Local convexity properties of quasihyperbolic balls in punctured space

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Abstract

This paper deals with local convexity properties of the quasihyperbolic metric in the punctured space. We consider convexity and starlikeness of the quasihyperbolic balls.
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1. Introduction

The *quasihyperbolic distance* between two points x and y in a proper subdomain G of the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, is defined by

$$k_G(x, y) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha \frac{|dz|}{d(z, \partial G)},$$

where $d(z, \partial G)$ is the (Euclidean) distance between the point $z \in G$ and the boundary of G and Γ_{xy} is the collection of all rectifiable curves in G joining x and y.

Since its introduction by F.W. Gehring and B.P. Palka [3] in 1976, the quasihyperbolic metric has been widely applied in geometric function theory and mathematical analysis in general, see e.g. [9,13]. Quasihyperbolic geometry has recently been studied by P. Hästö [4] and H. Lindén [5].

The purpose of this paper is to study the metric space (G, k_G) and especially local convexity properties of *quasihyperbolic balls* $D_G(x, M)$ defined by

$$D_G(x, M) = \{ z \in G : k_G(x, z) < M \}.$$

In the dimension $n = 2$ we call these balls *disks* and we often identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

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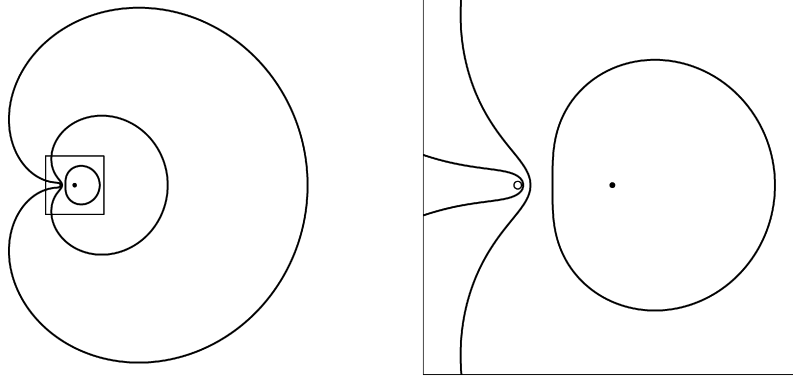
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Fig. 1. Boundaries of quasihyperbolic disks $D_{\mathbb{R}^2 \setminus \{0\}}(x, M)$ with radii $M = 1$, $M = 2$ and $M = \kappa$.

M. Vuorinen suggested in [14] a general question about the convexity of balls of small radii in metric spaces. Our work is motivated by this question and our main result Theorem 1.1 provides an answer in a particular case. For the definition of starlike domains see Definition 2.9.

Theorem 1.1.

(1) *For $x \in \mathbb{R}^n \setminus \{0\}$ the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ is strictly convex for $M \in (0, 1]$ and it is not convex for $M > 1$.*

(2) *For $x \in \mathbb{R}^n \setminus \{0\}$ the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ is strictly starlike with respect to x for $M \in (0, \kappa]$ and it is not starlike with respect to x for $M > \kappa$, where κ is defined by (4.1) and has a numerical approximation $\kappa \approx 2.83297$.*

Theorem 1.1 in the case $n = 2$ is illustrated in Fig. 1. O. Martio and J. Väisälä [7] have recently proved that if G is convex then $D_G(x, M)$ is also convex for all $x \in G$ and $M > 0$.

2. Quasihyperbolic balls with large and small radii

In this section we consider the behavior of quasihyperbolic balls with large and small radii.

Let us define ϕ-uniform domains, which were introduced by M. Vuorinen [12, 2.49], and consider quasihyperbolic balls with large radii in ϕ-uniform domains. We use notation $m(a, b) = \min\{d(a), d(b)\}$, where $d(x) = d(x, \partial G)$.

Definition 2.1. Let $\phi : [0, \infty) \to [0, \infty)$ be a homeomorphism. Then a domain $G \subsetneq \mathbb{R}^n$ is ϕ-uniform if

$$k_G(x, y) \leqslant \phi\left(\frac{|x - y|}{m(x, y)}\right)$$

for all $x, y \in G$.

