Journal of Computational and Applied Mathematics 235 (2010) 916–926

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

New steps on Sobolev orthogonality in two variables

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ARTICLE INFO

Article history:
Received 17 June 2009

MSC:
42C05
33C50

Keywords:
Orthogonal polynomials in two variables
Sobolev orthogonal polynomials
Classical orthogonal polynomials

ABSTRACT

Sobolev orthogonal polynomials in two variables are defined via inner products involving gradients. Such a kind of inner product appears in connection with several physical and technical problems. Matrix second-order partial differential equations satisfied by Sobolev orthogonal polynomials are studied. In particular, we explore the connection between the coefficients of the second-order partial differential operator and the moment functionals defining the Sobolev inner product. Finally, some old and new examples are given.

1. Introduction

The name Sobolev is associated with polynomials that are orthogonal with respect to an inner product involving both functions and their derivatives. This kind of polynomial has been widely studied during the last 20 years, and it constitutes the main subject of a vast literature (see, for instance, [1–3] and the references therein). However, as far as we know, Sobolev orthogonal polynomials in several variables have been studied only in a very few particular cases. At the time of writing this paper, the only references in the subject are [4–7].

The three last references [5–7] are related to orthogonal polynomials on the unit ball $B^d := \{x : \|x\| \le 1\}$ of the Euclidean space $\mathbb{R}^d$, $d \ge 1$.

In [6], the author considers an inner product motivated by an application in the numerical solution of the nonlinear Poisson equation $-\Delta u = f(\cdot, u)$ on the unit disk with zero boundary conditions (see [8]). This inner product is defined by

$$\langle f, g\rangle_{\Delta} = \alpha_d \int_{B^d} \Delta[(1-\|x\|^2)f(x)]\Delta[(1-\|x\|^2)g(x)]\mathrm{d}x, \tag{1}$$

where Δ is the usual Laplace operator, and $\alpha_d = 1/(4d^2\mathrm{vol}(B^d))$ so that $\langle 1, 1\rangle_{\Delta} = 1$. The central symmetry of the inner product plays an essential role in the construction of a basis of mutually orthogonal polynomials, which can be expressed in terms of *spherical harmonics*: each element in the basis is the product of a spherical harmonic and a radial part given by a Jacobi polynomial with parameters depending on its degree. The radial part of the polynomials turns out to be Sobolev orthogonal polynomials in one variable.

In [7], the author considers two different inner products involving the gradient operator on the ball. Using the same construction as above, a family of explicit orthonormal basis is constructed for both inner products. An interesting result

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.07.006

obtained by the author is that the orthogonal polynomials with respect to a Sobolev inner product satisfy a partial differential equation for non-standard values of the parameter.

In [4], Lee and Littlejohn consider polynomials in two variables which satisfy an *admissible* (as defined in [9]) second-order partial differential equation. They find conditions for the partial differential equation to have polynomial solutions which are orthogonal with respect to a symmetric bilinear form.

However, the Lee and Littlejohn approach to Sobolev orthogonal polynomials seems to be incomplete, since there exist *non-admissible* partial differential equations having Sobolev orthogonal polynomial solutions. Some interesting examples are the orthogonal polynomials constructed in [10] using Jacobi polynomials in one variable.

In this work, we study families of polynomials in two variables satisfying second-order partial differential equations, and we will connect this fact with Sobolev orthogonality. The structure of the paper is as follows. In Section 2, we recall some basic properties of orthogonal polynomials in two variables. Section 3 is devoted to relating classical orthogonal polynomials in two variables with a kind of Sobolev inner product.

A generalization of this Sobolev inner product is studied in Section 4, and we introduce the main results of this paper. The relation between Sobolev orthogonality and partial differential equations is analyzed. Finally, Section 5 contains several interesting examples. In particular, we deduce a Sobolev orthogonality for generalized classical families of polynomials in two variables, namely simplex, ball and Koornwinder polynomials of Class III for non-standard values of the parameters.

2. Orthogonal polynomials in two variables

First, we introduce some notation. Let $\mathcal{P}$ denote the linear space of real polynomials in two variables, and $\mathcal{P}_n$ the subspace of polynomials of total degree not greater than n.

Let $\mathcal{M}_{h\times k}(\mathbb{R})$ and $\mathcal{M}_{h\times k}(\mathcal{P})$ denote the linear spaces of $h\times k$ real and polynomial matrices, respectively. When $h = k$, the second index will be omitted.

Given a matrix A, we denote by A^t its transpose, and by $\det A$ its determinant. As usual, we say that A is non-singular if $\det A \neq 0$. Furthermore, we introduce I_h as the identity matrix of dimension h.

Moreover, we define the total *degree of* a polynomial matrix $A \in \mathcal{M}_{h\times k}(\mathcal{P})$ as

$$\deg A = \max\{\deg a_{i,j}(x, y),\ 1 \le i \le h,\ 1 \le j \le k\} \ge 0,$$

where $a_{i,j}(x, y)$ denotes the (i, j)-entry of A.

Before discussing our approach, we briefly give some general properties of bivariate orthogonal polynomials. For an exhaustive description of this and other related subjects see, for instance, [11–15,9,16,17].

Let $\{\mu_{h,k}\}_{h,k\ge 0}$ be a double-indexed sequence of real numbers, and let $u : \mathcal{P} \to \mathbb{R}$ be a functional defined by means of the moments $\mu_{h,k} = \langle u, x^h y^k\rangle$, $h, k = 0, 1, 2, \ldots$, and extended by linearity. Then we will say that u is a *moment functional*.

For any moment functional u, let us define the distributional partial derivatives and the product of a polynomial times u (see, for instance, [12]) by

$$\langle u_x, p\rangle = -\langle u, p_x\rangle, \qquad \langle u_y, p\rangle = -\langle u, p_y\rangle, \qquad \langle pu, q\rangle = \langle u, pq\rangle, \tag{2}$$

for any polynomials $p(x, y)$ and $q(x, y)$.

