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## TOUGH GRAPHS AND HAMILTONIAN CIRCUITS

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**Abstract.** The toughness of a graph  $G$  is defined as the largest real number  $t$  such that deletion of any  $s$  points from  $G$  results in a graph which is either connected or else has at most  $s/t$  components. Clearly, every hamiltonian graph is 1-tough. Conversely, we conjecture that for some  $t_0$ , every  $t_0$ -tough graph is hamiltonian. Since a square of a  $k$ -connected graph is always  $k$ -tough, a proof of this conjecture with  $t_0 = 2$  would imply Fleischner's theorem (the square of a block is hamiltonian). We construct an infinite family of  $(3/2)$ -tough nonhamiltonian graphs.

### 0. Introduction

In this paper, we introduce a new invariant for graphs. It measures in a simple way how tightly various pieces of a graph hold together; therefore we call it toughness. Our central point is to indicate the importance of toughness for the existence of hamiltonian circuits. Every hamiltonian graph is necessarily 1-tough. On the other hand, we conjecture that every graph that is more than  $\frac{3}{2}$ -tough is necessarily hamiltonian. This conjecture, if true, would strengthen recent results of Fleischner concerning hamiltonian properties of squares of blocks.

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We follow Harary's notation and terminology [11] with minor modifications. First of all, by a subgraph we always mean a spanning subgraph. Accordingly,  $G \subset H$  means that  $G$  is a spanning subgraph of  $H$ . As in [11],  $p(G)$  denotes the number of points,  $k(G)$  the number of com-

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ponents,  $\kappa(G)$  the point-connectivity,  $\lambda(G)$  the line-connectivity and  $\beta_0(G)$  the point-independence number of a graph  $G$ . By a *point-cutset* (resp. *line-cutset*) in  $G$  we mean a set  $S$  of points (resp. a set  $X$  of lines) of  $G$  whose removal results in a disconnected graph, i.e., for which  $k(G - S) > 1$  (resp.  $k(G - X) > 1$ ).

## 1. Toughness

Let  $G$  be a graph and  $t$  a real number such that the implication  $k(G - S) > 1 \Rightarrow |S| \geq t \cdot k(G - S)$  holds for each set  $S$  of points of  $G$ . Then  $G$  will be said to be *t-tough*. Obviously, a *t-tough* graph is *s-tough* for all  $s < t$ . If  $G$  is not complete, then there is a largest  $t$  such that  $G$  is *t-tough*; this  $t$  will be called the *toughness* of  $G$  and denoted by  $t(G)$ . On the other hand, a complete graph contains no point-cutset and so it is *t-tough* for every  $t$ . Accordingly, we set  $t(K_n) = +\infty$  for every  $n$ . Adopting the convention  $\min \emptyset = +\infty$ , we can write

$$(1) \quad t(G) = \min |S|/k(G - S),$$

where  $S$  ranges over all point-cutsets of  $G$ .

Using the obvious implication  $G \subset H \Rightarrow k(G) \geq k(H)$  and the definition of toughness we arrive at:

**Proposition 1.1.**  $G \subset H \Rightarrow t(G) \leq t(H)$ .

The toughness is a nondecreasing invariant whose values range from zero to infinity. A graph  $G$  is disconnected if and only if  $t(G) = 0$ ;  $G$  is complete if and only if  $t(G) = +\infty$ .

For every point-cutset  $S$  of  $G$ , we have  $|S| \geq \kappa(G)$  and  $k(G - S) \leq \beta_0(G)$ . Using (1), we readily obtain:

**Proposition 1.2.**  $t \geq \kappa/\beta_0$ .

If  $G$  is not complete (i.e.,  $\kappa \leq p(G) - 2$ ), then  $G$  has at least one point-cutset. Substituting the smallest point-cutset  $S$  of  $G$  into the right-hand side of (1), we derive:

**Proposition 1.3.** *If  $G$  is not complete, then  $t \leq \frac{1}{2}\kappa$ .*

Similarly, taking  $S$  to be the complement of a largest independent set of points of  $G$ , we deduce:

