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Cohomology of semi 1-coronae and extension of analytic subsets

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Abstract

We deal with the cohomology of semi 1-coronae. Semi 1-coronae are domains whose boundary is the union of a Levi flat part, a 1-pseudoconvex part and a 1-pseudoconcave part. Using the main result in [C. Laurent-Thiébaut, J. Leiterer, Uniform estimates for the Cauchy–Riemann equation on q-concave wedges, in: Colloque d'Analyse Complexe et Géométrie, Marseille, 1992, Astérisque 217 (7) (1993) 151–182], we prove a bump lemma for compact semi 1-coronae in $\mathbb{C}^n$ and then, applying Andreotti–Grauert theory, we get a cohomology finiteness theorem for coherent sheaves whose depth is at least 3. As an application we get an extension theorem for coherent sheaves and analytic subsets.

Résumé

On s'interesse à la cohomologie des semi 1-coronae. On appelle semi 1-corona tout domain dont la frontière a une partie 1-pseudoconvexe, une partie 1-pseudoconcave et une partie Levi plate. En utilisant le résultat principal de [C. Laurent-Thiébaut, J. Leiterer, Uniform estimates for the Cauchy–Riemann equation on q-concave wedges, in : Colloque d'Analyse Complexe et Géométrie, Marseille, 1992, Astérisque 217 (7) (1993) 151–182], on démontre un « bump lemma » pour des semi 1-coronae compactes dans $\mathbb{C}^n$, qui permet de démontrer, à l'aide de la théorie d'Andreotti–Grauert, un théorème de finitude pour la cohomologie d'un faisceau cohérent de profondeur au moins 3. On en déduit comme aplication un théorème d'extension pour les faisceaux cohérents et les sous-ensembles analytiques.

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1. Introduction and notations

Let X be a (connected and reduced) complex space. We recall that X is said to be *strongly q-pseudoconvex* in the sense of Andreotti–Grauert [1] if there exist a compact subset K and a smooth function $\varphi : X \to \mathbb{R}$, $\varphi \geqslant 0$, which is strongly q-plurisubharmonic on $X \setminus K$ and such that

(a) for every $b > \max_K \varphi$ the subset
$$B_b = \{x \in X\colon \varphi(x) < b\}$$
is relatively compact in X.

If $K = \emptyset$, X is said to be *q-complete*. We remark that, for a space X, being 1-complete is equivalent to be Stein. Replacing the condition (a) by

(a′) for every a, $\min_X \varphi < a < \min_K \varphi$, and every $b > \max_K \varphi$ the subset
$$C_{a,b} = \{x \in X\colon a < \varphi(x) < b\}$$
is relatively compact in X,

we obtain the notion of *q-corona* (see [1,2]).

A q-corona is said to be *complete* whenever $K = \emptyset$.

The cohomology of coherent sheaves defined on q-coronae was studied in [2].

In this paper we will focus on *semi q-coronae*, which are defined as follows.

Consider a strongly q-pseudoconvex space (or, more generally, a q-corona) X, and a smooth function $\varphi : X \to \mathbb{R}$ displaying the q-pseudoconvexity of X. Let $C_{a,b}$ be a q-corona of X and let $h : X \to \mathbb{R}$ be a pluriharmonic function (i.e. locally the real part of a holomorphic function) such that: $K \cap \{h = 0\} = \emptyset$, $\{h = 0\}$ and the boundaries of B_b and B_a are $\mathbb{C}$-transversal. A connected component $C^+_{a,b}$ of $C_{a,b} \setminus \{h = 0\}$ is, by definition, a *semi q-corona*.

At the origin of the interest for domains whose boundary contains a Levi flat part there is an extension theorem for CR-functions proved in [11] (see also [8,10,13]).

A more general type of semi q-corona is obtained by replacing the zero set of h with a Levi flat hypersurface H.

In both cases the semi q-coronae are differences $B^+_b \setminus \overline{B}^+_a$ where B^+_b, B^+_a are strongly q-pseudoconvex spaces. Indeed, the function $\psi = -\log h^2$ (respectively $\psi = -\log \delta_H(z)$, where $\delta_H(z)$ is the distance of z from H) is plurisubharmonic in $W \setminus \{h = 0\}$ (respectively in $W \setminus H$) where W is a neighborhood of $B_b \cap \{h = 0\}$ (respectively of $B_b \cap H$). Let $\chi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function such that $\chi \circ \varphi > \psi$ on a neighborhood of $B_b \setminus W$. The function $\Phi = \sup(\chi \circ \varphi, \psi) + \varphi$ is an exhaustion function for $B_b \setminus \{h = 0\}$ (respectively for $B_b \setminus H$) and it is strongly q-plurisubharmonic in $B_b \setminus (\{h = 0\} \cup K)$ (respectively in $B_b \setminus H \cup K$).

Results on the cohomology of coherent sheaves on semi q-coronae were obtained by the authors in [12] under the hypothesis that the sheaves are defined on the larger set B_b.

The aim of this paper is to give a generalization for coherent sheaves $\mathcal{F}$ defined only on the semi q-corona. For the sake of simplicity we restrict ourselves to the case of smooth semi 1-coronae.

Following Andreotti–Grauert (see [1]), given a semi 1-corona
$$C^+_{a,b} = C_{a,b} \cap \{h > 0\},$$

where h is pluriharmonic, and a coherent sheaf $\mathcal{F}$ on $C^+_{a,b}$ we consider the strongly plurisubharmonic functions $P_\varepsilon(z) = \varepsilon|z|^2 - h(z)$, $\varepsilon > 0$, and an exhaustion of $C^+_{a,b}$ by the following relatively compact domains

$$C^+_\varepsilon = \big\{z \in \mathbb{C}^n\colon\ P_\varepsilon(z) < 0\big\} \cap \overline{C}_{a+\varepsilon,b-\varepsilon}.$$

The idea is to prove for the domains C^+_ε a bump lemma and an approximation theorem as in the classical case of coronae. Here the situation is more complicated because of the presence of a non-empty pseudoconvex–pseudoconcave part in the boundary of each C^+_ε. In order to circumvent this difficulty, we work with the closed sets $\overline{C}^+_\varepsilon$ using in a crucial way a regularity result on $\bar\partial$-equation due to Laurent-Thiébaut and Leiterer (see Section 3). This enables us to prove the following results: assume that depth $\mathcal{F}_z \geqslant 3$ for z near to the pseudoconcave part of the boundary of $C^+_{a,b}$; then

1) if ε is sufficiently small and $\varepsilon' < \varepsilon$ is near ε
$$H^1(\overline{C}^+_{\varepsilon'}, \mathcal{F}) \simeq H^1(\overline{C}^+_\varepsilon, \mathcal{F})$$
2) the cohomology spaces $H^1(\overline{C}^+_\varepsilon, \mathcal{F})$ are finite dimensional

(see Lemmas 3.3, 3.9 and Proposition 3.10).

