ESSAY REVIEW

ABRÉGÉ D'HISTOIRE DES MATHÉMATIQUES, 1700-1900. Edited by J. Dieudonné. I. Algèbre, analyse classique, théorie des nombres. II. Fonctions elliptiques, analyse fonctionnelle, géométrie différentielle, topologie, probabilités, logique mathématique. Sous la direction de J. Dieudonné, de l'Institut. Avec la collaboration de P. Dugac, W. J. et Fern Ellison, J. Guérindon, M. Guillaume, G. Hirsch, C. Houzel, Paulette Libermann, M. Loève, et J.-L. Verley. Paris (Hermann), 1978. x + 385, vii + 472.

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The scholars who publish their works under the pen name of N. Bourbaki have paid due attention to the history of mathematics. Many volumes of their *Eléments des mathématiques* (published from 1939 onward) contain more or less detailed essays on the historical development of various branches of mathematics. These essays, put together in a somewhat revised form, made up the *Eléments d'histoire des mathématiques*, first published in 1960 and reprinted in 1974. It is generally known that A. Weil and J. Dieudonné wrote the greater part of this book.

The aim of this essay review is to analyze the recent two-volume study of mathematics in the 18th and 19th centuries, edited and coauthored by Jean Dieudonné. We begin by quoting from its preface:

> *On reproche souvent à l'enseignement actuel des mathématiques son caractère prématurément abstrait: on à tendance à introduire d'emblée les notions fondamentales sous leur aspect général, qui ne paraît avoir guère de points communs avec les objets des mathématiques traditionnelles. Si cette manière de faire est souvent justifiée par la nécessité d'arriver rapidement à des théorèmes assez généraux pour être utilisables dans des contextes variés, il n'en reste pas moins que ces notions générales peuvent être mieux comprises si l'on est conscient de leur origine et de la façon dont elles ont évolué à partir de concepts plus particuliers, mais plus proches de l'intuition.*

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> *Le présent ouvrage vise à faciliter cette compréhension en replaçant les notions les plus élémentaires des mathématiques contemporaines dans leur contexte historique, tant en ce qui concerne leur évolution interne que leurs rapports avec les problèmes posés par les applications des mathématiques aux sciences de la nature. On y retrace le développpement des principaux concepts et résultats dans les diverses branches des mathématiques durant la période qui va de 1700 à 1900 environ.* [The authors, however, do not strictly adhere to these limits.]
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> *Le choix de cette période est justifié tout d'abord par le fait que c'est seulement à la fin du dix-septième siècle que sont mis en place les outils fondamentaux qui ont dominé depuis lors toutes les techniques mathématiques: le Calcul infinitésimal et la méthode des corrdonnées cartésiennes, portant en germe la fusion de l'Algèbre, de la Géométrie et de l'Analyse qui caractérisera la mathématique de notre époque.* [Dieudonné 1978, I, ix]

The book is intended for readers whose background is equivalent to the first two or three years of mathematical study at a university. For this reason the authors have left out a number of topics, such as algebraic geometry, the spectral theory of operators, the ergodic theory, the theory of Lie groups, application of distributions to the theory of partial differential equations, and important branches of topology. The fact that the book is not restricted to the university program is a virtue rather than a shortcoming. And still, on the one hand, there are many sections which a third-year student, or even an ordinary graduate of a mathematical faculty, will hardly understand. On the other hand, in many instances, when no new subtle concepts had to be introduced, the authors have been able to overstep the limits of the 19th century and carry their exposition forward in some cases almost to the present time. And, for the benefit of those who wish to extend and broaden their knowledge of the subject still further, each of the thirteen chapters includes a list of references both to original (or, generally speaking, historical) sources and to textbooks, and the authors properly use their bibliographies for the purpose of documentation.

In his Preface and Introduction, Dieudonné states his concern for the principles of exposition and makes known his views on mathematics in general, as well as the mathematics of the period dealt with in the book. A special paragraph is devoted to the *communauté mathématique,* i.e., university education, including means and forms of personal contacts, and the emergence in the 19th century of mathematical schools in Paris, Berlin, and Göttingen, and, somewhat later in the century, in Petersburg and some Italian cities. Speaking about the 20th century, Dieudonné notes the loss suffered by mathematics in France during the military operations of World War I and, some fifteen or twenty years later, in Germany, where the Nazi régime all but quenched any mathematical activities. At the time, many German scholars escaped to the United States to the great advantage of American science (and mathematics in particular).

