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Two (Multi) Point Nonlinear Lyapunov Systems—Existence and Uniqueness

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1. INTRODUCTION

In this paper, we consider the following nonlinear Lyapunov systems of equations of the form

$$T'(t) = A(t)\,T(t) + T(t)\,B(t) + F(t, T(t)), \qquad a \leqslant t \leqslant b, \tag{1.1}$$

where $A(t)$, $B(t)$, and $T(t)$ are square matrices of order n. We assume that components of A, B, and T are continuous functions on $[a, b]$ and F: $[a, b] \times R^{n \times n} \to R^{n \times n}$ is continuous. We also assume that $F(t, 0) = 0$ on $[a, b]$. We seek a solution $T(t)$ of (1.1) satisfying the following general boundary conditions

$$MT(a) + NT(b) = \alpha, \tag{1.2}$$

(or $\sum_{i=1}^{n} M_i T(t_i) = \alpha$; here $a = t_1 < t_2 < \cdots < t_n = b$) where $M, N \in R^{n \times n}$ (constant square matrices of order n) and $\alpha \in R^{n \times n}$ (or $M_i \in R^{n \times n}$).

Boundary value problems (1.1) and (1.2) attracted the attention of such mathematicians as F. V. Atkinson [1], R. Bellman [2], *et al.*, but its closed form solution is as yet unavailable. The problem (1.1) satisfying the

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boundary conditions (1.2) naturally arises in a number of areas of Control Engineering, Dynamical Systems, Eco and Ecological Systems.

This paper is organized as follows: In Section 2, we develop the variation of parameters formula for (1.1). Using this formula, we obtain our main existence and uniqueness theorems in Section 3. The results obtained are for the most part not known although we obtain novel results in some special cases. The results obtained in Section 2 pave a way for studying stability, asymptotic stability, uniform (asymptotic) stability, strong stability, restrictive stability, Lipschitz stability, etc. for the nonlinear Lyapunov system of first order equations. Work in this direction is in progress.

2. General Solution of the Nonlinear System

In this section, we establish the general solution of the nonlinear Lyapunov system in terms of the fundamental matrices. Throughout this paper $Y(t)$ stands for a fundamental matrix solution of the system $T' = AT$ and $Z(t)$ stands for a fundamental matrix solution of $T'(t) = B^*(t)\,T(t)$ ($*$ refers to the transpose of the complex conjugate matrix). We now present the following theorems:

Theorem 2.1. *Any solution of*

$$T'(t) = A(t)\,T(t) + T(t)\,B(t) \tag{2.1}$$

is of the form $T(t) = Y(t)\,CZ^*(t)$, where C is a constant square matrix of order n.

Proof. It can easily be verified that $T(t) = Y(t)\,CZ^*(t)$ is a solution of (2.1). To prove that every solution of (2.1) is of this form, let T be a solution and let K be a square matrix of order n defined by $K(t) = Y^{-1}(t)\,T(t)$. Then $T(t) = Y(t)\,K(t)$. Now $T(t) = Y(t)\,K(t) \Leftrightarrow Y'(t)\,K(t) + Y(t)\,K'(t) = A(t)\,Y(t)\,K(t) + Y(t)\,K(t)\,B(t) \Leftrightarrow K'(t) = K(t)\,B(t) \Leftrightarrow K^{*\prime}(t) = B^*(t)\,K^*(t)$. Since Z is a fundamental matrix of $T'(t) = B^*(t)\,T(t)$, it follows that there exists a constant square matrix C such that $K^* = ZC$ or $K = C^*Z^*$. Hence $T(t) = YC^*Z^*$. (Take $C^* = C$.)

Theorem 2.2. *Any solution of*

$$T'(t) = A(t)\,T(t) + T(t)\,B(t) + F(t, T(T)) \tag{2.2}$$

is of the form $T(t) = YCZ^* + \bar{T}(t)$, where $\bar{T}(t)$ is a particular solution of (2.2).

Proof. It can easily be verified that $T(t)$ defined by $T(t) = YCZ^* + \bar{T}(t)$ is a solution of (2.2). Now to prove that every solution of (2.2) is of this form, let T be any solution of (2.2) and let $\bar{T}$ be a particular solution of (2.2). Then $T - \bar{T}$ is a solution of (2.1) and hence by Theorem 2.1, $T - \bar{T} = YCZ^*$ or $T = YCZ^* + \bar{T}$, and the proof of the theorem is complete.