We say that a polynomial $p \in \mathcal{P}_n$ is *orthogonal* with respect to u if

$$\langle u, pq\rangle = 0, \quad \forall q \in \mathcal{P}, \qquad \deg q < \deg p.$$

The action of a moment functional u over polynomial matrices is defined as follows [11–13,17]:

$$\langle u, A\rangle = (\langle u, a_{i,j}\rangle)_{i,j=1}^{h,k} \in \mathcal{M}_{h\times k}(\mathbb{R}), \quad \forall A \in \mathcal{M}_{h\times k}(\mathcal{P}).$$

Let $A \in \mathcal{M}_{h\times k}(\mathcal{P})$ be an arbitrary polynomial matrix. Then we define the *left product* of A times u in the following way:

$$\langle Au, B\rangle = \langle u, A^t B\rangle, \quad \forall B \in \mathcal{M}_{h\times l}(\mathcal{P}). \tag{3}$$

Definition 2.1 (*[14]*)**.** A *polynomial system* (PS) is a sequence of vectors $\{\mathbb{P}_n\}_{n\ge 0}$ of increasing size such that

$$\mathbb{P}_n = (P_{n,0}, P_{n-1,1}, \ldots, P_{0,n})^t \in \mathcal{M}_{(n+1)\times 1}(\mathcal{P}),$$

where $\{P_{n,0}, P_{n-1,1}, \ldots, P_{0,n}\}$ are polynomials of total degree n independent modulus $\mathcal{P}_{n-1}$.

Definition 2.2. We say that a moment functional u is quasi-definite if there exists a PS $\{\mathbb{P}_n\}_{n\ge 0}$ satisfying

$$\langle u, \mathbb{P}_n\mathbb{P}_m^t\rangle = 0, \quad n \neq m,$$
$$\langle u, \mathbb{P}_n\mathbb{P}_n^t\rangle = H_n, \quad n \ge 0,$$

where $H_n \in \mathcal{M}_{n+1}(\mathbb{R})$ is a non-singular matrix. In such a case, the PS $\{\mathbb{P}_n\}_{n\ge 0}$ is said to be a *weak orthogonal polynomial system* (WOPS) with respect to the quasi-definite moment functional u.

In the particular case where H_n is a diagonal matrix, we will say that the WOPS $\{\mathbb{P}_n\}_{n\ge 0}$ is an *orthogonal polynomial system* (OPS). Moreover, if $H_n = I_{n+1}$, we call $\{\mathbb{P}_n\}_{n\ge 0}$ an *orthonormal polynomial system.*

In addition, a WOPS is called a *monic* WOPS if every polynomial contains only one monic term of highest degree; that is,

$$\tilde{P}_{h,k}(x,y) = x^h y^k + R(x,y), \quad h+k=n,$$

where $R(x,y) \in \mathcal{P}_{n-1}$. Observe that, for a quasi-definite moment functional u, there exists a unique monic WOPS.

In this paper, we will need some differentiation tools. In fact, we will use the *gradient operator* ∇, and the *divergence operator* div, defined as usual. The extension of these operators for matrices is introduced in [18,19]. Let $A, B_0, B_1 \in \mathcal{M}_{h\times k}(\mathcal{P})$ be polynomial matrices. We define

$$\nabla A = \begin{pmatrix} \partial_x A \\ \partial_y A \end{pmatrix} \in \mathcal{M}_{2h\times k}(\mathcal{P}), \qquad \operatorname{div} \begin{pmatrix} B_0 \\ B_1 \end{pmatrix} = \partial_x B_0 + \partial_y B_1 \in \mathcal{M}_{h\times k}(\mathcal{P}).$$

The previous definitions can be translated to the linear space of moment functionals using duality. We define the *distributional gradient operator* acting over a moment functional in the following way:

$$\left\langle \nabla u, \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \right\rangle = -\left\langle u, \operatorname{div} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \right\rangle = -\langle u, \partial_x p_0\rangle - \langle u, \partial_y p_1\rangle. \tag{4}$$

Let $A \in \mathcal{M}_{2\times k}(\mathcal{P})$ be an arbitrary polynomial matrix. The *distributional divergence operator* acting over Au is defined as follows:

$$\langle \operatorname{div}(Au), p\rangle = -\langle Au, \nabla p\rangle = -\langle u, A^t \nabla p\rangle, \tag{5}$$

for any polynomial $p(x,y)$.

3. Motivation for Sobolev orthogonality: classical orthogonal polynomials in two variables

Definition 3.1 (*[19]*)**.** Let u be a quasi-definite moment functional defined on the linear space of real polynomials in two variables. Then u is said to be classical if there exist two matrix polynomials

$$\Phi = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \Psi = \begin{pmatrix} d \\ e \end{pmatrix},$$

with $\deg \Phi \le 2$, $\deg \Psi \le 1$, such that u satisfies the matrix Pearson-type distributional equation

$$\operatorname{div}(\Phi u) = \Psi^t u, \tag{6}$$

and $\det\langle u, \Phi\rangle \neq 0$.

Condition (6) is equivalent to

$$P_1[u] = (au)_x + (bu)_y - du = 0,$$
$$P_2[u] = (bu)_x + (cu)_y - eu = 0.$$

Let

$$\tilde{\Psi} = \Psi - (\operatorname{div}\Phi)^t = \begin{pmatrix} d - a_x - b_y \\ e - b_x - c_y \end{pmatrix}.$$

If we define the operator L acting over $\mathcal{P}$ by means of

$$L[p] \equiv \operatorname{div}(\Phi \nabla p) + \tilde{\Psi}^t \nabla p = a\partial_{xx}p + 2b\partial_{xy}p + c\partial_{yy}p + d\partial_x p + e\partial_y p,$$

then the *formal Lagrange adjoint* of L defined by

$$\langle L^*[u], p\rangle = \langle u, L[p]\rangle, \quad \forall p \in \mathcal{P}, \tag{7}$$

satisfies

$$L^*[u] \equiv \operatorname{div}(\Phi \nabla u) - \operatorname{div}(\tilde{\Psi} u). \tag{8}$$

It is easy to check that

$$\begin{aligned} L^*[u] &= (au)_{xx} + 2(bu)_{xy} + (cu)_{yy} - (du)_x - (eu)_y \\ &= (P_1[u])_x + (P_2[u])_y, \end{aligned}$$

and, as a consequence, if u is classical, then $L^*[u] = 0$.