Thus the function

$$d(\varepsilon) = \dim_{\mathbb{C}} H^1(\overline{C}^+_\varepsilon, \mathcal{F})$$

is piecewise constant, but, in general, it could have frequently a "jump-discontinuity" and it could happen that $d(\varepsilon) \to +\infty$ (see Remark 3.2). Nevertheless, the isomorphism 1) allows us to prove in the last section:

(1) the fact that Oka–Cartan–Serre Theorem A holds in semi 1-coronae for sheaves which satisfy the condition of Theorem 3.1 (see Theorem 4.3);
(2) an extension theorem for analytic subsets (see Corollary 4.4).

It is worth noticing that an extension theorem for codimension one analytic subsets of a non-singular semi 1-corona was proved in [12] and for higher codimensions, using different methods based on Harvey–Lawson's theorem [7], by Della Sala and the first author in [4].

2. Remarks on the proofs of theorems in [12]

Let X be a complex space. For every coherent sheaf $\mathcal{F}$ on X and every subset A of X we set

$$p(A;\mathcal{F}) = \inf_{x\in A} \operatorname{depth}(\mathcal{F}_x)$$
$$p(A) = p(A;\mathcal{O}).$$

Let $C = C_{a,b}$ be a q-corona of X. All the results in [12] on finite and/or vanishing cohomology for q-coronae and semi q-coronae are obtained using Andreotti–Grauert methods. They consist of two main point

(i) the bump lemma;

(ii) for every corona $C_{a',b'} \Subset C$ there exists a corona $C_{a'+\varepsilon,b'+\varepsilon} \Subset C$, $\varepsilon > 0$ such that the homomorphism

$$H^r(C_{a'-\varepsilon,b'+\varepsilon}, \mathcal{F}) \longrightarrow H^r(C_{a',b'}, \mathcal{F})$$

is bijective for $q \leqslant r \leqslant p(C; \mathcal{F})$.

As a matter of fact the method of proof shows that the condition on the depth is needed only on $C_{a,a'}$ i.e. the homomorphism

$$H^r(C_{a'-\varepsilon,b'+\varepsilon}, \mathcal{F}) \longrightarrow H^r(C_{a',b'}, \mathcal{F})$$

is bijective for $q \leqslant r \leqslant p(C_{a,a'}; \mathcal{F})$.

Let X be a strongly q-pseudoconvex space (respectively $X \subset \mathbb{C}^n$ be a strongly q-pseudoconvex open set) and $H = \{h = 0\}$ where h is pluriharmonic in X (respectively H Levi-flat), and $C = C_{a,b} = B_b \setminus \overline{B}_a$ a q-corona. We can suppose that $B_b \setminus H$ has two connected components, B^+ and B^-, and we define $C^+ = B^+ \cap C$, $C^- = B^- \cap C$.

From the above remark we derive the following improvements of Theorem 1, Corollary 2 and Theorem 3 in [12].

Theorem 2.1. *Let $\mathcal{F} \in \mathsf{Coh}(B_b)$. Then the image of the homomorphism*

$$H^r(\overline{B}^+, \mathcal{F}) \oplus H^r(\overline{C}, \mathcal{F}) \longrightarrow H^r(\overline{C}^+, \mathcal{F})$$

(*all closures are taken in B_b*), *defined by $(\xi \oplus \eta) \mapsto \xi_{|\overline{C}^+} - \eta_{|\overline{C}^+}$ has finite codimension provided that $q - 1 \leqslant r \leqslant p(\overline{B}_a; \mathcal{F}) - q - 2$.*

Corollary 2.2. *If $K \cap H = \emptyset$, under the same assumption of Theorem* 2.1

$$\dim_{\mathbb{C}} H^r(\overline{C}^+, \mathcal{F}) < \infty$$

for $q \leqslant r \leqslant p(\overline{B}_a; \mathcal{F}) - q - 2$.

Theorem 2.3. *If $\overline{B}_+$ is a q-complete space, then*

$$H^r(\overline{C}, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}^+, \mathcal{F})$$

for $q \leqslant r \leqslant p(\overline{B}_a; \mathcal{F}) - q - 2$ and the homomorphism

$$H^{q-1}(\overline{B}^+, \mathcal{F}) \oplus H^{q-1}(\overline{C}, \mathcal{F}) \longrightarrow H^{q-1}(\overline{C}^+, \mathcal{F}) \tag{1}$$

is surjective for $p(\overline{B}_a; \mathcal{F}) \geqslant 2q + 1$.

If $\overline{B}^+$ is a 1*-complete space and $p(\overline{B}_a; \mathcal{F}) \geqslant 3$, then*

$$H^0(\overline{B}^+, \mathcal{F}) \xrightarrow{\sim} H^0(\overline{C}^+, \mathcal{F}).$$

This implies the following. Let $C_1 = B_{b_1} \setminus \overline{B}_{a_1} \Subset C_2 = B_{b_2} \setminus \overline{B}_{a_2}$. Then

$$H^r(\overline{C}_1^+, \mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_2^+, \mathcal{F})$$

for $q \leqslant r \leqslant p(\overline{B}_{a_1}; \mathcal{F})$.
In particular, if $x \in C_2 \setminus \overline{B}_{a_1}$ and $\mathcal{M}_{\{x\}}$ denotes the sheaf of ideals of $\{x\}$, then

$$H^r(\overline{C}_2^+, \mathcal{M}_{\{x\}}\mathcal{F}) \xrightarrow{\sim} H^r(\overline{C}_1^+, \mathcal{F}),$$

for $q \leqslant r \leqslant p(\overline{B}_{a_1}; \mathcal{F})$.