THEOREM 2.3. *A particular solution of*

$$T'(t) = B^*(t)\, T(t) + F^*(t, T(t))\, Y^{*-1}(t) \tag{2.3}$$

is of the form

$$\bar{T}(t) = Z(t) \int_a^t Z^{-1}(s)\, F^*(s, T(s))\, Y^{*-1}(s)\, ds.$$

Proof. Let Z be a fundamental matrix solution of $T'(t) = B^*(t)\, T(t)$. Write $\bar{T}(t) = Z(t)\, L(t)$. Now

$$\begin{aligned}
\bar{T}(t) = Z(t)\, L(t) &\Leftrightarrow Z'(t)\, L(t) + Z(t)\, L'(t)\\
&= B^*(t)\, Z(t)\, L(t) + F^*(t, T(t))\, Y^{*-1}(t) \Leftrightarrow L'(t)\\
&= Z^{-1}(t)\, F^*(t, T(t))\, Y^*(t) \Leftrightarrow L(t)\\
&= \int_a^t Z^{-1}(s)\, F^*(s, T(s))\, Y^{*-1}(s)\, ds \text{ or } \bar{T}(t) = Z(t)\, L(t)\\
&= Z(t) \int_a^t Z^{-1}(s)\, F^*(s, T(s))\, Y^{*-1}(s)\, ds.
\end{aligned}$$

THEOREM 2.4. *A particular solution of the nonlinear Lyapunov system* (2.2) *is given by*

$$\bar{T}(t) = Y(t) \left[\int_a^t Y^{-1}(s)\, F(s, T(s))\, Z^{*-1}(s)\, ds \right] Z^*(t).$$

Proof. Let $Y(t)$ be a fundamental matrix solution of $T'(t) = AT + TB$. Then the matrix $\bar{T}(t) = Y(t)\, K(t)$ is a solution of (2.2) if, and only if, $K^{*\prime}(t) = B^*(t)\, K^*(t) + F^*(t, T(t))\, Y^{*-1}(t)$. Now by Theorem 2.3, a particular solution of $K^*(t)$ is of the form $K^*(t) = Z(t) \int_a^t Z^{-1}(s)\, F^*(s, T(s))\, Y^{*-1}(s)\, ds$. Hence

$$\bar{T}(t) = Y(t) \left[\int_a^t Y^{-1}(s)\, F(s, T(s))\, Z^{*-1}(s)\, ds \right] Z^*(t).$$

THEOREM 2.5. *Any solution of the nonlinear Lyapunov system* (2.2) *is of the form*

$$T(t)=Y(t)\,CZ^*(t)+Y(t)\left[\int_a^t Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right]^*(t). \quad (2.4)$$

Proof. It is easily verified that T defined by (2.4) is a solution of (2.2). Further, if T is any solution of (2.4), then

$$(T-\bar{T})'=A(T-\bar{T})+(T-\bar{T})B.$$

Hence $(T-\bar{T})$ is a solution of (2.1). It follows, that for some constant $(n\times n)$ square matrix C,

$$T-\bar{T}=YCZ^*.$$

Hence,

$$T=\bar{T}+YCZ^*.$$

By Theorem 2.3

$$T(t)=Y(t)\left[\int_a^t Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right]Z^*(t)+YCZ^*.$$

Substituting the general form of $T(t)$ in the boundary condition matrix (1.2), we get

$$MY(a)\,CZ^*(a)+NY(b)\,CZ^*(b)$$
$$=\alpha-NY(b)\left[\int_a^b Y^{-1}(s,T(s))\,Z^{*-1}(s)\,ds\right]z^*(b) \quad (2.5)$$

which is equivalent to

$$A_1CB_1+A_2CB_2=X, \quad (2.6)$$

where $A_1=MY(a)$, $A_2=NY(b)$, $B_1=Z^*(a)$, $B_2=Z^*(b)$, and

$$X=\alpha-NY(b)\left[\int_a^b Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right]Z^*(b)$$

are all known matrices of orders $(n\times n)$. Moreover B_1 and B_2 are non-singular square matrices. In the next section we consider two special cases of (2.6) and express the general solution of C in terms of the known matrices A_1, B_1, A_2, B_2, and X. In the case of the multipoint boundary value problem we get

$$\sum_{i=1}^{n} M_i Y(t_i)\, CZ^*(t_i) = \alpha$$

or

$$\sum_{i=1}^{n} A_i C B_i = X, \tag{2.7}$$

where $A_i = M_i Y(t_i)$, $B_i = Z^*(t_i)$, and

$$X = \alpha - \sum_{i=2}^{n} M_i Y(t_i) \left[\int_a^b y^{-1}(s)\, F(s, T(s))\, Z^{*-1}(s)\, ds \right] Z^*(t_i).$$