**Proposition 1.4.** *If  $G$  is not complete, then  $t \leq (p - \beta_0)/\beta_0$ .*

If  $G = K_{m,n}$  with  $m \leq n$ , then obviously  $\kappa(G) = m$ ,  $\beta_0(G) = n$  and  $p(G) = m + n$ . Combining Propositions 1.2 and 1.4, we obtain:

**Proposition 1.5.**  $m \leq n \Rightarrow t(K_{m,n}) = m/n$ .

Hence the equality in Propositions 1.2, 1.4 can be attained. In order to show that the equality in Proposition 1.3 can be attained as well, we shall prove:

**Theorem 1.6.**  $t(K_m \times K_n) = \frac{1}{2}(m + n) - 1 \quad (m, n \geq 2)$ .

**Proof.** Let  $S$  be a point-cutset of  $G = K_m \times K_n$  minimizing  $|S|/k(G - S)$ ; let us set  $k = k(G - S)$ . Then  $S$  is necessarily minimal with respect to the property  $k(G - S) = k$ . The point-set of  $G$  will be written as  $V \times W$  with  $|V| = m$ ,  $|W| = n$ . From the minimality of  $S$ , we easily conclude that the point-set of the  $j^{\text{th}}$  component of  $G - S$  is  $V_j \times W_j$  with  $V_j \subset V$  and  $W_j \subset W$ . Moreover,  $V_i \cap V_j = \emptyset$  and  $W_i \cap W_j = \emptyset$  whenever  $i \neq j$ . Thus, we have

$$(2) \quad |S| = mn - \sum_{i=1}^k m_i n_i,$$

where  $m_i = |V_i|$  and  $n_i = |W_i|$  for each  $i = 1, 2, \dots, k$ . The right-hand side of (2) is minimized by  $m_1 = m_2 = \dots = m_{k-1} = 1$ ,  $m_k = m - k + 1$  and  $n_1 = n_2 = \dots = n_{k-1} = 1$ ,  $n_k = n - k + 1$ . Hence

$$\begin{aligned} |S| &\geq mn - (k-1) - (m-k+1)(n-k+1) \\ &= (k-1)(m+n-k), \end{aligned}$$

and so

$$t(G) = |S|/k(G-S) \geq (k-1)(m+n-k)/k \geq \frac{1}{2}(m+n-2).$$

The opposite inequality follows from Proposition 1.3 as  $G$  is regular of degree  $m+n-2$ .

Propositions 1.2 and 1.3 indicate a relationship between toughness and connectivity. Another indication of this relationship is given by:

**Theorem 1.7.**  $t(G^2) \geq \kappa(G)$ .

**Proof.** Let  $G$  be a graph with connectivity  $\kappa$  and let  $S$  be a point-cutset in  $G^2$ . Let  $V_1, V_2, \dots, V_m$  be the point-sets of components of  $G^2 - S$ . For each point  $u \in S$  and each  $i = 1, 2, \dots, m$ , we set  $u \in S_i$  if and only if there is a point  $v \in V_i$  adjacent to  $u$  in  $G$ . Obviously, each  $S_i$  is a point-cutset of  $G$  (it separates  $V_i$  from the rest of  $G$ ). Hence

$$(3) \quad |S_i| \geq \kappa \text{ for each } i = 1, 2, \dots, m.$$

Moreover, each  $u \in S$  belongs to at most one  $S_i$ . Otherwise there would be points  $v_i \in V_i$  and  $v_j \in V_j$  with  $i \neq j$  such that  $u$  is adjacent in  $G$  to both  $v_i$  and  $v_j$ . Consequently, the points  $v_i$  and  $v_j$  would be adjacent in  $G^2$ , contradicting the fact that they belong to distinct components of  $G^2 - S$ . Thus we have

$$(4) \quad i \neq j \Rightarrow S_i \cap S_j = \emptyset.$$

Combining (3) and (4) we have

$$|S| \geq \sum_{i=1}^m |S_i| \geq \kappa m = \kappa k(G^2 - S).$$

Since  $S$  was an arbitrary set with  $k(G^2 - S) > 1$ ,  $G^2$  is  $\kappa$ -tough, which is the desired result.