The next theorem is devoted to characterizing multivariate classical orthogonal polynomials. The proof of these results can be found in [19].

Theorem 3.1. *Let u be a quasi-definite moment functional and let $\{\mathbb{P}_n\}_{n\ge 0}$ be a WOPS associated with it. The following statements are equivalent.*

(i) *u is classical in the sense of Definition 3.1.*

(ii) *There exist polynomial matrices $\Phi \in \mathcal{M}_2(\mathcal{P}_2)$, $\tilde{\Psi} \in \mathcal{M}_{2\times 1}(\mathcal{P}_1)$, and there exist matrices $\Lambda_n \in \mathcal{M}_{n+1}(\mathbb{R})$ such that*

$$L[\mathbb{P}_n^t] \equiv \operatorname{div}(\Phi \nabla \mathbb{P}_n^t) + \tilde{\Psi}^t \nabla \mathbb{P}_n^t = \mathbb{P}_n^t \Lambda_n^t,$$

and $\Lambda_1 \in \mathcal{M}_2(\mathbb{R})$ is a non-singular matrix.

(iii) *There exists a polynomial matrix $\Phi \in \mathcal{M}_2(\mathcal{P}_2)$, such that the $\{\nabla \mathbb{P}_n^t\}_{n\geq 1}$ satisfy the orthogonality relations*

$$\langle u, (\nabla \mathbb{P}_n^t)^t \Phi \nabla \mathbb{P}_m^t\rangle = 0, \quad n, m \geq 1,\ n \neq m,$$
$$\langle u, (\nabla \mathbb{P}_n^t)^t \Phi \nabla \mathbb{P}_n^t\rangle = K_n, \quad n \geq 1,$$

where $K_n \in \mathcal{M}_n(\mathbb{R})$ is a symmetric matrix, and K_1 is non-singular.

If one of these three conditions holds, then the $\{\mathbb{P}_n\}_{n\geq 0}$ satisfy the following Sobolev orthogonality relations:

$$(\mathbb{P}_n, \mathbb{P}_n^t)_S = \langle u, \mathbb{P}_n \mathbb{P}_n^t\rangle + \langle u, (\nabla \mathbb{P}_n^t)^t \Phi \nabla \mathbb{P}_n^t\rangle = H_n + K_n,$$
$$(\mathbb{P}_n, \mathbb{P}_m^t)_S = \langle u, \mathbb{P}_n \mathbb{P}_m^t\rangle + \langle u, (\nabla \mathbb{P}_n^t)^t \Phi \nabla \mathbb{P}_m^t\rangle = 0, \quad n \neq m.$$

The relations above can be seen as a Sobolev flavor for classical orthogonal polynomials in two variables.

4. Sobolev inner products in two variables

In the previous theorem, we observe that classical orthogonal polynomials in two variables are solutions of a second-order partial differential equation. Moreover, they satisfy some kind of Sobolev orthogonality. In this section, we generalize the definition of Sobolev inner product and we study Sobolev orthogonal polynomials as a solution of a second-order partial differential equation.

Definition 4.1. Let u, v be two moment functionals defined on the linear space of polynomials in two variables. A Sobolev bilinear form can be defined from u and v in the following way:

$$\begin{aligned}(f, g)_S &= \langle u, fg\rangle + \langle v, (\nabla f)^t \Theta \nabla g\rangle \\ &= \langle u, fg\rangle + \left\langle v, \begin{pmatrix}\partial_x f & \partial_y f\end{pmatrix} \begin{pmatrix}\theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11}\end{pmatrix} \begin{pmatrix}\partial_x g \\ \partial_y g\end{pmatrix}\right\rangle,\end{aligned}$$

where $\theta_{ij}(x, y)$ are fixed polynomials in two variables of degree less than or equal to 2.

In the following, u, v, and Θ will be chosen so that the Sobolev bilinear form $(\cdot, \cdot)_S$ is an inner product.

A WOPS with respect to the Sobolev inner product $(\cdot, \cdot)_S$ is called a Sobolev WOPS.

In this work, we are going to study the first interesting case of this kind of Sobolev inner product, where Θ is a non-scalar diagonal matrix; that is,

$$\begin{aligned}(f, g)_S &= \langle u, fg\rangle + \langle v, (\nabla f)^t \Theta \nabla g\rangle \\ &= \langle u, fg\rangle + \left\langle v, \begin{pmatrix}\partial_x f & \partial_y f\end{pmatrix} \begin{pmatrix}\theta_0 & 0 \\ 0 & \theta_1\end{pmatrix} \begin{pmatrix}\partial_x g \\ \partial_y g\end{pmatrix}\right\rangle,\end{aligned} \tag{9}$$

where $\theta_0(x, y)$ and $\theta_1(x, y)$ are fixed polynomials in two variables of degree less than or equal to 2.

Let us denote by $\{\mathbb{Q}_n\}_{n\geq 0}$ a Sobolev WOPS associated with the inner product $(\cdot, \cdot)_S$ defined in (9); that is, $\{\mathbb{Q}_n\}_{n\geq 0}$ satisfies

$$(\mathbb{Q}_n, \mathbb{Q}_n^t)_S = \langle u, \mathbb{Q}_n \mathbb{Q}_n^t\rangle + \langle v, (\nabla \mathbb{Q}_n^t)^t \Theta \nabla \mathbb{Q}_n^t\rangle = \tilde{H}_n,$$
$$(\mathbb{Q}_n, \mathbb{Q}_m^t)_S = \langle u, \mathbb{Q}_n \mathbb{Q}_m^t\rangle + \langle v, (\nabla \mathbb{Q}_n^t)^t \Theta \nabla \mathbb{Q}_m^t\rangle = 0, \quad n \neq m,$$

where $\tilde{H}_n \in \mathcal{M}_{n+1}(\mathbb{R})$ is non-singular.

