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Partial Matroid Representations

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A central theorem of matroid 3-connectivity is established that has a number of new and old connectivity results as corollaries. The proof of this theorem relies on a matrix theory developed here for partial matroid representations.

1. INTRODUCTION

In this paper we prove a central theorem of matroid 3-connectivity that has a number of useful corollaries, some old, some new, and some independently developed by others. We will leave the precise statement of the theorem to Section 3, but it implies the following statement: Let $\bar{M}$ be a 3-connected proper minor on at least four elements of a 3-connected matroid M. Then M has a 3-connected minor $\bar{\bar{M}}$ that (a) has $\bar{M}$ as a proper minor, and (b) has at most three additional elements beyond those of $\bar{M}$. The proof of the theorem relies on a matrix theory which we develop in Section 2 for partial matroid representations. The latter concept is nothing but a matrix representation of the fundamental circuit sets of H. Whitney [22], and can be defined as follows.

Let X be a base of a matroid M on a set S, and $Y=S-X$. Construct $\{0, 1\}$ valued $\hat{\mathbf{B}}=[\mathbf{I}|\mathbf{B}]$ as follows. S is to be the set of indices of the columns of $\hat{\mathbf{B}}$; in particular X is to index the columns of identity $\mathbf{I}$, say in the order $x_1, x_2, \ldots x_m$. Then we index the rows by $x_1, x_2, \ldots x_m$ as well. Let $y \in Y$, and suppose $\bar{X}$ is the subset of X that forms a circuit with y. Then in the column of $\mathbf{B}$ with index y, set element B_{xy} equal to 1 if $x \in \bar{X}$, and equal to 0 otherwise. Any $\hat{\mathbf{B}}$ that may be so constructed from M is a *partial representation* of M. Note that this construction can always be carried out unless M consists only of loops. In the latter case, we may formally take $\hat{\mathbf{B}}$ to be a matrix without rows (below we will call a matrix without rows or columns *empty*). Whenever we refer to matroid M below, we will assume that M has at least one independent element, or, equivalently, that $\hat{\mathbf{B}}$ of M is not empty.

Throughout we will assume a knowledge of elementary matroid definitions and results. A good reference is the book by D. J. A. Welsh [21].

2. RESULTS ON PARTIAL REPRESENTATIONS

In this section we define *determinant*, *pivot*, *cofactor* and *rank* for partial representations. With these concepts we then establish a matrix theory of partial representations which is a weaker form of the familiar matrix theory where the elements of the matrices are taken from a field. Due to space limitations we will only present results which were useful in connection with prior work [15, 16] and the development of Section 3.

A few conventions concerning notation will simplify the exposition. For any matrix $\mathbf{A}$ the matrix $\hat{\mathbf{A}}$ will denote $[\mathbf{I}|\mathbf{A}]$. Let $\hat{\mathbf{B}}$ be a partial representation of a matroid M on a set S, where an identity of $\hat{\mathbf{B}}$ corresponds to a base X. Define $Y=S-X$, and suppose $\mathbf{B}$ has

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a submatrix $\bar{\mathbf{B}}$. We then could display $\hat{\mathbf{B}}$ as

$$\hat{\mathbf{B}} = X \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{cccc} \overbrace{X_1 \quad X_2}^{X} & \overbrace{Y_1 \quad Y_2}^{Y} \\ \left[\begin{array}{cc|cc} 1 \ddots 1 & 0 & \bar{\mathbf{B}} & \\ \hline 0 & 1 \ddots 1 & & 0/1 \end{array}\right] \\ \underbrace{\qquad\qquad}_{\mathbf{I}} \underbrace{\qquad\qquad}_{\mathbf{B}} \end{array} \tag{2.1}$$

The index sets of rows and columns of $\hat{\mathbf{B}}$, here X, Y, X_i, Y_i, $i = 1, 2$, will always be shown in the indicated manner. Due to this mode of indexing, we can specify $\hat{\mathbf{B}}$ completely by just displaying $\mathbf{B}$ with row and column index sets. We will also make use of the following convention: Whenever the specified entries of a matrix or submatrix are all 1s, then the unspecified entries are taken to be zeros. In this section $\bar{\mathbf{B}}$ will always be a square submatrix of $\mathbf{B}$, and we usually will rearrange $\mathbf{B}$ so $\bar{\mathbf{B}}$ resides in the upper left corner of $\mathbf{B}$ as in (2.1). Submatrices that look exactly like $\bar{\mathbf{B}}$ of (2.1) may occur several times in $\hat{\mathbf{B}}$ or other partial representations of M, but we will not consider them to be the same matrix as $\bar{\mathbf{B}}$. Indeed, to be precise, we should denote $\bar{\mathbf{B}}$ of (2.1) as

$$\begin{array}{cc} & Y_1 \\ X_1 & \left[\ \bar{\mathbf{B}}\ \right] \\ & \qquad X_2 \end{array} \tag{2.2}$$

to point out the difference between $\bar{\mathbf{B}}$ and such seemingly identical submatrices. X_2 of (2.2) denotes the index set of rows of $\hat{\mathbf{B}}$ that do not intersect with $\bar{\mathbf{B}}$. We will rarely use notation (2.2), but the reader should be aware that when we write $\bar{\mathbf{B}}$, meaning a submatrix of a partial representation, we use a shorthand notation for (2.2). A number of matrix theory concepts will be used. We will say '$\bar{\mathbf{B}}$ is nonsingular', '$\bar{\mathbf{B}}$ has full rank', '$\bar{\mathbf{B}}$ has nonzero determinant', or '$\det \bar{\mathbf{B}} = 1$' if $X_2 \cup Y_1$ is a basis of M. Now and then we will employ determinants of a square matrix, say $\bar{\mathbf{A}}$, over a field $\mathscr{F}$. We write '$\det_{\mathscr{F}} \bar{\mathbf{A}}$' to denote such a determinant in case confusion with the above case is possible. The opposite case to '$\bar{\mathbf{B}}$ is nonsingular' is described by '$\bar{\mathbf{B}}$ is singular', etc. We can extend this definition to all square submatrices of $\mathbf{B}$ as follows. Let $\tilde{\mathbf{B}}$ be a square submatrix of $\hat{\mathbf{B}}$, say specified by $\tilde{X} \subseteq X$ and $\tilde{Z} \subseteq X \cup Y$. Define $\tilde{\mathbf{B}}$ to be *singular* if $(X \cap \tilde{Z}) - \tilde{X} \neq \varnothing$, or equivalently, if $\tilde{\mathbf{B}}$ has a 0 column with index in X. If $(X \cap \tilde{Z}) - \tilde{X} = \varnothing$, then $\tilde{\mathbf{B}}$ can be partitioned as

$$\tilde{\mathbf{B}} = \begin{array}{c} \\ X_1 \\ \bar{X}_2 \end{array} \begin{array}{c} \begin{array}{cc} \bar{X}_2 & Y_1 \end{array} \\ \left[\begin{array}{c|c} 0 & \bar{\mathbf{B}} \\ \hline 1 \ddots 1 & 0/1 \end{array}\right] \\ \qquad X_2 - \bar{X}_2 \end{array} \tag{2.3}$$

and is then defined to be *nonsingular* if $(X_2 - \bar{X}_2) \cup (\bar{X}_2 \cup Y_1) (= X_2 \cup Y_1)$ is a base of M, and is said to be *singular* otherwise. Clearly $\det \tilde{\mathbf{B}}$ (which is defined to be 0 or 1 in the obvious way) is equal to $\det \bar{\mathbf{B}}$ unless $X_1 = Y_1 = \varnothing$.

Elementary matroid operations manifest themselves in partial representations as follows.