3. An isomorphism theorem for semi 1-coronae

Our aim is to give a generalization of the above results for sheaves defined only on the semi q-coronae, i.e. for the case when the "hole" is real. For the sake of simplicity we will consider only complete 1-coronae in $\mathbb{C}^n$ with $n \geqslant 3$. So we consider connected 1-coronae of the form

$$C = \{z \in \mathbb{C}^n \colon 0 < \varphi(z) < 1\} \Subset \mathbb{C}^n,$$

where $\varphi : \mathbb{C}^n \to \mathbb{R}$ is a smooth strongly plurisubharmonic function in a Stein neighborhood U of $\{0 \leqslant \varphi \leqslant 1\}$, $d\varphi \neq 0$ on $\varphi = 0, 1$. Let h be a pluriharmonic function on U and H be the zero set of h. We assume that H is smooth and transversal to the hypersurfaces $\{\varphi = 0\}$, $\{\varphi = 1\}$, that $U \setminus H$ has two connected components $U^{\pm}$ and $h > 0$ on U^+. For $0 < a < b < 1$ we set

$$B_b = \{z \in U \colon \varphi < b\}, \quad B_b^+ = B_b \cap U^+, \quad C_{a,b} = (B_b \setminus \overline{B}_a), \quad C_{a,b}^+ = C_{a,b} \cap U^+.$$

Let $P_\varepsilon(z) = \varepsilon |z|^2 - h(z)$; then there is ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ the hypersurfaces $\{\varphi = \varepsilon\}$, $\{\varphi = 1 - \varepsilon\}$ meet $\{P_\varepsilon = 0\}$ transversally. Finally we define the following subsets (which are locally 1-*convex*, 1-*concave*, see [9] and Remark 3.1 below)

$$\overline{C}_\varepsilon^+ = \{z \in \mathbb{C}^n \colon P_\varepsilon(z) \leqslant 0\} \cap \overline{C}_{\varepsilon, 1-\varepsilon}.$$

We want to prove the following

Theorem 3.1. *Let C^+ be a semi* 1*-corona in $\mathbb{C}^n$. Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\bar{\varepsilon} \in [0, \varepsilon)$ such that for every $\mathcal{F} \in \mathsf{Coh}(C^+)$ satisfying*

$$\overline{\{z \in C^+ \colon \operatorname{depth}(\mathcal{F}_z) < 4\}} \cap B_{\varepsilon_0} = \emptyset$$

and every $\varepsilon' \in (\bar{\varepsilon}, \varepsilon)$ the homomorphism

$$H^1(C_{\varepsilon'}^+, \mathcal{F}) \longrightarrow H^1(\overline{C}_\varepsilon^+, \mathcal{F})$$

is an isomorphism.

The main ingredients for the proof are the bump lemma and a density theorem as in Andreotti–Grauert [1]. In order to treat points belonging to the pseudoconvex–pseudoconcave part of the boundary we work with closed bumps using the following result due to Laurent-Thiébaut and Leiterer (see [9, Proposition 7.5]):

Proposition 3.2. *Let $D \Subset \mathbb{C}^n$ be a* 1*-concave,* 1*-convex domain of order* 1 *of special type, and suppose that $n \geqslant 3$. If f is a continuous (n, r)-form in some neighborhood $U_{\overline{D}}$ of $\overline{D}$, $1 \leqslant r \leqslant n-2$, such that $\bar{\partial} f = 0$ in $U_{\overline{D}}$, then there exists a form $u \in \bigcap_{\varepsilon > 0} C_{n,r-1}^{1/2-\varepsilon}(\overline{D})$ such that $\bar{\partial} u = f$ in D.*

Remark 3.1. Proposition 7.5 in [9] is much more general, but we state it this way, since the semi 1-coronae we consider are locally 1-concave, 1-convex domain of order 1 of special type, i.e. they are locally biholomorpic to the set-difference of two convex domains.

The proof of Theorem 3.1 is a consequence of several intermediate results.

3.1. Bump lemma: surjectivity of cohomology

With the same notations as above let $\overline{D} = \overline{C}^+_\varepsilon$, $0 < \varepsilon < \varepsilon_0$ where $\varepsilon_0 < b$ is so chosen that for all $\varepsilon \in (0, \varepsilon_0)$ the hypersurfaces $\{\varphi = \varepsilon\}$, $\{\varphi = 1 - \varepsilon\}$ are $\mathbb{C}$-transversal to $\{P_\varepsilon = 0\}$. Let Γ_1, Γ_2 be respectively the pseudoconvex and the pseudoconcave part of the boundary $\mathrm{b}D$ of $\overline{D}$. Thus $\mathrm{b}D = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $\overline{\Gamma}_2$ is contained in the smooth hypersurface $\{\varphi = \varepsilon\}$.

Lemma 3.3 *(bump lemma). There exists a finite closed covering $\mathcal{U}$ of $\mathrm{b}D$, $\mathcal{U} = \{\overline{U}_j\}_{1 \leqslant j \leqslant m}$, and relatively compact domains $D_1, \ldots, D_m$ of $\mathbb{C}^n$ such that*

(i) $\overline{D} = \overline{D}_0 \subset \overline{D}_1 \subset \cdots \subset \overline{D}_m$;
(ii) $\overline{D} \subset D_m$;
(iii) $\overline{D}_j \setminus \overline{D}_{j-1} \subset \overline{U}_j$ *for* $1 \leqslant j \leqslant m$;
(iv) *if* $\mathcal{F} \in \mathsf{Coh}(C^+)$ *then*

$$H^r(\overline{U}_j \cap \overline{D}_k, \mathcal{F}) = 0$$

for every j, k *and* $1 \leqslant r \leqslant p(\overline{D}; \mathcal{F}) - 2$.

Moreover, the family of the coverings $\mathcal{U}$ as above is cofinal in the family of all finite coverings of $\mathrm{b}D$.

Proof. If $z^0 \in \Gamma_1 \cup \Gamma_2$ i.e. z^0 is a point of pseudoconvexity or pseudoconcavity we argue as in the proof of the classical Andreotti–Grauert bump lemma.

Assume that $z^0 \in \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. There exists a sufficiently small closed ball $\overline{B}$ of positive radius, centered at z^0 and a biholomorphism on $\Phi : \overline{B} \to \Phi(\overline{B})$ which transforms $\overline{B} \cap \{\varphi \geqslant \varepsilon\}$ and $\overline{B} \cap \{P_\varepsilon \leqslant 0\}$ respectively in a strictly concave and strictly convex set. We may also assume that $\mathcal{F}_{|\overline{B}}$ has a homological resolution

$$0 \to \mathcal{O}^{p_k} \to \cdots \to \mathcal{O}^{p_0} \to \mathcal{F} \to 0 \tag{2}$$

with $n - k \geqslant 3$. Choose a smooth function $\varrho \in C^\infty_0(B)$ such that $\varrho \geqslant 0$ and $\varrho(z^0) \neq 0$ and a positive number λ such that the closed domains

$$\overline{B}_1 = \{\varphi - \varepsilon + \lambda\rho \geqslant 0\} \cap \overline{B}, \qquad \overline{B}_2 = \{P_\varepsilon + \lambda\rho \leqslant 0\} \cap \overline{B}$$

are respectively strictly concave and strictly convex and contain z^0 as an interior point. Set $\overline{B}_3 = \overline{B}_1 \cap \overline{B}_2$ and $\overline{D}_1 = \overline{C}^+_\varepsilon \cup \overline{B}_3$; z^0 is an interior point of $\overline{D}_1$ and $\mathrm{b}\overline{B}_1 \smallsetminus \mathrm{b}\overline{B}_2 \Subset B$. By construction $\overline{D} \cap \overline{D}_1 \cap \overline{B} = \overline{D} \cap \overline{B}$ and $D \cap B$ is an intersection of two strictly convex domains with smooth boundaries; thus applying Proposition 3.2 we obtain