4.1. Particular cases

In [7], the author analyzes orthogonal polynomials for Sobolev inner products on the ball, involving the usual gradient operator ∇. In particular, he deals with two Sobolev inner products defined by

$$\langle f, g\rangle_{\mathrm{I}} := \frac{\lambda}{\omega_d} \int_{B^d} \nabla f(x) \cdot \nabla g(x) \mathrm{d}x + \frac{1}{\omega_d} \int_{S^{d-1}} f(x) g(x) \mathrm{d}\omega_d, \quad \lambda > 0 \tag{10}$$

and

$$\langle f, g\rangle_{\mathrm{II}} := \frac{\lambda}{\omega_d} \int_{B^d} \nabla f(x) \cdot \nabla g(x) \mathrm{d}x + f(0) g(0), \quad \lambda > 0, \tag{11}$$

where the normalizing constants are chosen in such a way that $\langle 1, 1\rangle_{\mathrm{I}} = \langle 1, 1\rangle_{\mathrm{II}} = 1$. These Sobolev inner products are particular cases of (9) with $\theta_1 = \theta_2 = 1$.

A family of explicit orthonormal bases is constructed for both inner products. The bases in [7] are similar to the one constructed in [6]. These bases depend again on Jacobi polynomials. An interesting consequence of those explicit formulas is that the Fourier expansion of a function f with respect to these orthogonal bases can be computed without the use of the derivatives of f.

It is well known (see [11]) that, for $\mu > -1$, orthogonal polynomials of degree n with respect to the weight function $W_\mu(x) = (1 - \|x\|^2)^\mu$ on the unit ball in $\mathbb{R}^d$ satisfy a partial differential equation, which can be written in the following compact form:

$$\left[\Delta - \langle x, \nabla\rangle^2 - (2\mu + d)\langle x, \nabla\rangle\right] p = -n(n + 2\mu + d)p. \tag{12}$$

The singular case of the values $\mu = -1, -2, \ldots$ is studied in [5]. Explicit polynomial solutions are constructed and the equation for $\mu = -2, -3, \ldots$ has complete polynomial solutions if the dimension d is odd. An interesting result obtained by the authors is that the orthogonal polynomials with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathrm{II}}$ which were studied in [7], satisfy (12) for $\mu = -1$.

In [4], Lee and Littlejohn consider polynomials in two variables which satisfy an *admissible* (as defined in [9]) second-order partial differential equation of the form

$$ap_{xx} + 2bp_{xy} + cp_{yy} + dp_x + ep_y = \lambda p, \tag{13}$$

where a, b, and c are second-degree polynomials in x and y, d and e are polynomials of total degree one, λ is an eigenvalue parameter, and such that the polynomial satisfying (13) are orthogonal with respect to a symmetric bilinear form defined by

$$\phi(f, g) = \langle\sigma, fg\rangle + \langle\tau, f_x g_x\rangle, \tag{14}$$

with σ and τ being moment functionals acting on polynomials. This bilinear form is a particular case of (9) with $\theta_1 = 1$ and $\theta_2 = 0$.

They find conditions for the partial differential equation (13) to have polynomial solutions which are orthogonal with respect to a symmetric bilinear form $\phi(\cdot,\cdot)$. From these results they deduce that the moment functionals σ and τ are closely connected. In fact, if both linear functionals are quasi-definite, they can prove that, under some additional hypotheses, there exists a polynomial $f(x, y)$ of degree ≤ 2 such that $\tau = f(x, y)\sigma$ and, if $\{\mathbb{P}_n\}_{n\ge0}$ is a Sobolev OPS with respect to $\phi(\cdot,\cdot)$, then $\{\mathbb{P}_n\}_{n\ge0}$ is a WOPS with respect to σ, and $\{\partial_x\mathbb{P}_n\}_{n\ge0}$ contains a WOPS relative to τ.

However, the result does not reduce to the quasi-definite situation as they show in one of their examples. The differential equation

$$xp_{xx} + p_{yy} + xp_x - yp_y + np = 0,$$

has a PS $\{\mathbb{P}_n\}_{n\ge0}$ as solutions, with $\mathbb{P}_n = (P_{n,0}, \ldots, P_{0,n})^t$, where every polynomial $P_{n-k,k}$ is the product of a generalized Laguerre polynomial (of parameter $\alpha = -1$) and an Hermite polynomial. In this case σ and τ get the distributional representations

$$\begin{cases} \sigma = \delta(x) \otimes \mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}x\,\mathrm{d}y, \\ \tau = \mathrm{e}^{-x}\mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}x\,\mathrm{d}y. \end{cases}$$

4.2. *Sobolev orthogonality and second-order partial differential equations*

Let u and v be two moment functionals, and let

$$\Theta = \begin{pmatrix} \theta_0 & 0 \\ 0 & \theta_1 \end{pmatrix}$$

be a diagonal polynomial matrix, where $\theta_0(x, y)$ and $\theta_1(x, y)$ are fixed polynomials in two variables of degree less than or equal to 2, such that the expression

$$(f, g)_S = \langle u, fg\rangle + \langle v, (\nabla f)^t\Theta\nabla g\rangle \tag{15}$$

defines a Sobolev inner product. In this section, we will use the Pearson-type operators

$$P_1[p] = (ap)_x + (bp)_y - dp,$$
$$P_2[p] = (bp)_x + (cp)_y - ep,$$

and the second-order partial differential operator introduced in Section 3:

$$L[p] \equiv ap_{xx} + 2bp_{xy} + cp_{yy} + dp_x + ep_y. \tag{16}$$

Theorem 4.1. *The following statements are equivalent.*

(i) *L is symmetric with respect to the Sobolev inner product; that is,*

$$(L[p], q)_S = (p, L[q])_S, \quad \forall p, q \in \mathcal{P}.$$

(ii) *v satisfies the Pearson-type equations*

$$P_1[\theta_0 v] = a_x(\theta_0 v), \qquad P_1[\theta_1 v] = c_x(\theta_0 v) + 2b_y(\theta_1 v),$$
$$P_2[\theta_0 v] = a_y(\theta_1 v) + 2b_x(\theta_0 v), \qquad P_2[\theta_1 v] = c_y(\theta_1 v),$$

and the functionals u and v are related by

$$2P_1[u] + (a_y(\theta_1 v)_x)_y - (c_x(\theta_0 v)_y)_y - (a_{xy} - d_x)(\theta_1 v)_y + (c_{xy} - e_y)(\theta_0 v)_y = 0,$$
$$2P_2[u] + (c_x(\theta_0 v)_y)_x - (a_y(\theta_1 v)_x)_x - (c_{xy} - e_x)(\theta_0 v)_x + (a_{xy} - d_y)(\theta_1 v)_x = 0.$$