The author also points out a highly remarkable event, viz., the rise of powerful schools in countries which formerly supported only a comparatively small number of scholars of international caliber:

> *Dès avant la fin de la première guerre mondiale, il faut d'abord citer l'U.R.S.S. et la Pologne, d'où surgit brisquement une pléiade de mathématiciens de premier ordre (Lusin, Souslin, puis Urysohn, P. Alexandrov, Kolmogorov, Vinogradov, Pontrjagin, Petrowski, Gelfand en U.R.S.S.; Sierpinski, Janiszewski, Kuratowski, Banach, puis Hurewicz, Eilenberg, Zygmund, Schauder en Pologne); c'est à leur efforts que l'on devra surtout le développement des fondements de la Topologie et de l'Analyse fonctionnelle modernes. En U.R.S.S., l'élan ainsi donné ne s'arrêtera pas, et a continué à produire de très nombreux mathématiciens de grand valeur; quant à la Pologne, dont la moitié des mathématiciens ont été massacrés par les nazis, elle n'a commencé que récemment à combler ses vides et reprendre sa marche en avant.* [Dieudonné 1978, I, 8]

Two remarks seem to be justified. First, Dieudonné does not mention several Soviet scholars no less outstanding. Second, while he sometimes describes the general social conditions for the rise or decline of scientific schools, in other cases he does not follow up this subject at all. Thus, Dieudonné does not say that the national revival of Italy in the second half of the last century and of Poland after World War I was the prerequisite for a rapid upsurge of science in these countries. Nor does he note the most profound social and cultural reorganization which took place in the former Russian Empire as a result of the Great October Socialist Revolution and which, in particular, brought about the flourishing of the Soviet mathematical school.

Dieudonné goes on to describe the development of mathematical traditions in the United States, the maturation of the American school and its prosperity, especially after 1933, when many European scholars moved to America. E. H. Moore, L. E. Dickson, and W. F. Osgood originated this school with G. D. Birkhoff, O. Veblen, J. W. Alexander, S. Lefschetz (of Russian extraction, but living in the United States from childhood), and M. Morse following in their steps.

The author also reports the ripening of the Japanese mathematical school, a process which accelerated from the end of World War II onward despite difficulties experienced by universities in Japan and the ensuing emigration of a number of Japanese mathematicians.

At present, as Dieudonné states in the Preface, only the Soviet, American, and Japanese schools are sufficiently numerous to include representatives of each branch of mathematics [Dieudonné 1978, I, 8-9]. Other nations have had to restrict themselves at one time or another to isolated areas in one direction or another.

Dieudonné more than once formulates his opinion on the motive forces for the advancement of mathematics. We quote two closely linked passages which distinctly explain his general idea. The former is preceded by examples of the connection of mathematics with problems posed by mechanics, astronomy, physics, etc. The view expressed in the latter passage is of utmost importance, having exerted a dominant influence on the subject-matter and exposition of the entire book. Here, then, are the passages:

> *(1) Mais ceci admis, il n'en reste pas moins que les domaines des mathématiques qui sont ainsi en contact permanent et fructueux avec les applications aux sciences de la nature sont loin, malgré leur importance, de constituer la majorité des branches des mathématiques actuelles.* [Dieudonné 1978, I, 9-10]

> *(2) Il semble que l'on a une vue assez juste du développement des mathématiques en considérant que son principal moteur est d'origine* interne, *la réflexion approfondie sur la nature des problèmes à résoudre, sans que l'origine de ces derniers exerce beaucop d'influence.* [Dieudonné 1978, I, 11]

We think that Dieudonné's conception is one-sided and therefore somewhat erroneous. Bearing in mind the aim of this review, we are not here calling into question his point of view. However, we have repeatedly formulated our own understanding of the subject elsewhere [1].

We now proceed to the discussion of the separate chapters of the *Abrégé*. We do not discuss all the chapters to the same extent; the unevenness of our exposition more or less corresponds to the unevenness with which various topics are treated in the book. Note that among its eleven authors one (Dugac) is a professional historian of science, while the rest are mathematicians extremely well acquainted with the history of the corresponding problems.