LEMMA 2.1. *Given a matroid M on a set S with partial representation $\hat{\mathbf{B}}$ of* (2.1). (a) $[\mathbf{B}^t|\mathbf{I}]$ *is a partial representation of the dual M^* of M, and a square submatrix $\bar{\mathbf{B}}$ of $\mathbf{B}$ is nonsingular if and only if related submatrix $(\bar{\mathbf{B}})^t$ of $\mathbf{B}^t$ is nonsingular.*
(b) *For $x \in X$, $\hat{\mathbf{B}}$ with row and column x deleted is a partial representation $[\mathbf{I}^1|\mathbf{B}^1]$ of $M/\{x\}$. A square submatrix $\bar{\mathbf{B}}$* of $\mathbf{B}^1$ *is nonsingular if and only if the related submatrix of $\mathbf{B}$ is nonsingular. Such x is a coloop if and only if $\mathbf{B}_{x.}$ (=row x of $\mathbf{B}$) is a zero vector.*
(c) *For $y \in Y$, $\hat{\mathbf{B}}$ with column y deleted is a partial representation $[\mathbf{I}|\mathbf{B}^1]$* of *$M\backslash\{y\}$. A square submatrix $\bar{\mathbf{B}}$ of $\mathbf{B}^1$ is nonsingular if and only if the related submatrix of $\mathbf{B}$ is nonsingular. Such y is a loop if and only if $\mathbf{B}_{.y}$(=column y of $\mathbf{B}$) is a zero vector.*
(d) *Elements $x \in X$ and $y \in Y$ are parallel if and only if $\mathbf{B}_{.y}$ is a unit vector with 1 in row x. Elements $y, z \in Y$ are parallel if and only if both $\mathbf{B}_{.y}$ and $\mathbf{B}_{.z}$ are not zero vectors and all 2×2 submatrices of $[\mathbf{B}_{.y}|\mathbf{B}_{.z}]$ are singular. (In the latter case we will say '$\mathbf{B}_{.y}$ and $\mathbf{B}_{.z}$ are parallel columns'.)*
(e) *Elements $x \in X$ and $y \in Y$ are in series if and only if $\mathbf{B}_{x.}$ is a unit vector with 1 in column y. Elements $x, z \in X$ are in series if and only if $\mathbf{B}_{x.}$ and $\mathbf{B}_{z.}$ are not zero vectors and all 2×2 submatrices of $\left[\frac{\mathbf{B}_{x.}}{\mathbf{B}_{z.}}\right]$ are singular. (In the latter case we will say '$\mathbf{B}_{x.}$ and $\mathbf{B}_{z.}$ are parallel rows'.)*

The proof of Lemma 2.1 is straightforward and hence omitted.

Pivots will play an important role since they will allow us to go from one partial representation to another one. Suppose we replace the 1s of a partial representation $\hat{\mathbf{B}}$ of a matroid M by nonzero elements of a field $\mathcal{F}$, getting $\hat{\mathbf{A}}$. We will say that $\hat{\mathbf{A}}$ is an *order k representation of M over $\mathcal{F}$*, for some integer $k \geq 1$, if for all $l \times l$ submatrices $\bar{\mathbf{A}}$ of $\mathbf{A}$, $l \leq k$, $\det_{\mathcal{F}} \bar{\mathbf{A}} \neq 0$ if and only if $\det \bar{\mathbf{B}} = 1$ for the related submatrix of $\mathbf{B}$. Matrix $\hat{\mathbf{A}}$ is an *order k representation of M over $\mathcal{F}$ relative to A_{xy}*, for some element A_{xy}, if the preceding requirement is satisfied by the $\bar{\mathbf{A}}$ containing A_{xy}. Clearly $\hat{\mathbf{B}}$ itself is an order 1 representation of M over every field. This definition raises a number of interesting questions of which we shall address only one here: Given a partial representation $\hat{\mathbf{B}}$ of a matroid M, does there always exist an order 2 representation derived from $\hat{\mathbf{B}}$, over some field? The answer is no. Consider the matroid M_1 represented by $\hat{\mathbf{A}}$ with

$$\mathbf{A} = \begin{array}{c} \\ X_1 \\ \\ X_2 \\ \end{array} \begin{array}{c} \begin{array}{cc|cc} \multicolumn{2}{c|}{Y_1} & \multicolumn{2}{c}{Y_2} \end{array} \\ \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

over some field $\mathcal{F}$. Now derive from M_1 an independence system M_2 with same groundset as M_1 and same set of bases, except for $X_2 \cup Y_1$, which is declared to be a base in M_2. In the terminology of [15] $\hat{\mathbf{A}}$ is an *almost representation* of M_2 over $\mathcal{F}$, and M_2 is easily verified to be a matroid by theorem 2 of that reference. If we view $\hat{\mathbf{A}}$ as a partial representation of M_2, then it is a trivial exercise to verify that $\hat{\mathbf{A}}$ cannot be turned into an order 2 representation of M_2 for any field. (Note that M_2 has at most $\binom{8}{4} - 9$ bases regardless of the field chosen, so M_2 cannot be the Vamos matroid, which has $\binom{8}{4} - 5$ bases (see Welsh [21, p. 140]).) However, we have the following easily proved lemma.

LEMMA 2.2. *Given a partial representation* $\hat{\mathbf{B}}$ *and indices* $x \in X$, $y \in Y$. *Then* $\hat{\mathbf{B}}$ *can be turned into an order* 1 *representation and order* 2 *representation* $\hat{\mathbf{A}}$ *relative to* A_{xy} *over any field other than* $GF(2)$.

We are interested in $\hat{\mathbf{A}}$ of Lemma 2.2 because of the following result, which again has a straightforward proof.

LEMMA 2.3. *Let* $\hat{\mathbf{A}}$ *be an order* $(k-1)$ *representation as well as an order k representation relative to a nonzero* A_{xy} *over some field* $\mathcal{F}$, *for a matroid M and some* $k \geq 2$. *Then* $(\hat{\mathbf{A}}^1)$ *derived from* $\hat{\mathbf{A}}$ *by a pivot (in* $\mathcal{F}$*) on* A_{xy} *is an order* $(k-1)$ *representation over* $\mathcal{F}$ *for M. In particular, the support of* $(\hat{\mathbf{A}}^1)$ *is a partial representation of M where the identity corresponds to basis* $(X-\{x\}) \cup \{y\}$ *of M.*

We now define a *pivot* on a nonzero B_{xy} of $\hat{\mathbf{B}}$ of (2.1) as follows: Replace $\hat{\mathbf{B}}$ by $\hat{\mathbf{A}}$ of Lemma 2.2 over some field $\mathcal{F} \neq \mathrm{GF}(2)$, pivot on A_{xy}, and define $(\hat{\mathbf{B}}^1)$ to be the support of the resulting $(\hat{\mathbf{A}}^1)$. By Lemma 2.3 the matrix $(\hat{\mathbf{B}}^1)$ resulting from such a pivot is the partial representation corresponding to basis $(X-\{x\}) \cup \{y\}$. If there is any chance of confusion of a pivot in a partial representation with a pivot in a matrix over a field $\mathcal{F}$, we will call the latter pivot an $\mathcal{F}$*-pivot.*

Note that the operation of taking the dual commutes with that of a pivot. Theorem 2.1 below summarizes elementary results for partial representations. There $rank(\bar{\bar{\mathbf{B}}})$, the *rank* of a submatrix $\bar{\bar{\mathbf{B}}}$ of $\hat{\mathbf{B}}$, denotes the order of a largest square nonsingular submatrix of $\bar{\bar{\mathbf{B}}}$. The *inverse* of nonsingular $\bar{\mathbf{B}}$ of (2.1), $(\bar{\mathbf{B}})^{-1}$, is the submatrix

$$\begin{array}{c|c|} & X_1 \\ \hline Y_1 & \left[\ \bar{\bar{\mathbf{B}}}\ \right] \\ \hline \end{array} \quad X_2$$

of the partial representation arising from basis $X_2 \cup Y_1$. The latter definition makes sense due to the following matrix multiplication rule. Let X_i, Y_i, $i=1, 2$ be as in (2.1), and suppose $\det \bar{\mathbf{B}} = 1$. Then for matrix

$$\begin{array}{c|c|} & Y_3 \\ \hline Y_1 & \left[\ \mathbf{C}\ \right] \\ \hline \end{array} \quad X_2$$

define $\bar{\mathbf{B}} \cdot \mathbf{C}$ to be the matrix

$$\begin{array}{c|c|} & Y_3 \\ \hline X_1 & \left[\ \mathbf{D}\ \right] \\ \hline \end{array} \quad X_2 \quad .$$

Thus

$$\bar{\mathbf{B}} \cdot (\bar{\mathbf{B}})^{-1} = \begin{array}{c|c|} & X_1 \\ \hline X_1 & \left[\ \mathbf{I}\ \right] \\ \hline \end{array} \quad X_2$$

and

$$(\bar{\bar{\mathbf{B}}})^{-1}\cdot\bar{\mathbf{B}} = \begin{array}{c|c|} & Y_1 \\ \hline Y_1 & \begin{bmatrix} I \end{bmatrix} \\ \hline \end{array} \; . \qquad X_2$$

Furthermore, if **C** is square, then $\det \mathbf{C} = \det \mathbf{D}$.