Proof. Using properties (3)–(5), we get

$$\begin{aligned}(L[p], q)_S &= \langle u, L[p]q\rangle + \langle v, (\nabla L[p])^t \Theta \nabla q\rangle \\ &= \langle L[p]u, q\rangle - \langle \operatorname{div}(\Theta \nabla L[p] v), q\rangle \\ &= \langle L[p]u - \operatorname{div}(\Theta \nabla L[p] v), q\rangle.\end{aligned}$$

On the other hand, using also (7), the second term expands as

$$\begin{aligned}(p, L[q])_S &= \langle u, pL[q]\rangle + \langle v, (\nabla p)^t \Theta \nabla L[q]\rangle \\ &= \langle L^*[pu], q\rangle - \langle L^*[\operatorname{div}(\Theta \nabla p v)], q\rangle \\ &= \langle L^*[pu] - L^*[\operatorname{div}(\Theta \nabla p v)], q\rangle.\end{aligned}$$

Therefore, $(L[p], q)_S = (p, L[q])_S$ is equivalent to

$$M[p] := L[p]u - \operatorname{div}(\Theta \nabla L[p] v) - L^*[pu] + L^*[\operatorname{div}(\Theta \nabla p v)] = 0, \quad \forall p \in \mathcal{P}.$$

Expanding $M[p]$ in terms of the partial derivatives of p, and after symbolic calculations, we get

$$M[p] = \sum_{m=0}^{3} \sum_{i=0}^{m} \alpha_i^m \partial_x^{m-i} \partial_y^i p,$$

where the coefficients of the partial derivatives of order three are

$$\alpha_0^3 = 2(P_1[\theta_0 v] - a_x \theta_0 v),$$
$$\alpha_1^3 = 2(P_2[\theta_0 v] - a_y \theta_1 v - 2b_x \theta_0 v),$$
$$\alpha_2^3 = 2(P_1[\theta_1 v] - c_x \theta_0 v - 2b_y \theta_1 v),$$
$$\alpha_3^3 = 2(P_2[\theta_1 v] - c_y \theta_1 v),$$

the coefficients of the partial derivatives of second order are

$$\alpha_0^2 = \frac{1}{2}(3\partial_x \alpha_0^3 + \partial_y \alpha_1^3),$$
$$\alpha_1^2 = \partial_x \alpha_1^3 + \partial_y \alpha_2^3,$$
$$\alpha_2^2 = \frac{1}{2}(\partial_x \alpha_2^3 + 3\partial_y \alpha_3^3),$$

the coefficients of the partial derivatives of first order are

$$\alpha_0^1 = \frac{1}{2}((\alpha_0^3)_{xx} + (\alpha_1^3)_{xy}) - 2P_1[u] - (a_y(\theta_1 v)_x)_y + (c_x(\theta_0 v)_y)_y + (a_{xy} - d_x)(\theta_1 v)_y - (c_{xy} - e_y)(\theta_0 v)_y,$$
$$\alpha_1^1 = \frac{1}{2}((\alpha_3^3)_{yy} + (\alpha_2^3)_{xy}) - 2P_2[u] - (c_x(\theta_0 v)_y)_x + (a_y(\theta_1 v)_x)_x + (c_{xy} - e_x)(\theta_0 v)_x - (a_{xy} - d_y)(\theta_1 v)_x,$$

and the term without derivatives is

$$\alpha_0^0 = -L^*[u].$$

Therefore, the condition $M[p] = 0$ is equivalent to $\alpha_i^m = 0$, for all $0 \le i \le m \le 3$, which is equivalent to (ii), and the theorem holds. □

The symmetry of the operator L and some additional hypotheses imply the Sobolev orthogonality of the solutions of the PDE.

Proposition 4.2. *Let L be a second-order partial differential operator as (16), and assume that it is symmetric with respect to the Sobolev inner product (15); that is,*

$$(L[p], q)_S = (p, L[q])_S, \quad \forall p, q \in \mathcal{P}. \tag{17}$$

Let $\{\mathbb{Q}_n\}_{n\ge 0}$ *be a PS and let* $\tilde{\Lambda}_n \in \mathcal{M}_{n+1}(\mathbb{R})$ *be a sequence of matrices with* $\tilde{\Lambda}_0 = (0)$ *satisfying*

$$\sigma(\tilde{\Lambda}_n) \cap \sigma(\tilde{\Lambda}_m) = \emptyset, \quad \forall n \ne m, \tag{18}$$

$$L[\mathbb{Q}_n] = \tilde{\Lambda}_n \mathbb{Q}_n, \quad n \ge 0, \tag{19}$$

where $\sigma(\tilde{\Lambda}_n)$ *denotes the spectrum of the matrix* $\tilde{\Lambda}_n$*. Then* $\{\mathbb{Q}_n\}_{n\ge 0}$ *is a WOPS for* $(\cdot, \cdot)_S$*.*

Proof. The symmetry of L on polynomials is equivalent to

$$(L[\mathbb{Q}_n], \mathbb{Q}_m^t)_S = (\mathbb{Q}_n, L[\mathbb{Q}_m^t])_S, \quad \forall m, n \geq 0;$$

therefore,

$$\tilde{\Lambda}_n (\mathbb{Q}_n, \mathbb{Q}_m^t)_S = (\mathbb{Q}_n, \mathbb{Q}_m^t)_S \tilde{\Lambda}_m^t, \quad \forall m, n \geq 0.$$

If $m \neq n$, the matrix $(\mathbb{Q}_n, \mathbb{Q}_m^t)_S$ is a solution of the equation $\tilde{\Lambda}_n X - X \tilde{\Lambda}_m^t = 0$. Then, using (18) and Theorem 4.4.6 in [20], we conclude that $(\mathbb{Q}_n, \mathbb{Q}_m^t)_S = 0$, $\forall n \neq m$.