$$H^r(\overline{D} \cap \overline{B}, \mathcal{O}) = \{0\}$$

for $1 \leqslant r \leqslant n - 2$ and consequently, in view of (2), the vanishing

$$H^r(\overline{D} \cap \overline{B}, \mathcal{F}) = \{0\}. \tag{3}$$

Iterating this procedure we get the conclusion. □

Proposition 3.4. *For every $\varepsilon \in (0, \varepsilon_0)$ there exists $\varepsilon' < \varepsilon$ such that the homomorphism*

$$H^r(\overline{C}^+_{\varepsilon'}, \mathcal{F}) \longrightarrow H^r(\overline{C}^+_\varepsilon, \mathcal{F})$$

is onto for $1 \leqslant r \leqslant p(\overline{C}^+_\varepsilon; \mathcal{F}) - 2$.

Proof. Keeping the notations of Lemma 3.3 we apply the Mayer–Vietoris exact sequence for closed sets to $\overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B})$. We get

$$\cdots \to H^r(\overline{D}_1, \mathcal{F}) \to H^r(\overline{D}, \mathcal{F}) \oplus H^r(\overline{D}_1 \cap \overline{B}, \mathcal{F}) \to H^r(\overline{D} \cap \overline{D}_1 \cap \overline{B}, \mathcal{F}) \to \cdots$$

thus in view of (3) the homomorphism

$$H^r(\overline{D}_1, \mathcal{F}) \to H^r(\overline{D}, \mathcal{F})$$

is onto for $1 \leqslant r \leqslant n-2$. By induction, we obtain that the homomorphism

$$H^r(\overline{D}_m, \mathcal{F}) \to H^r(\overline{D}, \mathcal{F})$$

is onto for $1 \leqslant r \leqslant p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. Since $\overline{C}_\varepsilon^+ \subset D_m$ if $\varepsilon' < \varepsilon$ is near ε one has $\overline{C}_\varepsilon^+ \subset C_{\varepsilon'}^+ \Subset D_m$, whence the homomorphism

$$H^r(\overline{C}_{\varepsilon'}^+, \mathcal{F}) \to H^r(\overline{C}_\varepsilon^+, \mathcal{F})$$

is onto for $1 \leqslant r \leqslant p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. In particular, the canonical homomorphism

$$H^r(C_{\varepsilon'}^+, \mathcal{F}) \xrightarrow{\delta} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \tag{4}$$

is onto for $1 \leqslant r \leqslant p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$. □

From Proposition 3.4 we derive

Proposition 3.5. *For every $\varepsilon \in (0, \varepsilon_0)$ there exists an $\bar{\varepsilon} < \varepsilon$ such that for every $\varepsilon' \in [\bar{\varepsilon}, \varepsilon)$ the homomorphism*

$$H^r(C_{\varepsilon'}^+, \mathcal{F}) \xrightarrow{\delta} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) \tag{5}$$

is onto for $1 \leqslant r \leqslant p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$.

Proof. We fix ε_0 as in Lemma 3.3. Let Λ be the (non-empty) set of the positive numbers $\varepsilon' < \varepsilon$ such that the homomorphism (4) is onto and $\bar{\varepsilon} = \inf \Lambda$. It follows (cf. [1, Lemma p. 241] for closed subsets) that the homomorphism (5) is onto. □

A second consequence of Proposition 3.4 is the following finiteness theorem

Theorem 3.6. *Under the conditions of Theorem 3.1, there exists $\varepsilon_1 \leqslant \varepsilon_0$ such that*

$$\dim_{\mathbb{C}} H^1(\overline{C}_\varepsilon^+, \mathcal{F}) < +\infty$$

for every $\varepsilon \in (0, \varepsilon_1)$.

Proof. We first observe the following. Let $\Omega \subset \mathbb{C}^n$ be a domain, $K \subset \Omega$ a compact subset. It is known that $\mathcal{F}(\Omega)$ is a Fréchet space. The space $\mathcal{F}(K)$ is an $\mathcal{LF}$-space i.e. a direct limit of Fréchet spaces and its topology is complete (cf. [6, p. 315]). Moreover, the restriction

$$\mathcal{F}(\Omega) \xrightarrow{\delta} \mathcal{F}(K)$$

is a compact map i.e. there exists a neighborhood U of the origin in $\mathcal{F}(\Omega)$ such that $\overline{\delta(U)}$ is a compact subset of $\mathcal{F}(K)$. This is a consequence of the following well-known fact: if Ω' is a relatively compact subdomain of Ω then the restriction $\mathcal{F}(\Omega) \to \mathcal{F}(\Omega')$ is a compact map. Take ε_0 as in Lemma 3.3. The proof is similar to that of Théorème 11 in [1] taking into account the following facts:

(1) Leray theorem for acyclic closed coverings (see Théorème 5.2.4 and Corollaire in [5]),
(2) the theorem of L. Schwartz on compact perturbations $u + v$ of a surjective linear operator $u : E \to F$ where E is a Fréchet (see [6, Corollaire 1]). □

We remark that, up to some modifications in the technical details of the proof, the finiteness result holds for all cohomology groups:

Theorem 3.7. *Under the conditions of Theorem* 3.1 *there exists* $\varepsilon_1 \leqslant \varepsilon_0$ *such that*

$$\dim_{\mathbb{C}} H^r(\overline{C}_\varepsilon^+, \mathcal{F}) < +\infty,$$

for every $\varepsilon \in (0, \varepsilon_1)$ *and* $1 \leqslant r \leqslant p(\overline{C}_\varepsilon^+; \mathcal{F}) - 2$.

3.2. Approximation

This subsection is devoted to approximation by global sections.

Lemma 3.8. *Let* $\operatorname{depth}(\mathcal{F}_z) \geqslant 4$ *for every* $z \in \{\varphi = \varepsilon\}$, $\varepsilon \in (0, \varepsilon_0)$. *Then, for every* $z^0 \in \mathrm{b}C_\varepsilon^+$ *there exists a closed neighborhood* $\overline{U}$ *of* z^0 *such that the homomorphism*

$$H^0(\overline{U}, \mathcal{F}) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})$$

is dense image.