**Corollary 1.8.** *If  $m$  is a positive integer and  $n = 2^m$ , then  $t(G^n) \geq \frac{1}{2}n\kappa(G)$ .*

**Proof.** We shall proceed by induction on  $m$ . The case  $m = 1$  is equivalent to Theorem 1.7. Next, if  $t(G^n) = +\infty$ , then  $t(G^{2n}) = +\infty$ . If  $t(G^n) < +\infty$ ,

then by Theorem 1.7 and Proposition 1.3 we have

$$t(G^{2n}) \geq \kappa(G^n) \geq 2t(G^n),$$

which is the induction step from  $m$  to  $m + 1$ .

Let us note that the inequality  $t(G^n) \geq \frac{1}{2}n\kappa(G)$  does not hold in general. The graph  $G$  in Fig. 1 is 1-connected but its cube  $G^3 = K_4 + \bar{K}_3$  is not  $\frac{3}{2}$ -tough. Actually,  $\beta_0(G^3) = 3$ ; using Proposition 1.4, we conclude that  $t(G^3) \leq \frac{4}{3}$ .

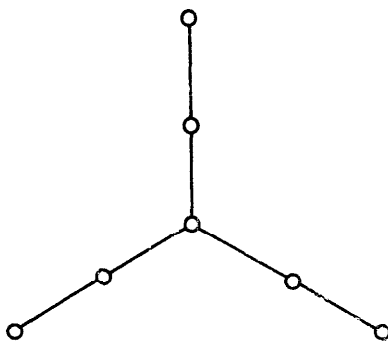


Fig. 1.

## 2. Toughness and hamiltonian graphs

It is easy to see that every cycle is 1-tough. This observation and Proposition 1.1 imply

**Proposition 2.1.** *Every hamiltonian graph is 1-tough.*

Unfortunately, the converse of Proposition 2.1 holds for graphs with at most six points only. The nonhamiltonian graph  $H$  in Fig. 2 is 1-tough. Let us note that  $H$  is a square of the graph  $G$  in Fig. 1; as  $\kappa(G) = 1$ , Theorem 1.6 yields  $t(H) \geq 1$ . Nevertheless, the graphs which are not 1-tough do play a special role among nonhamiltonian graphs. Let us say that a graph  $G$  is *degree-majorized* by a graph  $H$  if there is a one-to-one correspondence  $f$  between the points of  $G$  and those of  $H$  such that, for

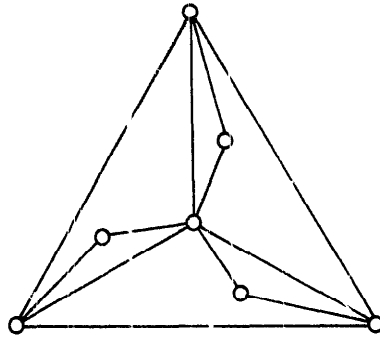


Fig. 2.

each point  $u$  of  $G$ , the degree of  $u$  in  $G$  does not exceed the degree of  $f(u)$  in  $H$ . Recently, I proved that every nonhamiltonian graph is degree-majorized by a graph which is not 1-tough [5] (in fact, by  $(\bar{K}_m \cup K_{p-2m}) + K_m$  with a suitable  $m < \frac{1}{2}p$ ). This is a strengthening of previous results due to Dirac [7], Pósa [14] and Bondy [1].

Now let us return to our Proposition 2.1. Even though its converse does not hold, one may wonder what additional conditions placed upon a 1-tough graph  $G$  would imply the existence of a hamiltonian cycle in  $G$ . As in our next conjecture, such conditions may have the flavour of Ramsey's theorem.

**Conjecture 2.2.** *If  $G$  is 1-tough, then either  $G$  is hamiltonian or its complement  $\bar{G}$  contains the graph  $F$  in Fig. 3.*

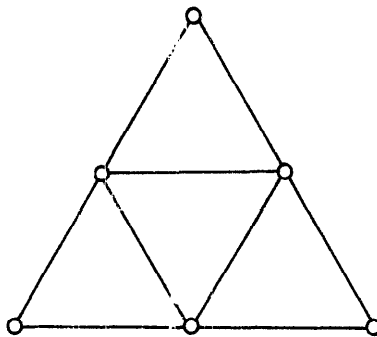


Fig. 3.

