Van der Waals black hole

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A B S T R A C T

In the context of extended phase space, where the negative cosmological constant is treated as a thermodynamic pressure in the first law of black hole thermodynamics, we find an asymptotically AdS metric whose thermodynamics matches exactly that of the Van der Waals fluid. We show that as a solution of Einstein's equations, the corresponding stress energy tensor obeys (at least for certain range of metric parameters) all three weak, strong, and dominant energy conditions.

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1. Introduction

Due to the AdS/CFT correspondence, there has been a revival of interest in the physics of asymptotically AdS black holes in recent years; the main focus being that of understanding strongly coupled thermal field theories living on the AdS boundary. Even from a bulk perspective such black holes are quite interesting, their thermodynamics exhibiting various phase transitions. A primary example is the thermal radiation/black hole first-order phase transition observed for Schwarzschild–AdS black hole spacetimes [1]. Interestingly, when a charge and/or rotation are added, behavior qualitatively similar to a Van der Waals fluid emerges [2–5]. This analogy becomes even more complete in the extended phase space [6,7], where the cosmological constant is treated as a thermodynamic pressure P,

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}, \qquad (1)$$

and is allowed to vary in the first law of black hole thermodynamics,

$$\delta M = T\delta S + V\delta P + \cdots, \qquad (2)$$

while a quantity thermodynamically conjugate to P is interpreted as a black hole thermodynamic volume [8,9]

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,\dots}. \qquad (3)$$

This allows one to write down a 'black hole equation of state' $P = P(V, T)$ and compare it to the corresponding fluid equation of state, while we identify the black hole and fluid temperatures $T \sim T_f$, the black hole and fluid volumes $V \sim V_f$, and the cosmological and fluid pressures $P \sim P_f$.

The Van der Waals fluid is described by the Van der Waals equation, which is a closed form two-parameter equation of state:

$$T = \left(P + \frac{a}{v^2}\right)(v - b), \qquad (4)$$

where v denotes the specific volume of the fluid, $v = V/N$ with N counting fluid's degrees of freedom. The parameter $a > 0$ measures the attraction in between the molecules of the fluid, and the parameter b measures their volume. The thermodynamics of charged and/or rotating AdS black holes have been shown to qualitatively mimic the behavior of this equation [6,7], including the existence of a small/large black hole first-order phase transition corresponding to a liquid/gas phase transition that eventually terminates at a critical point characterized by the standard mean field theory critical exponents. In both cases, the corresponding thermodynamic potential, the Gibbs free energy, displays the swallowtail catastrophe.

Although remarkable, the analogy between the thermodynamics of charged/rotating AdS black holes and that of the Van der Waals fluid is only *qualitative*—the corresponding equations of state are not identical and prevent one from identifying the black hole parameters such as charge Q or rotation J with the fluid parameters a and b. Unfortunately, this remains true for all other more complicated examples of black holes, possibly in higher or lower dimensions, which were found to demonstrate a qualitative Van der Waals behavior; for a recent review see [10].

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In this letter we turn the logic around and construct an asymptotically AdS black hole whose thermodynamics matches exactly that of the Van der Waals fluid and show that as a solution of Einstein's equations, and for sufficiently small pressures, the corresponding stress energy tensor obeys all three standard energy conditions.

2. Constructing the solution

In what follows we want to construct an asymptotically AdS black hole metric whose thermodynamics coincide exactly with the given fluid equation of state. For simplicity here we assume the static spherically symmetric ansatz

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2,$$

$$f = \frac{r^2}{l^2} - \frac{2M}{r} - h(r, P), \tag{5}$$

where the function h is to be determined.

We further assume that such a metric is a solution of the Einstein field equations with a given energy–momentum source, $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$. In order that the energy–momentum source be physically plausible, we require it to satisfy certain conditions such as positivity of energy density and dominance of the energy density over pressure, known as energy conditions, e.g. [11]. Namely, the minimal requirement we impose is the *weak energy condition*, demanding $T_{ab}\xi^a\xi^b \geq 0$ for any future-directed non-spacelike vector ξ. Writing the stress energy tensor in an orthonormal basis, $T^{ab} = \rho e_0^a e_0^b + \sum_i p_i e_i^a e_i^b$, where ρ stands for the energy density and p_i denote principal pressures, the energy conditions imply ($i = 1, 2, 3$)

Weak: $\quad \rho \geq 0, \quad \rho + p_i \geq 0,$ (6)

Strong: $\quad \rho + \sum_i p_i \geq 0, \quad \rho + p_i \geq 0,$ (7)

Dominant: $\quad \rho \geq |p_i|.$ (8)

In particular, for a metric ansatz we find (with prime denoting the derivative w.r.t. r)

$$\rho = -p_1 = \frac{1 - f - rf'}{8\pi r^2} + P,$$

$$p_2 = p_3 = \frac{rf'' + 2f'}{16\pi r} - P. \tag{9}$$

Once we determine f, we shall check the corresponding energy conditions above.