Lemma 2.2. *Fix ϕ, let G be ϕ-uniform, $x_0 \in G$ and $M > 0$. If $x \in G$ with $m(x, x_0) > |x - x_0|/\phi^{-1}(M)$ then $x \in D_G(x_0, M)$.*

Proof. Since ϕ is a homeomorphism $m(x, x_0) > |x - x_0|/\phi^{-1}(M)$ implies

$$\phi\left(\frac{|x - x_0|}{m(x, x_0)}\right) < M$$

and since G is ϕ-uniform

$$k_G(x, x_0) \leqslant \phi\left(\frac{|x - x_0|}{m(x, x_0)}\right) < M.$$

Therefore $x \in D_G(x_0, M)$. □

Definition 2.3. Let $\delta \in (0, 1)$ and $r_0 > 0$ be fixed and $G \subset \mathbb{R}^n$ be a bounded domain. We say that G satisfies the (δ, r_0)-*condition* if for all $z \in \partial G$ and $r \in (0, r_0]$ there exists $x \in B^n(z, r) \cap G$ such that $d(x) > \delta r$.

Theorem 2.4. *Assume G is a bounded ϕ-uniform domain and satisfies the (δ, r_0)-condition for a fixed $\delta \in (0, 1)$ and $r_0 > 0$. Let us assume $r_1 \in (0, r_0)$ and fix $x_0 \in G$ and $z \in \partial G$. Then $d(D_G(x_0, M), z) < r_1$ for*

$$M > \phi\left(\frac{|x_0 - z| + r_2}{\delta r_2}\right), \tag{2.5}$$

where $r_2 = \min\{r_1, d(x_0)/2\}$.

Proof. Since G satisfies the (δ, r_0)-condition and $r_2 < r_0$ we can choose $x \in B^n(z, r_2) \cap G$ with $d(x) > \delta r_2$. Now

$$m(x_0, x) = \min\{d(x_0), d(x)\} = d(x) > \delta r_2$$

and $|z - x| < r_2$. The inequality (2.5) is equivalent to

$$\delta r_2 > \frac{|x_0 - z| + r_2}{\phi^{-1}(M)}.$$

Since $|z - x| < r_2$ and by the triangle inequality

$$\frac{|x_0 - z| + r_2}{\phi^{-1}(M)} > \frac{|x_0 - z| + |z - x|}{\phi^{-1}(M)} \geqslant \frac{|x_0 - x|}{\phi^{-1}(M)}.$$

Now we have

$$m(x_0, x) > \delta r_2 > \frac{|x_0 - z| + r_2}{\phi^{-1}(M)} > \frac{|x_0 - x|}{\phi^{-1}(M)}$$

and by Lemma 2.2 we have $x \in G \cap D_G(x_0, M)$. Therefore

$$d(D_G(x_0, M), z) \leqslant |z - x| < r_2 \leqslant r_1$$

and the claim is clear. $\quad\square$

Corollary 2.6. *Let $G \subset \mathbb{R}^n$ be a bounded ϕ-uniform domain and let G satisfy the (δ, r_0)-condition. For a fixed $s \in (0, r_0)$ and $x \in G$*

$$\overline{G} \subset D_G(x, M(s)) + B^n(s) = \{y + z: y \in D_G(x, M(s)), |z| < s\},$$

where

$$M(s) > \max_{z \in \partial G} \phi\left(\frac{|x - z| + r}{\delta r}\right)$$

and $r = \min\{s, d(x)/2\}$.

Let us then point out that quasihyperbolic balls of small radii become more and more like Euclidean balls when the radii tend to zero. We shall study the local structure of the boundary of a quasihyperbolic ball and show that the boundary is round from the inside and cannot have e.g. outwards directed conical parts.

Definition 2.7. Let γ be a curve in domain $G \subsetneq \mathbb{R}^n$. If

$$k_G(x, y) + k_G(y, z) = k_G(x, z)$$

for all $x, z \in \gamma$ and $y \in \gamma'$, where γ' is the subcurve of γ joining x and z, then γ is a *geodesic segment* or briefly a *geodesic*. We denote a geodesic between x and y by $J_k[x, y]$.

Theorem 2.8. *For a proper subdomain G of $\mathbb{R}^n$, $M > 0$ and $y \in \partial D_G(x, M)$, let $J_k[x, y]$ be a geodesic segment of the quasihyperbolic metric joining x and y. For $z \in J_k[x, y]$ we have*

$$B^n\left(z, \frac{|z - y|}{1 + u}\right) \subset D_G(x, M),$$

where $u = |z - y|/d(z)$.