If this conjecture is true, then it is best possible in the sense that a replacement of  $F$  by any other graph  $F'$  results in a conjecture which is either weaker or false. To show this, it is sufficient to observe that the complement  $\bar{H}$  of the nonhamiltonian 1-tough graph  $H$  in Fig. 2 consists of the graph  $\bar{F}$  with an added isolated point.

As every 1-tough graph is 2-connected (see Proposition 1.3), our Proposition 2.1 is a strengthening of the obvious implication.

$$(5) \quad G \text{ is hamiltonian} \Rightarrow \kappa(G) \geq 2.$$

Even a weakened converse of (5), i.e. the implication

$$\kappa(G) \geq \kappa_0 \Rightarrow G \text{ is hamiltonian,}$$

does not hold. Indeed, the complete bipartite graphs  $K_{mn}$  with  $m < n$  are  $m$ -connected but not 1-tough (and therefore not hamiltonian) — see Proposition 1.5. However, it may well be that such a weakened converse of Proposition 2.1 holds.

**Conjecture 2.3.** *There exists  $t_0$  such that every  $t_0$ -tough graph is hamiltonian.*

It was conjectured independently by Nash–Williams [12] and Plummer [11, p. 69] that the square of every block (i.e., 2-connected graph) is hamiltonian. This has been proved only recently by Fleischner [9].

Theorem 1.7 implies that the square of every block is 2-tough. Thus a proof of Conjecture 2.3 with  $t_0 = 2$  would yield a strengthening of Fleischner's theorem. Actually, to strengthen Fleischner's theorem, it would suffice to prove the slightly weaker conjecture stated below. To formulate this one, we need the notion of a *neighborhood-connected* graph. This is a graph  $G$  such that the neighborhood of each point of  $G$  induces a connected subgraph of  $G$ . It is easy to see that the square of every graph is neighborhood-connected.

**Conjecture 2.4.** *Every 2-tough neighborhood-connected graph is hamiltonian.*

In Section 5, we shall construct  $\frac{3}{2}$ -tough nonhamiltonian graphs. The strongest form of Conjecture 2.3 for which I do not know any counterexample is the following:

**Conjecture 2.5.** *Every  $t$ -tough graph with  $t > \frac{3}{2}$  is hamiltonian.*

This conjecture is certainly valid for planar graphs. Indeed, every  $t$ -tough graph with  $t > \frac{3}{2}$  is 4-connected (Proposition 1.3) and by Tutte's theorem [16], every 4-connected planar graph is hamiltonian. By the theorem of Watkins and Mesner [17], every  $t$ -tough graph with  $t > 1$  is 3-cyclable (that is, every three points lie on a common cycle).

Recently, it has been proved that every graph with  $\kappa \geq \beta_0$  is hamiltonian [6]. Propositions 2.1 and 1.2 show how to relate this theorem to our concept of toughness. By Proposition 1.2, all graphs satisfy either  $\kappa/\beta_0 \leq t < 1$  or  $\kappa/\beta_0 < 1 \leq t$  or  $1 \leq \kappa/\beta_0 \leq t$ . By Proposition 2.1, graphs of the first kind are nonhamiltonian and, by the result of [6], graphs of the third kind are hamiltonian.

There may also be a relation between toughness and the concept of pancyclic graphs (i.e., graphs containing cycles of every length  $l$ ,  $3 \leq l \leq p$ ) introduced and studied in [2]. Actually, one can make

**Conjecture 2.6.** *There exists  $t_0$  such that every  $t_0$ -tough graph is pancyclic.*

### 3. Toughness and $k$ -factors

**Conjecture 3.1.** *Let  $G$  be a graph with  $p$  vertices and let  $k$  be a positive integer such that  $G$  is  $k$ -tough and  $kp$  is even. Then  $G$  has a  $k$ -factor.*

It follows from Tutte's matching theorem [15] that Conjecture 3.1 is valid with  $k = 1$ .