3. Analysis of the Matrix C and the General Solution of $T(t)$

In this section, we shall be concerned with the general form of the solution C satisfying condition (2.6) (or (2.7)). The problem in its full generality is far from tractable, although the transformation to a vector equation allows us to use currently available numerical weapons for the solution of the problem (2.6). We use the following notation:

If $A \in C^{n \times n}$ and $B \in C^{n \times n}$, then their direct product (or tensor product) A and B, denoted by $A \otimes B$, is defined to be the partitioned matrix [4]

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \tag{3.1}$$

and is in $C^{n^2 \times n^2}$. With this, one can easily verify that if $G = A_1 \otimes B_1^T + A_2 \otimes B_2^T$, then (2.6) is equivalent to a system of vector equations

$$Gc = x. \tag{3.2}$$

In fact, by viewing (2.6) as a system of n^2 scalar equations for the elements of C, (3.2) is exactly the same set of equations written in a vector system. In order to make pronouncements about existence, uniqueness, and techniques for the solution of (3.2), we need some information about the eigenvalues of G. We denote the set of all eigenvalues of the matrix A as $\sigma(A)$, the spectrum of A.

Case 1. If A_1 and B_1 are nonsingular, then (2.6) is equivalent to

$$C - ACB = Y, \tag{3.3}$$

where $A = -(A_1^{-1}A_2)$, $B = (B_2B_1^{-1})$, and $Y = A_1^{-1}XB_1^{-1}$.

Now to solve for C from (3.3), we have the following analysis:

$$\begin{aligned} C - ACB = Y &\Leftrightarrow [(I \otimes I) - (A \otimes B^T)]c = y \\ &\Leftrightarrow Ic - A \otimes B^T cy \\ \Leftrightarrow Ic - Gc = y &\qquad (\text{where } G = A \otimes B^T). \end{aligned}$$

Putting $C = Y + ACB$ in the second term on the LHS of (3.3), we get the following equivalent statements

$$\begin{aligned} C - A(Y + ACB)B = Y &\Leftrightarrow c - G(y + Gc) = y \\ C - A^2CB^2 = Y + AYB &\Leftrightarrow c - G^2c = y + Gy; \end{aligned}$$

similarly

$$C - A^3CB^3 = Y + AYB + A^2YB^2 \Leftrightarrow c - G^3c = y + Gy + G^2y$$

$$\cdots$$

$$\begin{aligned} C - A^nCB^n = Y + AYB + A^2YB^2 + \cdots + A^nYB^n &\Leftrightarrow c - G^nc \\ &= y + Gy + G^3y + \cdots + G^{n-1}y. \end{aligned}$$

If the spectral radius of A and B are such that

$$\rho(A)\,\rho(B) < 1,$$

then $A^nYB^n \to 0$ as $n \to \infty$. In this case

$$\begin{aligned} C &= Y + \sum_{j=1}^{\infty} A^jYB^j \\ &= (A_1^{-1}XB_1^{-1}) - \sum_{j=1}^{\infty} (A_1^{-1}XB_1^{-1})(B_2B_1^{-1})^j. \end{aligned}$$

Substituting the general form of C in (2.6), we get

$$T(t)=Y(t)[A_1^{-1}XB_1^{-1}]\,Z^*(t)$$

$$-\left[\sum_{j=1}^{\infty}(A_1^{-1}A_2)^j\,(A_1^{-1}XB_1^{-1})(B_2B_1^{-1})^j\right]Z^*(b)\,Z^*(t)$$

$$+\,Y(t)\left[\int_a^t Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right]Z^*(t)$$

$$=Y(t)\left\{A_1^{-1}\left[\alpha-NY(b)\right.\right.$$

$$\left.\left.\times\left(\int_a^b Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right)Z^*(b)\right]B_1^{-1}\right\}Z^*(t)$$

$$-\,Y(t)\sum_{j=1}^{\infty}(A_1^{-1}A_2)^j\left\{A_1^{-1}\left[\alpha-NY(b)\right.\right.$$

$$\left.\times\left(\int_a^b Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right)Z^*(b)\right]$$

$$\left.\times\,B_1^{-1}(B_2B_1^{-1})^j\right\}Z^*(t)$$

$$+\,Y(t)\left[\int_a^t Y^{-1}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right]Z^*(t).$$