THEOREM 2.1. *Let* $\hat{\mathbf{B}}$ *be a partial representation of a matroid M. Below* $\bar{\mathbf{B}}$ *is always a* $k\times k$ *submatrix of* $\hat{\mathbf{B}}$.
(a) Let

$$\hat{\mathbf{B}} = \begin{array}{c} \\ x \\ \\ \\ \end{array}\left[\begin{array}{c} \begin{array}{c|c} \multicolumn{1}{c}{y} & \\ 1 & c \\ \hline d & \bar{\mathbf{B}} \end{array} \\ \\ 0/1 \end{array}\right]$$

where **c** or **d** is a zero vector. If we pivot on the 1 of $\hat{B}_{xy}$, getting $(\hat{\mathbf{B}}^1)$, then $\bar{\mathbf{B}}$ is nonsingular if and only if the related submatrix of $(\hat{\mathbf{B}}^1)$ is nonsingular.
(b) If we pivot on a 1 of row x of $\hat{\mathbf{B}}$, getting $(\hat{\mathbf{B}}^1)$, then any $\bar{\mathbf{B}}$ intersecting row x and not containing the pivot element, is nonsingular if and only if the related submatrix of $(\hat{\mathbf{B}}^1)$ is nonsingular.
(c) (Schur complement) Let

$$\bar{\mathbf{B}} = \left[\begin{array}{cc} \bar{\bar{\mathbf{B}}} & \\ & 0/1 \end{array}\right],$$

where $\bar{\bar{\mathbf{B}}}$ is a square nonsingular proper submatrix of $\bar{\mathbf{B}}$. Assume we perform a sequence of pivots in the $\bar{\bar{\mathbf{B}}}$-part of $\bar{\mathbf{B}}$ such that $\bar{\mathbf{B}}$ becomes

$$\left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0/1 \\ \hline 0 & \bar{\mathbf{B}}/\bar{\bar{\mathbf{B}}} \end{array}\right],$$

where $\bar{\mathbf{B}}/\bar{\bar{\mathbf{B}}}$ is the *Schur complement*. Then

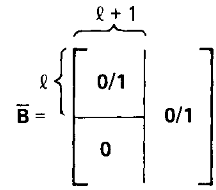
$$\det \bar{\mathbf{B}} = \det(\bar{\mathbf{B}}/\bar{\bar{\mathbf{B}}}). \tag{2.4}$$

(d) (Cofactor expansion) Define $\mathbf{C}^{ij}$ to be the submatrix of $\bar{\mathbf{B}}$ obtained by deleting row i and column j. Then

$$\det \bar{\mathbf{B}} = \sum_i \bar{B}_{ij} \det \mathbf{C}^{ij}, \tag{2.5}$$

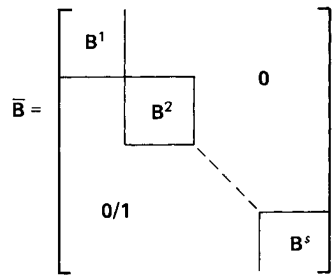
provided the right-hand side of (2.5) is equal to 0 or 1.

(e) If



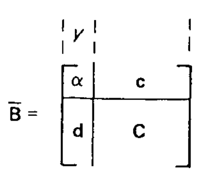
where $l \geq 0$, then $\det \bar{\mathbf{B}} = 0$. In particular, $\det \bar{\mathbf{B}} = 0$ if $\bar{\mathbf{B}}$ has a zero row or column.

(f) If



where each $\mathbf{B}^i$ is square, then $\det \bar{\mathbf{B}} = \prod_{i=1}^{s} \det \mathbf{B}^i$.
(g) In (g1)-(g3) below, let $\bar{\bar{\mathbf{B}}}$ be an arbitrary submatrix of $\hat{\mathbf{B}}$ with $\operatorname{rank}(\bar{\bar{\mathbf{B}}}) = l$, for some $l \geq 0$.
(g1) If $\bar{\bar{\mathbf{B}}}$ is a column submatrix of $\hat{\mathbf{B}}$, say specified by an index set $\bar{\bar{S}}$, then $\bar{\bar{S}}$ has rank equal to l in M.
(g2) Every nonsingular submatrix $\bar{\mathbf{B}}$ of $\bar{\bar{\mathbf{B}}}$ is contained in an $l \times l$ nonsingular submatrix of $\bar{\bar{\mathbf{B}}}$.
(g3) Let $\mathbf{B}^r$ with $\operatorname{rank}(\mathbf{B}^r) = l$ consist of l rows of $\bar{\bar{\mathbf{B}}}$, and $\mathbf{B}^c$ be a column submatrix of $\bar{\bar{\mathbf{B}}}$. Further define $\mathbf{B}^{rc}$ to be the submatrix of $\bar{\bar{\mathbf{B}}}$ specified by the row indices of $\mathbf{B}^r$ and the column indices of $\mathbf{B}^c$. Then $\operatorname{rank}(\mathbf{B}^c) = \operatorname{rank}(\mathbf{B}^{rc})$.
(h) (Cofactor inversion formula) Let $\bar{\mathbf{B}}$ be nonsingular and $\mathbf{C}^{ij}$ be the matrix of part (d). Then $(\bar{\mathrm{B}})_{ji}^{-1} = \det \mathbf{C}^{ij}$, for all i and j.

Proof. Almost all statements follow by routine arguments from the previous lemmas. We only note that it is convenient to reduce each case to the situation where $\bar{\mathbf{B}}$ and $\bar{\bar{\mathbf{B}}}$ are submatrices of $\mathbf{B}$ and not just $\hat{\mathbf{B}}$. Then the only instance possibly requiring details is case (d), where we may suppose $\bar{\mathbf{B}}$ to be $\mathbf{B}$ itself. The result is obviously correct for $k = 2$, and inductively we may assume it to hold for $k - 1 \geq 2$. Also, we may suppose that $i = 1$. If all $\mathbf{C}^{1j}$ are singular, then the right-hand side of (2.5) is equal to 0. Let $x \in X$ correspond to the first unit vector of $\mathbf{I}$. Then M has no basis $\{x\} \cup \bar{Y}$ where $\bar{Y} \subseteq Y$ [see (2.1)], so every basis of M contains at least two elements of X if it contains x. If Y is a basis, it can be obtained from X by a sequence of 1-element exchanges such that (a) all intermediate sets are bases as well, and (b) x is exchanged at the last step. But this is clearly not possible, so Y is not a base, and $\det \bar{\mathbf{B}} = 0$. So suppose $\mathbf{C} = \mathbf{C}^{11}$ is nonsingular, and



If the right hand side of (2.5) is equal to 1, then in addition we may choose $\mathbf{C}$ such that $\alpha = 1$. Note that $\mathbf{C}$ cannot contain a zero column or row since else a minor or the dual of a minor of M has a base containing a loop. If c contains a 0, say in column $\bar{y}$, then we pivot on a 1 in column $\bar{y}$ of $\mathbf{C}$, and by induction and part (c) the desired result follows. Hence we may suppose $\mathbf{c}$ to be a vector of 1s. This implies that all $\mathbf{C}^{1j} \neq \mathbf{C}$ are singular. We now pivot within the $\mathbf{C}$-part to obtain

$$\bar{\mathbf{B}}^1 = \begin{array}{c} \quad y \\ \left[\begin{array}{c|c} \alpha^1 & \mathbf{0} \\ \hline \mathbf{d}^1 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right] \end{array}$$

from $\bar{\mathbf{B}}$. Clearly $\mathbf{d}^1$ is a zero vector since else some $\mathbf{C}^{1j} \neq \mathbf{C}$ is nonsingular. If $\alpha^1 = 0$, then y is a loop of M, in $\mathbf{B}$ we must have $\alpha = 0$, and $\mathbf{d}$ must be a zero vector. Thus the right hand side of (2.5) must have been equal to 0, and $\det \bar{\mathbf{B}} = 0$ as desired. If $\alpha^1 = 1$, then x and y are parallel in M, and in $\mathbf{B}$ we must have $\alpha = 1$ and $\mathbf{d}$ must be a zero vector. This implies that the right-hand side of (2.5) is equal to 1, and also that $\det \bar{\mathbf{B}} = 1$.

The reader will surely have recognized many links between Theorem 2.1 and well-known matroid results. For example, the case of part (d) where the right-hand side of (2.5) is equal to 0, is a special case of a base exchange result of C. Greene [7], and (g2) is based on J. Edmonds' [5] independence axioms. We shall not go into details here, or show how the various axiomatic systems for matroids relate to our matrix approach here; the reader may want to fill in the details and see how these axioms manifest themselves in our framework. One can, however, specify a matroid directly in terms of partial representations, and we shall include such axioms here. To this end we define $\mathcal{A}_s$ to be the set of square $\{0, 1\}$ matrices with permanent equal to s, for $s \geq 0$. (The permanent is the number of transversals; see H. J. Ryser [11].)