If Conjecture 2.5 is true, then every graph that is more than  $\frac{3}{2}$ -tough has a 2-factor. Actually, I even do not know any counterexample to the following:

**Conjecture 3.2.** *Every  $\frac{3}{2}$ -tough graph has a 2-factor.*



If this conjecture is true, then it is certainly the best possible as the following set of examples shows.

**Theorem 3.3.** *Given any  $t < \frac{3}{2}$ , there is a  $t$ -tough graph having no 2-factor.*

**Proof.** Let  $t < \frac{3}{2}$  be given. Then there is a positive integer  $n$  such that  $3n/(2n + 1) > t$ . Take pairwise disjoint sets  $S = \{s_1, s_2, \dots, s_n\}$ ,  $T = \{t_1, t_2, \dots, t_{2n+1}\}$ ,  $R = \{r_1, r_2, \dots, r_{2n+1}\}$ , join each  $s_i$  to all the other points and each  $r_i$  to every other  $r_j$  as well as to the point  $t_i$  with the same subscript  $i$ . Call the resulting graph  $H$ . (If  $n = 1$ , we obtain the graph  $H$  in Fig. 2.)

Let  $W$  be a point-cutset in  $H$  which minimizes  $|W|/k(H - W)$ . Let  $k = k(H - W)$  and  $m = |W \cap R|$ . Obviously,  $W$  is a minimal set whose removal from  $H$  results in a graph with  $k$  components. As  $W$  is a cutset, we have  $S \subset W$  and  $m \geq 1$ . From the minimality of  $W$  we then easily conclude that  $T \cap W = \emptyset$  and  $m \leq 2n$ . Then we have  $|W| = n + m$  and  $k(H - W) = m + 1$ . Hence

$$t(H) = \frac{|W|}{k(H - W)} = \min_{1 \leq m \leq 2n} \frac{n + m}{m + 1} = \frac{3n}{2n + 1} > t.$$

It is straightforward to see that  $H$  has no 2-factor. Indeed, let us assume the contrary, i.e., let  $F \subset H$  be regular of degree 2. Let us denote by  $X$  the set of lines of  $F$  having at least one endpoint in  $T$ . Since  $T$  is independent, we have  $|X| = 2|T|$ . On the other hand, there are at most  $2|S|$  lines in  $X$  having one endpoint in  $S$  and at most  $|R|$  lines in  $X$  having one endpoint in  $R$ . Thus

$$4n + 2 = 2|T| = |X| \leq 2|S| + |R| = 4n + 1$$

which is a contradiction.

#### 4. Line-toughness

Looking at our definition of toughness from a merely formal point of view, one could wonder why we did not define a *line-toughness*  $t^*(G)$  of  $G$  by

$$t^*(G) = \min \{ |X| / k(G - X) \},$$

where  $X$  ranges over all the line-cutsets of  $G$ . The answer is given by the following theorem; line-toughness is exactly one half of line-connectivity.

**Theorem 4.1.**  $t^* = \frac{1}{2}\lambda$ .

**Proof.** Let  $G$  be a graph with line-connectivity  $\lambda$ . Then there is a line-cutset  $X_0$  of  $G$  with  $|X_0| = \lambda$  and we have

$$t^*(G) \leq |X_0| / k(G - X_0) \leq \frac{1}{2}\lambda.$$

On the other hand, let  $X$  be a line-cutset of  $G$  minimizing  $|X| / k(G - X)$ . Let the components of  $G - X$  be  $H_1, H_2, \dots, H_k$ . For each  $i = 1, 2, \dots, k$ , let us denote by  $X_i$  the set of lines in  $X$  having an endpoint in  $H_i$ . Obviously, each  $X_i$  is a line-cutset of  $G$  and so we have  $|X_i| \geq \lambda$  for each  $i = 1, 2, \dots, k$ .

Moreover,  $X$  is a minimal line-cutset of  $G$  whose removal results in a graph with  $k$  components. Hence no line in  $X$  has both endpoints in the same  $H_i$  and so we have

$$2|X| = \sum_{i=1}^k |X_i| \geq \lambda k$$

or

$$t^*(G) = |X| / k \geq \frac{1}{2}\lambda.$$

## 5. Toughness of inflations

Let  $G$  be an arbitrary graph. By the *inflation*  $G^*$  of  $G$  we mean the graph whose points are all ordered pairs  $(u, x)$ , where  $x$  is a line of  $G$  and  $u$  is an endpoint of  $x$ ; two points of  $G^*$  are adjacent if they differ in exactly one coordinate.