Proof. By [2, Lemma 1] there exists $J_k[x, y]$. By the choice of z we have

$$M = k_G(x, y) = k_G(x, z) + k_G(z, y)$$

and by the triangle inequality for $w \in D_G(z, k_G(z, y))$ we have

$$k_G(x, w) \leqslant k_G(x, z) + k_G(z, w) < M.$$

Now

$$D_G(z, k_G(z, y)) \subset D_G(x, M).$$

By [11, p. 347]

$$B^n(z, (1 - e^{-k_G(z,y)})d(z)) \subset D_G(z, k_G(z, y))$$

and therefore

$$B^n(z, (1 - e^{-k_G(z,y)})d(z)) \subset D_G(x, M).$$

By [3, Lemma 2.1] $k_G(z, y) \geqslant \log(1 + \frac{|z-y|}{d(z)})$ and therefore

$$(1 - e^{-k_G(z,y)})d(z) \geqslant \left(1 - \frac{d(z)}{d(z) + |z - y|}\right)d(z) = \frac{|z - y|}{1 + u}$$

for $u = \frac{|z-y|}{d(z)}$. Now

$$B^n\left(z, \frac{|z - y|}{1 + u}\right) \subset B^n(z, (1 - e^{-k_G(z,y)})d(z))$$

and the claim is clear. □

Now we have found a Euclidean ball $B^n(z, r)$ inside the quasihyperbolic ball $D_G(x, M)$ with the following property:

$$\frac{r}{d(z, \partial D_G(x, M))} \to 1, \quad \text{when } z \to \partial D_G(x, M).$$

Geometrically this convergence means that the boundary of the quasihyperbolic ball must be round from the interior. The boundary cannot have any cone shaped corners pointing outwards from the ball. However, there can be corners in the boundary pointing inwards to the ball. An example in $\mathbb{R}^2 \setminus \{0\}$ is the quasihyperbolic disk with $M > \pi$. This example is considered in more detail in Remark 4.8.

Definition 2.9. Let $G \subset \mathbb{R}^n$ be a domain and $x \in G$. We say that G is *starlike with respect to* x if each line segment from x to $y \in G$ is contained in G. The domain G is *strictly starlike with respect to* x for $x \in G$ if G is bounded and each ray from x meets ∂G at exactly one point.

The following result considers starlikeness of quasihyperbolic balls in starlike domains. The same result was independently obtained by J. Väisälä [10].

Theorem 2.10. *If $G \subsetneq \mathbb{R}^n$ is a starlike domain with respect to x, then the quasihyperbolic ball $D_G(x, M)$ is starlike with respect to x.*

Proof. We need to show that the function $f(y) = k_G(x, y)$ is increasing along each ray from x to ∂G. To simplify notation we may assume $x = 0$.

Let $y \in G \setminus \{x\}$ be arbitrary and denote a geodesic segment from x to y by γ. Let us choose any $y' \in (x, y)$ and denote

$$\gamma' = \frac{|y'|}{|y|}\gamma = c\gamma.$$

Since G is starlike with respect to x the path γ' from x to y' is in G. Therefore

$$k_G(x, y') \leqslant \int_{\gamma'} \frac{|dz|}{d(z)} = \int_\gamma \frac{c|dz|}{d(cz)}.$$

Since G is starlike with respect to x we have $d(cz) \geqslant cd(z)$ which is equivalent to

$$\frac{c}{d(cz)} \leqslant \frac{1}{d(z)}.$$

Now

$$k_G(x, y') \leqslant \int_\gamma \frac{c|dz|}{d(cz)} \leqslant \int_\gamma \frac{|dz|}{d(z)} = k_G(x, y)$$

and f is increasing along each ray from x to ∂G.   □

For a domain $G \subset \mathbb{R}^n$ and quasihyperbolic ball $D_G(x, M)$, $x \in G$ and $M > 0$, we define *the points that can affect the shape of* $D_G(x, M)$ to be the set

$$\big\{z \in \partial G \colon |z - y| = d(y) \text{ for some } y \in D_G(x, M)\big\}.$$

Let G be a domain and fix $x \in G$ and $M > 0$. Now by [11, p. 347] we know that $D_G(x, M) \subset B^n(x, Rd(x))$, for $R = e^M - 1$, and therefore for each $y \in D_G(x, M)$ we have $d(y) \leqslant d(x) + 2Rd(x) = d(x)(2e^M - 1)$. This fact is generalized in the following lemma.

Lemma 2.11. *Let $G \subsetneq \mathbb{R}^n$ be a domain, $x \in G$ and $y \in \partial G$. Then the points that can affect the shape of the quasihyperbolic ball $D_G(x, M)$ for $M \in (0, 1]$ are in the closure of the set*

$$U_y = B^n\big(x, |x - y|(2e^M - 1)\big) \setminus \big\{z \in \mathbb{R}^n \setminus \{y\} \colon \measuredangle x'yz \leqslant \pi/2 - 1, x' = 2y - x\big\},$$

where $\measuredangle x'yz$ is the angle between line segments $[x', y]$ and $[z, y]$ at y.