THEOREM 2.2. *Let S be a set of $n \geq 1$ elements and $\mathcal{B}$ be a nonempty set of $m \times n$ $\{0, 1\}$ matrices such that the columns of each $\hat{\mathbf{B}} \in \mathcal{B}$ are indexed by S, and each such $\hat{\mathbf{B}}$ contains at least one $m \times m$ identity. Then $\mathcal{B}$ is the collection of partial representations of a matroid if and only if* (a) *and* (b) *below hold for every $\hat{\mathbf{B}} \in \mathcal{B}$, where we suppose $\hat{\mathbf{B}}$ to be partitioned and indexed as in* (2.1).
(a) *If a submatrix $\bar{\mathbf{B}}$ of $\mathbf{B}$ is a member of $\mathcal{A}_0$, then there is no matrix in $\mathcal{B}$ having an identity in columns $X_2 \cup Y_1$.*
(b) *If a submatrix $\bar{\mathbf{B}}$ of $\mathbf{B}$ is a member of $\mathcal{A}_1$, then there is a matrix in $\mathcal{B}$ having an identity in columns $X_2 \cup Y_1$.*
Further, (a) *can be weakened by requiring the condition only for $\bar{\mathbf{B}} \in \mathcal{A}_0$ with a zero row, and* (b) *can be weakened by requiring the condition only for $\bar{\mathbf{B}} = [1] \in \mathcal{A}_1$.*

The proof involves routine checking of the base axioms for matroids and is omitted.

One way to look at Theorem 2.2 is as follows. Let $\mathcal{D}_0$ ($\mathcal{D}_1$) be the sets of square singular (nonsingular) matrices $\mathbf{D}$ over some field which always remain singular (nonsingular) when the nonzero entries are replaced by other nonzero entries, possibly by those of another field. Then $\mathbf{D} \in \mathcal{D}_0$ ($\mathbf{D} \in \mathcal{D}_1$) if and only if the support of $\mathbf{D}$ is in $\mathcal{A}_0$ ($\mathcal{A}_1$). Theorem 2.2 says that the entire matroid structure is specified via the supports of matrices in $\mathcal{D}_0 \cup \mathcal{D}_1$. Finally, the reader may wonder whether (a) above could be weakened, by requiring the condition of (a) only for $\bar{\mathbf{B}} = [0] \in \mathcal{A}_0$. This is not possible, as is evident from the collection

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}.$$

Finally, we will make repeated use of a simple characterization of matroid connectivity.

We recall that a matroid M on a set S is *connected* if for every nontrivial partition of S into sets S_1 and S_2 we have $r(S_1)+r(S_2)>r(S)$, where $r(\cdot)$ is the rank function of M. The graph $G(\mathbf{B})$ referred to below is defined as follows. Each row and each column of $\mathbf{B}$ corresponds to a node, and an undirected arc connects nodes i and j if $B_{ij}=1$.

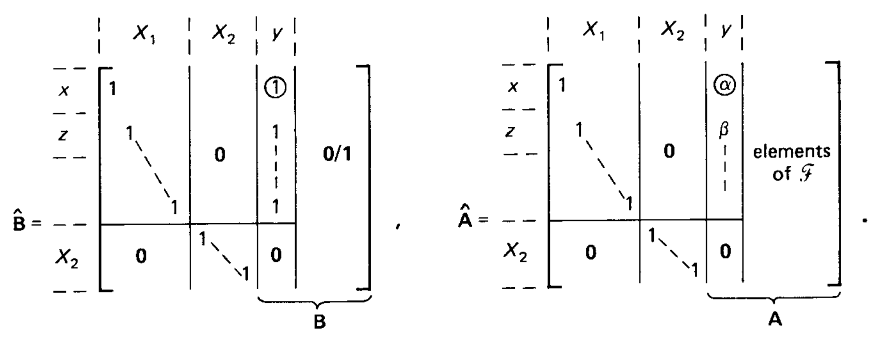
THEOREM 2.3 (W. H. Cunningham [2], S. Krogdahl [9]). *A matroid M on a set S with a partial representation $\hat{\mathbf{B}}$ is connected if and only if $G(\hat{\mathbf{B}})$ is connected.*

The preceding theorems lead to almost trivial proofs for a number of important matroid results. As a demonstration we will prove a theorem about matroid representability due to W. T. Tutte [18]. First we introduce the relevant definitions. A matroid is *representable* over a field $\mathscr{F}$ if one can replace the 1s of a partial representation $\hat{\mathbf{B}}$ by nonzero elements of the field, producing, say, a matrix $\hat{\mathbf{A}}$, such that every square submatrix of $\hat{\mathbf{B}}$ is nonsingular if and only if the related submatrix of $\hat{\mathbf{A}}$ is nonsingular. Equivalently we may require that (a) to each partial representation $\hat{\mathbf{B}}$ there corresponds a matrix $\hat{\mathbf{A}}$ over $\mathscr{F}$ whose support is $\hat{\mathbf{B}}$; (b) when we $\mathscr{F}$-pivot in $\hat{\mathbf{A}}$ on A_{ij} and pivot in $\hat{\mathbf{B}}$ on B_{ij}, producing, say, $(\hat{\mathbf{A}}^1)$ and $(\hat{\mathbf{B}}^1)$, respectively, then $(\hat{\mathbf{A}}^1)$ is the matrix over $\mathscr{F}$ corresponding to $(\hat{\mathbf{B}}^1)$. We can weaken the second requirement by demanding that $(\hat{\mathbf{A}}^1)$ can be scaled to the matrix over $\mathscr{F}$ corresponding to $(\hat{\mathbf{B}}^1)$. A *hyperplane* of a matroid of rank m is a maximal subset of rank $m-1$. Let H_x be the index set of the 0s in row x of a partial representation $\hat{\mathbf{B}}$. We see that the column submatrix of $\hat{\mathbf{B}}$ with index set H_x observes this rank condition, so H_x is a hyperplane. It is also easy to see that each hyperplane occurs as an H_x in some $\hat{\mathbf{B}}$. (Since the 1s of row x correspond to a dual circuit—look at $[\mathbf{B}^t|\mathbf{I}]$—we also have immediately that each dual circuit is the complement of a hyperplane.) With these observations we are ready to prove the following theorem.

THEOREM 2.4 (W. T. Tutte [18]). *A simple matroid M (i.e., without loops or parallel elements) on a set S is representable over a field $\mathscr{F}$ if and only if for every hyperplane H of M there exists a function f_H such that*
(a) $\operatorname{Ker} f_H = H$;
(b) *For any three hyperplanes H_1, H_2, and H_3 of M which intersect in a coline (=flat of M of rank $r(S)-2$), there exist constants a_1, a_2, $a_3 \in \mathscr{F}$, all nonzero, such that*

$$a_1 f_{H_1} + a_2 f_{H_2} + a_3 f_{H_3} = 0.$$

PROOF. The 'only if' part is easily verified, so we will only prove the converse. For each partial representation $\hat{\mathbf{B}}$ of M we first establish the following matrix $\hat{\mathbf{A}}$. If the 0s of row $\hat{\mathbf{B}}_{x.}$ correspond to hyperplane H, then define $\hat{\mathbf{A}}_{x.}=\mathbf{f}_H$, where for convenience we view $\mathbf{f}_H$ as a vector with elements $f_H(z)$, $z\in S$. We will show that the previously described pivot condition is satisfied by these $\hat{\mathbf{A}}$ matrices. Consider the following $\hat{\mathbf{B}}$ and $\hat{\mathbf{A}}$ pair:

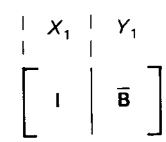


By (a) the 0s of $\hat{\mathbf{B}}$ must agree with the 0s of $\hat{\mathbf{A}}$. If we pivot on the circled 1 in $\hat{\mathbf{B}}$ and the circled α in $\hat{\mathbf{A}}$, getting $(\hat{\mathbf{B}}^1)$ and $(\hat{\mathbf{A}}^1)$, respectively, then the rows of $\{x\} \cup X_2$ undergo at most scaling, so for these rows we have the desired agreement between $(\hat{\mathbf{B}}^1)$ and $(\hat{\mathbf{A}}^1)$. For the remainder we examine a typical row, say row z. Let the 0s of rows x and z of $\hat{\mathbf{B}}$ correspond to hyperplanes H_1 and H_2, respectively, and the 0s of row z of $(\hat{\mathbf{B}}^1)$ to hyperplane H_3. The three hyperplanes contain the set $[X_1 \cup X_2] - \{x, z\}$, so they do intersect on a coline. By (b) and scaling we have $a_1 \neq 0$ such that $f_{H_3} = a_1 f_{H_1} + f_{H_2}$. Now H_3 contains y, so $0 = a_1\alpha + \beta$, i.e., $a_1 = -\beta/a$. But then row z of $(\hat{\mathbf{A}}^1)$, which is $f_{H_2} - \beta \cdot f_{H_1}/\alpha$, is equal to f_{H_3}, which was to be shown.