**Theorem 5.1.** *Let  $G$  be an arbitrary graph without isolated points and  $G^*$  its inflation. If  $G \neq K_2$ , then  $t(G^*) = \frac{1}{2}\lambda(G)$  and  $\kappa(G^*) = \lambda(G^*) = \lambda(G)$ .*

**Proof.** Let  $S$  be a point-cutset of  $G^*$  minimizing  $|S| / k(G^* - S)$ ; set  $k = k(G^* - S)$ . Obviously,  $S$  is a minimal set whose removal from  $G^*$

yields a graph with at least  $k$  components. From this we easily conclude that for each line  $x$  of  $G$ ,  $S$  contains at most one point  $(u, x)$  of  $G^*$ . Denoting by  $X$  the set of all the lines  $x$  of  $G$  with  $(u, x) \in S$  for some  $u$ , we then have  $|X| = |S|$ . If two points  $(u, x), (v, y)$  of  $G^*$  belong to distinct components of  $G^* - S$ , then necessarily  $u \neq v$  and  $u, v$  belong to distinct components of  $G - X$ . Hence  $k(G - X) \geq k(G^* - S)$  and Theorem 4.1 implies

$$(6) \quad t(G^*) = |S|/k(G^* - S) \geq |X|/k(G - X) \geq t^*(G) = \frac{1}{2}\lambda(G).$$

Next, if  $G \neq K_2$ , then  $G^*$  is not complete and so, by Proposition 1.3,  $t(G^*) \leq \frac{1}{2}\kappa(G^*)$ . By Whitney's inequality [18],  $\kappa(G^*) \leq \lambda(G^*)$ . Moreover, there is a natural one-to-one mapping  $f$  from the line-set of  $G$  into the line-set of  $G^*$ . If  $X$  is a cutset of  $G$  then  $f(X)$  is a cutset of  $G^*$ . Hence  $\lambda(G^*) \leq \lambda(G)$  and we have

$$(7) \quad t(G^*) \leq \frac{1}{2}\kappa(G^*) \leq \frac{1}{2}\lambda(G^*) \leq \frac{1}{2}\lambda(G).$$

Combining (6) and (7), we obtain the desired result.

It is quite easy to see that a hamiltonian circuit in  $G^*$  induces a closed spanning trail in  $G$  and vice versa. Hence we have:

**Proposition 5.2.**  *$G^*$  is hamiltonian if and only if  $G$  has an eulerian spanning subgraph.*

This proposition and Theorem 5.1 yield:

**Corollary 5.3.** *Let  $G$  be a cubic nonhamiltonian graph with  $\lambda(G) = 3$ . Then its inflation  $G^*$  is a cubic nonhamiltonian graph with  $t(G^*) = \frac{3}{2}$  and  $\lambda(G^*) = 3$ .*

Indeed, the inflation of a regular graph of degree  $n$  is a regular graph of degree  $n$ . Moreover, an eulerian spanning subgraph of a cubic graph is necessarily a hamiltonian cycle.

In particular, denoting by  $G_0$  the Petersen graph and setting  $G_{k+1} = G_k^*$  we obtain an infinite family  $G_1, G_2, \dots$  of cubic nonhamiltonian  $\frac{3}{2}$ -tough graphs. The Petersen graph  $G_0$  is not  $\frac{3}{2}$ -tough; one can show that  $t(G_0) = \frac{4}{3}$ . In the next section, we will prove that the number of points of any  $\frac{3}{2}$ -tough cubic graph  $G$  with  $G \neq K_4$  is divisible by six.

## 6. Toughness of regular graphs

Let  $G$  be a regular graph of degree  $n$  with  $p$  points, where  $p > n + 1$  (so that  $G$  is not complete). Then  $\kappa(G) \leq n$  and by Proposition 1.3,  $t(G) \leq \frac{1}{2}n$ . One may ask for which choice of  $n$  and  $p$  the equality  $t(G) = \frac{1}{2}n$  can be attained. If  $n$  is even, then every  $p$  works. Indeed, it is easy to see that the graph  $C_p^{n/2}$  is  $\frac{1}{2}n$ -tough. Now, let  $n$  be odd and greater than one; then the situation is different.