Proof. Let us consider $G' = \mathbb{R}^n \setminus \{y\}$. Now $G \subset G'$ and therefore $D_G(x, M) \subset D_{G'}(x, M)$. Now the points that can affect the shape of $D_G(x, M)$ need to be inside $\overline{B^n(x, |x - y|(2e^M - 1))}$.

Let $z \in \partial D_{G'}(x, M)$. Because $M \leqslant 1$ we have by (3.1) $\measuredangle xyz \leqslant 1$. Therefore the points in

$$\big\{z \in \mathbb{R}^n \setminus \{y\} \colon \measuredangle x'yz \leqslant \pi/2 - 1, \ x' = 2y - x\big\}$$

do not affect the shape of $D_{G'}(x, M)$. Since $D_G(x, M) \subset D_{G'}(x, M)$, the claim is clear.   □

Theorem 2.12. *For a domain $G \subsetneq \mathbb{R}^n$, $M \in (0, 1]$ and $x \in G$ the quasihyperbolic ball $D_G(x, M)$ is starlike with respect to x.*

Proof. We denote

$$V_x = G \cap \left(\bigcap_{y \in \partial G} U_y \right).$$

We will show that V_x is strictly starlike with respect to x. By definition of V_x and U_y

$$V_x = B^n\big(x, d(x)(2e^M - 1)\big) \setminus \bigcup_{y \in \partial G} S_y,$$

where $S_y = \{z \in \mathbb{R}^n \setminus \{y\} \colon \measuredangle x'yz \leqslant \pi/2 - 1, \ x' = 2y - x\} \cup \{y\}$. Let $u \in V_x$ and denote $R(u) = \{z \in \mathbb{R}^n \colon z = x + t(u - x), \ t \geqslant 1\}$ and $v \in R(u) \cap \partial V_x$ such that $v' \in R(u) \cap \partial V_x$ implies $|x - v| \leqslant |x - v'|$. If $v \in S^{n-1}(x, d(x)(2e^M - 1))$ then $V_x \cap \partial G = \{v\}$. Otherwise $v \in \partial S_y$ for some $y \in \partial G$ and $V_x \cap \partial G = \{v\}$. Therefore V_x is strictly starlike with respect to x.

The set $\overline{V_x}$ contains all of the boundary points of G that affect the shape of $D_G(x, M)$. Therefore for fixed $x \in G$ we have $D_G(x, M) = D_{V_x}(x, M)$ and $D_G(x, M)$ is starlike with respect to x by Theorem 2.10, because V_x is starlike with respect to x.   □

Remark 2.13. In Lemma 2.11 and Theorem 2.12 we could replace $M \in (0, 1]$ by $M \in (0, \alpha]$ and $\angle x' yz \leqslant \pi/2 - 1$ by $\angle x' yz \leqslant \pi/2 - \alpha$ for any $\alpha \in [1, \pi/2)$. This modified version of Theorem 2.12 was also proved by J. Väisälä [9, Theorem 3.11].

3. Convexity of quasihyperbolic balls in punctured space

The set $\mathbb{R}^n \setminus \{z\}$, $z \in \mathbb{R}^n$, is called a punctured space. To simplify notation we may assume $z = 0$. In this section we will find values M such that the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ is convex for all $x \in \mathbb{R}^n \setminus \{0\}$.

Let us assume that $x, y \in \mathbb{R}^n \setminus \{0\}$ and that the angle φ between segments $[0, x]$ and $[0, y]$ satisfies $0 < \varphi \leqslant \pi$. It can be shown [6, p. 38] that

$$k_{\mathbb{R}^n \setminus \{0\}}(x, y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}}. \tag{3.1}$$

In particular, we see that $k_{\mathbb{R}^n \setminus \{0\}}(x, y) = k_{\mathbb{R}^n \setminus \{0\}}(x, y_1)$, where y_1 is obtained from y by the inversion with respect to $S^{n-1}(|x|)$, i.e. $y_1 = y|x|^2/|y|^2$. Hence this inversion maps the quasihyperbolic sphere $\{z \in \mathbb{R}^n \setminus \{0\}: k_{\mathbb{R}^n \setminus \{0\}}(x, z) = M\}$ onto itself.

Quasihyperbolic balls are similar in $\mathbb{R}^n \setminus \{0\}$ for fixed M. In other words any quasihyperbolic ball of radius M can be mapped onto any other quasihyperbolic ball of radius M by rotation and stretching.