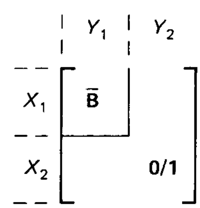
3. Matroid 3-Connectivity

The theorems of the preceding section are now utilized to prove a central theorem of matroid 3-connectivity. This theorem has a number of useful corollaries, some of which are new while the rest are well-known or have recently been independently proved by others.

We have chosen to first introduce helpful lemmas before stating the main result (Theorem 3.1). All proofs are so simple that we will either sketch them or omit them entirely. Some terminology beyond that of deletion and contraction is used to simplify the exposition. If $M_2 = M_1 \backslash \{z\}$, we say that we *add* z to M_2 to obtain M_1. The case of $M_2 = M_1/\{z\}$ is covered by saying that we *expand* M_2 by z to obtain M_1. Both cases are handled by saying that we *extend* M_2 by z to produce M_1. Finally *reduction* by z is a deletion or contraction of z. Analogously we *extend* (*reduce*) a partial representation to another partial representation by adjoining (removing) rows and/or columns. We say that $\mathbf{B}$ of a partial representation $\hat{\mathbf{B}}$ of a matroid M *contains* $\bar{\mathbf{B}}$ of a partial representation



of a matroid $\bar{M}$ if $\mathbf{B}$ can be rearranged to



and the related square submatrices of the two $\bar{\mathbf{B}}$ matrices have identical determinants. (The two $\bar{\mathbf{B}}$ matrices are not identical according to (2.2)!) Note that $M/X_2 \backslash Y_2 = \bar{M}$. A matroid M on a set S is *k-separable* [19] for given $k \geq 1$ if there exists a partition S_1, S_2 of S with $|S_1|$, $|S_2| \geq k$ for which

$$r(S_1) + r(S_2) \leq r(S) + k - 1, \tag{3.1}$$

where $r(\cdot)$ is the rank function of M. We call $\{S_1, S_2\}$ a *k-separation* of M. A matroid M is *k-connected* if any l-separation necessarily has $l \geq k$. By the previous definition M is connected if and only if M is not 1-separable, i.e., if and only if M is 2-connected.

LEMMA 3.1. *A matroid M with partial representation $\hat{\mathbf{B}}$ has a k-separation if and only if $\mathbf{B}$ has a partition*

$$\mathbf{B} = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & \mathbf{B}^{11} & \mathbf{B}^{12} \\ \hline X_2 & \mathbf{B}^{21} & \mathbf{B}^{22} \\ \hline \end{array}, \tag{3.2}$$

for which $|X_i \cup Y_i| \geq k$, $i = 1, 2$, and $rank(\mathbf{B}^{12}) + rank(\mathbf{B}^{21}) \leq k-1$.

Of particular interest are 2-separations. We call a 2-separation $\{S_1, S_2\}$ a *split*, and specifically an *l-split* if $\min\{|S_1|, |S_2|\} = l$. By the previous definition $l \geq 2$ for any l-split. We call each S_i a *splitting set.*

LEMMA 3.2. *Let M be a connected matroid on a set S of at least four elements with partial representation $\hat{\mathbf{B}}$.*
(a) *M has a 2-split if and only if $\mathbf{B}$ contains a column unit vector, or a row unit vector, or two parallel rows or columns.*
(b) *If M has an l-split for some $l \geq 3$, but no 2-split, then in every partition of $\hat{\mathbf{B}}$ as in Lemma 3.1 (with $k = 2$) the sets X_i and Y_i are all nonempty.*
(c) *Suppose $\mathbf{I}$ of $\hat{\mathbf{B}}$ corresponds to $X \subseteq S$, and $\mathbf{B}$ of $\hat{\mathbf{B}}$ to $Y \subseteq S$. Assume there exist disjoint $\bar{X}_i$ and $\bar{Y}_i$, $i = 1, 2$ and a submatrix $\mathbf{D}$ of $\mathbf{B}$ such that*
 (i) *$|\bar{X}_i \cup \bar{Y}_i| \geq 2$, $i = 1, 2$,*
 (ii) *the column indices of $\mathbf{D}$ are all in $\bar{Y}_1$,*
 (iii) *$rank\,\mathbf{D} \leq 1$, and*
 (iv) *every path in $G(\mathbf{B})$ from a node of $\bar{X}_1 \cup \bar{Y}_1$ to a node of $\bar{X}_2 \cup \bar{Y}_2$ involves an arc corresponding to a 1 of $\mathbf{D}$.*
Then M has a split $\{S_1, S_2\}$ for which $S_i \supseteq \bar{X}_i \cup \bar{Y}_i$, $i = 1, 2$.

Below we repeatedly use the following observations. Let $\bar{M}$ be a minor of a matroid M on a set S. If X_1 is a base of $\bar{M}$ and Y_1 consists of the remaining elements of $\bar{M}$, then there exists a set X_2 disjoint from X_1 such that $X = X_1 \cup X_2$ is a base of M and $\bar{M} = M/X_2 \backslash (S - [X \cup Y_1])$. This fact implies that for any partial representation $[\mathbf{I}|\bar{\mathbf{B}}]$ of $\bar{M}$ there always exists a partial representation $\hat{\mathbf{B}}$ where $\mathbf{B}$ contains $\bar{\mathbf{B}}$.

LEMMA 3.3. *Let $\bar{M}$ be a 3-connected matroid on a set S of at least four elements, with partial representation $[\mathbf{I}|\bar{\mathbf{B}}]$.*
(a) *A matroid M^1 produced by a 1-element extension of $\bar{M}$ is 3-connected if and only if every partial representation $(\hat{\mathbf{B}}^1)$ of M^1 containing $\bar{\mathbf{B}}$ is specified by one of the following matrices, where in both cases z is the additional element.*

$$\mathbf{B}^1 = \begin{array}{c|c|c|} & Y_1 & z \\ \hline X_1 & \bar{\mathbf{B}} & \mathbf{c} \\ \hline \end{array} \tag{3.3}$$

where $\mathbf{c}$ is not a zero or unit vector, and is not parallel to a column of $\bar{\mathbf{B}}$.

$$\mathbf{B}^1 = \begin{array}{c|c|} & Y_1 \\ \hline X_1 & \bar{\mathbf{B}} \\ \hline z & \mathbf{c} \\ \hline \end{array} \tag{3.4}$$

where **c** *is not a zero or unit vector, and is not parallel to a row of* $\bar{\mathbf{B}}$.

(b) *Let* M^2 *be produced by a 2-element extension of* $\bar{M}$ *such that every minor of* M^2 *which is a* 1*-element extension of* $\bar{M}$, *is 2-separable. Then* M^2 *is 3-connected if and only if every partial representation* $(\hat{\mathbf{B}}^2)$ *containing* $\bar{\mathbf{B}}$ *has* $\mathbf{B}^2$ *of the following type, where x and y are the two additional elements.*

$$B^2 = \begin{array}{c|c|c|} & Y_1 & y \\ \hline X_1 & \bar{B} & c \\ \hline x & d & \alpha \end{array}, \qquad (3.5)$$

where

(*i*) **c** (**d**) *is a unit vector, or is parallel to a column (row) of* $\bar{\mathbf{B}}$.

(ii) *each of* $\mathbf{c}^1 = \begin{bmatrix} \mathbf{c} \\ \hline \alpha \end{bmatrix}$, $\mathbf{d}^1 = [\mathbf{d}|\alpha]$ *is not a unit vector, and it is not parallel to a column/row of the remainder of* $\mathbf{B}^2$;

(iii) *if* **c** (**d**) *is parallel to the jth column (row) of* $\bar{\mathbf{B}}$, *then* **d** (**c**) *is not the jth unit vector.*

(c) *Let* M^3 *be produced by a 3-element extension of* $\bar{M}$ *such that every minor of* M^3 *which is a 1- or 2-element extension of* $\bar{M}$, *is 2-separable. Then* M^3 *is 3-connected if and only if every partial representation* $(\hat{\mathbf{B}}^3)$ *containing* $\bar{\mathbf{B}}$ *has* $\mathbf{B}^3$ *as one of the matrices below, where x, y, and z are the additional elements.*

$$B^3 = \begin{array}{c|c|c|c|} & Y_1 & y & z \\ \hline X_1 & \bar{B} & c & e \\ \hline x & 0 & 1 & 1 \end{array}, \qquad (3.6)$$

where

(i) *each of* **c**, **e** *is a unit vector or parallel to a column of* $\bar{\mathbf{B}}$;

(ii) **c** *is not parallel to* **e**.

$$B^3 = \begin{array}{c|c|c|} & Y_1 & y \\ \hline X_1 & \bar{B} & 0 \\ \hline x & c & 1 \\ \hline z & e & 1 \end{array}, \qquad (3.7)$$

where

(i) *each of* **c**, **e** *is a unit vector or parallel to a row of* $\bar{\mathbf{B}}$;

(ii) **c** *is not parallel to* **e**.