Chapter I, "Mathematical Analysis in the 18th Century" (by J. Dieudonné), is not a systematic account but rather an essay, divided into several sections, on the development of some important and typical branches of analysis of the century (mostly from 1725 onward). The reader will find essential sections devoted to functions of complex variables and elliptic integrals in the 18th century, Chapters IV and VII, respectively.

In a sense, the author acquits, and quite properly so, procedures used by the analysts of those times who lacked exact definitions of some fundamental concepts and had to make up for this deficiency by sound and clear intuition [2]. Dieudonné discusses examples of the treatment of infinite series by analysts of the 18th century.

Leaving this preliminary section on "Rigour and formalism," he discusses functions of large numbers, the Euler-Maclaurin summation formula, first studies of trigonometric series, infinite continued fractions, important special functions, methods of solving ordinary and partial differential equations, calculus of variations, and numerical methods.

Dieudonné's skillful exposition of this large number of problems occupies only 34 pages. Even so, he leaves out several important topics. In particular, he does not take up the attempts at providing more rigorous foundations for analysis in the 18th century, so that P. Dugac, who wrote Chapter VI on the foundations of analysis (see below), had to refer not to this Chapter I but rather to Bourbaki's *Eléments d'histoire des mathématiques*.

We note one inaccuracy: it is not true that the relation between exponential and trigonometric functions in the complex domain is due to Euler [Dieudonné 1978, I, 32]; Cotes formulated this relation in his *Harmonia Mensurarum* published posthumously in 1722. In general, however, Dieudonné's chapter, as well as the book as a whole, is almost free of such insignificant and unavoidable mistakes and they do not affect the very high level of exposition.

Chapters II and III, treating algebra and geometry before 1840 and algebra after 1840, are written by J. Guérindon and J. Dieudonné in the style of Bourbaki's *Eléments*. Essentially they are an elaboration and generalization of the corresponding sections of the latter. The manner of writing in these (but not only in these) chapters is borrowed from mathematics proper, and yet is intended to suffice for historical studies as well. The unimpeachably strict and distinct exposition which thus emerges is peculiar to the French mathematical school.

The division of the history of algebra into two periods, 1700-1840 and 1840-1900, adopted by the authors, seems somewhat dubious. They explain that before 1840 algebra was mainly a science restricted to the solution of equations, whereas, beginning with the 1840s, new ideas and objects of research penetrated the field as a result of the works of W. R. Hamilton, A. Cayley, H. Grassmann, and G. Boole. These brought about a new understanding of algebra as being "a science of operations performed on abstract objects" [Dieudonné 1978, I, 93]. However, the most important object studied by algebra, the group, first appeared in the works of Lagrange in 1770-1771 and was subsequently investigated by P. Ruffini, C. F. Gauss, N. H. Abel, A. L. Cauchy, and E. Galois. Therefore, if ideas and methods are to be given priority over applications in the separation of consecutive periods, then 1770 would be the beginning of the new period. Exactly 1770, rather than 1840, was the turning point in the evolution of algebra.

Especially interesting and abundant in their subject-matter are the sections on linear and multilinear algebra. Here the authors discuss the introduction of the concepts of vectors, matrices, quaternions, and hypercomplex numbers, as well as the emergence of new calculi. Also in the same sections is a fine exposition of the theory of invariants. Our only objection concerns the absence of Hilbert's second main theorem, an omission which might well create a distorted notion about his general contribution to this theory.

Generally speaking, both chapters contain much new and important information, and mathematicians and historians of science would be interested in, and profit from, reading them.

Chapter IV is closely connected with Chapter VII and we discuss them together (see below).

Chapter V, "Theory of Numbers" (W. J. and Fern Ellison), the longest chapter in the book (it comprises almost 170 pages), covers the period from 1700 to the 1960s. Its authors study both algebraic and analytic number theory and, also, the geometry of numbers, Diophantine analysis, and Diophantine approximations.