We will first consider convexity of the quasihyperbolic disks in the punctured plane $\mathbb{R}^2 \setminus \{0\}$ and then extend the results to the punctured space $\mathbb{R}^n \setminus \{0\}$.

By (3.1) we have a coordinate representation in the case $n = 2$

$$x = \left(|x| \cos \varphi, |x| \sin \varphi\right) = \left(e^{\pm \sqrt{M^2 - \varphi^2}} \cos \varphi, e^{\pm \sqrt{M^2 - \varphi^2}} \sin \varphi\right) \tag{3.2}$$

for $x \in \partial D_{\mathbb{R}^2 \setminus \{0\}}(1, M)$ and $-M \leqslant \varphi \leqslant M$. By using this presentation we will prove the following result.

Theorem 3.3. *For $M > 1$ and $z \in \mathbb{R}^2 \setminus \{0\}$ the quasihyperbolic disk $D_{\mathbb{R}^2 \setminus \{0\}}(z, M)$ is not convex.*

Proof. We may assume $z = 1$ and let $x \in \partial D_{\mathbb{R}^2 \setminus \{0\}}(z, M)$ be arbitrary. Assume $M > 1$. By (3.2) we have

$$x = \left(e^{\pm \sqrt{M^2 - \varphi^2}} \cos \varphi, e^{\pm \sqrt{M^2 - \varphi^2}} \sin \varphi\right),$$

where $-M \leqslant \varphi \leqslant M$.

If $M > \pi/2$, then the claim is clear by symmetry because $\operatorname{Re} x = e^{-M} > 0$ for $\varphi = 0$ and $\operatorname{Re} x < 0$ for $\varphi = \pm M$.

We will show that the function

$$f(\varphi) = e^{-\sqrt{M^2 - \varphi^2}} \cos \varphi$$

is concave in the neighborhood of $\varphi = 0$ and the function

$$g(\varphi) = e^{-\sqrt{M^2 - \varphi^2}} \sin \varphi$$

is increasing in $(0, \min\{M, \frac{\pi}{2}\})$. This will imply non-convexity of $D_{\mathbb{R}^2 \setminus \{0\}}(z, M)$.

First,

$$g'(\varphi) = e^{-\sqrt{M^2 - \varphi^2}} \left(\cos \varphi + \frac{\varphi \sin \varphi}{\sqrt{M^2 - \varphi^2}} \right)$$

and this is clearly non-negative for $0 < \varphi < \min\{M, \frac{\pi}{2}\}$. Therefore $g(\varphi)$ is increasing.

Second, by a straightforward computation we obtain

$$f'(\varphi) = e^{-\sqrt{M^2 - \varphi^2}} \left(\frac{\varphi \cos \varphi}{\sqrt{M^2 - \varphi^2}} - \sin \varphi \right)$$

and

$$f''(\varphi) = \frac{e^{-\sqrt{M^2-\varphi^2}}((M^2 - \sqrt{M^2-\varphi^2}(M^2 - 2\varphi^2)))\cos\varphi + 2\varphi(\varphi^2 - M^2)\sin\varphi}{(\sqrt{M^2-\varphi^2})^3}.$$

Now $f'(0) = 0$ and $f''(0) = e^{-M}(1/M - 1) < 0$ and therefore $f(\varphi)$ is concave in the neighborhood of $\varphi = 0$. $\square$

Theorem 3.3 can easily be extended to the case $n \geqslant 3$.

Corollary 3.4. *If $M > 1$ and $z \in \mathbb{R}^n \setminus \{0\}$, then the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(z, M)$ is not convex.*

Proof. Let us choose any $y \in \mathbb{R}^n \setminus \{0\}$ such that $y \neq tz$ for all $t \in \mathbb{R}$. Now $D_{\mathbb{R}^n \setminus \{0\}}(z, M) \cap \mathrm{span}(0, y, z)$ is not convex by Theorem 3.3 and therefore the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(z, M)$ cannot be convex. $\square$

Let us now consider the convexity of the quasihyperbolic balls in the case $M \leqslant 1$ and $n = 2$.