In the statement of the main Theorem 3.1 as well as later on $\mathbf{B}_{\bar{X},\bar{Y}}$ denotes the submatrix of **B** specified by the row indices of $\bar{X}$ and the column indices of $\bar{Y}$. The Theorem also mentions an *independence black box* of a matroid. This device decides dependence/independence of a set in the matroid in unit time. Finally, the term *efficient algorithm* refers to an appropriate Turing machine for the given matroid problem whose running time is bounded by a polynomial in the cardinality of the groundset of the matroid.

THEOREM 3.1. *Let $\bar{M}$ be a 3-connected proper minor on at least four elements of a 3-connected matroid M with a given partial representation $\hat{\mathbf{B}}$, where*

$$
\mathbf{B} = \begin{array}{cc|c|c|c|}
 & & Y_1 & Y_2 & Y_3 \\
\hline
 & X_1 & \bar{\mathbf{B}} & \mathbf{W}^2 & 0 \\
\hline
X & X_2 & \mathbf{W}^1 & 0/1 & 0/1 \\
\hline
 & X_3 & 0 & 0/1 & \mathbf{K} \\
\hline
\end{array} \qquad (3.8)
$$

and $\bar{M} = M/(X_2 \cup X_3)\backslash(Y_2 \cup Y_3)$. Each row of W_1 and each column of W_2 is assumed to be nonzero. Suppose in addition we have an independence black box for M. Then there exists an efficient algorithm that locates a partial representation $(\hat{\mathbf{B}}^1)$ for M by pivoting at most in the K-part of B (so the structure of matrices $\mathbf{B}_{X,Y_1}$ and $\mathbf{B}_{X_1,Y}$, including determinants of square submatrices, is unchanged), with

$$
\mathbf{B}^1 = \begin{array}{c|c|c|c|}
 & Y_1 & Y_2 & Y_3^1 \\
\hline
X_1 & \bar{\mathbf{B}} & \mathbf{W}^2 & 0 \\
\hline
X_2 & \mathbf{W}^1 & 0/1 & 0/1 \\
\hline
X_\beta^1 & 0 & 0/1 & \mathbf{K}^1 \\
\hline
\end{array}, \qquad (3.9)
$$

such that $\mathbf{B}^1$ contains a submatrix of type (3.3)-(3.7).

PROOF. We may suppose that $\mathbf{B}$ of (3.8) does not contain a matrix of (3.3)-(3.7). Let $\mathbf{W}^2_{X_1,\bar{Y}}$, $\bar{Y} \subseteq Y_2$, be nonempty and consist of the columns of $\mathbf{W}^2$ parallel to one column $\mathbf{b}$ of $\bar{\mathbf{B}}$, say with index $z \in Y_1$. Clearly $\bar{\mathbf{W}} = [\mathbf{b}|\mathbf{W}_{X_1,\bar{Y}}]$ has rank equal to 1. Adjoin to $\bar{\mathbf{W}}$ all rows of $\mathbf{B}_{X,\{z\}\cup\bar{Y}}$ that are parallel to a nonzero row of $\bar{\mathbf{W}}$. Let $\bar{\bar{\mathbf{W}}}$ be the resulting rank 1 matrix. From at least one node of $\bar{Y} \cup \{z\}$ of $G(\mathbf{B})$ there must be a path to at least one node of X_1 that does not involve arcs corresponding to any 1s of $\bar{\bar{\mathbf{W}}}$ since otherwise M is 2-separable by Lemma 3.2(c). Choose a shortest such path. If this path makes use of arcs corresponding to 1s of $\mathbf{K}$, reduce its length by suitable pivots in $\mathbf{K}$. A straightforward process then derives one of the matrices (3.5)-(3.7) from the resulting $\mathbf{B}^1$ of (3.9). If $\mathbf{W}^2$ consists only of unit vectors, then a pivot in $\bar{\mathbf{B}}$ produces the above case. A final pivot in $\mathbf{B}^1$ then 'undoes' the initial pivot. Finally, suppose $\mathbf{W}^2$ to be empty. Then $\mathbf{W}^1$ must be nonempty since otherwise $\mathbf{M}$ is separable, and passage to the dual $\mathbf{M}^*$ of $\mathbf{M}$ produces the above case. When the $\mathbf{B}^1$ for M^* has been found, we revert back to M.

Theorem 3.1 implies a number of interesting results, some old and some new. Here we restrict ourselves to corollaries concerning matroid connectivity. We use the concept of a *nested* 3-*connected extension sequence* of matroids $M_0, M_1, \ldots, M_s$, where M_i is a proper 3-connected minor of M_{i+1}, for all $i < s$. Symbol $\cong$ denotes 'is isomorphic to'. The *wheel*

$$\mathbf{B} = \begin{array}{c} Y \\ \begin{bmatrix} 1 & & & & 1 \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{bmatrix} \end{array},$$

W_m (*whirl* $\mathcal{W}_m$) is the matroid on $2m \geqslant 6$ elements with a partial representation $\hat{\mathbf{B}}$ where and where Y is dependent (independent). (It is easily checked that this definition is consistent with the original one [19] using a theorem on almost representability [15]). It will be convenient for us to consider U_4^2, the rank 2 uniform matroid on 4 elements, to be a whirl, say $\mathcal{W}_2$.

A *subdivision* of a matroid M is obtained from M by possibly repeated application of the following process: select an element z and replace it by the elements of a series class that includes z.

COROLLARY 3.1.1. *Let M be a 3-connected matroid on a set S with a 3-connected minor $\bar{M} = M/\tilde{X}\backslash\tilde{Y}$ on at least four elements. Then there exists at least one 3-connected extension sequence $M_0, M_1, \ldots, M_s = M$ for each of (a), (b), (c) below, where in all cases k_i^a and k_i^e are the number of 1-element additions and expansions, respectively, required to go from M_i to M_{i+1}, for all $i < s$. Let k_i be the total number of elements M_{i+1} has beyond those of M_i, for all $i < s$.*
(a) *$M_0 = \bar{M}$; k_i^a, $k_i^e \leqslant 2$ and $k_i \leqslant 3$, for all $i < s$.*
(b) *$M_0 \cong \bar{M}$; $k_i^a \leqslant 1$ and $k_i \leqslant 3$, for all $i < s$. If there exists a $\bar{Z} \subseteq S$ such that $M \times \bar{Z}$ is a subdivision of $\bar{M}$, then there exists a $Z_0 \subseteq S$ such that $M \times Z_0$ is a subdivision of M_0.*
(c) *$M_0 \cong \bar{M}$; k_i^a, $k_i^e \leqslant 1$ and $k_i \leqslant 2$. If $k_i = 2$, then either $M_i \cong \mathcal{W}_m$ and $M_{i+1} \cong \mathcal{W}_{m+1}$, for some $m \geqslant 2$, or $M_i \cong W_m$ and $M_{i+1} \cong W_{m+1}$, for some $m \geqslant 3$.*

If we have an independence black box for M, and if $\bar{M}$ is specified via two sets X_2, Y_2 as $M/X_2\backslash Y_2$, then there exists an efficient algorithm that locates such a sequence for any one of the three cases.

The proof of Corollary 3.1.1 makes use of the following lemma.

LEMMA 3.4. *Let a 3-connected matroid M have a 3-connected minor $\bar{M}$ that has at least four elements such that*
(a) *Every 1-element extension of every minor of M isomorphic to $\bar{M}$ produces a 2-separable matroid;*
(b) *M has two elements beyond those of $\bar{M}$.*
Then $\bar{M} = \mathcal{W}_m$ and $M = \mathcal{W}_{m+1}$, some $m \geqslant 2$, or $\bar{M} = W_m$ and $M = W_{m+1}$ for some $m \geqslant 3$.