Proof. This is known if $z^0 \in \Gamma_1 \cup \Gamma_2$ i.e. when z^0 is a point of pseudoconvexity or pseudoconcavity (see [1]), thus we may assume that $z^0 \in \overline{\Gamma}_1 \cap \overline{\Gamma}_2$. First we consider the case $\mathcal{F} = \mathcal{O}$. We may suppose that there exists a sufficiently small closed ball $\overline{B}$ of positive radius, centered at z^0 such that $\overline{B} \cap \{\varphi \geqslant \varepsilon\}$ and $\overline{B} \cap \{P_\varepsilon \leqslant 0\}$ respectively are strictly concave and strictly convex (again, locally, up to a biholomorphism). Take a real hyperplane with equation $l = 0$ such that $z^0 \in \{l > 0\}$ and $\{l = 0\} \cap \{\varphi \leqslant \varepsilon\} \Subset B$. Let $\psi = \alpha(\varphi - \varepsilon) + \beta l$, α, β positive real numbers; ψ is strongly plurisubharmonic. For α, β sufficiently small the hypersurface $\{\psi = 0\} \cap \{l < 0\}$ is a portion of a compact smooth hypersurface which bounds a domain $D \Subset B$. Set

$$\overline{V} = \{P_\varepsilon \leqslant 0\} \cap \overline{D}, \qquad \overline{W} = \overline{D} \smallsetminus \{\varphi < \varepsilon\}$$

and $\overline{U}' = \overline{V} \cap \overline{W}$. We are going to prove that $H^1(\overline{V} \cup \overline{W}, \mathcal{O}) = 0$. Let $R = \overline{D} \smallsetminus \overline{V} \cup \overline{W}$. Since $\overline{D}$ is a Stein compact, from the exact sequence of cohomology relative to the closed subspace $\overline{V} \cup \overline{W}$ we get the isomorphism

$$H^r(\overline{V} \cup \overline{W}, \mathcal{O}) \simeq H_c^{r+1}(R, \mathcal{O}) \tag{6}$$

for $r \leqslant n - 2$. R is an open subset of $S = \overline{D} \cap \{\varphi < \varepsilon\}$. Set $R' = S \smallsetminus R$. Again, we consider the exact sequence of cohomology with compact supports relative to the closed subspace $R' = S \smallsetminus R$

$$\cdots \longrightarrow H_c^r(S, \mathcal{O}) \longrightarrow H_c^r(R', \mathcal{O}) \longrightarrow H_c^{r+1}(R, \mathcal{O}) \longrightarrow H_c^{r+1}(S, \mathcal{O}) \longrightarrow \cdots.$$

Since S and R' have a fundamental system of Stein neighborhoods (see [14]) and $n \geqslant 4$, we have

$$H_c^r(S, \mathcal{O}) = H_c^r(R', \mathcal{O}) = 0,$$

for $1 \leqslant r \leqslant n - 2$ and consequently $H_c^r(R, \mathcal{O}) = 0$ for $1 \leqslant r \leqslant n - 3$. In view of the isomorphism (6) we obtain

$$H^r(\overline{V} \cup \overline{W}, \mathcal{O}) = 0$$

for $1 \leqslant r \leqslant n-3$. In particular, since $n \geqslant 4$, (6) implies that

$$H^1(\overline{V} \cup \overline{W}, \mathcal{O}) = 0,$$

so that every function $f \in \mathcal{O}(\overline{U}')$ is a difference of two functions $f_1 - f_2$ where $f_1 \in \mathcal{O}(\overline{V})$, $f_2 \in \mathcal{O}(\overline{W})$. Since $\overline{V}$ is Runge in $\overline{D}$ there exists a sequence of holomorphic functions $f_\nu \in \mathcal{O}(\overline{D})$ such that $f_\nu \to f_1$ in $\mathcal{O}(\overline{V})$. Moreover, by the extension theorem in [11] the function f_2 extends holomorphically to $W \cap \{l \geqslant 0\}$. Choose a smooth function $\varrho \in C_0^\infty(D)$ such that $\varrho \geqslant 0$ and $\varrho(z^0) \neq 0$ and a positive number λ such that the closed domains

$$\overline{D}_1 = \{\varphi - \varepsilon + \lambda\varrho \leqslant 0\} \cap \overline{D}, \qquad \overline{D}_2 = \{P_\varepsilon - \lambda\rho \leqslant 0\} \cap \overline{D}$$

are respectively strongly pseudoconcave and strongly pseudoconvex, both contain z^0 as an interior point, $\mathrm{b}D_1 \setminus \{\varphi = \varepsilon\} \cap D$ is relatively compact in $D \cap \{l > 0\}$ and $\mathrm{b}D_2 \setminus \{P_\varepsilon = 0\}$ is relatively compact in D. Then we define $\overline{U} = \overline{D}_1 \cap \overline{D}_2$.

Observe that, by construction, Proposition 3.2 applies, thus $H^r(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}) = 0$ for $1 \leqslant r \leqslant n-2$.

In the general case, since $\overline{D}$ is Stein, we have on $\overline{D}$ an exact sequence

$$0 \longrightarrow \mathcal{H} \xrightarrow{\alpha} \mathcal{O}^q \xrightarrow{\beta} \mathcal{F} \longrightarrow 0.$$

Consider the following commutative diagram of continuous maps

$$\begin{tikzcd}
H^0(\overline{U}, \mathcal{O}^q) \arrow[r, "\beta_0"] \arrow[d, "\mathsf{r}"'] & H^0(\overline{U}, \mathcal{F}) \arrow[d, "\mathsf{r}"'] \\
H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q) \arrow[r, "\beta_1"] & H^1(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})
\end{tikzcd}$$

where r denotes the natural restriction. Then, since $\operatorname{depth}(\mathcal{F}_z) \geqslant 4$ for every $z \in D$, we have $\operatorname{depth}(\mathcal{H}_z) \geqslant 5$ for every $z \in D$. Again by Proposition 7.5 in [9] we have $H^1(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F}) = 0$ whence the homomorphism

$$H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})$$

is onto. Let $\sigma \in H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{F})$ and N a neighborhood of σ. Let $g \in H^0(\overline{U}, \mathcal{O}^q)$ such that $\beta_0(g) = \sigma$. Since the homorphism

$$H^0(\overline{U}, \mathcal{O}^q) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q)$$

is dense image there exists $h \in H^0(\overline{U}, \mathcal{O}^q)$ such that $\mathsf{r}(h) \in \beta_0^{-1}(N)$. Then $\mathsf{r}(\beta_0(h)) \in N$ with $\beta_0(h) \in H^0(\overline{U}, \mathcal{F})$. This shows that the homomorphism

$$H^0(\overline{U}, \mathcal{O}^q) \longrightarrow H^0(\overline{U} \cap \overline{C}_\varepsilon^+, \mathcal{O}^q)$$

is dense image. $\square$

Lemma 3.9. *Let $\mathcal{F}$ and ε_0 be as in Lemma* 3.8. *Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\varepsilon_2 \in (0, \varepsilon)$ such that for every $\varepsilon' \in (\varepsilon_2, \varepsilon)$ the homomorphism*

$$H^0(\overline{C}_{\varepsilon'}^+, \mathcal{F}) \longrightarrow H^0(\overline{C}_\varepsilon^+, \mathcal{F})$$

is dense image.