Displaying considerable methodological skill, the authors first introduce such concepts and objects as are needed for readers who have graduated from technical colleges. Then they illustrate their general propositions and definitions by appropriate numerical examples. Last, describing the works of mathematicians of the 18th and the beginning of the 19th century, they try to use the language of those times rather than subsequent mathematical terminology. However, the explication suffers from essential shortcomings. Important topics are lacking, while inaccuracies and unsubstantiated opinions appear occasionally. Also, the authors study mainly the achievements of the French mathematical school, leaving aside comparable work of German and Russian mathematicians, and thus somewhat corrupting the general historical picture.

For example, the authors obviously underestimate the importance of Gauss' work. They dismiss his first proof of the law of quadratic reciprocity as "at least unreadable" [Dieudonné 1978, I, 176]. They erroneously attribute the main achievements in cyclotomy to Vandermonde, and they call Gauss' proofs of the theorems on quadratic forms "*horribles*" [Dieudonné 1978, I, 222] only because Hilbert, working a hundred years later and using the language of quadratic fields, managed to demonstrate these theorems in a version that was ten times shorter.

The first proof of the law of quadratic reciprocity is especially meaningful. It was exactly this proof that recently enabled J. Tate to solve one of the currently most interesting problems of homological algebra, the calculation of the group $K_2(Q)$. As to the theory of cyclotomy, a correct historical account of Gauss' and Vandermonde's studies is given in Chapter II of the *Abrégé* [Dieudonné 1978, I, 73-74]. The approach of Chapter V, however, founded on a comparison of the lengths of proofs due to Gauss and Hilbert, is absolutely antiscientific. Indeed, how much more *horrible* were the proofs given by Archimedes! We also note that the authors attribute to Legendre Euler's criterion for the determination of quadratic residues [Dieudonné 1978, I, 170].

As to lacunas, of all the members of the celebrated Russian number-theoretic school of the 19th century, the authors only give an account of P. L. Chebyshev; E. I. Zolotarev is referred to only on page 7 of the Introduction; A. A. Markov and G. F. Voronoy are not mentioned at all. Given the contemporary level of research in the history of mathematics, such omissions are inadmissible.

It seems that the authors are somewhat better informed about Soviet mathematicians of the present, but even in this respect their exposition is rather fragmentary. Thus having considered Hilbert's ninth problem, they fail to say that the Soviet mathematician I. R. Shafarevitch solved it in 1950. There are also other minor inaccuracies in the chapter.

As a whole, this chapter, despite its one-sidedness, which, we hope, will be overcome in later editions, is very interesting. It will undoubtedly become a valuable aid for those studying the theory of numbers of the last century.

Chapter VI, "Foundations of Analysis" (P. Dugac), is the last chapter of the first volume. It does not overstep the limits of the 19th century. Containing historical information quite sufficient for a university education, it might be read easily and beneficially, not only by a student-mathematician, but even by students in technical colleges after two years of study.

Compactness of exposition enabled the author to discuss practically all the main historical problems concerning functions of one variable in a comparatvely short chapter (58 pages). Starting with the first steps of the reform in the foundation of analysis, i.e., from the basic works of Gauss, Bolzano, Cauchy, and Abel, he goes on to consider trigonometric series and more subtle criteria of convergence, the theory of integration before Riemann, Riemann's theory, the theory of real numbers, "Weierstrassian rigorism," elements of set theory and general topology, the theory of measure, and, last, various theories of the real numbers.

Describing the history of set theory, Dugac stresses the importance of Dedekind's ideas (he begins the corresponding section of the chapter with Dedekind), and he notes the difference between the approaches used by the latter (algebra and theory of numbers) and by Cantor (trigonometric series). F. A. Medvedev studied this problem in detail, and a reference to him on page 373 of the *Abrégé* would have been quite appropriate. Also, Medvedev's book, published in 1965, is regrettably not included in the list of references: Ф. А. Медведев. Развитие теории множеств в XIX веке *(The Development of Set Theory in the 19th Century)* Москва, 1965.

The last section of the chapter devoted to the theory of integers seems somewhat out of place. Bearing in mind the general structure of Chapter XIII (see below), this section should have been included there, the more so because this chapter also describes the axiomatic construction of the arithmetic of integral numbers [Dieudonné 1978, II, 332-333].