PROOF. Due to (3.5) and pivots M has a partial representation $\hat{\mathbf{B}}$ where $m \times n$ $\mathbf{B}$, $m + n \geqslant 6$, is

$$\mathbf{B} = \begin{array}{c|c|c|} & \bar{Y} & y \\ \hline & c & 1 \\ \bar{X} & \bar{\bar{B}} & 0 \\ \hline x & d & 1 \\ \hline \end{array}, \qquad (3.10)$$

Here $\mathbf{d}$ is not parallel to $\mathbf{c}$, and $\bar{M} = M/\{x\}\backslash\{y\}$. In each partial representation $\hat{\mathbf{B}}$ of M displayed below, the $(m-1)\times(n-1)$ submatrix in the upper left corner is always called $\bar{\mathbf{B}}$, and it corresponds to a minor isomorphic to $\bar{M}$. When we use the same letter for two vectors we imply that they are parallel. With these conventions we see that the last column (row) of any $\mathbf{B}$ after deletion of the mth (nth) element must always be a unit vector or it must be parallel to a column (row) of $\bar{\mathbf{B}}$. When we pivot in $\hat{\mathbf{B}}$, say on B_{ij}, we also exchange columns i and j of the resulting new partial representation to determine the new $\mathbf{B}$. Also note that m and n must be at least three. We are now ready for the proof. If $\mathbf{d}$ is a unit vector, we can change it to a non-unit vector by a pivot in $\bar{\bar{\mathbf{B}}}$ without changing column y. If this is not possible, M is clearly 2-separable. So let $\mathbf{d}$ be parallel to row $\bar{x} = 2$ of $\bar{\mathbf{B}}$. An exchange of rows x and $\bar{x}$ converts the second element in column y to a 1. That column minus the last element must be parallel to a column of $\bar{\mathbf{B}}$, say column $\bar{y} = 1$. An exchange of columns y and $\bar{y}$ results in

$$
\begin{array}{c|c|c|c|}
 & y & & \bar{y} \\
\hline
\bar{\bar{x}} & 1 & \bar{\mathbf{c}} & 1 \\
\hline
x & 1 & \bar{\mathbf{d}} & 1 \\
\hline
 & 0 & 0/1 & 0 \\
\hline
\bar{x} & 0 & \bar{\mathbf{d}} & 1 \\
\hline
\end{array}
\tag{3.11}
$$

The subvectors $[\bar{\mathbf{d}}|1]$ of rows x and $\bar{x}$ in (3.11) are the $\mathbf{d}$s of (3.10), so they are parallel. The 2×2 submatrix specified by indices x, $\bar{\bar{x}}$, y, $\bar{y}$ is singular, so a pivot on B_{xy} changes the first element in column y to a 0. Inductively we thus may assume that $\mathbf{B}$ is of the form

$$
\mathbf{B} = \begin{array}{c|c|c|c|}
 & Y_1 & Y_2 & y \\
\hline
 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} \mathbf{c} \\ \mathbf{0} \\ \mathbf{d} \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \\
X_1 & & & \\
\hline
X_2 & 0 & 0/1 & 0 \\
\hline
x & 0 & \mathbf{d} & 1 \\
\hline
\end{array}
\tag{3.12}
$$

where the 2 subvectors of type $[\mathbf{d}|1]$ are parallel.

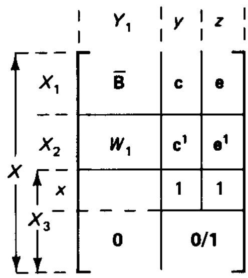
If $\mathbf{d}$ is not a unit vector, then it is parallel to a row of $\bar{\mathbf{B}}$ with index in X_2. We then do the same arguments as before to produce an instance of (3.12) where $|Y_1|$ is increased by 1. If $\mathbf{d}$ is a unit vector, say with the 1 in column $\bar{y} \in Y_2$, and if the vector defined by X_2 and column $\bar{y}$ is not zero, then we pivot on a 1 in that vector and produce another instance of (3.12) where $|Y_1|$ is as before, but $\mathbf{d}$ has become a non-unit vector. If this pivot is not possible, Y_2 must be $\{\bar{y}\}$ and $X_2 = \varnothing$ since otherwise M is 2-separable. By induction we may thus assume $Y_2 = \{\bar{y}\}$ and $X_2 = \varnothing$. The first element of column $\bar{y}$ must be a 1 since otherwise row 1 of $\mathbf{B}$ is a unit vector. We conclude that $\bar{M}$ is isomorphic to a wheel or whirl. A pivot on B_{xy} then shows M to be isomorphic to a wheel if $\bar{M}$ is, and to be isomorphic to a whirl otherwise.

PROOF OF COROLLARY 3.1.1. By Theorem 3.1 we may take **B** to be the matrix of (3.8) and $\bar{M}$ to be $M/(X_2 \cup X_3)\backslash(Y_2 \cup Y_3)$. Furthermore we may assume **B** to contain one of the matrices of (3.3)–(3.7). Case (a) is then trivial. If in case (b) some $M \times \bar{Z}$ is a subdivision of $\bar{M}$ for some $\bar{Z} \subseteq S$, then $\bar{Z} = X_1 \cup X_2 \cup Y_1$, and each row of $\mathbf{W}^1$ is a unit vector or it is parallel to some row of $\bar{\mathbf{B}}$. We always suppose that there exists such $\bar{Z}$ for part (b) since otherwise (b) is subsumed by (c).

Initially we designate M_0 to be $\bar{M}$ and for (b) we define Z_0 to be $\bar{Z}$. We first show how to find M_1 for (b) and (c). The only cases deserving detailed discussion are (3.6) for (b) and (3.5)–(3.7) for (c).

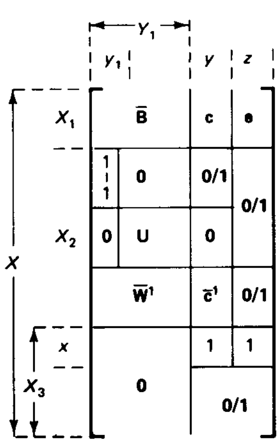
(3.5): (c) Implement the proof procedure of Lemma 3.4. If we discover M_0 and the matroid defined by $\mathbf{B}^2$ of (3.5) to be one of the pairs specified in (c), then let M_1 be the latter matroid. Otherwise a 1-element extension of a minor isomorphic to M_0 produces a 3-connected matroid. Let M_1 be this matroid, and redefine M_0 to be the minor. The latter step amounts to a relabelling of elements of M_0 according to the isomorphism.

(3.6): (b) **B** contains a column submatrix



(3.13)

By assumption each row of $\mathbf{W}^1$ must be a unit vector or be parallel to a row of $\bar{\mathbf{B}}$. If both **c** and **e** are unit vectors, we pivot on the (x, z) entry to get a new **B** where **c** has become a vector with two 1s. Note that this pivot leaves the determinantal structure of $\mathbf{B}_{X,Y_1}$ unchanged. If the new **c** is not parallel to a column of $\bar{\mathbf{B}}$, we have case (3.3); otherwise, we are still in (3.6). Hence we may suppose that **c** is parallel to a column of $\bar{\mathbf{B}}$, say the first one. Next we analyze the rows of $\mathbf{W}^1$. If such a row, say with index w, is a unit vector other than the first one, and $\mathbf{c}^1_w = 1$, then row w and column y of **B** produce case (3.5). Similarly if row w of $\mathbf{W}^1$ is parallel to a row of $\bar{\mathbf{B}}$, but row w of $[\mathbf{W}^1|\mathbf{c}^1]$ is not parallel to the related row of $[\bar{\mathbf{B}}|\mathbf{c}]$, then again we have an instance of (3.5). Thus the submatrix (3.13) of **B** must be



(3.14)

where each row of **U** is a unit vector, and where each row of $[\bar{\mathbf{W}}^1|\bar{\mathbf{c}}^1]$ is parallel to a row of $[\bar{\mathbf{B}}|\mathbf{c}]$. Suppose we relabel y_1 to y in M_0. The new M_0 is isomorphic to $\bar{M}$, and there exists a new Z_0 such that $M \times Z_0$ is a series extension of the new M_0 by (3.14). Clearly the correspondingly redefined (3.8) leads to case (3.5).

(3.6): (c) As in (b) we may suppose that **c** is parallel to the first column of $\bar{\mathbf{B}}$, say with index y_1. Relabel y_1 to y in M_0. The related change in **B** produces case (3.5).

(3.7): (c) Analogous to (3.6) we may suppose that **c** is parallel to a row of $\bar{\mathbf{B}}$, say with index x_1. Relabel x_1 to x in M_0. The related change in **B** produces case (3.5).