DEFINITION 3.1. The condition numbers of the fundamental matrix solutions $Y(t)$ and $Z(t)$ are defined as

$$\operatorname{cond}(Y)=\max_{t\in[a,b]}\|Y(t)\|\max_{t\in[a,b]}\|Y^{-1}(t)\|$$

and

$$\operatorname{cond}(Z)=\max_{t\in[a,b]}\|Z(t)\|\max_{t\in[a,b]}\|Z^{-1}(t)\|.$$

To obtain a unique solution of the two-point BVP, we define the iterations

$$T^i(t)=Y(t)\,BZ^*(t)-Y(t)\,A_1^{-1}NY(b)$$

$$\times\left(\int_a^b Y^{-1}(s)\,F(s,T^{i-1}(s))\,Z^{*-1}(s)\,ds\right)$$

$$\times\,B_1^{-1}(B_2B_1^{-1})^j\,Z^*(t)$$

$$+\int_a^b Y(t)\,Y^{-1}(s)\,F(s,T^{i-1}(s))\,Z^{*-1}(s)\,ds\,Z^*(t),$$

where

$$B = A_1^{-1}\alpha B_1^{-1} - \sum_{j=1}^{\infty} (A_1^{-1}A_2)^j A_1^{-1}\alpha B_1^{-1}(B_2B_1^{-1})^j.$$

At this stage, we assume that F satisfies a Lipschitz condition on $[a, b] \times R^{n \times n}$, i.e.,

$$\|F(s, T, (s)) - F(s, T_2(s))\| \leqslant K \|T_1 - T_2\| \qquad (K > 0).$$

Then

$$\begin{aligned}\|T^i(t) - T^{i-1}(t)\| \leqslant{}& \|Y(t)\| \, \|A_1^{-1}\| \, \|NY(b)\| \\ & \times \left(\int_a^b \|Y^{-1}(s)\| \, K \, \|T^{i-1}(s) - T^{i-2}(s)\| \, ds \right) \\ & \times \|Z^{*-1}(s)\| \, \|Z^*(b)\| \, \|B_1^{-1}\| \, \|Z^*(t)\| \\ & + \|Y(t)\| \sum_{j=1}^{\infty} \|A_1^{-1}A_2\|^j \, \|A_1^{-1}\| \, \|NY(b)\| \\ & \times \left(\int_a^b \|Y^{-1}(s)\| \, K \, \|T^{i-1}(s) - T^{i-2}(s)\| \, ds \right) \\ & \times \|Z^{*-1}(s)\| \, \|Z^*(b)\| \, \|B_1^{-1}\| \, \|B_2B_1^{-1}\|^j \, \|Z^*(t)\| \\ & + \|Y(t)\| \left(\int_a^t \|Y^{-1}(s)\| \, K \, \|T^{i-1}(s) - T^{i-2}(s)\| \, ds \right) \\ & \times \|Z^{*-1}(s)\| \, \|Z^*(t)\|.\end{aligned}$$

Using definition (3.1), we get

$$\begin{aligned}\|T^i(t) - T^{i-1}(t)\| \leqslant{}& \operatorname{cond}(Y) \operatorname{cond}(Z) \\ & \times \left\{ \sum_{j=1}^{\infty} \|A_1^{-1}A_2\|^j \, \|A_1^{-1}\| \, \|NY(b)\| \, \|B_1^{-1}\| \right. \\ & \left. \times \|B_2B_1^{-1}\|^j + 1 \right\} K(b-a).\end{aligned}$$

Let

$$\begin{aligned}\alpha ={}& \frac{\operatorname{cond}(Y) \operatorname{cond}(Z)}{(1 - \|A_1^{-1}A_2\| \, \|B_2B_1^{-1}\|)} \\ & \times K[\|A_1^{-1}\| \, \|B_1^{-1}\| \, \|NY(b)\| + 1] < 1.\end{aligned}$$

Then

$$\begin{aligned}\|T^i(t)-T^{i-1}(t)\| &\leqslant \alpha\, \|T^{i-1}-T^{i-2}\|\, (b-a)\\ &\leqslant \alpha^2\, \|T^{i-2}-T^{i-3}\|\, (b-a)^2\\ &\quad \cdots\\ &\alpha^{i-1}\, \|T^1-T^0\|\, (b-a)^{i-1}.\end{aligned}$$

Thus T is a contraction operator and hence by the Banach fixed point theorem T has a unique fixed point and this fixed point is the unique solution of TPBVP. The above theory can easily be generalized to multipoint BVPs. In order to avoid monotony, we omit the proof.