Finally, since $(\cdot, \cdot)_S$ is an inner product and $\{\mathbb{Q}_n\}_{n\geq 0}$ is a PS, we can easily deduce that the matrix $\tilde{H}_n = (\mathbb{Q}_n, \mathbb{Q}_n^t)_S$ is non-singular. □

The next theorem provides a partial reciprocal for the above proposition.

Theorem 4.3. *Let $\{\mathbb{Q}_n\}_{n\geq 0}$ be a Sobolev WOPS associated with the Sobolev inner product (15). Then the following statements are equivalent.*

(i) *L is symmetric; that is,*

$$(L[p], q)_S = (p, L[q])_S, \quad \forall p, q \in \mathcal{P}.$$

(ii) *For $n \geq 1$, there exist $\tilde{\Lambda}_n \in \mathcal{M}_{n+1}(\mathbb{R})$ satisfying*

$$L[\mathbb{Q}_n] = \tilde{\Lambda}_n \mathbb{Q}_n,$$

and $\tilde{\Lambda}_n \tilde{H}_n = \tilde{H}_n \tilde{\Lambda}_n^t$, where $\tilde{H}_n = (\mathbb{Q}_n, \mathbb{Q}_n^t)_S$.

Proof. (i) ⇒ (ii) Since the operator L preserves the degree of the polynomials,

$$L[\mathbb{Q}_n] = \sum_{i=0}^{n} A_i \mathbb{Q}_i,$$

where $A_i \in \mathcal{M}_{(n+1)\times(i+1)}(\mathbb{R})$. For $m \leq n$, using the Sobolev inner product, we obtain

$$(L[\mathbb{Q}_n], \mathbb{Q}_m^t)_S = \sum_{i=0}^{n} A_i (\mathbb{Q}_i, \mathbb{Q}_m^t)_S = A_m \tilde{H}_m.$$

On the other hand, using the symmetry of L, we get

$$(L[\mathbb{Q}_n], \mathbb{Q}_m^t)_S = (\mathbb{Q}_n, L[\mathbb{Q}_m^t])_S = 0, \quad \text{for } m < n.$$

This implies that $A_m = 0$, for $m < n$, and so $L[\mathbb{Q}_n] = A_n \mathbb{Q}_n$. Then (ii) holds with $\tilde{\Lambda}_n = A_n$.

(ii) ⇒ (i) It suffices to prove (i) for a basis of polynomials. Obviously,

$$(L[\mathbb{Q}_n], \mathbb{Q}_m^t)_S = \tilde{\Lambda}_n (\mathbb{Q}_n, \mathbb{Q}_m^t)_S = \begin{cases} 0, & n \neq m, \\ \tilde{\Lambda}_n \tilde{H}_n, & n = m, \end{cases}$$

$$(\mathbb{Q}_n, L[\mathbb{Q}_m^t])_S = (\mathbb{Q}_n, \mathbb{Q}_m^t)_S \tilde{\Lambda}_m^t = \begin{cases} 0, & n \neq m, \\ \tilde{H}_n \tilde{\Lambda}_n^t, & n = m, \end{cases}$$

and (i) clearly follows. □

5. Examples

5.1. Classical orthogonal polynomials on the simplex

Classical polynomials on the simplex (see [11]) are orthogonal with respect to the weight function $\omega(x, y) = x^\alpha y^\beta (1 - x - y)^\gamma$, with $\alpha, \beta, \gamma > -1$, on the standard simplex

$$T = \{(x, y) : x, y \geq 0, 1 - x - y \geq 0\}.$$

They satisfy the partial differential equation

$$\begin{aligned} L[p] \equiv\ & x(1-x)\partial_{xx} p - 2xy\partial_{xy} p + y(1-y)\partial_{yy} p + ((\alpha+1) - (\alpha+\beta+\gamma+3)x)\, \partial_x p \\ & + ((\beta+1) - (\alpha+\beta+\gamma+3)y)\, \partial_y p = \lambda_n p, \end{aligned} \tag{20}$$

where $\lambda_n = -n(n + \alpha + \beta + \gamma + 3)$ and $n = \deg\ p$.

An orthogonal basis of polynomials is given by the following set of monic polynomials [11]:

$$V_{n-k,k}^{(\alpha,\beta,\gamma)}(x,y) = \sum_{i=0}^{n-k}\sum_{j=0}^{k}(-1)^{(n+i+j)}\binom{n-k}{i}\binom{k}{j}\frac{(\alpha+i+1)_{n-k-i}(\beta+j+1)_{k-j}}{(\alpha+\beta+\gamma+n+i+j+2)_{n-i-j}}x^i y^j, \tag{21}$$

for $n \geq k \geq 0$. From this expression, we can easily deduce that

$$\frac{\partial}{\partial x}V_{n-k,k}^{(\alpha,\beta,\gamma)}(x,y) = (n-k)V_{n-k-1,k}^{(\alpha+1,\beta,\gamma+1)}(x,y), \tag{22}$$

$$\frac{\partial}{\partial y}V_{n-k,k}^{(\alpha,\beta,\gamma)}(x,y) = kV_{n-k,k-1}^{(\alpha,\beta+1,\gamma+1)}(x,y), \tag{23}$$

and therefore they are orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int_T fgx^\alpha y^\beta (1-x-y)^\gamma \,\mathrm{d}x\,\mathrm{d}y + \int_T (\nabla f)^t \Theta (\nabla g) x^\alpha y^\beta (1-x-y)^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\Theta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

In the singular case $\gamma = -1$, the monic PS

$$\mathbb{Q}_n = (V_{n,0}^{(\alpha,\beta,-1)}, V_{n-1,1}^{(\alpha,\beta,-1)}, \ldots, V_{0,n}^{(\alpha,\beta,-1)})^t, \quad n \geq 0,$$

defined by (21), still satisfies the partial differential equation (20), with $\tilde{\Lambda}_n = \lambda_n I_{n+1}$. However, $\{\mathbb{Q}_n\}_{n\geq 0}$ is not a sequence of orthogonal polynomials on the simplex since $\omega(x,y)$ for $\gamma = -1$ is not a weight function on T.