We now return to Chapter IV, "Analytic Functions" (J.-L. Verley). Beginning with an essay on the prehistory of the general theory of analytic functions which dates back to the 18th century, the author takes a look at the relevant contributions of Cauchy, Riemann, and Weierstrass. As a rule, he does not supply more historical information than is already included, sometimes even in more detail, in generally known treatises on the theory of analytic functions, as, for example, in A. Dinghas' *Vorlesungen über Functionentheorie* (Springer-Verlag, 1961).

Verley carries his exposition to the 1870s, and we must make one point here. Discussing the first edition of Briot's and Bouquet's *Théorie des fonctions doublement périodiques* ... (1859) the author asserts that it contains an "improper formulation of a statement which subsequently became Picard's theorem" [Dieudonné 1978, I, 149]. This, of course, is a misunderstanding. What is actually meant is an "improper formulation" of a theorem due to Weierstrass. Neither Briot and Bouquet, nor their contemporaries, distinguished between the value of a function at a point and its limiting value at a (singular) point. Picard's problem, i.e., the study of the behavior of a function in the neighborhood of an essential singularity, could have been raised only when the distinction just mentioned was completely understood.

The historical sequence of events in the exposition ends with Mittag-Leffler's theorem on the representation of a meromorphic function by the sum of its principal parts. The rest of the chapter is given over to the theory of analytic functions of several complex variables. Without mentioning Osgood's classical *Lehrbuch der Funktionentheorie*, Bd. 2, 1 (first edition, 1924), which summed up a semicentennial period in the evolution of the theory, a period marked by the works of K. T. W. Weierstrass, H. Poincaré, P. Cousin, F. Hartogs, E. E. Levi, and Osgood himself, Verley begins straightaway with H. Behnke and P. Thulen's *Theorie der Funktionen mehrer komplexen Veränderlichen* (Berlin, 1934), calling it the first monograph in the field. He then formulates some problems peculiar to the theory, for example, problems on the domain of holomorphy and Cousin's problems and propositions on conformal mapping. The author also describes the most important relevant achievements, mainly those pertaining to the period before the 1950s.

However, the *Abrégé* as a whole is much richer in materials on the development of ideas and methods of the theory of analytic functions than is its Chapter IV, and in this connection we turn our attention to the extensive commentary in Chapter VII, "Elliptic Functions and Abelian Integrals" (C. Houzel). With its 112 pages, it is more than three times longer than Chapter IV. The author succeeds in his difficult task of combining a comprehensive description of a field which boasts an immense literature, one going back for more than two hundred years, with an utterly distinct exposition, and yet this study of the historical process by which the ideas and methods of the theory of analytic functions was framed is not sufficiently vivid. As we see it, the situation would have been different had not the general theory been somewhat separated from those branches of analysis which used and perfected methods worked out by the theory of functions and from which even new branches detached themselves. In the 19th Century, the most important of these branches was the theory of elliptic functions and Abelian integrals. Mathematicians of the time understood perfectly well the connections between the general theory of analytic functions and the separate branches of analysis. An external manifestation of this fact is seen in university courses and monographs written in the second half of the 19th century: as a rule, the theory of elliptic functions (or Abelian integrals) was then studied together with the elements of the general theory. However, we do not direct our reproach, or, rather, regret, to C. Houzel. As the author of Chapter VII, he did all he could do to reveal the guiding ideas.

The chapter is divided into two parts devoted to elliptical functions and Abelian integrals, respectively, with a short introduction listing the main landmarks, including Euler's acquaintance with Fagnano's memoirs (1751) and his subsequent discovery of the addition theorem for elliptic integrals, the appearance of Abel's and Jacobi's first publications (1827), Riemann's memoir (1857), and Poincaré's study of Fuchsian functions (1884 and later). The main body of the chapter, however, is arranged according to separate topics which are important in themselves, irrespective of the landmarks just mentioned. The history of elliptic integrals, elliptic functions, and theta functions of one variable, as well as an essay on the history of modular and automorphic functions, is included in the first part of the chapter. Also discussed here are the applications of these functions to the solution of equations (the modular equations), to the study of algebraic curves of genus 1, and to mechanics, differential geometry, and the theory of numbers.