The ansatz (5) implies that the "mass" of the black hole M is related to the horizon radius r_+ according to

$$M = \frac{4}{3}\pi r_+^3 P - \frac{h(r_+, P)r_+}{2}. \tag{10}$$

Imposing further the first law, (2), the thermodynamic volume V is determined from (3). Since the horizon area is given by $A = 4\pi r_+^2$, we now define the black hole 'specific volume' as

$$v = k\frac{V}{N}, \qquad N = \frac{A}{L_{pl}^2}, \tag{11}$$

where in accord with previous papers we set the constant (in $d = 4$ dimensions) to be $k = \frac{4(d-1)}{d-2} = 6$, and interpret N as the number of degrees of freedom associated with the black hole horizon, with L_{pl} being the Planck length [6,7,10], giving

$$v = \frac{k}{4\pi r_+^2}\left[\frac{4}{3}\pi r_+^3 - \frac{r_+}{2}\frac{\partial h(r_+, P)}{\partial P}\right]. \tag{12}$$

Since we are in Einstein gravity, the entropy and the horizon area are related as $S = A/4$. We also know that the black hole temperature reads

$$T = \frac{f'(r_+)}{4\pi} = 2r_+ P - \frac{h(r_+, P)}{4\pi r_+} - \frac{1}{4\pi}\frac{\partial h(r_+, P)}{\partial r_+}. \tag{13}$$

This can now be compared to any desired fluid equation of state, $T = T(v, P)$.

3. Van der Waals black hole

The discussion in the previous section can be applied to any desired equation of state. Let us now specify to the Van der Waals case (4). That is, we compare the expression for T (13) with (4), to get

$$2r_+ P - \frac{h}{4\pi r_+} - \frac{h'}{4\pi} - \left(P + \frac{a}{v^2}\right)(v - b) = 0, \tag{14}$$

where we substitute for v the expression (12). This represents a partial differential equation for $h(r, P)$ which gives a solution to our problem.

In particular, we find a solution of this equation by employing the following ansatz:

$$h(r, P) = A(r) - PB(r). \tag{15}$$

With this ansatz the above PDE becomes of the form $F_1(r)P + F_2(r) = 0$, where F_1 and F_2 depend on functions A and B and their derivatives. Since both these parts have to vanish separately, we get a system of two ODEs for unknown functions A and B. Solving first $F_1 = 0$ we find that (setting $k = 6$)

$$B = \left(C_1 - \frac{8\pi}{3}\right)r^2 + 4\pi b r. \tag{16}$$

Setting now the integration constant $C_1 = \frac{8\pi}{3}$ [so that we preserve the AdS structure postulated in (5)], we find that $F_2 = 0$ gives a solution

$$A = -2\pi a + \frac{3\pi ab^2}{r(2r + 3b)} + \frac{4\pi ab}{r}\log\left(\frac{2r}{r_0} + 3\frac{b}{r_0}\right), \tag{17}$$

where r_0 is an integration constant with dimensions of length. For simplicity setting $r_0 = 2b$ yields

$$f = 2\pi a - \frac{2M}{r} + \frac{r^2}{l^2}\left(1 + \frac{3}{2}\frac{b}{r}\right) - \frac{3\pi ab^2}{r(2r + 3b)}$$
$$- \frac{4\pi ab}{r}\log\left(\frac{r}{b} + \frac{3}{2}\right) \tag{18}$$

for the 'Van der Waals black hole metric'. For $b \ll r$ this expands as

$$f = 2\pi a - \frac{2M}{r} + \frac{8\pi P}{3}r^2\left(1 + \frac{3}{2}\frac{b}{r}\right)$$
$$- \frac{4\pi ab\log(r/b)}{r} - \frac{15\pi a}{2}\frac{b^2}{r^2} + O\left[(b/r)^3\right]. \tag{19}$$