Theorem 3.5. *For $0 < M \leqslant 1$ and $z \in \mathbb{R}^2 \setminus \{0\}$ the quasihyperbolic disk $D_{\mathbb{R}^2 \setminus \{0\}}(z, M)$ is strictly convex.*

Proof. Let $z = 1$ and $x \in \partial D_{\mathbb{R}^2 \setminus \{0\}}(z, M)$. By symmetry it is sufficient to consider the upper half C of $\partial D_{\mathbb{R}^2 \setminus \{0\}}(z, M)$, which is given by

$$x = x(s) = \left(e^s \cos\varphi, e^s \sin\varphi\right), \tag{3.6}$$

where $M \in (0, \pi)$, $s \in [-M, M]$ and $\varphi = \varphi(s) = \sqrt{M^2 - s^2}$. Now $\varphi'(s) = -s/\varphi(s)$ and therefore for $s \in (-M, M)$

$$x'(s) = \frac{e^s}{\varphi(s)}\left(a(s), b(s)\right),$$

where $a(s) = \varphi(s)\cos\varphi(s) + s\sin\varphi(s)$ and $b(s) = \varphi(s)\sin\varphi(s) - s\cos\varphi(s)$. Now $t(s) = (a(s), b(s))$ is a tangent vector of C for $s \in [-M, M]$. Equality $t(s) = 0$ is equivalent to $s^2 = -\varphi(s)^2$, which never holds. Since $t(s) \neq 0$ for all $s \in [-M, M]$ the angle $\alpha(s) = \arg t(s)$ is a continuous function on $(-M, M)$. We need to show that $\alpha(s)$ is strictly decreasing on $[-M, M]$.

Since $\tan\alpha(s) = b(s)/a(s)$ and $\cot\alpha(s) = a(s)/b(s)$ it suffices to show that $d(s) = b'(s)a(s) - a'(s)b(s) < 0$ on $(-M, M)$. By a straightforward computation

$$d(s) = -\frac{M^2(1 + s)}{\phi(s)} \tag{3.7}$$

and $\alpha(s)$ is strictly decreasing on $[-M, M]$. Since $t(s) \to (0, M)$ as $s \to -M$ and $t(s) \to (0, -M)$ as $s \to M$, $\alpha(s)$ decreases from $\pi/2$ to $-\pi/2$ on C and the assertion follows. $\square$

Remark 3.8. The boundary $\partial D_{\mathbb{R}^2 \setminus \{0\}}(1, M)$ is smooth since $\alpha(s)$ is continuous,

$$t(M) = (0, -M) \quad \text{and} \quad t(-M) = (0, M).$$

By using the symmetry of the quasihyperbolic balls we can extend Theorem 3.5 to the case of punctured space.

Lemma 3.9. *Let the domain $G \subset \mathbb{R}^n$ be symmetric about a line l (i.e. is invariant under any rotation about l) and $G \cap L$ be strictly convex for any plane L with $l \subset L$. Then G is strictly convex.*

Proof. We may assume that the line l is the first coordinate axis of $\mathbb{R}^n$ to simplify notation. Let $x, y \in \partial G$ and $z \in (x, y)$. Define $A_x = \{u \in \overline{G}: u_1 = x_1\}$ and $A_y = \{u \in \overline{G}: u_1 = y_1\}$. Denote by H the convex hull of $A_x \cup A_y$. Now $z \in \overline{H}$ and $\overline{H} \setminus (A_x \cap A_y) \subset G$ and hence the assertion follows. $\square$

Corollary 3.10. *For $0 < M \leqslant 1$ and $z \in \mathbb{R}^n \setminus \{0\}$ the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(z, M)$ is strictly convex.*

Proof. By (3.1) the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ is symmetric about the line that contains x and 0. By Lemma 3.9 and Theorem 3.5 $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ is strictly convex for $0 < M \leqslant 1$. $\square$

4. Starlikeness of quasihyperbolic balls in punctured space

In this section we will find the maximum value of the radius M for which the quasihyperbolic ball $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ is strictly starlike with respect to x. As in the previous section we will first consider the quasihyperbolic disks in the punctured plane and then extend the results to the punctured space.

Let us define a constant κ as the solution of the equation

$$\cos \sqrt{p^2 - 1} + \sqrt{p^2 - 1} \sin \sqrt{p^2 - 1} = e^{-1} \tag{4.1}$$

for $p \in [1, \pi]$. The proof of the next theorem shows that Eq. (4.1) has only one solution κ on $[1, \pi]$ with numerical approximation

$$\kappa \approx 2.83297.$$

Remark 4.2. According to [1] the number κ was first introduced by P.T. Mocanu in 1960 [8]. Later V. Anisiu and P.T. Mocanu showed [1, p. 99] that if f is an analytic function in the unit disk, $f(0) = 0$ and

$$\left| \frac{f''(z)}{f'(z)} \right| \leqslant \kappa,$$

then f is starlike with respect to 0.