Proof. With the notations of Lemma 3.3 we have

$$\overline{D}=\overline{C}^+_\varepsilon, \qquad \overline{D}_1=\overline{D}\cup\overline{B}, \qquad \overline{D}_1=\overline{D}\cup(\overline{D}_1\cap\overline{B})$$

and we set $\overline{V}=\overline{D}_1\cap\overline{B}$. In view of Lemma 3.8 we may assume that the homomorphism

$$H^0(\overline{V},\mathcal{F})\longrightarrow H^0(\overline{V}\cap\overline{D},\mathcal{F})$$

is dense image. Moreover, $H^1(\overline{V},\mathcal{F})=0$. Let $\overline{\mathcal{U}}$ be the closed covering $\{\overline{D},\overline{V}\}$ of $\overline{D}_1$, $\mathsf{Z}^1(\overline{\mathcal{U}},\mathcal{F})$ and $\mathsf{B}^1(\overline{\mathcal{U}},\mathcal{F})$ respectively be the space of cocycles and the space of coboundaries of $\overline{\mathcal{U}}$ with values in $\mathcal{F}$. Since $H^1(\overline{\mathcal{U}},\mathcal{F})$ is a subgroup of $H^1(\overline{D}_1,\mathcal{F})$ which is of finite dimension (cf. Theorem 3.6) we have

$$\dim_{\mathbb{C}} H^1(\overline{\mathcal{U}},\mathcal{F})<+\infty.$$

It follows that $H^1(\overline{\mathcal{U}},\mathcal{F})$ is of finite dimension in the $\mathcal{LF}$-space $\mathsf{Z}^1(\overline{\mathcal{U}},\mathcal{F})$, thus an $\mathcal{LF}$-space for the induced topology. Moreover, in view of the Banach open mapping theorem the surjective map

$$H^0(\overline{D},\mathcal{F})\oplus H^0(\overline{V},\mathcal{F})\longrightarrow \mathsf{B}^1(\overline{\mathcal{U}},\mathcal{F})$$

given by $s\oplus\sigma\mapsto s_{|\overline{D}\cap\overline{V}}-\sigma_{|\overline{D}\cap\overline{V}}$ is a topological homomorphism.

Let $s\in H^0(\overline{D},\mathcal{F})$; $s_{|\overline{V}\cap\overline{D}}\in\mathsf{B}^1(\overline{\mathcal{U}},\mathcal{F})$. By Lemma 3.8, there exists a sequence $\{s_\nu\}\subset H^0(\overline{V},\mathcal{F})$ such that

$$s_{\nu|\overline{V}\cap\overline{D}}-s_{|\overline{V}\cap\overline{D}}\longrightarrow 0.$$

In view of Banach theorem there exist two sequences $\sigma^1_\nu\in H^0(\overline{D},\mathcal{F})$, $\sigma^2_\nu\in H^0(\overline{V},\mathcal{F})$ such that

$$\sigma^1_{\nu|\overline{D}\cap\overline{V}}-\sigma^2_{\nu|\overline{D}\cap\overline{V}}=s_{\nu|\overline{D}\cap\overline{V}}-s_{|\overline{D}\cap\overline{V}}, \qquad \sigma^1_\nu\to 0, \qquad \sigma^2_\nu\to 0.$$

It follows that for every ν

$$\tilde{s}_\nu=\begin{cases} s-\sigma^1_\nu & \text{on } \overline{D}\\ s_\nu-\sigma^2_\nu & \text{on } \overline{V}\end{cases}$$

is a section of $\mathcal{F}$ on $\overline{D}_1$ and that $\tilde{s}_\nu\to s$. In order to ends the proof we apply this procedure a finite numbers of times. □

As a corollary we get the following

Proposition 3.10. *Let $\mathcal{F}$ and ε_0 be as in Theorem* 3.1. *Then for every $\varepsilon\in(0,\varepsilon_0)$ there exists $\bar{\varepsilon}_0\in[0,\varepsilon)$ such that for every $\varepsilon'\in(\bar{\varepsilon}_0,\varepsilon]$ the homomorphism*

$$H^0(C^+_{\varepsilon'},\mathcal{F})\longrightarrow H^0(\overline{C}^+_\varepsilon,\mathcal{F})$$

is dense image.

Proof. Let $I\subset(0,\varepsilon_0)$ be the (non-empty) set of $\varepsilon'<\varepsilon$ such that the homomorphism

$$H^0(\overline{C}^+_{\varepsilon'},\mathcal{F})\longrightarrow H^0(\overline{C}^+_\varepsilon,\mathcal{F})$$

is dense image. Let $\bar{\varepsilon}=\inf I$ and $\{\varepsilon_\nu\}$ be a decreasing sequence with $\varepsilon_0=\varepsilon$, $\varepsilon_\nu\to\bar{\varepsilon}$ and set $F_\nu=H^0(\overline{C}^+_{\varepsilon_\nu},\mathcal{F})$. The topology of F_ν can be defined by an increasing sequence $\{p^{(\nu)}_j\}_{j\in\mathbb{N}}$ of translation invariant seminorms. Let for $\nu\geqslant 1$

$$\mathsf{r}_\nu: F_\nu\longrightarrow F_{\nu-1}$$

be the restriction map; then

$$H^0(\overline{C}^+_{\bar{\varepsilon}}, \mathcal{F}) = \varprojlim_{\{\mathsf{r}_\nu\}} F_\nu$$

and denote $\pi_\nu : H^0(\overline{C}^+_{\varepsilon}, \mathcal{F}) \to F_\nu$ the natural map. We have to show that π_0 is dense image.