Apart from the strange terms logarithmic and linear in r, we observe that the requirement for positivity of a, signifying the attraction in between the molecules of the fluid, implies 'spherical' horizon topology of the VdW black hole. Without loss of generality we can set $a = 1/(2\pi)$.
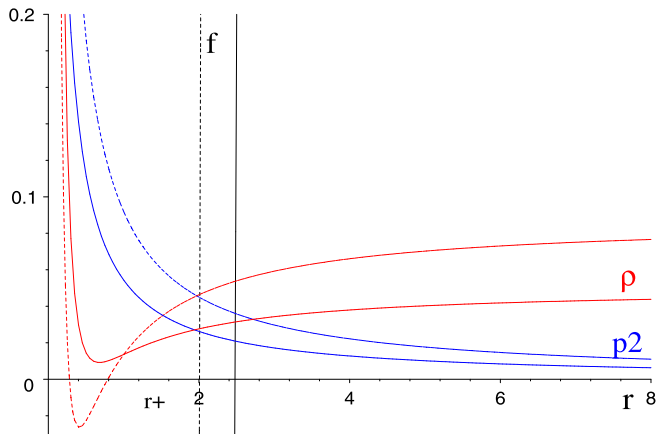
Fig. 1. *Horizon and energy conditions*. We display the energy density ρ (red curves) and the principal pressure p_2 (blue curves). The x-axis represents the radial coordinate r, the black curves depict metric function f, which is positive everywhere outside of r_+, giving rise to a black hole horizon. For small pressures P, and since $p_1 = -\rho$, all three energy conditions are satisfied (not displayed). As P increases, first the dominant energy condition is violated, illustrated by thick curves for $P = 0.1$, followed by the violation of the weak energy condition, illustrated by dotted curves for $P = 0.17$. All dimensionful quantities are measured in units of the Van der Waals parameter b, while we have set $M = 10$, $a = \frac{1}{2\pi}$.

4. Conclusions

The obtained metric (5) with f given by (18) reproduces exactly the Van der Waals equation (4) with the specific volume v, (12), given by

$$v = 2r_+ + 3b. \tag{20}$$

It is straightforward to show that the corresponding thermodynamic volume V satisfies the reverse isoperimetric inequality [12]. We find that for sufficiently small pressures P, the corresponding energy–momentum source satisfies all three energy conditions (6), (7), and (8). As the pressure increases, ρ diminishes at small radii and eventually becomes negative, resulting in first violation of the dominant condition (8), followed by violation of the weak condition (6), see Fig. 1; the strong condition (7) is always satisfied.

We emphasize that our ansatz (15) is essentially unique. It is straightforward to check that inclusion of higher powers of P does not yield solutions consistent with the asymptotic AdS structure postulated in (5). Indeed our ansatz furnishes a general procedure for constructing a metric from a given equation of state, at least in the spherically symmetric case, and our approach can easily be extended to other dimensions. One can, for example, construct a black hole whose thermodynamics coincides with the virial expansion of equation of state to an arbitrary order.

However our results also indicate that the asymptotic fall-off behavior of the metric and matter fields will in general differ from that of standard electrovacuum and perfect fluid cases. For the particular Van der Waals case we consider here, an expansion of the stress-energy tensor at large r yields

$$\rho = \frac{P}{2} - \frac{Pb}{2r} + \frac{1 - 2\pi a}{16\pi r^2} + \frac{ab}{4r^3} + \cdots,$$
$$p_2 = \frac{Pb}{2r} + \frac{ab}{4r^3} + \cdots, \tag{21}$$

whose corresponding exact matter content remains to be found. We also remark that, apart from the logarithmic term, the obtained metric is qualitatively similar to an exact solution to the field equations of a certain class of conformal gravity theories [13] with vanishing stress-energy, affording an alternate interpretation of our results.

Since the first law (2) is satisfied by (5) with f given by (18), we can regard M as a "mass". An independent evaluation of this mass via either conformal [14,15] or counter term [16] methods is problematic due to the weaker subleading falloff behavior of the metric and the stress-energy. It will be necessary to modify these approaches (somewhat along the lines of Dilaton gravity [17] or asymptotically Lifshitz metrics [18,19]) to cancel the divergent terms that arise.

Acknowledgements

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