Theorem 4.3. *The quasihyperbolic disk* $D_{\mathbb{R}^2 \setminus \{0\}}(x, M)$ *is strictly starlike with respect to x for $0 < M \leqslant \kappa$ and is not starlike with respect to x for $M > \kappa$.*

Proof. Because of symmetry we will consider $\partial D_{\mathbb{R}^2 \setminus \{0\}}(x, M)$ only above the real axis and by the similarity it is sufficient to consider only the case $x = 1$. By Theorem 3.5 we need to consider $M \in (1, \pi)$.

Let us denote by $l(s)$ a tangent line of the upper half of $\partial D_{\mathbb{R}^2 \setminus \{0\}}(1, M)$. The angle between $l(s)$ and the real axis is described by the function $\alpha(s)$ defined in the proof of Theorem 3.5. The function $\alpha'(s)$ has the same sign than $d(s)$ in (3.7) and therefore $\alpha'(s)$ is positive on $(-M, -1)$ and negative on $(-1, M)$. We need to find M such that $l(s)$, $s \in [-M, M]$, goes through point 1 exactly once. In other words, we need to find M such that $l(-1)$ goes through 1.

Let us define functions $a(s)$ and $b(s)$ as in the proof of Theorem 3.5. Since $a(-1) \neq 0$ for $M \in (1, \pi)$, the tangent line $l(-1)$ goes through 1 if and only if

$$\frac{b(-1)}{a(-1)} = \frac{x_2}{x_1 - 1}, \tag{4.4}$$

where $x_1 = e^s \cos \varphi(s)$ and $x_2 = e^s \sin \varphi(s)$. Eq. (4.4) is equivalent to

$$\frac{e \cos \sqrt{M^2 - 1} + e \sqrt{M^2 - 1} \sin \sqrt{M^2 - 1} - 1}{(e - \cos \sqrt{M^2 - 1})(\sqrt{M^2 - 1} \cos \sqrt{M^2 - 1} - \sin \sqrt{M^2 - 1})} = 0,$$

which holds if and only if $M = \kappa$.

We will finally show that $M = \kappa$ is the only solution of (4.1) on $(1, \pi)$. We define function $h(x) = \cos x + x \sin x - e^{-1}$ and show that it has only one root on $(0, \sqrt{\pi^2 - 1})$. Since $h'(x) = x \cos x$, $h(0)1 - e^{-1} > 0$ and $h(\sqrt{\pi^2 - 1}) < h(11\pi/12) < 0$ the function h has only one root on $(0, \sqrt{\pi^2 - 1})$ and the assertion follows. $\quad\square$

Corollary 4.5. *The quasihyperbolic ball* $D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ *is strictly starlike with respect to x for $0 < M \leqslant \kappa$ and is not starlike with respect to x for $M > \kappa$.*

Proof. By Theorem 4.3 the claim is true for $n = 2$. Let us assume $n > 2$ and choose $x \in \mathbb{R}^n \setminus \{0\}$ and $M \in (0, \kappa]$. Let us assume, on the contrary, that there exist $y \in \partial D_{\mathbb{R}^n \setminus \{0\}}(x, M)$ and $z \in (x, y)$ such that $z \in \partial D_{\mathbb{R}^n \setminus \{0\}}(x, M)$. Now $z \in \partial D_{\mathbb{R}^n \setminus \{0\}}(x, M) \cap \operatorname{span}(0, x, y)$ and therefore $D_{\mathbb{R}^2 \setminus \{0\}}(x, M)$ is not strictly starlike with respect to x. This is a contradiction by Theorem 4.3. $\quad\square$

Remark 4.6. Let us consider the starlikeness property of the quasihyperbolic disk $D_{\mathbb{R}^2\setminus\{0\}}(x, M)$ with respect to any point $z \in D_{\mathbb{R}^2\setminus\{0\}}(x, M)$. For $M > 1$ and $z = (e^{-M} + \varepsilon)x/|x|$, where $\varepsilon > 0$, we can choose ε so small that $D_{\mathbb{R}^2\setminus\{0\}}(x, M)$ is not starlike with respect to z. On the other hand for $M < \lambda \approx 2.9648984$, where λ is a solution of

$$\cos\sqrt{p^2 - 1} + \sqrt{p^2 - 1}\sin\sqrt{p^2 - 1} = e^{-1-p}, \tag{4.7}$$

$D_{\mathbb{R}^2\setminus\{0\}}(x, M)$ is starlike with respect to $z = (e^M - \varepsilon)x/|x|$ for small enough $\varepsilon > 0$. This is also true for quasihyperbolic balls $D_{\mathbb{R}^n\setminus\{0\}}(x, M)$. Eq. (4.7) can be obtained by similar computations as in the proof of Theorem 4.3.