Let $s \in F_0 = H^0(\overline{C}^+_{\varepsilon}, \mathcal{F})$ and N a neighborhood of s_0. We may assume that

$$N = \{ s \in F_0 \colon\ p_0^{(0)}(s - s_0) < \varepsilon \}.$$

Since the maps r_ν are continuous and dense image we can choose elements $s_\nu \in F_\nu$, for $\nu \geqslant 0$, satisfying the following conditions:

$$\begin{array}{ll}
s_1 \in F_1 & p_0^{(0)}\big(\mathsf{r}_1(s_1) - s_0\big) < \varepsilon/2 \\
s_2 \in F_2 & p_0^{(1)}\big(\mathsf{r}_2(s_2) - s_1\big) < \varepsilon/2 \\
 & p_1^{(0)}\big(\mathsf{r}_1\mathsf{r}_2(s_2) - \mathsf{r}_1(s_1)\big) < \varepsilon/2^2 \\
s_3 \in F_3 & p_0^{(2)}\big(\mathsf{r}_3(s_3) - s_2\big) < \varepsilon/2 \\
 & p_1^{(1)}\big(\mathsf{r}_2\mathsf{r}_3(s_3) - \mathsf{r}_2(s_2)\big) < \varepsilon/2^2 \\
 & p_2^{(0)}\big(\mathsf{r}_1\mathsf{r}_2\mathsf{r}_3(s_3) - \mathsf{r}_1\mathsf{r}_2(s_2)\big) < \varepsilon/2^3
\end{array}$$

and so on. Then, for every $\nu \in \mathbb{N}$, the series

$$s_\nu + \big(\mathsf{r}_{\nu+1}(s_{\nu+1}) - s_\nu\big) + \big(\mathsf{r}_{\nu+1}\mathsf{r}_{\nu+2}(s_{\nu+2}) - \mathsf{r}_{\nu+1}(s_{\nu+1})\big) + \cdots$$

is convergent in F_ν and $\mathsf{r}_\nu(\sigma_\nu) = \sigma_{\nu-1}$. Hence $\sigma = \{\sigma_\nu\}_{\nu\in\mathbb{N}}$ belongs to $H^0(C^+_{\bar{\varepsilon}}, \mathcal{F})$ and, by definition, $p_0^{(0)}(\sigma_0 - s_0) < \varepsilon$, i.e. $\pi_0(\sigma_0) \in N$. $\square$

Proof of Theorem 3.1. The proof uses Proposition 3.4 and Lemma 3.8. With the notations of Lemma 3.3 we have

$$\overline{D} = \overline{C}^+_{\varepsilon}, \qquad \overline{D}_1 = \overline{D} \cup \overline{B}, \qquad \overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B}), \qquad \overline{D} \cap (\overline{D}_1 \cap \overline{B}) = \overline{D} \cap \overline{B}.$$

We may assume that the homomorphism

$$H^1(\overline{D}_1, \mathcal{F}) \longrightarrow H^1(\overline{D}, \mathcal{F})$$

is onto and

$$H^0(\overline{D}_1, \mathcal{F}) \longrightarrow H^0(\overline{D}, \mathcal{F})$$

is dense image. Moreover, $H^1(\overline{B} \cap \overline{D}_1, \mathcal{F}) = 0$. Thus it is sufficient to show that the homomorphism

$$H^1(\overline{D}_1, \mathcal{F}) \longrightarrow H^1(\overline{D}, \mathcal{F})$$

is injective.

Since $H^1(\overline{D}_1 \cap \overline{B}, \mathcal{F}) = 0$ the Mayer–Vietoris exact sequence applied to $\overline{D}_1 = \overline{D} \cup (\overline{D}_1 \cap \overline{B})$ gives the exact sequence

$$H^0(\overline{D} \cap \overline{B}, \mathcal{F}) \xrightarrow{\ \mathsf{a}\ } H^1(\overline{D}_1, \mathcal{F}) \xrightarrow{\ \mathsf{b}\ } H^1(\overline{D}, \mathcal{F}) \longrightarrow 0.$$

Let $\xi \in \operatorname{Ker} \mathsf{b} = \operatorname{Im} \mathsf{a}$, $\xi = \mathsf{a}(\eta)$ with $\eta \in H^0(\overline{D} \cap \overline{B}, \mathcal{F})$. By Lemma 3.8 η is approximated by a sequence $\{\eta_\nu\} \subset H^0(\overline{D}_1 \cap \overline{B}, \mathcal{F})$. Each η_ν is a 1-coboundary of the closed covering $\mathcal{U} = \{\overline{D}, \overline{D}_1 \cap \overline{B}\}$ with values in $\mathcal{F}$ and such a space is closed in the space $\mathsf{Z}^1(\mathcal{U}, \mathcal{F})$ of the 1-cocycles. This proves that η is a 1-coboundary of $\{\mathcal{U}, \mathcal{F}\}$, whence $\xi = \mathsf{a}(\eta) = 0$. $\square$

Remark 3.2. For a full q-corona the cohomology groups of all the approximating coronae are isomorphic, in the expected range (see [1]). Differently, in the case of a semi 1-corona the cohomology groups of the approximating spaces C_ε^+ (which are not semi 1-coronae) are all isomorphic up to a critical value $\bar{\varepsilon}$, where the dimension of the cohomology spaces jumps. In particular their dimensions must not be bounded.

4. Extension of coherent sheaves and analytic subsets

An interesting consequence is that on a semi 1-corona $C^+ = C_{0,1}^+$ Theorem A of Oka–Cartan–Serre holds for a coherent sheaf $\mathcal{F}$ satisfying the conditions of Theorem 3.1. We first prove the following

Lemma 4.1. *Let X be a complex space, $\mathcal{F} \in \mathsf{Coh}(X)$ satisfying the following property: for every $x \in X$ there exists a subset $Y \not\ni x$ of X such that:*

(i) $H^1(X, \mathcal{F}) \simeq H^1(Y, \mathcal{F})$
(ii) *if $\mathcal{M}_{[x]}$ denotes the ideal of $\{x\}$ the homomorphism*

$$\alpha : H^1(X, \mathcal{M}_{[x]}\mathcal{F}) \longrightarrow H^1(Y, \mathcal{M}_{[x]}, \mathcal{F})$$

is injective.

Then, for every $x \in X$ the space $H^0(X, \mathcal{F})$ of the global sections of $\mathcal{F}$ generates $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$.

Proof. Let $x \in X$ and Y satisfying the conditions of the lemma. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{M}_{[x]}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{M}_{[x]}\mathcal{F} \longrightarrow 0$$

and the associated diagram

$$\begin{tikzcd} H^0(X, \mathcal{F}) \arrow[r] & H^0(X, \mathcal{F}/\mathcal{M}_{[x]}\mathcal{F}) \arrow[r] & H^1(X, \mathcal{M}_{[x]}\mathcal{F}) \arrow[r, "\delta"] \arrow[d, "\alpha"'] & H^1(X, \mathcal{F}) \arrow[d, "\beta"'] \\ & & H^1(Y, \mathcal{M}_{[x]}\mathcal{F}) \arrow[r, "\gamma"] & H^1(Y, \mathcal{F}). \end{tikzcd}$$

The homomorphism α is injective and β is an isomorphism, by hypothesis, moreover, since $\mathcal{M}_{[x]|Y} \simeq \mathcal{F}_{|Y}$, γ is an isomorphism. It follows that δ is injective and consequently that the homomorphism

$$H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}/\mathcal{M}_{[x]}\mathcal{F}) \simeq \mathcal{F}_x/\mathcal{M}_{[x],x}\mathcal{F}_x$$

is onto. Then the Lemma of Nakayama implies that

$$H^0(X, \mathcal{F}) \longrightarrow \mathcal{F}_x$$

is onto and this proves the lemma. □

Keeping the notations of the proof of Theorem 3.1, we deduce the following

Corollary 4.2. *Under the conditions of Theorem* 3.1 *for every compact subset*

$$K \subset C_{\varepsilon'}^+ \cap \{\varphi > \varepsilon'\} \cap \{P_{\varepsilon'} < 0\}$$

there exist sections $s_1, \dots, s_k \in H^0(C^+_{\varepsilon'}, \mathcal{F})$ *which generate* $\mathcal{F}_z$ *for every* $z \in K$.