The second part is given over to Abel's theorem, to Jacobi's inversion problem and achievements due to Rosenhain and Göpel, to the corresponding problem of division of elliptic integrals, and to works of Weierstrass on hyperelliptic integrals and of Riemann and again Weierstrass on Abelian integrals. The ending sections of the chapter are devoted to the application of the Riemannian theory of Abelian integrals to the study of algebraic curves (from Clebsch and Gordan to Poincaré), and to a very short account of the theory of Abelian functions and manifolds (up to the middle of this century).

Chapter VIII, "Functional Analysis" (J. Dieudonné), is a very comprehensive essay on the development of disciplines and of ideas pertaining to analysis, function theory, algebra, etc., the synthesis of which at the turn of the 19th century led to the establishment of functional analysis. The author begins with a general description of the prehistory of functional analysis, and then discusses in consecutive order local existence theorems for ordinary and partial differential equations from Cauchy to Sophie Kovalevskaya, including some additions due to H. Lewy (1956), Pfaffian systems, the analytic theory of linear differential equations from Riemann and Fuchs to Poincaré and Birkhoff, and nonlinear equations in the complex domain up to Painlevé and Gambier. Dieudonné next studies Poincaré's qualitative theory of differential equations and Ljapunov's theory of stability. His remarks on the latter should have been extended somewhat to allow for achievements due to the Soviet school of the theory. In a separate section devoted to Hamiltonian systems the author again discusses the work of Poincaré, this time on the problem of three bodies which is closely interwoven with studies made by Hill and continued by Birkhoff. The section ends with a description of a new method for studying Hamiltonian systems due to A. N. Kolmogorov, whose discoveries were developed by V. I. Arnold and J. Moser. Regrettably, Dieudonné did not say that the works of the latter three authors originated from long-standing efforts of astronomers to overcome the difficulties connected with the so-called small denominators in asymptotic expansions used in celestial mechanics.

Dieudonné goes on to describe the enormous body of work devoted to the solution of the three main types of linear boundary value problems in mathematical physics, and, also, Fourier's achievements in the theory of heat, the Sturm-Liouville problem, and methods due to Poincaré, C. Neumann, and Schwarz. As we have indicated above, the author as a rule pays due attention to achievements of Russian and Soviet mathematicians. Thus, he mentions that Bouniakovsky was the first to publish the integral inequality which often goes under Schwarz' name [Dieudonné 1978, II, 148]. Still, Dieudonné does not mention several important works of Russian methematicians. Thus, he fails to point out that A. M. Ljapunov essentially improved the classical methods of solving boundary value problems and that V. A. Steklov introduced his *théorie de la fermeture*. The latter achieved expansions of very general classes of functions into series of eigenfunctions of a linear operator determined by a given equation. The situation changed only when the theory of integral equations (unused by Steklov) advanced to a high level.

Dieudonné also traces the origins of developments in functional analysis which were brought about at the end of the 19th century by Volterra's work on the calculus of variations and studies of integral equations due to Volterra and Fredholm. These were followed by Hilbert, E. Schmidt, and Fréchet, who accomplished the transition from "algebra of the infinite" to "geometry of the infinite." The titles of the next sections speak for themselves: "Metric spaces"; "Normed spaces and spectral theory." Dieudonné concludes with a sketch of several developments of the last fifty years, such as the theory of distributions originated by S. L. Sobolev in 1937 and the theory of normed algebras studied by I. M. Gelfand from 1941 onward.

Dieudonné omits several topics; for example, the calculus of variations in the 19th century is all but ignored. He also fails to name some active participants in the creation of functional analysis. The importance of this chapter in which clarity of exposition competes with conciseness (it is only 63 pages long) will undoubtedly compel many readers to study the literature included in the appended list of references.

Chapter IX, "Differential Geometry" (Paulette Libermann), describes the history of this discipline. After a short introduction devoted to the 17th and 18th centuries, the author describes Gauss' intrinsic geometry of surfaces, the evolution of the theory of superposition of surfaces, and the study of surfaces of constant curvature. This, of course, leads to a discussion of the interpretation of hyperbolic geometry, the discovery of which is treated in Chapter XIII. Libermann then discusses the origin of Riemann's n-dimensional geometry and the creation of tensor analysis by Ricci and Levi-Civita, and she ends by studying E. Cartan's theory of exterior differential forms. We note that she supplies no information on the work of K. M. Peterson and does not refer to the generally known source on the history of differential geometry published by D. J. Struik.