On the other hand, $\lambda_n \neq \lambda_m$ for $n \neq m$, and then Theorem 4.1 and Proposition 4.2 provide the orthogonality with respect to the Sobolev inner product

$$(f,g)_S = \langle u, fg\rangle + \langle v, (\nabla f)^t \Theta (\nabla g)\rangle,$$

where u and v satisfy

$$\begin{cases} (x-x^2)u_x - xyu_y = (\alpha - (\alpha+\beta-1)x)u, \\ -xyu_x + (y-y^2)u_y = (\beta - (\alpha+\beta-1)y)u, \end{cases} \tag{24}$$

and

$$\begin{cases} (x-x^2)v_x - xyv_y = (\alpha - (\alpha+\beta)x)v, \\ -xyv_x + (y-y^2)v_y = (\beta - (\alpha+\beta)y)v. \end{cases} \tag{25}$$

A solution for (25) is the usual moment functional on the simplex for $\gamma = 0$:

$$\langle v, f\rangle = \int_T f(x,y)x^\alpha y^\beta \,\mathrm{d}x\,\mathrm{d}y,$$

and a very simple computation shows that a solution of (24) can be given by means of

$$\langle u, f\rangle = \int_0^1 f(x, 1-x)x^\alpha (1-x)^\beta \,\mathrm{d}x.$$

Observe that u is non-positive definite and $(1-x-y)u = 0$. Then, the *generalized simplex polynomials* $\{V_{n-k,k}^{(\alpha,\beta,-1)}(x,y)\}_{n\geq k\geq 0}$ are orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int_0^1 f(x,1-x)g(x,1-x)x^\alpha(1-x)^\beta \,\mathrm{d}x + \int_T (\nabla f)^t \Theta (\nabla g) x^\alpha y^\beta \,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\Theta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

5.2. Sobolev orthogonal polynomials on the ball

Now, we consider classical orthogonal polynomials on the unit ball. Using our results, we will provide a Sobolev orthogonality for those PSs corresponding to non-standard values of the parameter.

Typically, classical orthogonal polynomials on the unit ball in $\mathbb{R}^2$ are related to the weight function defined by

$$\omega_\mu(x,y) = (1-x^2-y^2)^\mu, \quad \mu > -1.$$

In [10], an orthogonal basis for this weight function is given, and it can be written in terms of monic Gegenbauer polynomials as

$$P_{n,k}^{(\mu)}(x,y) = C_{n-k}^{\left(\mu+k+\frac{1}{2}\right)}(x)(1-x^2)^{k/2} C_k^{(\mu)}(y(1-x^2)^{-1/2}).$$

Here, $C_n^{(\mu)}(x)$ stands for the n-th monic Gegenbauer polynomial which is orthogonal with respect to the weight function $(1-x^2)^\mu$ on the interval $[-1,1]$.

From this expression, and using the well-known differential properties of Gegenbauer polynomials, it is easy to see that the partial derivatives with respect to x and y are

$$\frac{\partial}{\partial x} P_{n,k}^{(\mu)}(x,y) = (n-k)P_{n-1,k}^{(\mu+1)}(x,y) + \alpha_{n,k}^{(\mu)} P_{n-1,k-2}^{(\mu+1)}(x,y),$$
$$\frac{\partial}{\partial y} P_{n,k}^{(\mu)}(x,y) = k P_{n-1,k-1}^{(\mu+1)}(x,y),$$

and so the PS $\{\mathbb{Q}_n^{(\mu)}\}_{n\ge 0}$, given by $\mathbb{Q}_n^{(\mu)} = (P_{n,0}^{(\mu)}, \ldots, P_{n,n}^{(\mu)})^t$, is a WOPS associated with the Sobolev inner product

$$(f,g)_S = \int_{B^2} fg(1-x^2-y^2)^\mu \mathrm{d}x\mathrm{d}y + \int_{B^2} (f_x g_x + f_y g_y)(1-x^2-y^2)^{\mu+1} \mathrm{d}x\mathrm{d}y.$$

Moreover, it is well known (see [11]) that $P_{n,k}^{(\mu)}$ satisfies the partial differential equation

$$L[p] \equiv (x^2-1)p_{xx} + 2xyp_{xy} + (y^2-1)p_{yy} + (3+2\mu)xp_x + (3+2\mu)yp_y = \lambda_n p, \tag{26}$$

with $\lambda_n = n(n+2\mu+2)$.

Now, let $\mu = -1$. Then, classical orthogonality on the ball does not hold any more, since ω_μ is not a weight function. Nevertheless, taking into account the hypergeometric representation for Gegenbauer polynomials [21]

$$C_n^{(\mu)}(x) = \frac{n!}{2^n \Gamma(n+\mu+1/2)} \sum_{m=0}^{[n/2]} \frac{(-1)^m \Gamma(n-m+\mu+1/2)}{m!(n-2m)!}(2x)^{n-2m}, \quad n \ge 0,$$

the definition of $P_{n,k}^{(\mu)}(x,y)$ makes perfect sense even in the case when $\mu = -1$. Also, the polynomials $P_{n,k}^{(\mu)}(x,y)$ still satisfy (26).

Then, using Theorem 4.1 for the Sobolev inner product (15), with $\Theta = I_2$, the symmetry of L with respect to $(\cdot,\cdot)_S$ holds if and only if the functionals u and v satisfy

$$\begin{cases} (x^2-1)u_x + xyu_y = -2xu, \\ xyu_x + (y^2-1)u_y = -2yu, \end{cases} \tag{27}$$

$$\begin{cases} (x^2-1)v_x + xyv_y = 0, \\ xyv_x + (y^2-1)v_y = 0. \end{cases} \tag{28}$$

Moreover, since $\lambda_n \neq \lambda_m$ for $n \neq m$, Proposition 4.2 guarantees that $\{\mathbb{Q}_n^{(-1)}\}_{n\ge 0}$ is a WOPS with respect to $(\cdot,\cdot)_S$.