We already have two methods for constructing  $\frac{1}{2}n$ -tough regular graphs of degree  $n$ . Firstly, if  $p = rs$  with  $r + s - 2 = n$ , then the graph  $K_r \times K_s$  with  $p$  points is regular of degree  $n$  and  $\frac{1}{2}n$ -tough (see Theorem 1.6). Secondly, if  $p = nk$  for an even integer  $k \geq n + 1$ , then there is a regular graph  $H$  of degree  $n$  with  $k$  points and  $\lambda(H) = n$  (the existence of  $H$  follows from [8] or [4]). Its inflation  $H^*$  has  $p$  points, is regular of degree  $n$  and  $\frac{1}{2}n$ -tough (see Theorem 5.1).

However, it seems likely that for  $p$  sufficiently large and not divisible by  $n$  there is no graph  $G$  with  $p$  points which is regular of degree  $n$  and  $\frac{1}{2}n$ -tough. We will prove this for  $n = 3$  and leave the cases  $n \geq 5$  open.

Let us call a coloring of  $G$  *balanced* if all of its color classes have the same size; otherwise the coloring is *unbalanced*.

**Theorem 6.1.** *No cubic  $\frac{3}{2}$ -tough graph admits an unbalanced 3-coloring.*

**Proof.** Let  $G$  be a cubic  $\frac{3}{2}$ -tough graph and let the point-set of  $G$  be partitioned into color classes  $R, S, T$  with

$$(8) \quad |R| \leq |S| \leq |T|.$$

Let  $|R|$  be as small as possible. Then each  $u \in R$  is adjacent to some  $v \in S$  (otherwise  $R^* = R - \{u\}$ ,  $S^* = S \cup \{u\}$  and  $T^* = T$  would be color classes with  $|R^*| < |R|$ ) and similarly, each  $u \in R$  is adjacent to some  $v \in T$ . Hence there is a partition  $R = R_S \cup R_T$  such that each  $u \in R_S$  is adjacent to exactly one point in  $S$  and each  $u \in R_T$  is adjacent to exactly one point in  $T$ . Obviously, the subgraph of  $G$  induced by  $S \cup R_S$  has exactly  $|S|$  components. Thus,

$$k(G - (T \cup R_T)) = |S|,$$

and similarly

$$k(G - (S \cup R_S)) = |T|.$$

We have  $|S| \geq 2$  (otherwise (8) implies  $|R \cup S| \leq 2$ , which is impossible since each point in  $T$  is adjacent to three points in  $R \cup S$ ) and by (8) also  $|T| \geq 2$ . Since  $G$  is  $\frac{3}{2}$ -tough, we have

$$|T \cup R_T| \geq \frac{3}{2}|S|$$

and

$$|S \cup R_S| \geq \frac{3}{2}|T|.$$

Adding these two inequalities we obtain  $|R| + |S| + |T| \geq \frac{3}{2}(|S| + |T|)$  or  $|R| \geq \frac{1}{2}(|S| + |T|)$  which together with (8) implies  $|R| = |S| = |T|$ .

**Corollary 6.2.** *A necessary and sufficient condition for the existence of a cubic  $\frac{3}{2}$ -tough graph with  $p$  points is that either  $p = 4$  or  $p$  is divisible by six.*

Indeed,  $K_4$  and  $K_2 \times K_3$  are  $\frac{3}{2}$ -tough and we can construct cubic  $\frac{3}{2}$ -tough graphs with  $6k$  points ( $k > 1$ ) by inflations as described above. On the other hand, let  $G$  be a cubic  $\frac{3}{2}$ -tough graph with more than four points. Obviously, the number  $p$  of points of  $G$  must be even. By Brooks' theorem [3],  $G$  admits a 3-coloring. By Theorem 5.4, this 3-coloring must be balanced and therefore  $p$  divisible by 3.

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