Chapter X, "Topology" (G. Hirsch), occupies a somewhat special place in the *Abrégé*. Beginning with a concise review of topology from Euler to the end of the 19th century, the author adduces a number of interesting facts and remarks on the development of combinatorial and algebraic topology, mainly concerning events in this century. Sections devoted to Hopf and even to more recent research are the most interesting. However, set-theoretic topology is to all practical purposes left out.

Paracompactness is about as much as the reader will find of general topology, and even it is discussed in complete isolation from such fundamental issues as the problem of metrization, the main subject which makes paracompactness important! Hirsch does not mention any noteworthy achievements connected with paracompactness, as, for example, the quite profound studies of A. Stown, E. Michel, and V. I. Ponomarev. He also leaves out the topology of continuums. The reader will find nothing here about pseudo-arcs and their topological uniqueness (Bing's theorem), or about locally connected continuums and the reason why they are called Jordan's and Peano's continuums.

Absolutely lacking as well is geometric topology. Retracts and shapes are not treated. Hirsch does not even mention K. Borsuk or for that matter any other Polish mathematician. Taken in itself, this single fact is a telltale proof of the imperfection of this exposition, even though the author considers algebraic topology to be his main subject [3].

Chapter XI, "Integration and Measure" (J. Dieudonné), the shortest chapter of the book--only ten pages long--is devoted to Lebesgue and Stieltjes integrals and to the theory of measure. In essence, it serves as an introduction to the chapter which follows.

In Chapter XII, "Calculus of Probabilities" (M. Loève), regrettably only six pages are devoted to the theory of probability in the 17th through 19th centuries. As an unavoidable consequence, the account is of little interest. Even so, Loève presents a correct description of the "classical" period in the formation of the theory, i.e., the period from J. Bernoulli to Chebyshev. Works which appeared at the beginning of the 20th century constituted a natural completion of the classical period and in this connection Loeve should have mentioned S. N. Bernstein.

The author devotes the next 28 pages to the new period of the development of probability theory, a period characterized by general use of the theory of measure and by the dominant position of stochastic processes and random fields as concrete objects of research. Loève has written an interesting appraisal of the contemporary problems concerning the theory of probability in a masterly fashion. We recommend it to every young mathematician eager to study the contemporary theory. However, being an able specialist in the subject, the author is interested mainly in Markov processes and he scarcely traces the spectral theory of stationary processes. Nothing at all is to be found about problems linked with random fields or about connections of the theory of probability with contemporary statistical physics.

During the 19th century mathematical statistics was nothing but a collection of applications of probability theory to isolated practical problems. However, in the 20th century it developed into a vast science with its own robust mathematical backbone, and the omission of this aspect of the theory from Loève's account is disappointing.

Chapter XIII, "Axiomatics and Logic" (M. Guillaume), contains four main sections following a short introduction (Section 1); Section 2, "The formation of the axiomatic method in the 19th century"; Section 3, "The development of formalization and the understanding of its importance at the end of the 19th century"; Section 4, "Mathematical logic in the 19th century"; Section 5, "The grand ideas of the 20th century."

Written in an inventive manner--and lengthy (116 pages)--the chapter makes interesting reading. However, the author's desire to dispose of the 18th and 19th centuries in order to reach the 20th century as quickly as possible is somewhat unfortunate. Section 2 is devoted mainly to the axiomatics of geometry; historically speaking, this approach is justified. The section begins with a concise explication of the theory of parallels and of the discovery of non-Euclidean geometries [4]. From a general point of view the inclusion of Sections 2 and 3 before those devoted to logic in the proper sense create a broad outlook and might be considered a fortunate attempt. Sections 4 and 5 briefly indicate sources of mathematical logic which date back to the 19th century.

To summarize, the *Abrégé d'histoire des mathematiques, 1700-1900,* is largely devoted to the evolution of the chief directions of mathematics of the 19th, and in many instances, of the 20th, century, and this in itself constitutes its undeniable value. Before its publication there existed only one book concerned exclusively with mathematics of the 19th century, the first volume of F. Klein's splendid *Vorlesungen über die Geschichte der Mathematik im 19. Jahrhundert.* However, Klein compiled these *Vorlesungen* in the interval 1914-1919; and R. Courant and O. Neugebauer published them in 1926. They conveyed the outlook attained by mathematicians by the end of the 19th century when many trends now prevalent in the development of algebra, geometry, and function theory were still rather vague. A reappraisal of the evolution of mathematics during the 19th (and part of the 20th) century from a current point of view late in the 1970s has long been needed.