In such a case, the linear functional v, as a solution of (28), is still classical, and it is associated with the weight function $\omega_0(x,y) = 1$,

$$\langle v, p\rangle = \int_{B^2} p(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

However, the Pearson-type equation for the linear functional u does not admit a classical solution of this kind, but we can give a solution of (27) via a line integral over the unit sphere on $\mathbb{R}^2$, $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$:

$$\langle u, f\rangle = \int_{S^1} f(x,y)\,\mathrm{d}\omega = \int_0^{2\pi} f(\cos(t), \sin(t))\mathrm{d}t,$$

where $\mathrm{d}\omega$ stands for the measure on S^1.

In that way, generalized ball polynomials $\{\mathbb{Q}_n^{(-1)}\}_{n\ge 0}$ can be seen as a WOPS associated with the Sobolev inner product

$$(f,g)_S = \int_{S^1} f(x,y)g(x,y)\mathrm{d}\omega + \int_{B^2} (\nabla f)^t(\nabla g)\,\mathrm{d}x\,\mathrm{d}y.$$

We should mention here that this Sobolev inner product is a particular case of (10), which was previously studied in [7,5].

5.3. Class III of Koornwinder polynomials

In [10], Koornwinder studied seven *examples of two-variable analogues of the Jacobi polynomials*. For $\alpha, \beta > -1$, if we denote by $P_n^{(\alpha,\beta)}$ the standard Jacobi polynomial on $[-1, 1]$, the Koornwinder polynomials of *Class* III are given by

$$P_{n,k}^{(\alpha,\beta)}(x, y) = P_{n-k}^{\left(\alpha,\beta+k+\frac{1}{2}\right)}(2x-1)x^{\frac{1}{2}k}P_k^{(\beta,\beta)}\left(x^{-\frac{1}{2}}y\right), \quad n \geq k \geq 0,$$

and they are orthogonal with respect to the weight function $\omega(x, y) = (1-x)^{\alpha}(x-y^2)^{\beta}$, on the region

$$R = \{(x, y) : y^2 < x < 1\},$$

which is bounded by a straight line and a parabola.

Koornwinder showed that these polynomials satisfy the second-order partial differential equation

$$L[p] \equiv 2x(1-x)p_{xx} + 2y(1-x)p_{xy} + \frac{1}{2}(1-x)p_{yy} + (2\beta+3-(2\alpha+2\beta+5)x)p_x - (\alpha+1)yp_y = \lambda_{n,k}p, \tag{29}$$

where $\lambda_{n,k} = -(n-k)(2n+2\alpha+2\beta+3) - (1+\alpha)k$, $k = 0, 1, \ldots, n$.

Moreover, using the well-known property for the derivatives of Jacobi polynomials, we get

$$\frac{\partial}{\partial y}P_{n,k}^{(\alpha,\beta)}(x, y) = \frac{1}{2}(k+2\beta+1)P_{n-1,k-1}^{(\alpha,\beta+1)}(x, y). \tag{30}$$

Then, Koornwinder polynomials of Class III are orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int_R fg(1-x)^{\alpha}(x-y^2)^{\beta}\, dx\, dy + \int_R (\nabla f)^t \Theta (\nabla g)(1-x)^{\alpha}(x-y^2)^{\beta+1}\, dx\, dy,$$

where

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, for $\alpha > -1$ and $\beta = -1$, the polynomials

$$P_{n,k}^{(\alpha,-1)}(x, y) = P_{n-k}^{(\alpha,k-1/2)}(2x-1)x^{k/2}P_k^{(-1,-1)}(x^{-1/2}y), \quad n \geq k \geq 0,$$

where

$$P_n^{(-1,-1)}(x) = (x-1)^n {}_2F_1\left(-n, -n+1; -2n+2; \frac{2}{1-x}\right),$$

define a PS which even satisfies (29) and (30).

Consider the Sobolev inner product

$$(f, g)_S = \langle u, fg\rangle + \langle v, (\nabla f)^t \Theta \nabla g\rangle,$$

where

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using Theorem 4.1, L is symmetric if and only if u and v satisfy

$$\begin{cases} 2x(1-x)u_x + y(1-x)u_y = -2((\alpha-1)x+1)u, \\ y(1-x)u_x + \dfrac{1}{2}(1-x)u_y = -\alpha yu \end{cases}$$

and

$$\begin{cases} 2x(1-x)v_x + y(1-x)v_y = -2\alpha xv, \\ y(1-x)v_x + \dfrac{1}{2}(1-x)v_y = -\alpha yv. \end{cases}$$

A solution for the above equations is

$$\langle v, f\rangle = \int_R f(x, y)(1-x)^{\alpha}\, dx\, dy,$$

and

$$\langle u, f\rangle = \int_{-1}^{1} f(y^2, y)(1-y^2)^{\alpha}\, dy,$$

and a direct computation shows that the PS $\{P_{n,k}^{(\alpha,-1)}\}_{n\geq k\geq 0}$ is orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int_{-1}^{1} f(y^2, y)g(y^2, y)(1-y^2)^{\alpha}\, dy + \int_R f_y(x, y)g_y(x, y)(1-x)^{\alpha}\, dx\, dy.$$

Acknowledgements

Cleonice F. Bracciali was partially supported by grants from CAPES and CNPq from Brazil. Antonia M. Delgado was partially supported by the Spanish Ministry of Education and Science, grant MTM2006-13000-C03-02, and Junta de Andalucía, Grupo de Investigación FQM 0229. Lidia Fernández, Teresa E. Pérez and Miguel A. Piñar were partially supported by the Ministerio de Ciencia e Innovación (Micinn) of Spain and by the European Regional Development Fund (ERDF), grant MTM2008-06689-C02-02, and Junta de Andalucía, Grupo de Investigación FQM 0229.

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