Remark 4.8. For $M \leqslant \pi$ we note that

$$\lim_{\varphi \to M} \frac{b(s)}{a(s)} = -\infty \quad \text{and} \quad \lim_{\varphi \to -M} \frac{b(s)}{a(s)} = \infty$$

and therefore $D_{\mathbb{R}^n\setminus\{0\}}(x, M)$ smooth. For $M > \pi$ the boundary $\partial D_{\mathbb{R}^n\setminus\{0\}}(x, M)$ has two components. The upper half of the larger boundary component is defined by (3.6) for $s \in [m, M]$, where $m = \max\{t \in (-M, M): \sin\sqrt{M^2 - t^2} = 0\}$. Therefore

$$\lim_{\varphi \to M} \frac{b(s)}{a(s)} = -\infty \quad \text{and} \quad \lim_{\varphi \to m} \frac{b(s)}{a(s)} = \frac{-m\cos\varphi(m)}{\varphi(m)\cos\varphi(m)} = -\frac{m}{\varphi(m)},$$

where $|-m/\varphi(m)| < \infty$, and $D_{\mathbb{R}^n\setminus\{0\}}(x, M)$ is not smooth at $(e^m \sin\varphi(m), 0)$. Similarly the smaller boundary component of $\partial D_{\mathbb{R}^n\setminus\{0\}}(x, M)$ is not smooth at the point where it intersects the negative real axis. Note that by (3.1) $D_{\mathbb{R}^2\setminus\{0\}}(x, M)$ is not simply connected for $M > \pi$ and is simply connected for $M \in (0, \pi]$.

Proof of Theorem 1.1. The claim is clear by Corollaries 3.4, 3.10 and 4.5. $\quad\square$

The following lemma shows a property of the Euclidean radius of a quasihyperbolic ball.

Lemma 4.9. *Let* $M \in (0, \kappa]$, $z \in \mathbb{R}^n \setminus \{0\}$ *and* $x, y \in \partial D_{\mathbb{R}^n\setminus\{0\}}(z, M)$. *Then* $\measuredangle xz0 < \measuredangle yz0$ *implies* $|x - z| < |y - z|$.

Proof. Since $M \leqslant \kappa$ the quasihyperbolic ball $D_{\mathbb{R}^n\setminus\{0\}}(z, M)$ is strictly starlike with respect to z by Corollary 4.5 and the angle $\measuredangle xz0$ determines the point x uniquely up to a rotation about the line through 0 and z. By symmetry and similarity it is sufficient to consider only the case $n = 2$ and $z = 1$. We will show that the function

$$f(s) = |x(s) - 1|^2$$

is strictly increasing on $(-M, M)$, where $x(s)$ defined by (3.6). Now

$$f(s) = |x(s)|^2 + 1 - 2|x(s)|\cos p(s) = e^{2s} + 1 - 2e^s\cos\varphi(s)$$

for $s \in [-M, M]$ and

$$f'(s) = 2e^s\left(e^s - \cos\varphi(s) - \frac{s\sin\varphi(s)}{\varphi(s)}\right).$$

If $s \in (0, M)$, then

$$e^s - \cos\varphi(s) - \frac{s\sin\varphi(s)}{\varphi(s)} \geqslant e^s - \cos\varphi(s) - s \geqslant e^s - 1 - s > 0$$

and $f'(s) > 0$.

If $s \in [-M, 0)$, then $e^s - \cos\varphi(s) - s\sin\varphi(s)/\varphi(s) > 0$ is equivalent to $e^{-t} - \cos\varphi(t) + t\sin\varphi(t)/\varphi(t) > 0$ for $t \in (0, M]$. Because $M < 3$, by elementary calculus

$$e^{-t} - \cos\varphi(t) + \frac{t\sin\varphi(t)}{\varphi(t)} \geqslant \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6}\right) - \left(1 - \frac{\varphi(t)^2}{2} + \frac{\varphi(t)^4}{24}\right) + \left(t - t\frac{\varphi(t)^2}{6}\right)$$

$$= \frac{1}{24}\left(12M^2 - M^4 - 4M^2 t + 2M^2 t^2 - t^4\right) > 0$$

and also $f'(s) > 0$. Therefore f is strictly increasing and the assertion follows. $\quad\square$

Finally we pose an open problem concerning the uniqueness of short geodesics: are quasihyperbolic geodesics with length less than π always unique?

Acknowledgments

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