Case 2. Suppose A_1 is invertible. Then the system of equations (2.6) is equivalent to

$$AC+CB=Y, \tag{3.4}$$

where $A=A_1^{-1}A_2$, $B=B_1B_2^{-1}$, and $Y=A_1^{-1}XB_2^{-1}$. One of the most effective methods of solving the matrix equation (3.4) is the Bartels–Stewart algorithm [5]. Key to this technique is the orthogonal reduction of A and B to triangular form using the QR-algorithm for eigenvalues. The method of finding the general solution to the system (3.4) is the following:

Let $A\in R^{n\times n}$, $B\in R^{n\times n}$ be given matrices and define the linear transformation ϕ: $R^{n\times n}\rightarrow R^{n\times n}$ by [3]

$$\phi(C)=AC+CB=Y. \tag{3.5}$$

This linear transformation is nonsingular if and only if A and $-B$ have no eigenvalues in common; i.e., if λ is an eigenvalue of A with corresponding eigenvector u and μ is an eigenvalue of B with corresponding eigenvector v, then

$$Auv^T+uv^TB=(\lambda+\mu)\, uv^T.$$

Thus $(\lambda+\mu)$ is an eigenvalue of the system (3.5), which can therefore be solved if and only if

$$\lambda_i+\mu_j\neq 0$$

for all $i, j=1, 2, ..., n$. When A and B can be reduced to the diagonal form by similarity transformations, i.e.,

$$U^{-1}AU=\operatorname{diag}(\lambda_1, \lambda_2, ..., \lambda_n)=A_1$$

$$V^{-1}BV=\operatorname{diag}(\mu_1, \mu_2, ..., \mu_n)=B_1,$$

then (3.5) is equivalent to

$$(U^{-1}AU)(U^{-1}CV)+(U^{-1}CV)(V^{-1}BV)=U^{-1}YN.$$

Solving this system involves the following four steps:

Step 1. Transform A and B into diagonal forms by similarity transformations to get

$$A_1=U^{-1}AU \qquad \text{and} \qquad B_1=V^{-1}BV.$$

Step 2. Solve $UF=YV$ for F.

Step 3. Solve the transformed system

$$A_1X_1+X_1B_1=F \qquad \text{for } X_1.$$

Step 4. Solve the system

$$CV=UX$$

for C.

From these four steps the solution of the system (3.5) is easily obtained as

$$C=UX_1V^{-1},$$

where

$$(X_{ij})_1=\frac{\tilde{f}_{ij}}{\lambda_i+l_{ij}}$$

and

$$\tilde{F}=U^{-1}YV.$$

Now substituting the general form of C in the variation of parameters formula, we obtain

$$T(t)=YuX_1Y^{-1}Z^*+Y(t)\left[\int_a^t Y^{-11}(s)\,F(s,T(s))\,Z^{*-1}(s)\,ds\right]Z^*(t).$$

Assuming that F satisfies a Lipschitz condition on $[a,b]\times R^{n\times n}$, we get as before

$$\|T^i(t) - T^{i-1}(t)\| \leqslant \text{cond}(Y)\,\text{cond}(Z) \int_a^b K\, \|T^{i-1} - T^{i-2}\|\, ds$$
$$\leqslant \text{cond}(Y)\,\text{cond}(Z) K\, \|T^{i-1} - T^{i-2}\|\, (b-a)^2$$
$$\leqslant \text{cond}(Y)\,\text{cond}(Z)\, K^2\, \|T^{i-2} - T^{i-3}\|\, (b-a)^2$$
$$\cdots$$
$$\leqslant \text{cond}(Y)\,\text{cond}(Z)\, K^{i-1}\, \|T^1 - T^0\|\, (b-a)^{i-1}.$$

If $\alpha = \text{cond}(Y)\,\text{cond}(Z)\, K^{i-1}(b-a)^{i-1} < 1$, then T is a contraction map and hence by Banach fixed point theorem T has a unique solution of the TPBVP.

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