The authors of the *Abrégé* have isolated very skillfully those problems, ideas, and methods which have led to the establishment of contemporary mathematics. However, they have not tried to attune themselves to the intellectual world of mathematicians of the 19th century to determine what guided them in their selection of problems or methods of research. The authors discuss processes studied many generations ago almost exclusively from the point of view of contemporary 20th-century mathematics, and, moreover, from a position peculiar to those who write under the name of N. Bourbaki. This circumstance imparts some subjectivism and one-sidedness to the selection, distribution, and description of historical facts throughout all their work. Of course, historical research is fraught with divergence in the interpretation of many problems and we do not profess to know the objective historical truth. But, as we have stressed above, our understanding of the developmental process of mathematics and our interpretation of a number of particulars do not coincide with those prevalent in the *Abrégé* in a number of instances.

As also stated above, the principles which J. Dieudonné sets forth in the Preface and Introduction have predetermined the general structure of the *Abrégé*. Correspondingly, the authors of this book were interested mainly in theories least connected with applications.

The reader will search in vain for the history of mathematics in its connections with natural science or other aspects of social activity; isolated particular references to links of this kind do not change the general picture. As expounded in the Introduction, the authors intended to treat the main concepts of contemporary mathematics not only from the point of view of their intrinsic evolution but also in their connections with natural science. This aim remains unfulfilled. Except for a few words in the introduction, the text also ignores problems of education, organization of research, or the history of scientific schools. Although many scholars are mentioned in the book, this is done primarily for the purpose of designating theories, concepts, and theorems. People as such, their interrelations and specific working conditions, are just as lacking. Reading the book, we were unable to trace the connections between the mathematical ideas of individual scholars and their general views; we learned nothing of what psychological inertia they sometimes had to overcome, or how violent the battles fought between various directions of mathematical thought actually were. One cannot help but regret the absence of such human aspects in the *Abrégé,* which in this respect distinguishes itself unfavorably from Klein's *Vorlesungen.* Note, however, that the *Abrégé* ends with a short guide [Vol. 2, pp. 431-459] including nearly 250 savants mentioned in the book, with references, where possible, to more comprehensive biographical dictionaries.

There is no index of names, a fact which is extremely disappointing. To make up for this, the subject index, frequent cross-references in the main text, and detailed lists of contents at the beginning of each chapter help somewhat to guide the reader through the book, while the references accompanying each chapter provide suggestions for additional reading.

Actually, the *Abrégé* is not a unified entity. It breaks up into essays on the development of isolated mathematical sciences, essays which supplement (and sometimes overlap) each other. Some topics are treated more than once, but not always in the same manner. Not infrequently a panorama of the 20th century is substituted for the history of the evolution of one or another branch of mathematics in the 18th and 19th centuries. The history of mathematics as a whole in the last hundred and fifty years is still unwritten. Moreover, the kind of disintegration reflected in the structure of the *Abrégé* seems unavoidable. But the separate essays do constitute an essential step toward the compilation of the history as a whole.

The publication of the *Abrégé d'histoire des mathématiques, 1700-1900* is a significant event in both mathematical and historical-mathematical literature. Not only students, but scholars of every age, will benefit from reading it, and we hope that it will induce many scholars to participate in research on the history of modern mathematics.

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NOTES

1. See, for example, A. N. Kolmogorov's article "Mathematics" in any of the three editions of the *Greater Soviet Encyclopedia,* or the editors' "Preface" to Математика XIX века *(Mathematics of the 19th Century)*, eds, A. N. Kolmogorov and A. P. Youshkevitch (Moscow, 1978).

2. G. H. Hardy, in his generally known book on divergent series (1949), adhered to this opinion even before Dieudonné.

3. This analysis of Chapter X of the *Abrégé* was written by P. S. Alexandrov.

4. Note, however, that the non-Euclidean geometry has its own concrete subject-matter, and it is unfortunate that the authors of the *Abrégé* found no place for a coherent description of the history of this important science.