vol. 21, Springer-Verlag, Berlin, 1975.
[11] J.C. Jantzen, Representations of Algebraic Groups, in: Pure and Applied Math., vol. 131, Academic Press, San Diego, 1987.
[12] M. Krämer, Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen, Compositio Math. 38 (1979) 129–153.
[13] R. Lawther, Finiteness of double coset spaces, Proc. London Math. Soc. 79 (3) (1999) 605–625.
[14] M.V. Nori, On subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$, Invent. Math. 88 (1987) 257–275.
[15] G. Seitz, Double cosets in algebraic groups, in: R.W. Carter, J. Saxl (Eds.), Algebraic Groups and Their Representations, Kluwer, Dordrecht, 1998, pp. 214–257.
[16] T. Springer, R. Steinberg, Conjugacy classes, in: Seminar in Algebraic Groups and Related Finite Groups, in: Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin, 1970.
[17] E. Vinberg, Complexity of action of reductive groups, Funct. Anal. Appl. 20 (1986) 1–11.