Inductively, suppose we have obtained $M_0, M_1, \ldots, M_i$, and we want to produce M_{i+1}. The appropriate procedure follows almost immediately from the above discussion if we take $\bar{\mathbf{B}}$ of (3.8) to correspond to M_i instead of M_0. Again Theorem 3.1 guarantees one of (3.3)–(3.7), and we select a case with smallest equation number. Part (a) then goes through without modification. For (3.6), part (b), $\mathbf{W}^1$ still has all rows as unit vectors or parallel to a row of $\bar{\mathbf{B}}$ since otherwise we have case (3.4). Now any relabelling is done not just in M_i, but also in $M_{i-1}, \ldots, M_0$. The new M_0 is isomorphic to $\bar{M}$, and there exists a new Z_0 such that $M \times Z_0$ is a series extension of the new M_0 as before. For part (c) just one modification is needed, namely any relabelling in M_i is also extended to a relabelling in $M_{i-1}, \ldots, M_0$. By induction we obtain a sequence $M_0, M_1, \ldots, M_s = M$ having the desired properties in each case of (a), (b), (c).

Corollary 3.11(c) is equivalent to P. D. Seymour's splitter theorem [12]; a binary version of the latter theorem as well as of the equivalent of our Lemma 3.4, (published in the preprint of [12]) precedes our work. When this work was completed, we received a preprint by S. Negami [10] in which the graph case of Corollary 3.1.1(c) is proved, and subsequently we obtained a copy of the thesis by J. J.-M. Tan [13] in which Corollary 3.1.1(c) and Lemma 3.4 are established.

LEMMA 3.5. *Every* 3*-connected matroid* M *on at least four elements has a minor isomorphic to* U_4^2 *or* W_3.

PROOF. The lemma follows from characterizations of binary and graphic matroid due to W. T. Tutte [18] and of series-parallel graphs by G. A. Dirac [4], but proving the lemma that way is like firing a cannon to swat a fly. Suppose M has no U_4^2 minor, so every 2×2 submatrix with four 1s of any partial representation $\hat{\mathbf{B}}$ is singular. Trivial graph arguments on $G(\mathbf{B})$ and pivots then produce one of the following submatrices:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

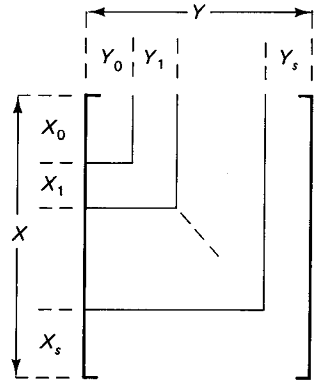
Both cases establish W_3 as minor.

A few arguments could be added to the above proof to establish the stronger result (also implied by [18] and [4]) that every matroid without minors isomorphic to U_4^2 or W_3 is the forest matroid of a series-parallel graph.

COROLLARY 3.1.2 (W. T. Tutte [19]). *Let* M *be a* 3*-connected matroid such that every* 1*-element reduction results in a* 2*-separable matroid. Then* M *is isomorphic to a wheel or whirl.*

PROOF. We may suppose that M has at least five elements. By Lemma 3.5 M is isomorphic to W_3 or it has a proper 3-connected minor $\bar{M}$ isomorphic to U_4^2 or W_3. In the latter case M must be isomorphic to a wheel or whirl by Corollary 3.1.1(c) since in any extension sequence $M_0 \cong \bar{M}, M_1, \ldots, M_{s-1}, M_s = M$ matroids M_s and M_{s-1} must differ by two elements.

We gain additional insight into the sequence $M_0, M_1, \ldots, M_s$ of Corollary 3.1.1 by analyzing the **B** of (3.8) attained when M_s is reached. By the procedure of that corollary we may partition **B** as



where $\bigcup_{j=0}^{i} X_j = \bar{X}_i$ is a base of M_i and the related nonbasic elements are given by $\bar{Y}_i = \bigcup_{j=0}^{i} Y_j$. Note that $\bar{X}_i \cup \bar{Y}_i \neq \emptyset$, for all i. Let $\mathbf{B}^i$ be the submatrix $B_{\bar{X}_i, \bar{Y}_i}$, and collect the nonzero rows of $B_{X-\bar{X}_i, \bar{Y}_i}$ in a matrix $\mathbf{R}^i$, say, specified by $B_{\bar{\bar{X}}_i, \bar{Y}_i}$. Define $\tilde{\mathbf{B}}^i = \left[\frac{\mathbf{B}^i}{\mathbf{R}^i}\right]$ and $\tilde{M}_i$ to be the matroid partially represented by $(\hat{\tilde{\mathbf{B}}}^i)$. Thus the $\tilde{M}_i$ may be efficiently derived from the M_i if the latter matroids are determined from $\bar{M}$ and M by the procedure of Corollary 3.1.1. The $\tilde{M}_i$ have the following properties.

COROLLARY 3.1.3. *Let $\bar{M}$, $M_0, M_1, \ldots, M_s = M$ be the matroids of Corollary* 3.1.1(*a*), (*b*), *or* (*c*). *Then the sequence $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_s$ satisfies the following*:
(a) *Each $\tilde{M}_i$ is equal to $M \times Z_i$ for $Z_i = \bar{X}_i \cup \bar{\bar{X}}_i \cup \bar{Y}_i$, and it is a subdivision of a* 3*-connected matroid. Furthermore $\tilde{M}_i$ may be obtained from M_i by expansion steps, and $\tilde{M}_s = M_s$. In case of Corollary* 3.1.1 *part* (a) [*part* (*b*)], *$\tilde{M}_0$ is a subdivision of $\bar{M}$ (of a minor of M isomorphic to $\bar{M}$) provided there exists a $\bar{\bar{Z}}$ such that $M \times \bar{\bar{Z}}$ is a subdivision of $\bar{M}$.*
(b) *$\tilde{M}_{i+1}$ is either equal to $\tilde{M}_i$, or it is obtained from $\tilde{M}_i$* by *$l \geq 1$ repeated applications of the following procedure: Add a new element y, then replace y by one or more elements of a series class which includes y. Here $l \leq 2$ for part* (a), *and $l = 1$ for parts* (*b*) *and* (*c*).

PROOF. (a) follows from the definition of $\tilde{M}_i$, Theorem 3.1, and Corollary 3.1.1. That corollary also establishes $|Y_i| \leq 2$ for part (a) and $|Y_i| \leq 1$ for parts (b), (c), for all $i > 0$. The rows of $\mathbf{R}_{i+1}$ with index not in $\bar{\bar{X}}_i$ (i.e., rows of $\mathbf{R}^{i+1}$ that do not intersect $\tilde{\mathbf{B}}^i$) must all be unit vectors if $Y_{i+1} = \{y\}$. We thus can add y and (possibly) additional elements in series with y, to produce $\tilde{M}_{i+1}$ from $\tilde{M}_i$. A minor modification of this argument proves the case when $|Y_{i+1}| = 2$, so (b) holds as well.

For convenience of exposition we have allowed the sequence $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_s$ to contain duplicate matroids, but we could strike out such duplicates and get after renumbering a new sequence $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_t$. Due to this notation we have lost the connection between M_i and $\tilde{M}_i$, but from now on we will not be interested in that relationship. Hence $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_t$ will be the sequence derived from $M_0, M_1, \ldots, M_s$ by deleting duplicates from and renumbering the $\tilde{M}_i$ of Corollary 3.1.3.

D. W. Barnette and B. Grünbaum [1] first proved existence of $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_t$ arising from Corollary 3.1.1(b) for graphs, and P. D. Seymour [12] extended this result to matroids. Here we have seen that it is just one more consequence of Theorem 3.1.

It is trivial but true that we could view any of the extension sequences $M_0, M_1, \ldots, M_s = M$ or $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_t = M$ as a reduction sequence from M to M_0 or $\tilde{M}_0$. Sometimes this view gives a result that looks superficially different as seen below.

COROLLARY 3.1.4. *Every 3-connected matroid M on a set S of at least four elements contains an element y such that $M\backslash\{y\}$ becomes 3-connected once the elements of each series class of $M\backslash\{y\}$ are contracted to one element each.*

PROOF. By Lemma 3.5 M has a minor $\bar{M}$ isomorphic to U_4^2 or W_3, where we prefer U_4^2 if there is a choice. Take $\tilde{M}_0, \tilde{M}_1, \ldots, \tilde{M}_t$ to be the sequence derived from a sequence $M_0, M_1, \ldots, M_s$ of Corollary 3.1.1(c). If $t > 1$, the element added to $\tilde{M}_{t-1}$ to get $\tilde{M}_t$ is the desired y. The case $t = 1$ is trivial.

Corollary 3.1.4 was previously proved by W. H. Cunningham [3], and the dual version restricted to graphs has appeared in a paper by C. Thomassen [14].

A final comment seems in order. Though we have produced quite a few corollaries from Theorem 3.1, many more can be derived from that theorem, as we have discovered in more recent work. There is no need to include such results here since the reader can easily produce them by rather simple arguments and routine operations. Finally, we should mention that translation of the above results (and their dual versions) into graph language is quite straightforward since only one graph corresponds to a 3-connected graphic matroid.

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