Theorem 4.3. *Let* $C^+ = (B_1 \smallsetminus B_0) \cap \{h \geqslant 0\}$ *and* $\mathcal{F} \in \mathsf{Coh}(C^+)$. *If* $\operatorname{depth}(\mathcal{F}_z) \geqslant 3$ *on* $\{\varphi = 0\}$ *then for every* $a > 0$ *near* 0 $\mathcal{F}_{|B_1 \smallsetminus \overline{B}_a}$ *extends on* $B_1 \cap \{h \geqslant 0\}$ *by a coherent sheaf* $\widetilde{\mathcal{F}}_a$.

Proof. With the usual notations choose $\varepsilon_0 \in (0, a)$, and $c_0 > 0$ such that

(i) $\mathcal{F}$ is defined on the semi 1-corona $(B_1 \smallsetminus B_{-\varepsilon}) \cap \{h > -c\}$
(ii) $\{z \in B_1\colon\ h(z) \geqslant c\} \Subset \{z \in B_1\colon\ P_\varepsilon(z) < 0\}$
(iii) for every $\varepsilon \in (0, \varepsilon_0)$, and $c \in (0, c_0)$ the hypersurface $\{P_\varepsilon = -c\}$, $P_\varepsilon(z) = \varepsilon|z|^2 - h(z)$, meet the hypersurfaces $\{\varphi = \varepsilon\}$, $\{\varphi = -\varepsilon\}$ transversally.

Let $Y^+_{l,s}$ denote the semi 1-corona $\{l < \varphi < s\} \cap \{h > c\}$, with $l < s$. In view of Corollary 4.2 applied to the semi 1-corona $Y^+_{\varepsilon,a}$ there exist $\alpha, \beta, \gamma \in (0, a)$ with $\alpha < \beta < \gamma$ such that $H^0(Y^+_{\alpha,\gamma}, \mathcal{F})$ generates $\mathcal{F}$ on $K_{\beta,\gamma} = \overline{Y}^+_{\beta,\gamma} \cap \{h \geqslant 0\}$. Thus on $K_{\beta,\gamma}$ there exists an exact sequence

$$\mathcal{O}^p \xrightarrow{\ \lambda\ } \mathcal{F} \longrightarrow 0\,.$$

Since, by hypothesis, $\operatorname{depth}(\mathcal{F}_z) \geqslant 3$ for every $z \in K_{\beta,\gamma}$ we have

$$\operatorname{depth}(\mathsf{Ker}\,\lambda) \geqslant 4$$

on $K_{\beta,\gamma}$ (cf. [3]). Again by Corollary 4.2 there exist $\beta_1, \gamma_1 \in (\beta, \gamma)$, $\beta_1 < \gamma_1$ and sections $\sigma_1, \dots, \sigma_l$ on $K_{\beta_1,\gamma_1} = \overline{Y}^+_{\beta_1,\gamma_1} \cap \{h \geqslant 0\}$ which generate $(\mathsf{Ker}\,\lambda)_z$ for every $z \in V$. Since $\mathsf{Ker}\,\lambda$ is a subsheaf of $\mathcal{O}^p$, by the theorem in [11] the sections $\sigma_1, \dots, \sigma_l$ extend holomorphically on

$$\{\varphi \leqslant \gamma_1\} \cap \{h \geqslant 0\}$$

and their extensions $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ generate a coherent sheaf $\mathcal{H}$ on

$$\{\varphi \leqslant \gamma_1\} \cap \{h \geqslant 0\}.$$

Let $\widetilde{\mathcal{F}}'_a$ be the sheaf defined by

$$\widetilde{\mathcal{F}}'_{a,z} = \begin{cases} \mathcal{F}_z & \text{for } z \in \{\varphi > \gamma_1\} \cap \{h \geqslant 0\} \\ \mathcal{O}_z/\mathcal{H}_z & \text{for } z \in \{\varphi \leqslant \gamma_1\} \cap \{h \geqslant 0\}; \end{cases}$$

$\widetilde{\mathcal{F}}'_\varepsilon$ is a coherent sheaf on $B^+_c \cap \{y_n > \varepsilon\}$ extending $\mathcal{F}$. $\square$

Corollary 4.4. *Let* $X^+ = (B_1 \smallsetminus \overline{B}_0) \cap \{h > 0\}$ *and* Y *be an analytic subset of* X^+ *such that* $\operatorname{depth}(\mathcal{O}_{Y,z}) \geqslant 3$ *for* z *near* $\{\varphi = 0\}$. *Then* Y *extends on* $B_1 \cap \{h \geqslant 0\}$ *by an analytic subset.*

Proof. We apply Theorem 4.3 to $X^+ \cap \{h \geqslant \varepsilon\}$, where $\varepsilon \sim 0$ is positive. Then, for $\nu \in \mathbb{N}$ there exists a coherent sheaf $\widetilde{\mathcal{O}}^{(\nu)}_Y$ on $B_1 \cap \{h \geqslant 0\}$ which extends $\mathcal{O}_Y$; $\widetilde{Y}^{(\nu)} = \operatorname{supp} \widetilde{\mathcal{O}}^{(\nu)}_Y$ is an analytic subset extending $Y \cap (B_1 \smallsetminus B_{1/\nu}) \cap \{h \geqslant \varepsilon\}$. In view of the strong pseudoconvexity of $\mathrm{b}B_{1/\nu}$, the subset $F_\nu = \widetilde{Y}^{(\nu)} \smallsetminus \widetilde{Y}^{(\nu+1)}$ is a finite set of points which is contained in $B_{1/\nu}$. Start by $\nu = 2$ and consider the first extension $\widetilde{Y}^{(2)}$. Then $\widetilde{Y}^{(2)} \smallsetminus F_2 \cap (B_{1/2} \smallsetminus B_{1/3})$ coincide with Y on $(B_1 \smallsetminus B_{1/3})$ and so on. To handle different extensions depending on ε we argue in the same way. $\square$

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